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Crack propagation from a pre-existing flaw at a notch root – II: Detailed form of the stress intensity factors at the initial crack tip and conclusion.

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Abstract. This paper pursues the study of crack kinking from a pre-existing crack emanating from some notch root. It was shown in Part I that the stress intensity factors at the tip of the *small* initial crack are given by *universal* (that is, applicable in all situations, whatever the geometry of the body and the loading) formulae; they depend only on the 'stress intensity factor of the notch' (the multiplicative coefficient of the singular stress field near the apex of the notch in the absence of the crack), the length of the crack, the aperture angle of the notch and the angle between its bisecting line and the direction of the crack. Here we identify the universal functions of the two angles just mentioned which appear in these formulae, by considering the model problem of an infinite body endowed with a notch with *straight* boundaries and a *straight* crack of *unit* length. The treatment uses Muskhelishvili's complex potentials formalism combined with some conformal mapping. The solution is expressed in the form of an infinite series involving an integral operator, which is evaluated numerically. Application of Goldstein and Salganik's *principle of local symmetry* then leads to prediction of the kink angle of the crack extension. It is found that although the direction of the crack is closer to that of the bisecting line of the notch after kinking than before it, the kink angle is not large enough for the crack tip to get closer to this line after kinking, except perhaps in some special situations.

Key words: Notch, pre-existing crack, mixed mode, stress intensity factors, complex potentials, conformal mapping, crack kinking, principle of local symmetry

1. Introduction

The reader is referred to the Introduction and Fig. 1 of Part I for a general statement of the problem envisaged and notations. It was shown in Section 4 of this first part that the stress intensity factors (SIFs) $K_p(a)$ $(p = I, II)$ (where a is recalled to denote the length of the initial crack) at the tip of this crack admit the following asymptotic expression for $a \to 0$:

$$K_p(a) \equiv K_p^{(\beta)} a^\beta + o(a^\beta). \tag{1}$$

In this expression β is an exponent given by

$$\beta = \alpha - \tfrac{1}{2}, \tag{2}$$

where α denotes the 'exponent of the displacements' in their asymptotic expression near the notch root in the absence of the crack; α is the smallest positive solution of the equation

$$\sin(2\alpha\psi) + \alpha \sin(2\psi) = 0, \tag{3}$$

where it is recalled that $2(\pi - \psi)$ represents the aperture angle of the notch. Since α varies from 1 to $\frac{1}{2}$ when ψ varies from $\pi/2$ to π, β is positive except for $\psi = \pi$ (case where the notch is itself a crack), so that except in this case, the SIFs $K_p(a)$ go to zero with a. Furthermore the quantities $K_p^{(\beta)}$ $(p = I, II)$ are given by the following *universal* (that is, applicable in all circumstances, whatever the geometry of the body, the notch and the crack and whatever the loading) formula:

$$K_p^{(\beta)} \equiv L_p(\psi, \varphi)\kappa, \tag{4}$$

where φ is recalled to denote the angle between the bisecting line Ox of the notch and the initial tangent to the crack, the $L_p(\psi, \varphi)$ some yet unknown functions of the angles ψ and φ and κ the 'mode I stress intensity factor of the notch', that is, the multiplicative coefficient of the singular stress field near the notch root, in the absence of the crack. It is recalled that the definition of the 'stress intensity factor κ of the notch' chosen, implicitly implied by Equations (1) of Part I, possesses the nice properties that κ reduces to the ordinary SIF K_I for $\psi = \pi$ (case of a crack) and to the uniform tensile stress σ_{yy} for $\psi = \pi/2$ (case of a regular, corner-free boundary).

It was also shown, in Section V of Part I, that the function $L_p(\psi, \varphi)$ $(p = I, II)$ can be identified with the p-th SIF at the crack tip in a 'model problem' involving an *infinite* body endowed with a notch with *straight* boundaries and a *straight* crack originating from the notch root, of *unit* length (Fig. 1c). The conditions on the notch boundaries, the crack lips and at infinity in this problem read as follows:

$$\begin{cases} \boldsymbol{\sigma}(r, \theta = \pm\psi) \cdot \mathbf{e}_\theta(\theta = \pm\psi) = \mathbf{0}, \forall r > 0 \text{ (conditions on the notch boundaries)} \\ \\ \boldsymbol{\sigma}(r, \theta = \varphi^+).(-\mathbf{e}_\theta(\theta = \varphi)) = \boldsymbol{\Sigma}(r, \theta = \varphi).\mathbf{e}_\theta(\theta = \varphi) \\ \quad \forall r, 0 < r < 1 \quad \text{(condition on the upper lip of the crack)} \\ \\ \boldsymbol{\sigma}(r, \theta = \varphi^-).\mathbf{e}_\theta(\theta = \varphi) = -\boldsymbol{\Sigma}(r, \theta = \varphi).\mathbf{e}_\theta(\theta = \varphi) \\ \quad \forall r, 0 < r < 1 \quad \text{(condition on the lower lip of the crack)} \\ \\ \boldsymbol{\sigma}(r, \theta) = O(r^{-\alpha-1}) \quad \text{for} \quad r \to +\infty, \forall \theta, -\psi < \theta < + \qquad \text{(condition at infinity).} \end{cases} \tag{5}$$

