Second-order theory for nonlinear composites and application to isotropic constituents

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New prescriptions are proposed for the 'reference' fields in the context of the 'second-order' nonlinear homogenization method [P. Ponte Castañeda, Second-order homogenization estimates for nonlinear composites incorporating field fluctuations: I—Theory, J. Mech. Phys. Solids 50 (2002) 737–757], and are used to generate estimates for the effective behavior and first moments of the local fields in nonlinear composites. The new prescriptions yield simple, analytical expressions not only for the effective potentials, but also for the macroscopic stress-strain relation, as well as for the phase averages of the strain and stress fields. For illustrative purposes, 'second-order' estimates of the Hashin–Shtrikman type are provided for two-phase, transversely-isotropic composites with power-law phases, and are compared with exact results available for power-law, multiple-rank, sequential laminates. The agreement is found to be quite good for all ranges of nonlinearities and inclusion concentrations considered.

Méthode du « second ordre » pour les composites non linéaires et applications aux matériaux isotropes. On utilise la méthode d'homogénéisation non linéaire proposée par Ponte Castañeda [P. Ponte Castañeda, Second-order homogenization estimates for nonlinear composites incorporating field fluctuations: I—Theory, J. Mech. Phys. Solids 50 (2002) 737–757], dite du « second ordre », pour générer des estimations pour le comportement effectif et les premiers moments des champs locaux dans des composites non-linéaires. Des expressions analytiques simples sont données non seulement pour les potentiels effectifs mais également pour la relation contrainte-déformation macroscopique, aussi bien que pour les moyennes par phase des champs de contrainte et de déformation. Des estimations du type de Hashin–Shtrikman sont données pour des composites biphasés, isotropes avec des phases suivant une loi puissance, et sont comparées aux résultats exacts disponibles pour les matériaux laminés. L'accord s'avère bon pour toutes les valeurs de la non-linéarité et de concentration d'inclusion considérées.

Keywords: Computational solid mechanics; Homogenization; Nonlinear behavior; Ideal plasticity

Mots-clés : Mécanique des solides numérique ; Homogénéisation ; Comportement non linéaire ; Plasticité parfaite

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1. Introduction

This work is concerned with the problem of estimating the effective (or homogenized) behavior of *nonlinear* composites [1]. The so-called 'second-order' homogenization method [2] is used to generate estimates for the effective potentials of nonlinear composites with isotropic constituents. In addition, simple expressions of practical importance are provided for the resulting effective stress-strain relations and the first moments of the local fields in each constituent. The accuracy of these estimates is then assessed by comparing them with exact results available for a special class of nonlinear composites.

We consider composite materials made of N different homogeneous constituents, or *phases*, which are assumed to be *randomly* distributed in a specimen occupying a volume Ω, at a length scale that is much smaller than the size of Ω and the scale of variation of the loading conditions. The constitutive behavior of each phase is characterized by *isotropic* incompressible strain potentials $w^{(r)}$ $(r = 1, \ldots, N)$ such that

$$\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} w^{(r)}(\boldsymbol{\varepsilon}), \quad w^{(r)}(\boldsymbol{\varepsilon}) = \phi^{(r)}(\varepsilon_e) \tag{1}$$

where the von Mises equivalent strain is defined in terms of the deviatoric strain tensor by

$$\varepsilon_e = \sqrt{(2/3)\boldsymbol{\varepsilon}_d \cdot \boldsymbol{\varepsilon}_d}$$

$\partial_{\boldsymbol{\varepsilon}}$ denotes differentiation with respect to $\boldsymbol{\varepsilon}$, and $\operatorname{tr} \boldsymbol{\varepsilon} = 0$. This constitutive relation can be used within the context of the deformation theory of plasticity, where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represent the infinitesimal stress and strain, respectively. Relation (1) applies equally well to viscoplastic materials, in which case $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represent the Cauchy stress and Eulerian strain rate, respectively.

