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# Planar crack identification for the transient heat equation

A. BEN ABDA\* and H.D. BUI†

**Abstract** — We consider the inverse problem of crack determination related to the non-destructive thermal testing of materials process. Using arbitrary transient heat flux applied to the external boundary and measuring the induced planar cracks.

## 1. INTRODUCTION

The problem under consideration arises in non-destructive testing of materials. Up to our knowledge, theoretical results on identifiability of cracks are seldom and focus on the Laplace equation (i. e. the steady state thermal equation or electrical equation). The uniqueness (identifiability) result for buried crack has been first given by Friedman and Vogelius [11]. They proved that two appropriate current fluxes, together with the corresponding voltages, are necessary and suffice to uniquely determine a single crack. Bryan and Vogelius [10], followed by Alessandrini and Diaz Valenzuela [1] examined the multiple crack problem always in 2D situation. In 3D situation, a complete result has been given by Alessandrini and DiBeneditto [2]. When complete data is available on the boundary Andrieux and Ben Abda introduced in [4,5] the reciprocity gap concept which turned out to be a relevant tool for recovering 3D planar cracks in the case of Laplace equation. An analogous result has been proven by Andrieux, Ben Abda and Bui [6] for the electrostatic system. Recently Bui, Constantinescu and Maigre [9] proved an identifiability result for the problem of planar crack recovery in 3D transient acoustic problem. For given Neumann boundary condition we shall suppose that Dirichlet boundary condition is measured for finite time (or all times). The crack will be considered fixed over the time and it will be simultaneously characterized by a free boundary condition and a jump of the thermal at its passage. The normal to the host plane

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inversion process has been given by Andrieux in [3], and is recalled here, the plane identification is then achieved. The last section is devoted to the complete identification process.

## 2. THE RECIPROCITY GAP FUNCTIONAL IN THE TRANSIENT CASE

Let us consider the direct modelling problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 & \text{in } (\Omega \setminus \sigma) \times [0, T] \\ u(x, 0) &= 0 & \text{in } (\Omega \setminus \sigma) \\ u &= u_n & \text{on } \partial\Omega \\ \partial_n u &= 0 & \text{on } \sigma. \end{aligned}$$

The overdetermined condition is given by:

$$\frac{\partial u}{\partial n}(x, 0) = \Phi_n \quad \text{on } \sigma.$$

Let  $w$  be a scalar field satisfying the following backward heat conduction problem

$$\begin{aligned} \frac{\partial w}{\partial t} + \Delta w &= 0 & \text{in } \Omega \times [0, T] \\ w(x, T) &= 0 & \text{in } \Omega. \end{aligned}$$

The reciprocity gap function is therefore defined by :

$$RG(w) := \int_0^T \int_{\partial\Omega} \left( \Phi_m w - u_m \frac{\partial w}{\partial n} \right). \quad (1)$$

The following lemma express  $RG(w)$  in terms of an integral involving the unknown geometry  $\sigma$ .

**Lemma 2.1.**

$$RG(w) := \int_0^T \int_{\sigma} [u] \frac{\partial w}{\partial n}. \quad (2)$$

The proof is achieved by Green's formula.

## 3. THE HOST PLANE DETERMINATION

As for the steady state heat conduction control, the host plane determination is explicit, nevertheless there is no inversion formulae in that case.

Let us recall that in the laplacian situation the inversion formulae rely on the assumption that the main value of the solution's jump on the crack does not vanish.

In the transient case the assumption on the jump is more severe. In all this section, we suppose, that the heat flux  $\Phi$  induces a solution  $u$  such that  $(A_1)$   $[u] > 0$  on the crack. For the determination of the normal of the plane we refer to [3]. This result is recalled in the following proposition:

**Proposition 3.1.** *Under the assumption  $(A_1)$ , the normal  $\vec{N}$  is given by:*

$$\vec{N} = \text{Arg} \min_{|\vec{n}|=1} \left[ \max_{|\vec{m}|=1, \vec{n} \cdot \vec{m}=0} F(\vec{n} \wedge \vec{m}) \right]$$

where  $F(n) = RG(v_{\vec{n}})$  and  $v_{\vec{n}}(x, t) = 1 - \text{erf}(\vec{x} \cdot \vec{n} / (2\sqrt{T-t}))$  where erf is the error function.

Once the normal to the host plane is determined, let us operate a frame change such that the  $Ox_3$  axes is directed by  $\vec{n}$ . In the new framework one has  $\pi : x_3 - C = 0$ .

**Proposition 3.2.** *Consider the auxiliary scalar field*

$$w_\beta(x, t; T) = (1/\sqrt{4\pi(T-t)}) \exp(-(x_3 - \beta)^2 / (4(T-t))).$$

Denote by  $f(\beta) := RG(w_\beta)$ . Assuming (A),  $C$  is a simple zero of  $f$ .

#### 4. COMPLETE IDENTIFICATION

As in the steady state heat conduction problem, we establish the next lemma relating the crack to the support of the jump of the heat solution.