In these equations, r and θ are recalled to denote polar coordinates with origin at the notch root, $\mathbf{e}_\theta(\theta)$ the unit orthoradial vector and $\{\boldsymbol{\Sigma}(r, \theta)\}$ the stress field defined by Equations (1) of Part I, except for the omission of the factor κ:

$$\begin{cases} \Sigma_{rr}(r, \theta) = -\dfrac{\alpha}{2}(2\pi r)^{\alpha-1}\left[\dfrac{(3-\alpha)\sin((1+\alpha)\psi)}{(1-\alpha)\sin((1-\alpha)\psi)}\cos((1-\alpha)\theta) + \cos((1+\alpha)\theta)\right], \\ \Sigma_{\theta\theta}(r, \theta) = \dfrac{\alpha}{2}(2\pi r)^{\alpha-1}\left[-\dfrac{(1+\alpha)\sin((1+\alpha)\psi)}{(1-\alpha)\sin((1-\alpha)\psi)}\cos((1-\alpha)\theta) + \cos((1+\alpha)\theta)\right], \\ \Sigma_{r\theta}(r, \theta) = \dfrac{\alpha}{2}(2\pi r)^{\alpha-1}\left[-\dfrac{\sin((1+\alpha)\psi)}{\sin((1-\alpha)\psi)}\sin((1-\alpha)\theta) + \sin((1+\alpha)\theta)\right]. \end{cases} \tag{6}$$

This plane elasticity problem is solved here with the aid of Muskhelishvili's (1952) complex potentials formalism and conformal mapping. The conformal mapping used maps the lower half-plane of some z-plane onto the physical domain in some Z-plane. To the best of the authors' knowledge, its expression has not previously been available in the literature. It is

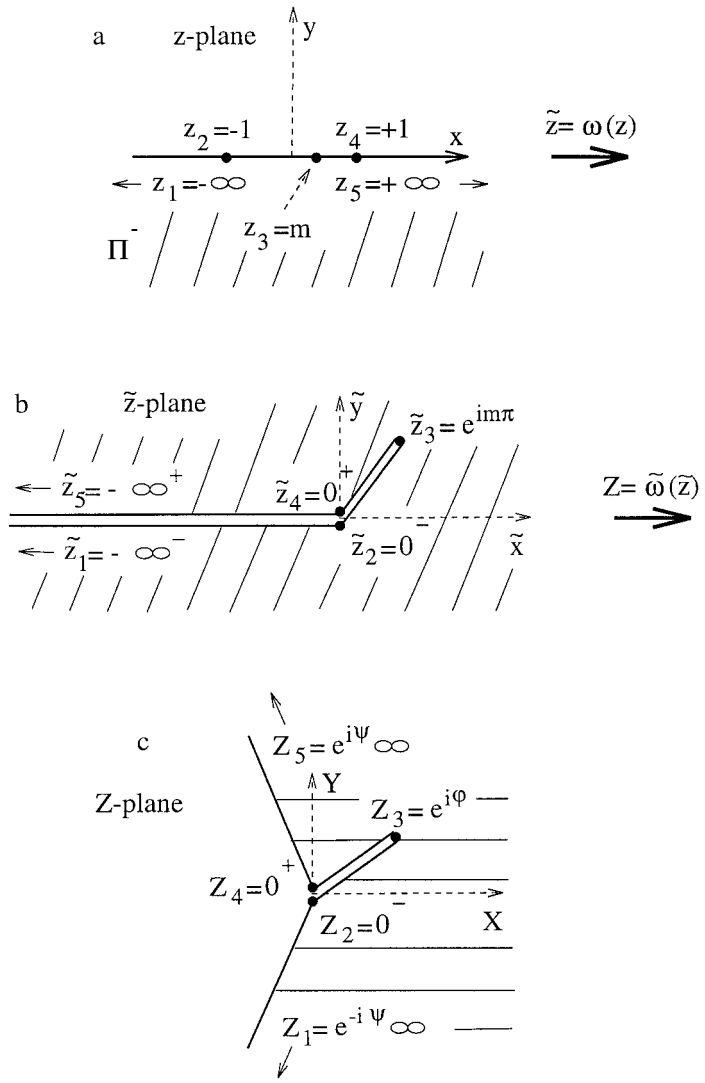

Figure 1. Conformal mapping: the z-plane (a); the $\tilde{z}$-plane (b); the (physical) Z-plane (c).

deduced here from the thesis of Mouchrif (1994), who has considered the special case where $= \pi$. The problem is reduced to an integral equation on Muskhelishvili's potential $\varphi'(z)$. The solution is given in the form of an infinite series involving the integral operator appearing in the integral equation. This series is evaluated numerically. Finally, the SIFs and hence the functions $L_p(\psi, \varphi)$, are deduced from Andersson's (1969) formula which relates them to the value of the potential $\varphi'(z)$ at that point corresponding to the crack tip in the z-plane.