We are concerned with the problem of finding the effective behavior of the composite, which is defined as the relation between the average stress $\overline{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma} \rangle$ and the average strain $\overline{\boldsymbol{\varepsilon}} = \langle \boldsymbol{\varepsilon} \rangle$, and can also be characterized [1] by an effective strain potential $\widetilde{W}$, such that

$$\overline{\boldsymbol{\sigma}} = \partial_{\overline{\boldsymbol{\varepsilon}}} \widetilde{W}(\overline{\boldsymbol{\varepsilon}}), \quad \widetilde{W}(\overline{\boldsymbol{\varepsilon}}) = \min_{\boldsymbol{\varepsilon} \in \mathcal{K}(\overline{\boldsymbol{\varepsilon}})} \sum_{r=1}^{N} c^{(r)} \big\langle w^{(r)}(\boldsymbol{\varepsilon}) \big\rangle^{(r)} \tag{2}$$

Here, $\langle \cdot \rangle$ and $\langle \cdot \rangle^{(r)}$ denote the volume averages over the composite (Ω) and over phase r $(\Omega^{(r)})$, respectively, $c^{(r)}$ is the volume fraction of phase r, and $\mathcal{K}(\overline{\boldsymbol{\varepsilon}}) = \{\boldsymbol{\varepsilon} \mid$ there is $\mathbf{u}$ such that $\boldsymbol{\varepsilon} = (1/2)[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$ in Ω, $\mathbf{u} = \overline{\boldsymbol{\varepsilon}}\mathbf{x}$ on $\partial\Omega\}$ is the set of kinematically admissible strain fields.

A relation completely equivalent to $(2)_1$ results from a dual formulation which makes use of stress potentials $u^{(r)}$, such that locally $\boldsymbol{\varepsilon} = \partial_{\boldsymbol{\sigma}} u^{(r)}(\boldsymbol{\sigma})$. For materials characterized by (1), the stress potentials are of the form $u^{(r)}(\boldsymbol{\sigma}) = \psi^{(r)}(\sigma_e)$, where the von Mises equivalent stress is $\sigma_e = \sqrt{(3/2)\boldsymbol{\sigma}_d \cdot \boldsymbol{\sigma}_d}$. The effective behavior is then given in terms of an effective stress potential $\widetilde{U}$, such that $\overline{\boldsymbol{\varepsilon}} = \partial_{\overline{\boldsymbol{\sigma}}} \widetilde{U}(\overline{\boldsymbol{\sigma}})$ (see for instance [1]). Thus, the problem of estimating the effective behavior of the composite reduces to that of estimating the effective potentials $\widetilde{W}$ or $\widetilde{U}$.

2. Second-order homogenization method

A fairly general nonlinear homogenization method has been introduced by Ponte Castañeda [2], which delivers estimates for the effective potentials $\widetilde{W}$ and $\widetilde{U}$ which are exact to second order in the heterogeneity contrast. The central idea behind this method is the introduction of a *linear comparison composite* (LCC), with the same microstructure as the nonlinear composite, and with phase potentials $w_L^{(r)}$ given by second-order Taylor-type expansions of the nonlinear potentials $w^{(r)}$,

$$w_L^{(r)}\big(\boldsymbol{\varepsilon}; \check{\boldsymbol{\varepsilon}}^{(r)}, \mathbf{L}_0^{(r)}\big) = w^{(r)}\big(\check{\boldsymbol{\varepsilon}}^{(r)}\big) + \partial_{\boldsymbol{\varepsilon}} w^{(r)}\big(\check{\boldsymbol{\varepsilon}}^{(r)}\big) \cdot \big(\boldsymbol{\varepsilon} - \check{\boldsymbol{\varepsilon}}^{(r)}\big) + \frac{1}{2}\big(\boldsymbol{\varepsilon} - \check{\boldsymbol{\varepsilon}}^{(r)}\big) \cdot \mathbf{L}_0^{(r)}\big(\boldsymbol{\varepsilon} - \check{\boldsymbol{\varepsilon}}^{(r)}\big) \tag{3}$$

where the $\check{\boldsymbol{\varepsilon}}^{(r)}$ are reference strains, and $\mathbf{L}_0^{(r)}$ are symmetric, fourth-order tensors (of moduli), uniform in each phase. For isotropic, incompressible phases characterized by potentials of the form (1), the (anisotropic) tensors $\mathbf{L}_0^{(r)}$ are assumed to be of the form:

$$\mathbf{L}_0^{(r)} = 2\lambda_0^{(r)}\mathbf{E}^{(r)} + 2\mu_0^{(r)}\mathbf{F}^{(r)}, \quad \text{with } \mathbf{E}^{(r)} = \frac{2}{3}\frac{\check{\boldsymbol{\varepsilon}}_d^{(r)}}{\check{\varepsilon}_e^{(r)}} \otimes \frac{\check{\boldsymbol{\varepsilon}}_d^{(r)}}{\check{\varepsilon}_e^{(r)}},\ \mathbf{F}^{(r)} = \mathbf{K} - \mathbf{E}^{(r)} \tag{4}$$

where $\mathbf{K}$ denotes the standard, fourth-order, isotropic, shear projection tensor, and the subscript d denotes the deviatoric part. Then, the second-order method delivers the following estimate for the effective strain potential of a general N-phase composite with isotropic constituents:

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \underset{\lambda_0^{(s)},\,\mu_0^{(s)}}{\operatorname{stat}} \left\{ \widetilde{W}_L\big(\bar{\boldsymbol{\varepsilon}}; \check{\boldsymbol{\varepsilon}}^{(s)}, \mathbf{L}_0^{(s)}\big) + \sum_{r=1}^{N} c^{(r)} V^{(r)}\big(\check{\boldsymbol{\varepsilon}}^{(r)}, \mathbf{L}_0^{(r)}\big) \right\} \tag{5}$$

where the stationary operation consists in setting the partial derivative of the argument with respect to the variable equal to zero. In this expression, $\widetilde{W}_L$ is the effective potential of the above mentioned LCC, and the functions $V^{(r)}$ are defined as

$$V^{(r)}\big(\check{\boldsymbol{\varepsilon}}^{(r)}, \mathbf{L}_0^{(r)}\big) = \underset{\hat{\boldsymbol{\varepsilon}}^{(r)}}{\operatorname{stat}} \big\{ w^{(r)}\big(\hat{\boldsymbol{\varepsilon}}^{(r)}\big) - w_L^{(r)}\big(\hat{\boldsymbol{\varepsilon}}^{(r)}; \check{\boldsymbol{\varepsilon}}^{(r)}, \mathbf{L}_0^{(r)}\big) \big\} \tag{6}$$

where the $\hat{\boldsymbol{\varepsilon}}^{(r)}$ are uniform (strain) tensors in each phase. Making use of the symmetry of the tensors $\mathbf{L}_0^{(r)}$, we can define two components of the tensors $\hat{\boldsymbol{\varepsilon}}^{(r)}$ that are 'parallel' and 'perpendicular' to the corresponding reference tensor $\check{\boldsymbol{\varepsilon}}^{(r)}$, respectively, $\hat{\varepsilon}_{\parallel}^{(r)} = \sqrt{(2/3)\hat{\boldsymbol{\varepsilon}}^{(r)} \cdot \mathbf{E}^{(r)}\hat{\boldsymbol{\varepsilon}}^{(r)}}$ and $\hat{\varepsilon}_{\perp}^{(r)} = \sqrt{(2/3)\hat{\boldsymbol{\varepsilon}}^{(r)} \cdot \mathbf{F}^{(r)}\hat{\boldsymbol{\varepsilon}}^{(r)}}$. The stationary operation in (6) then leads to the following two conditions in each phase:

$$3\lambda_0^{(r)}\big(\hat{\varepsilon}_{\parallel}^{(r)} - \check{\varepsilon}_e^{(r)}\big) = \phi^{(r)'}\big(\hat{\varepsilon}_e^{(r)}\big)\frac{\hat{\varepsilon}_{\parallel}^{(r)}}{\hat{\varepsilon}_e^{(r)}} - \phi^{(r)'}\big(\check{\varepsilon}_e^{(r)}\big), \quad 3\mu_0^{(r)} = \frac{\phi^{(r)'}(\hat{\varepsilon}_e^{(r)})}{\hat{\varepsilon}_e^{(r)}} \tag{7}$$

Relations (7) state that the tensors $\mathbf{L}_0^{(r)}$ correspond to 'generalized secant' approximations to the nonlinear stress-strain relations. In turn, the stationary operations in (5) lead to the conditions

$$\hat{\varepsilon}_{\parallel}^{(r)} - \check{\varepsilon}_e^{(r)} = \pm\sqrt{\frac{2}{3}\frac{1}{c^{(r)}}\frac{\partial \widetilde{W}_L}{\partial \lambda_0^{(r)}}} = \pm\sqrt{\frac{2}{3}\big\langle\big(\boldsymbol{\varepsilon}_L - \check{\boldsymbol{\varepsilon}}^{(r)}\big) \cdot \mathbf{E}^{(r)}\big(\boldsymbol{\varepsilon}_L - \check{\boldsymbol{\varepsilon}}^{(r)}\big)\big\rangle^{(r)}} \tag{8}$$

$$\hat{\varepsilon}_{\perp}^{(r)} = \pm\sqrt{\frac{2}{3}\frac{1}{c^{(r)}}\frac{\partial \widetilde{W}_L}{\partial \mu_0^{(r)}}} = \pm\sqrt{\frac{2}{3}\big\langle\boldsymbol{\varepsilon}_L \cdot \mathbf{F}^{(r)}\boldsymbol{\varepsilon}_L\big\rangle^{(r)}} \tag{9}$$