**Lemma 4.1.**

$$\text{supp} [u_\sigma] = \sigma.$$

By  $\text{supp} [u_\sigma]$  we mean the spatial support of  $[u_\sigma]$  defined as follows

$$\text{supp} [u_\sigma] = \overline{\{x \in \mathcal{R}^2 / \exists t \in [0, \infty[, [u(x, t)] \neq 0\}}$$

**Proof.** 1. Obviously  $\text{supp} [u_\sigma] \subset \sigma$ . 2. Assume that  $\sigma \not\subset \text{supp} [u_\sigma]$ . Let  $x_0 \in \sigma \setminus \partial\sigma$  be given. Then, there is an open submanifold  $\sigma_0 \subset \sigma \setminus \partial\sigma$  and an open ball  $B$  such that  $\partial(B \cap \sigma) = \sigma_0$  and  $\partial B^\pm = \sigma_0^\pm$ ,  $B = B^+ \cup B^- \cup \sigma_0$ .  $u|_{B^\pm}$  are solutions of the homogeneous heat equation with homogeneous Meumann data on  $\sigma_0^\pm$ , which is assumed to be real analytic (planar or linear manifold). Therefore,  $u|_{\sigma_0}$  are real analytic functions. By the Cauchy—Kovalevskaya theorem, one deduces that  $[u]$  is real analytic on  $\sigma$  and vanishes on  $\sigma_0$ , that is  $[u] = 0$  on  $\sigma_0$  unless  $\text{supp} [u] = \sigma$ .  $\square$

Let us consider the auxiliary scalar field  $w$  :

$$w(x, t, s_1, s_2, q) = \exp(iqt) \exp(-(s_1x_1 + s_2x_2)) \exp(ix_3(s_1^2 + s_2^2 - iq)^{1/2})$$

where  $s_1, s_2$  are two real parameters and  $q$  is a complex one  $q = q_R + iq_I$ , with  $q_I = 0$ .

**Lemma 4.2.**

$$\begin{aligned} RG(w) &= \int_0^\infty \int_{\mathcal{R}^2} [[u_\sigma]] \exp(-igt) \exp(-i(s_1x_1 + s_2x_2)) \\ &= (s_1^2 + s_2^2 - iq)^{1/2} \int_0^\infty \int_{\partial\Omega_{ext}} (u^n \partial_n w - w \Phi^n). \end{aligned}$$

Denote by  $D(x, t)$  the jump of  $u_\sigma$  across  $\sigma$

$$D(x, t) = [u_\sigma]$$

and let us consider the Laplace transform of  $[u_\sigma]$

$$H(x, q) = \int_0^\infty [u_\sigma] \exp(iqt) dt.$$

**Lemma 4.3.**

$$\text{supp } H(x, q) = \text{supp } D(x, t),$$

**Proof.** Let  $x \notin \text{supp } D(x, t)$ , therefore  $[u_\sigma] = 0$  and  $H(x, q) = 0$ . Therefore  $\text{supp } D(x, t) \subset \text{supp } H$ . The inverse Laplace transform gives the other inclusion.

□

**Theorem 4.1.** *The crack of  $\sigma$  is uniquely determined by measurement  $(u_n, \Phi_n)$ .*

**Proof.** Recalling lemma 2.1, one obtains the following equation

$$\begin{aligned} \int_{\mathcal{R}^\epsilon} H(x, t) \exp(-i(s_1x_1 + s_2x_2)) \\ = (s_1x_1 + s_2x_2 - iq)^{1/2} \int_{[0, \infty]} dt \int_{\partial\Omega_{ext}} (u^n \partial_n w - w \Phi^n), \quad (3) \end{aligned}$$

as we have chosen  $q = q_R + iq_I$ ,  $q_I > 0$ , the right hand side of the previous identity is therefore a “smooth” function  $F(s)$ . Let us check that it is a Fourier transform of a distribution  $H(x_1, x_2, q)$  belonging to  $S'_{\mathcal{R}^2}$ . Since  $\Omega$  is a bounded domain,  $\Omega$  and  $\partial\Omega$  are included in a cube  $C = \{|x_1| \leq a, |x_2| \leq a, |x_3| \leq a\}$ . Consider the complex extension  $F(z)$  of the function  $F, z = (z_1, z_2) = (s_1 + iv_1, s_2 + iv_2)$ . The singularities of  $F(z)$  do not meet the real axes. For large  $|z|$ , one has the following majoration

$$\begin{aligned} |F(z)| &\leq C_0 \exp(a(|z_1| + |z_2|)) \exp(a(|z_1|^2 + |z_2|^2)^{1/2}) \\ &\leq C_0 \exp(2a(|z_1| + |z_2|)), \end{aligned}$$

where  $C_0$  is a non negative constant.

Therefore  $F(z)$  is of exponential type and the equation (3) is invertible by the Paley—Wiener theorem. Hence,  $H(x_1, x_2, q_0)$  is uniquely determined and so is  $D(x, t)$  (by Laplace inversion). □

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