The results are then compared with those available in the literature. The latter results all pertain to special cases involving a notch with straight boundaries and a straight crack; that is, the *universal* applicability of the functions $L_p(\psi, \varphi)$ has not been perceived. As already mentioned in the Introduction of Part I, four special cases have been previously envisaged. The first one is quite classical and concerns the case where $\psi = \pi/2$ and $\varphi = 0$ (edge crack originating perpendicularly from a regular free surface). The second one, envisaged by many authors (notably Bilby and Cardew, 1975; Wu, 1978; Amestoy et al., 1979; Amestoy and

Leblond, 1992), is that where $\psi = \pi$ but φ is arbitrary (semi-infinite crack with an arbitrarily kinked extension). The third one, also considered by several authors (Wilson, 1969; Sha and Yang, 1985; Aliabadi, 1987; Fett and Munz, 1997), is that where $\psi = \pi/2$ but φ is arbitrary (oblique edge crack emanating from a regular free surface). Finally, the fourth one, which has been treated by Fett and Munz (1997), using some previous works of Gross (1970), Hasebe and Isida (1978) and Iida (1979), concerns the situation where $\varphi = 0$ but ψ is arbitrary (crack lying on the bisecting line of an arbitrary notch). In all of these special cases, we find results in good agreement with those of the literature.

We finally envisage the problem of determining the kink angle χ of the crack extension (see Fig. 1 of Part I). This kink angle is deduced from the values of the SIFs of the initial crack using some suitable criterion, namely Goldstein and Salganik's (1974) well-known and widely accepted *principle of local symmetry*. As a consequence of the fact that the notch is always asymptotically predominantly loaded in 'mode I', the angles φ and χ are found to be of opposite signs; that is, *the direction of the crack is closer to that of the bisecting line of the notch after kinking than before it*. It remains to see whether or not the kink angle is large enough for the distance from the crack tip to the bisecting line to decrease after the kink. It is found that it is generally *not* sufficiently large for this possibility to occur, that is, *the crack tip generally still tends to move away from the bisecting line of the notch after the kink*, although less markedly than prior to it.

2. The problem in the physical Z-plane

Let $\Phi(Z)$ and $\Psi(Z)$ denote Muskhelishvili's potentials in the physical Z-plane. Then, the boundary $Z_1Z_2Z_3Z_4Z_5$ (see Fig. 1c) of the body being oriented in that way, the boundary conditions read

$$i\frac{\mathrm{d}}{\mathrm{d}S}\left[\Phi(Z) + Z\overline{\Phi'(Z)} + \overline{\Psi(Z)}\right] = T_X(Z) + iT_Y(Z) \quad \text{on} \quad Z_1Z_2Z_3Z_4Z_5, \tag{7}$$

where S denotes the curvilinear length along the arc $Z_1Z_2Z_3Z_4Z_5$ and $T_X(Z)$, $T_Y(Z)$ the components of the traction $\mathbf{T}(Z)$ exerted on the boundary[1]. Also, since the stresses are $O\left(|Z|^{-\alpha-1}\right)$ at infinity (Equation $(5)_4$), the conditions at infinity for the potentials $\Phi(Z)$, $\Psi(Z)$ read

$$\Phi(Z) = O\left(|Z|^{-\alpha}\right), \quad \Psi(Z) = O\left(|Z|^{-\alpha}\right) \quad \text{for } |Z| \to +\infty, \tag{8}$$

which implies that *they vanish at infinity* (since α is positive).

The traction $\mathbf{T}(Z)$ being zero on the notch boundaries Z_1Z_2 and Z_4Z_5 (Equation $(5)_1$), Equation (7) implies that $\Phi(Z) + Z\overline{\Phi'(Z)} + \overline{\Psi(Z)}$ is constant along these two half-straight lines. Since both $\Phi(Z)$ and $\Psi(Z)$ vanish at infinity, one must in fact have

$$\Phi(Z) + Z\overline{\Phi'(Z)} + \overline{\Psi(Z)} = 0 \quad \text{for} \quad Z \in Z_1Z_2 \cup Z_4Z_5. \tag{9}$$

Let us now calculate the traction $\mathbf{T}(Z)$ on the lower lip Z_2Z_3 of the crack. By Equation $(5)_3$, one has on this lip $\mathbf{T}(Z) = -\boldsymbol{\Sigma}(r,\varphi).\mathbf{e}_\theta(\varphi) = -\Sigma_{r\theta}(r,\varphi)\mathbf{e}_r(\varphi) - \Sigma_{\theta\theta}(r,\varphi)\mathbf{e}_\theta(\varphi)$ so that $T_X(Z) + iT_Y(Z) = [T_r(r,\varphi) + iT_\theta(r,\varphi)]\,e^{i\varphi} = -[\Sigma_{r\theta}(r,\varphi) + i\Sigma_{\theta\theta}(r,\varphi)]\,e^{i\varphi}$. A straightforward calculation based on Equations $(6)_2$ and $(6)_3$ then leads to $T_X(Z) + iT_Y(Z) = -i\alpha\mathcal{S}(2\pi|Z|)^{\alpha-1}$ where

[1] There is a difference of sign here with respect to Muskhelishvili's classical formula, which arises from the fact that the unit normal exterior vector to the boundary is oriented to the left, instead of to the right as usual, of the unit tangent vector to the boundary.