where $\boldsymbol{\varepsilon}_L$ denotes the strain field in the LCC. The sign of the square roots in these expressions should be taken to be positive if $\check{\varepsilon}_e^{(r)} \leqslant \bar{\varepsilon}_{Le}^{(r)}$, and negative otherwise, for consistency of (10) with the case of uniform fields (e.g., laminate, homogeneous limit). It is worth noting that the right-hand sides of these relations depend on the (intraphase) field fluctuations in the LCC, through certain projections of the phase covariance tensors $\mathbf{C}_{\boldsymbol{\varepsilon}_L}^{(r)} = \langle\boldsymbol{\varepsilon}_L \otimes \boldsymbol{\varepsilon}_L\rangle^{(r)} - \bar{\boldsymbol{\varepsilon}}_L^{(r)} \otimes \bar{\boldsymbol{\varepsilon}}_L^{(r)}$.

Then, using the fact that (5) is stationary with respect to the moduli $\lambda_0^{(r)}$ and $\mu_0^{(r)}$, we can rewrite the estimate (5) in the simpler form

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \sum_{r=1}^{N} c^{(r)}\big[w^{(r)}\big(\hat{\boldsymbol{\varepsilon}}^{(r)}\big) - \partial_{\boldsymbol{\varepsilon}} w^{(r)}\big(\check{\boldsymbol{\varepsilon}}^{(r)}\big) \cdot \big(\hat{\boldsymbol{\varepsilon}}^{(r)} - \bar{\boldsymbol{\varepsilon}}_L^{(r)}\big)\big] \tag{10}$$

where the $\bar{\boldsymbol{\varepsilon}}_L^{(r)} = \langle\boldsymbol{\varepsilon}_L\rangle^{(r)}$ are the phase averages of the strain in the LCC. Relations (7)–(9) determine the variables $\hat{\boldsymbol{\varepsilon}}^{(r)}$ and $\mathbf{L}_0^{(r)}$, for any choice of the reference tensors $\check{\boldsymbol{\varepsilon}}^{(r)}$, which remain to be specified. Unfortunately, enforcing stationarity of (5) with respect to the tensors $\check{\boldsymbol{\varepsilon}}^{(r)}$, as suggested in [2], leads to conditions that cannot be satisfied together with (7)–(9), in general. Motivated by the findings of Idiart and Ponte Castañeda [3], we propose here the following prescription:

$$\check{\boldsymbol{\varepsilon}}^{(r)} = \bar{\boldsymbol{\varepsilon}}_d, \quad \text{for all } r \tag{11}$$

where the subscript d has been used to denote deviatoric part. In addition to giving sensible results in the case of isotropic composites, as will be seen in the next section, this prescription has the advantage of simplicity. In fact, with this choice of reference tensors, the 'second-order' estimates for the effective behavior, which follow from differentiation of (10), can be shown [4] to be given by

$$\overline{\boldsymbol{\sigma}} = \partial_{\bar{\boldsymbol{\varepsilon}}} \widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \overline{\boldsymbol{\sigma}}_L + \sum_{r=1}^{N} c^{(r)} \boldsymbol{\rho}^{(r)} \tag{12}$$

where $\overline{\boldsymbol{\sigma}}_L$ denotes the macroscopic stress in the LCC, and the (incompressible) tensors $\boldsymbol{\rho}^{(r)}$ are

$$\begin{aligned} \boldsymbol{\rho}^{(r)} = {} & \big(\mathbf{L}_0^{(r)} - \mathbf{L}_t^{(r)}\big)\big(\hat{\boldsymbol{\varepsilon}}^{(r)} - \bar{\boldsymbol{\varepsilon}}_L^{(r)}\big) + \frac{4}{3}\frac{\lambda_0^{(r)} - \mu_0^{(r)}}{\bar{\varepsilon}_e^2} \\ & \times \Big[\big\langle (\boldsymbol{\varepsilon}_{L_d} - \bar{\boldsymbol{\varepsilon}}_d) \otimes (\boldsymbol{\varepsilon}_{L_d} - \bar{\boldsymbol{\varepsilon}}_d)\big\rangle^{(r)} - \big(\hat{\boldsymbol{\varepsilon}}_d^{(r)} - \bar{\boldsymbol{\varepsilon}}_d\big) \otimes \big(\hat{\boldsymbol{\varepsilon}}_d^{(r)} - \bar{\boldsymbol{\varepsilon}}_d\big)\Big]\bar{\boldsymbol{\varepsilon}}_d \end{aligned} \tag{13}$$

In this last expression, $\mathbf{L}_t^{(r)} = \partial^2_{\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}} w^{(r)}(\bar{\boldsymbol{\varepsilon}})$ are the tangent moduli of the phases, evaluated at $\bar{\boldsymbol{\varepsilon}}$. Thus, while the LCC is subjected to the same macroscopic strain as the nonlinear composite, the macroscopic stress exhibits a 'correction' term due the fact that the estimates (5) are not stationary with respect to the variables $\check{\boldsymbol{\varepsilon}}^{(r)}$. In addition, corresponding estimates for the phase averages of the local fields are given by

$$\bar{\boldsymbol{\varepsilon}}^{(r)} = \bar{\boldsymbol{\varepsilon}}_L^{(r)}, \qquad \overline{\boldsymbol{\sigma}}^{(r)} = \overline{\boldsymbol{\sigma}}_L^{(r)} + \boldsymbol{\rho}^{(r)} \tag{14}$$

where the subscript L denotes quantities in the LCC. These results follow from the rigorous procedure described in [4], making use of suitably perturbed phase potentials to extract estimates for the pertinent phase averages via differentiation. Again, while the estimates for $\bar{\boldsymbol{\varepsilon}}^{(r)}$ coincide with those in the associated LCC, the estimates for $\overline{\boldsymbol{\sigma}}^{(r)}$ do not. However, it is emphasized that these estimates are entirely consistent with (12), in the sense that they satisfy the relations $\overline{\boldsymbol{\sigma}} = \sum_{r=1}^{N} c^{(r)} \overline{\boldsymbol{\sigma}}^{(r)}$.

Completely analogous expressions may be developed [2] starting from the dual formulation in terms of the stress potentials $u^{(r)}$. This formulation involves a LCC with phase potentials $u_L^{(r)}$, given by second-order Taylor approximations to $u^{(r)}$ of the same form as (3), in terms of reference stresses $\check{\boldsymbol{\sigma}}^{(r)}$ and compliance tensors $\mathbf{M}_0^{(r)}$, and generates the following estimate for the effective stress potential

$$\widetilde{U}(\overline{\boldsymbol{\sigma}}) = \sum_{r=1}^{N} c^{(r)}\big[u^{(r)}\big(\hat{\boldsymbol{\sigma}}^{(r)}\big) - \partial_{\boldsymbol{\sigma}} u^{(r)}\big(\check{\boldsymbol{\sigma}}^{(r)}\big) \cdot \big(\hat{\boldsymbol{\sigma}}^{(r)} - \overline{\boldsymbol{\sigma}}_L^{(r)}\big)\big] \tag{15}$$

where $\overline{\boldsymbol{\sigma}}_L^{(r)} = \langle \boldsymbol{\sigma}_L \rangle^{(r)}$ are the phase averages of the stress in the associated LCC, and the tensors $\hat{\boldsymbol{\sigma}}^{(r)}$ and $\mathbf{M}_0^{(r)}$ depend on the reference tensors $\check{\boldsymbol{\sigma}}^{(r)}$ and the second moments of the stress fluctuations (in the LCC) through equations analogous to (7)–(9). (The same sign convention should also be used for the equivalents of relations (8) and (9).) Again, the reference stresses $\check{\boldsymbol{\sigma}}^{(r)}$ need to be specified, and prescription [3] is proposed:

$$\check{\boldsymbol{\sigma}}^{(r)} = \overline{\boldsymbol{\sigma}}_d, \quad \text{for all } r \tag{16}$$

which is the counterpart of (11) in this context. Note that, in spite of the symmetry of the prescriptions (11) and (16), the corresponding homogenized estimates are not expected to give identical results.

Finally, it is worth emphasizing that nonlinear homogenization methods based on LCCs involve *two* different levels of approximation [5,6]. The first level consists in the generation of the LCC by linearizing the behavior of each phase in the nonlinear composite, while the second level consists in the computation of the effective behavior of the LCC, which in general cannot be done exactly and therefore requires the use of suitable *linear* homogenization estimates.