$$\mathcal{S} \equiv \frac{1}{2}\left[-\frac{\sin((1+\alpha)\psi)}{(1-\alpha)\ \sin((1-\alpha)\psi)}\left(e^{i\alpha\varphi}+\alpha\ e^{i(2-\alpha)\varphi}\right)+e^{-i\alpha\varphi}\right]. \tag{10}$$

Since $\mathrm{d}S \equiv \mathrm{d}|Z|$ on Z_2Z_3, Equation (7) reads

$$\frac{\mathrm{d}}{\mathrm{d}|Z|}\left[\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)}\right]=-\alpha\mathcal{S}(2\pi|Z|)^{\alpha-1},$$

which yields upon integration

$$\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)}=-\frac{\mathcal{S}}{2\pi}(2\pi|Z|)^{\alpha} \quad \text{for} \quad Z\in Z_2Z_3, \tag{11}$$

the constant of integration being zero since by Equation (9), $\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)}$ is zero at the point $Z_2 \equiv 0^-$.

On the upper lip Z_3Z_4 of the crack, the traction $\mathbf{T}(Z)$ is opposite to that on the lower lip Z_2Z_3 by Equations $(5)_2$ and $(5)_3$. It follows that $T_X(Z)+iT_Y(Z)=i\alpha\mathcal{S}(2\pi|Z|)^{\alpha-1}$ on Z_3Z_4. On the other hand, $\mathrm{d}S \equiv -\mathrm{d}|Z|$ instead of $\mathrm{d}|Z|$, so that Equation (7) reads for $Z\in Z_3Z_4$:

$$-\frac{\mathrm{d}}{\mathrm{d}|Z|}\left[\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)}\right]=\alpha\mathcal{S}(2\pi|Z|)^{\alpha-1},$$

which is the same equation as on Z_2Z_3 and leads upon integration to the same result:

$$\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)}=-\frac{\mathcal{S}}{2\pi}(2\pi|Z|)^{\alpha} \quad \text{for } Z\in Z_3Z_4, \tag{12}$$

the constant of integration being again zero because $\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)}$ is zero at the point $Z_4 \equiv 0^+$ by Equation (9).

Equations (9), (11) and (12) can be summarized as follows:

$$\Phi(Z)+Z\overline{\Phi'(Z)}+\overline{\Psi(Z)}=-\frac{\mathcal{S}}{2\pi}(2\pi|Z|)^{\alpha}I(Z) \qquad \text{on } Z_1Z_2Z_3Z_4Z_5, \tag{13}$$

where $I(Z)$ denotes the characteristic function of the arc $Z_2Z_3Z_4$:

$$I(Z)=\begin{cases}1 & \text{if } Z\in Z_2Z_3Z_4,\\ 0 & \text{if } Z\notin Z_2Z_3Z_4.\end{cases} \tag{14}$$

The problem is to find complex potentials $\Phi(Z)$, $\Psi(Z)$ analytical in the physical domain and satisfying conditions (13) on $Z_1Z_2Z_3Z_4Z_5$ and (8) at infinity.

3. Conformal mapping – the problem in the z-plane

Mouchrif (1994) has provided the expression of the conformal mapping which maps the lower half Π^- of some z-plane onto the exterior of a kinked crack composed of a principal semi-infinite branch and an extension of unit length making an arbitrary angle $m\pi$ $(-1<m<+1)$ with the principal branch in some $\tilde{z}$-plane (Figs. 1a and 1b). The expression of this conformal mapping is

$$\tilde{z}\equiv\omega(z)\equiv A(1-z^2)\left(\frac{z+1}{z-1}\right)^m, \quad A\equiv\frac{1}{1-m^2}\left(\frac{1-m}{1+m}\right)^m. \tag{15}$$

Defining

$$Z \equiv \Omega(z) \equiv \tilde{\omega}(\tilde{z}) \equiv (\tilde{\omega} \circ \omega)(z) \equiv \tilde{z}^n, \tag{16}$$

where n is a parameter such that $\frac{1}{2} \leq n \leq 1$, we 'open the principal branch of the crack' and thus obtain, in the Z-plane, the geometry we are interested in, with a notch of aperture angle $2(\pi - \psi)$ and a crack of unit length making an angle φ with the bisecting line of the notch (Figs. 1b and 1c). The a priori free parameters m and n must be chosen in order for the angles and φ to take the desired values. It is evident that $\psi = n\pi$; furthermore the point $\tilde{z}_3 \equiv e^{im\pi}$ in the $\tilde{z}$-plane becomes $z_3 \equiv e^{imn\pi} \equiv e^{i\varphi}$ in the z-plane, which implies that $\varphi = mn\pi$. Thus the parameters m and n are given by

$$n \equiv \frac{}{\pi}\ ;\ m \equiv \frac{\varphi}{n\pi} = \frac{\varphi}{}. \tag{17}$$

We shall need the expression of the quantity $\Omega(t)/\overline{\Omega'(t)}$ for $t \in \mathbb{R}$ in the sequel. After a somewhat lengthy but straightforward calculation, one gets

$$\frac{\Omega(t)}{\overline{\Omega'(t)}} = Q(t)[1 + j(t)], \quad \forall t \in \mathbb{R}, \tag{18}$$

where

$$Q(t) \equiv \frac{t^2 - 1}{2(t - m)} \tag{19}$$

and

$$j(t) \equiv \begin{cases} e^{-2i\psi}/n\ - 1 & \text{if } t < -1 \\ e^{2i\varphi}/n\ - 1 & \text{if } -1 < t < +1 \\ e^{+2i\psi}/n\ - 1 & \text{if } t > +1. \end{cases} \tag{20}$$

(Note that the quantity $Q(t)[1 + j(t)]$ is continuous at the points $t = \pm 1$ in spite of the discontinuities of $j(t)$, since $Q(t)$ vanishes there.)

In the z-plane, Equation (13) takes the form

$$\varphi(t) + \frac{\Omega(t)}{\overline{\Omega'(t)}}\overline{\varphi'(t)} + \overline{\psi(t)} = -\frac{\mathcal{S}}{2\pi}(2\pi|\Omega(t)|)^{\alpha} i(t)\ , \quad \forall t \in \mathbb{R}$$

where $\varphi(z) \equiv \Phi(Z)$, $\psi(z) \equiv \Psi(Z)$ and $i(t)$ denotes the characteristic function of the interval $(-1, +1)$. By Equation (18), this equation reads

$$\varphi(t) + Q(t)\overline{\varphi'(t)} + Q(t)j(t)\overline{\varphi'(t)} + \overline{\psi(t)} = -\frac{\mathcal{S}}{2\pi}(2\pi|\Omega(t)|)^{\alpha} i(t), \quad \forall t \in \mathbb{R}. \tag{21}$$

Furthermore, by Equations (8), (15) and (16),

$$\varphi(z) = O\left(|z|^{-2n\alpha}\right)\ , \quad \psi(z) = O\left(|z|^{-2n\alpha}\right) \quad \text{for } |z| \to +\infty. \tag{22}$$

The problem is reduced to finding some complex potentials $\varphi(z)$, $\psi(z)$ analytical in the lower half-plane Π^- and satisfying conditions (21) on $\mathbb{R}$ and (22) at infinity.

4. Solution of the problem

4.1. Reduction of the problem to an integral equation

Let us define the function

$$\chi(z) \equiv \frac{1}{2i\pi}\int_{-\infty}^{+\infty} Q(t)\,j(t)\overline{\varphi'(t)}\frac{\mathrm{d}t}{t-z} \equiv \frac{1}{4i\pi}\int_{-\infty}^{+\infty} \frac{(t^2-1)\,j(t)\overline{\varphi'(t)}\mathrm{d}t}{(t-m^-)(t-z)}\,, \quad \forall z \in \Pi^- \qquad (23)$$

where $m^- \equiv m - i\epsilon$, $\epsilon > 0$, $\epsilon \to 0$. (Note that the integrand behaves like $|t|^{-2n\alpha-1}$ for $t \to \pm\infty$; hence, since n and α are positive, the integral is convergent at infinity. It appears essential here to have removed the tractions imposed from infinity to the crack lips; otherwise the integrand would behave like $|t|^{2n\alpha-1}$ for $t \to \pm\infty$, and the integral would diverge at infinity). By Plemelj's formula,

$$\chi(t^+) - \chi(t^-) = Q(t)\,j(t)\overline{\varphi'(t)}\,, \quad \forall t \in \mathbb{R}$$

where $t^\pm \equiv t \pm i\epsilon$, $\epsilon > 0$, $\epsilon \to 0$. Also, let us define the function

$$\gamma(z) \equiv \frac{1}{2i\pi}\int_{-1}^{+1} \frac{\mathscr{J}}{2\pi}(2\pi|\Omega(t)|)^\alpha \frac{\mathrm{d}t}{t-z}\,, \quad \forall z \in \Pi^-. \qquad (24)$$

Then Plemelj's formula again yields

$$\gamma(t^+) - \gamma(t^-) = \frac{\mathscr{J}}{2\pi}(2\pi|\Omega(t)|)^\alpha i(t), \quad \forall t \in \mathbb{R}\,.$$

These elements and the obvious property $Q(t) = \overline{Q(t)}$, $\forall t \in \mathbb{R}$, imply that Equation (21) may be written as

$$\varphi(t) + \overline{Q(t)\varphi'(t)} + \chi(t^+) - \chi(t^-) + \overline{\psi(t)} + \gamma(t^+) - \gamma(t^-) = 0\,, \quad \forall t \in \mathbb{R}.$$

Now, $\overline{f}(z) \equiv \overline{f(\overline{z})}$ denoting the conjugate of $f(z)$ in the sense of Muskhelishvili, one has $\chi(t^+) = \chi(\overline{t^-}) = \overline{\chi(t^-)}$ and similarly $\gamma(t^+) = \gamma(\overline{t^-}) = \overline{\gamma(t^-)}$, so that the preceding equation may be re-written as

$$\varphi(t) - \chi(t^-) - \gamma(t^-) = -\overline{Q(t)\varphi'(t)} - \overline{\chi(t^-)} - \overline{\psi(t)} - \overline{\gamma(t^-)}\,, \quad \forall t \in \mathbb{R}. \qquad (25)$$

We shall now make use of the following classical result, the (elementary) proof of which is given for instance in Mouchrif's (1994) thesis:

Lemma. Let $f(z)$ and $g(z)$ denote two functions defined and continuous on $\Pi^- \cup \mathbb{R}$, analytic on Π^-, bounded at infinity and such that $f(t) = \overline{g(t)}$, $\forall t \in \mathbb{R}$. Then $f(z)$ and $g(z)$ are constant and conjugate to each other.