3. Two-phase, power-law composites

We consider here two-phase, fiber composites with *random* microstructures exhibiting overall transversely isotropic symmetry that are loaded in transverse shear. The phases are characterized by isotropic, incompressible, power-law potentials

$$\phi^{(r)}(\varepsilon_e) = \frac{\varepsilon_0 \sigma_0^{(r)}}{1+m}\left(\frac{\varepsilon_e}{\varepsilon_0}\right)^{1+m}, \qquad \psi^{(r)}(\sigma_e) = \frac{\varepsilon_0 \sigma_0^{(r)}}{1+n}\left(\frac{\sigma_e}{\sigma_0^{(r)}}\right)^{1+n} \tag{17}$$

where $\sigma_0^{(r)}$ is the flow stress of phase r, $m = 1/n$ is the strain-rate sensitivity, such that $0 \leqslant m \leqslant 1$, ε_0 is a reference strain rate. Note that $m = 1$ and $m = 0$ correspond to linear and rigid-ideally plastic behaviors, respectively. For simplicity, both phases are assumed to have the same exponent m and reference strain ε_0. It then follows that, under isochoric *plane-strain* conditions, the effective potentials can be written as

$$\widetilde{W}(\bar{\boldsymbol{\varepsilon}}) = \frac{\varepsilon_0 \widetilde{\sigma}_0}{1+m}\left(\frac{\bar{\varepsilon}_e}{\varepsilon_0}\right)^{1+m}, \qquad \widetilde{U}(\bar{\boldsymbol{\sigma}}) = \frac{\varepsilon_0 \widetilde{\sigma}_0}{1+n}\left(\frac{\bar{\sigma}_e}{\widetilde{\sigma}_0}\right)^{1+n} \tag{18}$$

where $\widetilde{\sigma}_0$ is the *effective flow stress* of the composite, and $\bar{\varepsilon}_e$ and $\bar{\sigma}_e$ are the equivalent macroscopic strain and stress. The effective behavior is thus completely characterized by $\widetilde{\sigma}_0$.

The extreme cases of infinite contrast are of particular interest, and are given in Figs. 1 and 2 corresponding respectively to porous and rigidly reinforced composites. The matrix phase, labeled 1, has flow stress $\sigma_0^{(1)} = \sigma_0$, and the randomly distributed voids or rigid fibers (phase 2) have circular cross-section, and volume fraction $c^{(2)} = c$. 'Second-order' estimates are generated by making use of the Hashin–Shtrikman (HS) estimates of Willis [7] to determine the effective behavior of the associated LCC. The HS estimates are known to be appropriate for (linear) particulate media at low to moderate concentrations, and are exact to second-order in the heterogeneity contrast. Both the strain (W) and the stress (U) versions of the 'second-order' (SO) estimates of the previous section are provided, making use of the simple prescriptions (11) and (16). In addition, the earlier 'variational' (VAR) bounds of Ponte Castañeda [8], also of the HS type, are included for comparison purposes.

It should be mentioned at this stage that for the cases considered here, i.e., incompressible, transversely isotropic composites under plane-strain loadings, expression (13) for the tensor $\boldsymbol{\rho}^{(1)}$ in the matrix phase simplifies to

$$\boldsymbol{\rho}^{(1)} = (2/\bar{\varepsilon}_e)\big(\lambda_0^{(1)} - \lambda_t^{(1)}\big)\big(\hat{\varepsilon}_{\parallel}^{(1)} - \bar{\varepsilon}_{Le}^{(1)}\big)\bar{\boldsymbol{\varepsilon}}_d \tag{19}$$

where $2\lambda_t^{(1)} = \mathbf{E}^{(1)} \cdot \mathbf{L}_t^{(1)} = (2/3)\phi^{(1)\prime\prime}(\bar{\boldsymbol{\varepsilon}})$, while in the inclusion phase, $\boldsymbol{\rho}^{(2)} = \mathbf{0}$. Because the tensor $\boldsymbol{\rho}^{(1)}$ is 'aligned' with the macroscopic strain $\bar{\boldsymbol{\varepsilon}}$, so is then the macroscopic stress given by (12), as it should be for transversely isotropic composites under in-plane loading.

In order to assess the accuracy of these estimates, *exact* results have been generated for power-law composites with a special class of transversely isotropic, 'particulate' microstructures known as multiple-rank sequential laminates (LAM), following the procedure described by deBotton and Hariton [9]. The rank of these laminates has been set sufficiently high so that the effective behavior exhibits transverse isotropy up to a certain tolerance. The interest in composites with this type of microstructures is that, in the *linear* case, their effective behavior is given *exactly* by the HS estimates, for *any* modulus tensors of the phases. For this reason, LCC-based homogenization estimates of the HS type are particularly appropriate for nonlinear composites with this class of microstructures, since the effective behavior of the LCC is being computed exactly in that case, and therefore there is only one level of approximation involved, namely, in the linearization. In addition, since this holds for any linearization, exact results for nonlinear composites with this class of microstructures provide an ideal test bed to compare different LCC-based homogenization methods. A peculiarity of these nonlinear composites is that, by construction, the fields in the inclusion phase are uniform, independently of the behavior of the phases.