It is straightforward, though somewhat tedious, to check that all the hypotheses of this lemma are verified by the left- and right-hand sides of Equation (25). (Here again, it appears as essential to have removed the remote tractions from infinity to the lips of the crack; otherwise $\varphi(z)$ would be $O(|z|^{2n\alpha})$ for $|z| \to +\infty$ so that the left-hand side of Equation (25) would not be bounded at infinity). Hence the lemma yields

$$\varphi(z) - \chi(z) - \gamma(z) = 0 \;\Rightarrow\; \varphi(z) = \chi(z) + \gamma(z)\,, \quad \forall z \in \Pi^-, \qquad (26)$$

the additive constant being zero in the present case since by Equations $(22)_1$, (23) and (24), all functions $\varphi(z)$, $\chi(z)$, $\gamma(z)$ vanish at infinity. Differentiating Equation $(26)_2$ and accounting for the expressions $(23)_2$, (24) of the functions $\chi(z)$ and $\gamma(z)$, one gets the following integral equation on the unknown potential $\varphi'(z)$:

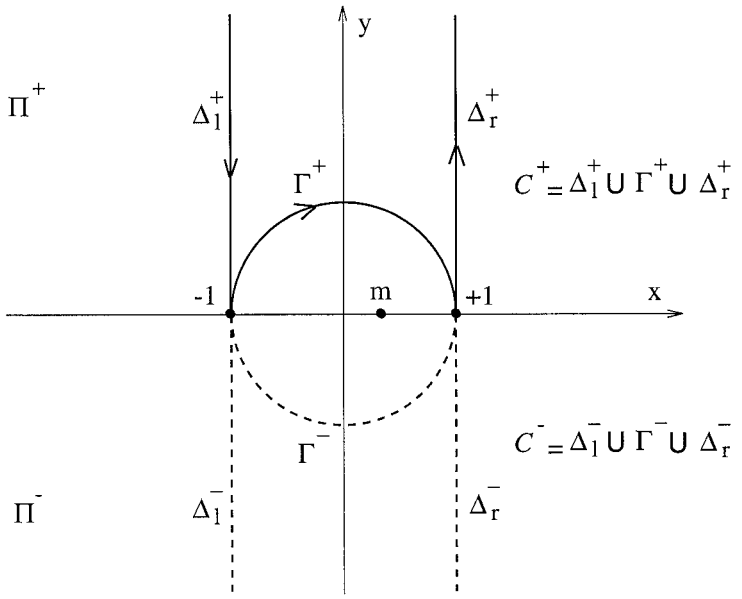

Figure 2. Figure 2. Integration paths in the z-plane.

$$\varphi'(z) = \frac{1}{2i\pi}\int_{-1}^{+1} \frac{\mathcal{S}}{2\pi}(2\pi|\Omega(t)|)^{\alpha}\frac{dt}{(t-z)^2} + \frac{1}{4i\pi}\int_{-\infty}^{+\infty}\frac{(t^2-1)j(t)\overline{\varphi'(t)}dt}{(t-m^-)(t-z)^2}\,,$$

$$\forall z \in \Pi^-, \tag{27}$$

where, by Equations (15) and (16),

$$|\Omega(t)| = A^n(1-t^2)^n\left(\frac{1+t}{1-t}\right)^{mn}, \quad \forall t \in (-1,+1). \tag{28}$$

The function $j(t)$ is constant over each interval $(-\infty,-1)$, $(-1,+1)$, $(+1,+\infty)$ (see Equation (20)). Furthermore, for any real t, $\overline{\varphi'(t)} = \overline{\varphi}'(t)$; hence $\overline{\varphi'(t)}$ may be replaced by $\overline{\varphi}'(t)$ in the second integral of the right-hand side of Equation (27). Provided that this replacement is made, the integrand corresponding to each interval $(-\infty,-1)$, $(-1,+1)$, $(+1,+\infty)$ is then analytic over the upper half-plane Π^+. Hence these intervals may be displaced within Π^+ : see Fig. 2. Thus the interval $(-\infty,-1)$ may be replaced by the half-straight line Δ_ℓ^+ of equation $\mathrm{Re}(\lambda) = -1$, $\mathrm{Im}(\lambda) > 0$, oriented from $-1+i\infty$ to -1; indeed the integral over the great quarter of a circle which closes the integration path is zero since the integrand is $O(|\lambda|^{-2-2n\alpha})$. Similarly, the interval $(+1,+\infty)$ may be replaced by the half-straight line $\Delta_r^+ : \mathrm{Re}(\lambda) = +1$, $\mathrm{Im}(\lambda) > 0$, oriented from $+1$ to $+1+i\infty$. Finally, the interval $(-1,+1)$ may be replaced by the semi-circle $\Gamma^+ : |\lambda| = 1$, $\mathrm{Im}(\lambda) > 0$, oriented from -1 to $+1$ (since the pole at m^- is not crossed in the process). Hence the second integral in the right-hand side of Equation (27) may be written as

$$\mathcal{L}\varphi'(z) \equiv \frac{1}{4i\pi}\int_{\mathcal{C}^+}\frac{(\lambda^2-1)j(\lambda)\overline{\varphi}'(\lambda)d\lambda}{(\lambda-m)(\lambda-z)^2}\,, \quad j(\lambda) \equiv \begin{cases} e^{-2i\psi}/n - 1 & \text{on} \quad \Delta_\ell^+ \\ e^{2i\varphi}/n - 1 & \text{on} \quad \Gamma^+ \\ e^{+2i\psi}/n - 1 & \text{on} \quad \Delta_r^+, \end{cases} \tag{29}$$

where $\mathcal{C}^+$ denotes the path $\Delta_\ell^+ \cup \Gamma^+ \cup \Delta_r^+$, oriented like Δ_ℓ^+, Γ^+ and Δ_r^+. Similarly, taking advantage of the fact that the function $(1-\lambda^2)^{n\alpha}\,((1+\lambda)/(1-\lambda))^{mn\alpha}$ is analytic on Π^+, one may write the first integral of the right-hand side of Equation (27), using Equation (28), in the form

$$\varphi_0'(z) \equiv \frac{1}{2i\pi}\int_{\Gamma^+}\frac{\mathcal{S}}{2\pi}\left[2\pi A^n(1-\lambda^2)^n\left(\frac{1+\lambda}{1-\lambda}\right)^{mn}\right]^\alpha \frac{d\lambda}{(\lambda-z)^2}\,. \tag{30}$$

The integral equation (27) on the potential $\varphi'(z)$ can therefore be written as

$$\varphi'(z) = \varphi_0'(z) + \mathcal{L}\varphi'(z)\,,\ \forall z \in \Pi^- \tag{31}$$

where the integral operator $\mathcal{L}$ and the function $\varphi_0'(z)$ are defined by Equations (29) and (30).

4.2. Solution in the form of an infinite series

The solution of the integral equation (31) is unique and can be written as

$$\varphi'(z) = \sum_{p=0}^{+\infty} \mathcal{L}^p \varphi_0'(z), \tag{32}$$

where $\mathcal{L}^p$ denotes the p-th iterate of the integral operator $\mathcal{L}$ defined by Equation (29), provided that this operator is *contractant*, that is, that it reduces some norm (for instance the $\|\ \|_\infty$ norm) of the function it is applied on. Since it involves the function $j(\lambda)$ (defined on $\mathcal{C}^+$ by Equation $(29)_2$), the contractant character will be ensured by the smallness of this function. Now $j(\lambda)$ is small when ψ is close to π (sharp notch) and φ close to 0 (crack lying almost along the bisecting line of the notch); indeed, by Equation $(17)_1$, n is then close to 1 so that $e^{-2i\psi}/n - 1$, $e^{2i\varphi}/n - 1$, $e^{+2i\psi}/n - 1$ are all small. Hence the convergence of the series (32) is guaranteed under these conditions. Numerical experience (see Section 5 below) confirms that convergence can indeed be achieved except when ψ is close to $\pi/2$ ('open notch') and φ large (crack lying away from the bisecting line).

In the integral defining the operator $\mathcal{L}$ (see Equation $(29)_1$), the only values of the function $\varphi'(z)$ it acts on which are involved are those on the conjugate $\mathcal{C}^-$ of the arc $\mathcal{C}^+$ (see Fig. 2). Therefore the values of the successive iterates $\varphi_0'(z)$, $\mathcal{L}\varphi_0'(z)$, $\mathcal{L}^2\varphi_0'(z)$, etc. can be calculated (numerically) inductively on the sole arc $\mathcal{C}^-$, and the values of the unknown potential $\varphi'(z)$ on the same arc follow from there and application of Equation (32). The value of $\varphi'(z)$ at any point $z \in \Pi^-$ can then be obtained from Equation (31).

4.3. Stress intensity factors

The values of the SIFs at the crack tip $K_p \equiv L_p(\psi,\varphi)$ $(p = I, II)$ can finally be deduced from Andersson's (1969) formula, which involves the value of the potential φ' at that point $z_3 \equiv m$ in the z-plane corresponding to the crack tip $Z_3 \equiv e^{i\varphi}$ in the physical Z-plane (see Fig. 1):

$$K_I - iK_{II} \equiv L_I(\psi,\varphi) - iL_{II}(\psi,\varphi) = 2\sqrt{\pi}\ e^{-i\varphi/2}[\Omega''(m)]^{-1/2}\varphi'(m).$$

The value of $\varphi'(m)$ here can be obtained as explained above. Calculation of the quantity $\Omega''(m)$ leads to the following final expression:

$$L_I(\psi,\varphi) - iL_{II}(\psi,\varphi) = -i\sqrt{\frac{2\pi}{n}(1-m^2)}\ e^{-i\varphi}\varphi'(m)\,. \tag{33}$$