Figs. 1 and 2 present results for the effective flow stress $\widetilde{\sigma}_0$ as functions of the strain-rate sensitivity m and concentration c, for both the porous and rigidly reinforced composites. They also present results for the averages of the strain $\bar{\varepsilon}_e^{(2)}$ and stress $\bar{\sigma}_e^{(2)}$ over the inclusion phase for the porous and rigidly reinforced composites, respectively. (Note that $\bar{\sigma}_e^{(2)} = 0$ in the pores, and $\bar{\varepsilon}_e^{(2)} = 0$ for rigid particles.) The main observations from these figures are: (i) The agreement of the SO estimates—both W and U versions—with the LAM exact results is quite good, and certainly much

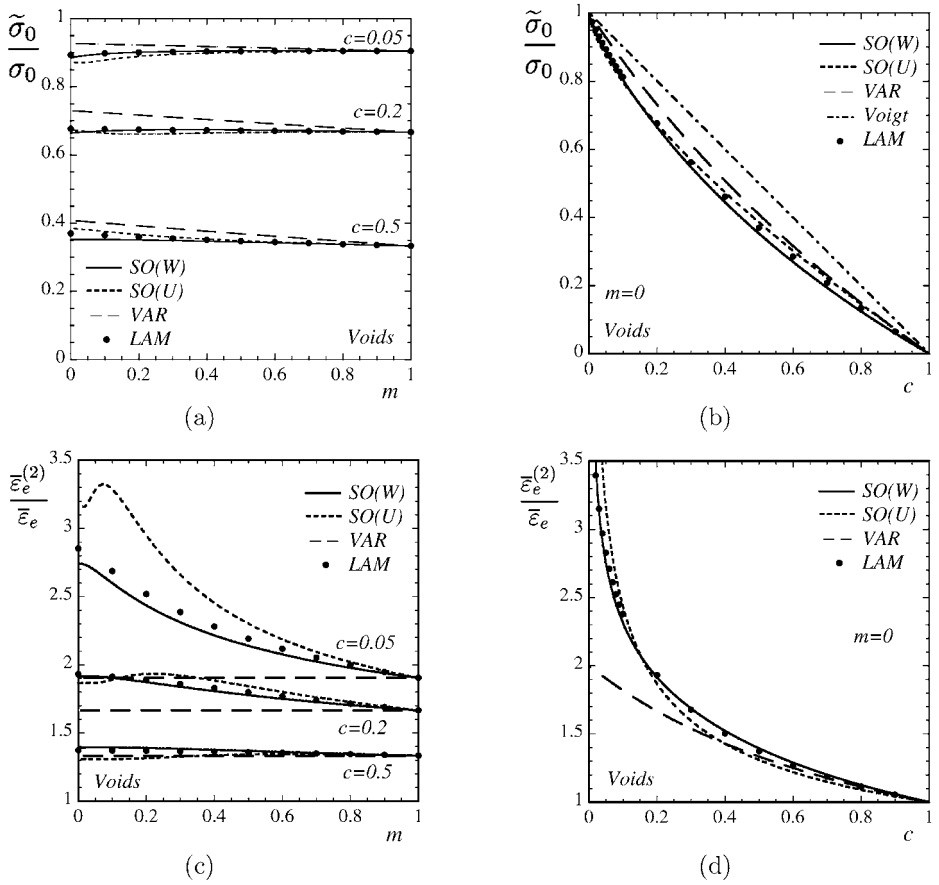

Fig. 1. Estimates and exact results for power-law porous materials subject to in-plane shear. Effective flow stress $\widetilde{\sigma}_0$, normalized by the flow stress of the matrix σ_0, (a) as a function of the power m, for several porosities c, and (b) as a function of the porosity c, in the case of an ideally-plastic matrix. (c) and (d) Corresponding equivalent average strains in the porous phase $\overline{\varepsilon}_e^{(2)}$, normalized by the equivalent macroscopic strain $\overline{\varepsilon}_e$.

better than the *VAR* estimates. (*ii*) Globally speaking, the two versions of the *SO* estimates perform equally well (sometimes the W version is better, sometimes, the U version is better). (*iii*) While the improvement in the predictions for the effective behavior of the *SO* estimates over the *VAR* estimates is relatively modest, the corresponding improvement in the phase averages of the fields in the inclusion phase can be quite significant. This is especially the case for small concentrations of pores or rigid fibers, where huge differences in the predictions are observed. In particular, for the case of a porous materials with an ideally plastic matrix ($m = 0$) (see Fig. 1d), the *SO* predictions and *LAM* results for the average strain in the pores blow up as the porosity c tends to zero, while the corresponding *VAR* estimates remain finite. (In fact, the *SO* estimates behave as $\overline{\varepsilon}_e^{(2)}/\overline{\varepsilon}_e \sim c^{-1/3}$ as $c \to 0$.)

4. Concluding remarks

It has been shown that the use of the macroscopic strain and stress as references in the context of the 'second-order' homogenization method leads to simple and accurate estimates for the effective response and field averages in the phases of nonlinear composites, even at large heterogeneity contrast and nonlinearity. One advantage of the new prescription is that the phase averages of both the strain and stress fields can be computed explicitly using only one version (W, or U) of the method. This is in contrast with the methodology proposed recently by Idiart et al. [6], which requires the computation of both the W and U versions of the second-order estimate (and closing the gap between them) to be able to generate consistent estimates for the phase averages of the stress and strain fields. Although comparisons between these two approaches were not shown here, both approaches give similar predictions, but the new approach is much simpler from a computational point of view.

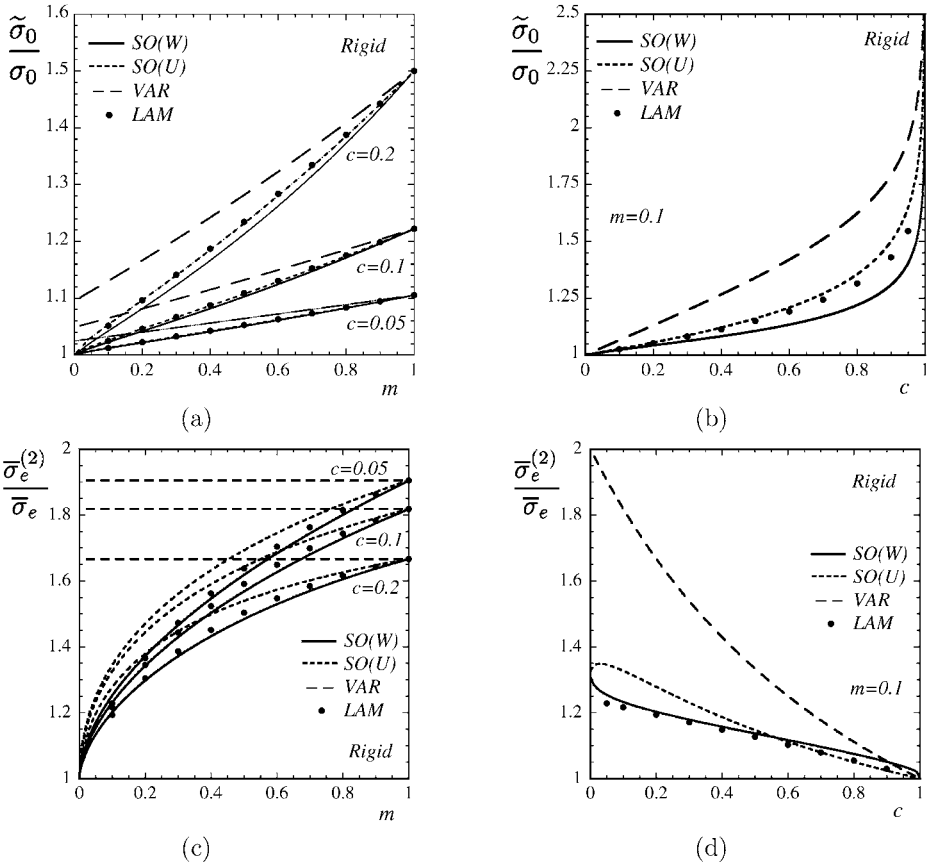

Fig. 2. Estimates and exact results for power-law rigidly-reinforced materials subject to in-plane shear. Effective flow stress $\widetilde{\sigma}_0$, normalized by the flow stress of the matrix σ_0, (a) as a function of the power m, for several fiber concentrations c, and (b) as a function of the fiber concentration c, for a power $m = 0.1$. (c) and (d) Corresponding equivalent average stresses in the rigid phase $\overline{\sigma}_e^{(2)}$, normalized by the equivalent macroscopic stress $\overline{\sigma}_e$.

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