Table 1. Value of L_I as a function of ψ (1st row) and φ (1st column)

	$= 90°$	100°	110°	120°	130°	140°	150°	160°	170°
$\varphi = 0°$	1.99	1.29	0.99	0.86	0.82	0.83	0.87	0.93	0.98
10°	1.95	1.27	0.97	0.85	0.81	0.82	0.86	0.92	0.97
20°	1.83	1.20	0.93	0.81	0.78	0.79	0.84	0.89	0.94
30°	1.65	1.10	0.86	0.76	0.73	0.75	0.79	0.84	0.89
40°	–	0.98	0.78	0.69	0.67	0.69	0.73	0.78	0.82
50°	–	–	0.68	0.61	0.60	0.62	0.66	0.70	0.74
60°	–	–	–	0.53	0.52	0.54	0.58	0.62	0.65
70°	–	–	–	–	0.44	0.46	0.50	0.53	0.56

Table 2. Value of L_{II} as a function of ψ (1st row) and φ (1st column)

	$= 90°$	100°	110°	120°	130°	140°	150°	160°	170°
$\varphi = 0°$	0	0	0	0	0	0	0	0	0
10°	0.19	0.12	0.09	0.08	0.07	0.07	0.07	0.08	0.08
20°	0.36	0.23	0.17	0.15	0.14	0.14	0.14	0.15	0.16
30°	0.48	0.31	0.24	0.20	0.19	0.19	0.20	0.22	0.23
40°	–	0.36	0.28	0.25	0.24	0.24	0.25	0.27	0.29
50°	–	–	0.30	0.27	0.26	0.27	0.29	0.31	0.33
60°	–	–	–	0.27	0.27	0.29	0.31	0.33	0.35
70°	–	–	–	–	0.27	0.28	0.31	0.34	0.36

5. Numerical calculation of the functions $L_p(\psi, \varphi)$

In order to perform the numerical integrations, the arc $\mathcal{C}^+$ is divided into 400 intervals: 100 on each half-straight line Δ_ℓ^+, Δ_r^+ (numerical integration being 'cut off' on these half-lines at a distance of 100 from the x-axis) and 200 on the semi-circle Γ^+. The 'mesh' is refined near the singular points ± 1 and enlarged near 'infinity' on the half-straight lines Δ_ℓ^+, Δ_r^+. Gaussian integration is used, with 2 Gauss points in each interval (which makes a total of 800 Gauss points). The values of the successive iterates $\varphi_0'(z)$, $\mathcal{L}\varphi_0'(z)$, $\mathcal{L}^2\varphi_0'(z)$, etc., are calculated at the conjugates (lying on $\mathcal{C}^-$) of the Gauss points of the arc $\mathcal{C}^+$. The number of terms in the series (32) which is necessary to reach convergence is highly dependent upon the values of the angles ψ and φ : it varies from just a few ones when ψ is close to π and φ to 0 to several hundreds when ψ is closer to $\pi/2$ and φ larger.

Tables 1 and 2 show the results derived in that way for the functions $L_p(\psi, \varphi)$ $(p = I, II)$, for $90° \leq \psi \leq 170°$ and $0° \leq \varphi \leq 70°$. Blank spaces correspond to those cases (involving values of ψ close to 90° and large values of φ) where satisfactory convergence of the series (32) could not be achieved. Results corresponding to negative values of the angle φ can also be obtained using the fact that for obvious symmetry reasons, $L_I(\psi, \varphi)$ and $L_{II}(\psi, \varphi)$ are even and odd functions, respectively, of this argument. The (absolute) accuracy is better than 10^{-2}, and in many cases than 10^{-3}.

Of course, these results could also be obtained through more conventional techniques, such as the finite element method. The advantages of the approach used here are three-fold. First, the numerical accuracy which can be achieved is better than that currently encountered when using the finite element method, except in those cases where convergence of the series (32) is difficult to obtain, that is, when ψ is close to $\pi/2$ and φ large. Second, the computation time is shorter, except in the same cases as before. Third, changing the values of ψ and φ only requires to modify the values of the input parameters of the program instead of demanding a new mesh to be designed for each case. On the other hand, the versatility of the finite element method is obviously superior since it can deal with finite bodies; but consideration of an infinite one is sufficient to obtain the values of the functions $L_I(\psi, \varphi)$ and $L_{II}(\psi, \varphi)$ looked for here.

6. Partial checks on the results

As mentioned in the Introduction, there are four special cases which have been treated by former authors, thus allowing for a comparison with our results.

The first one corresponds to the situation where $\psi = \pi/2$ and $\varphi = 0$ (edge crack emanating perpendicularly from a regular, corner-free surface). It that case it is well known that $K_I(a) \simeq 1.12\, \sigma_{yy}^{\infty} \sqrt{\pi a}$ where σ_{yy}^{∞} denotes the remote tensile stress ($K_{II}(a)$ being zero). Now for this value of ψ, the 'exponent of the displacements' α in the absence of the crack is unity; hence, by Equation (2), $\beta = \frac{1}{2}$ and Equations (1) and (4) read $K_I(a) \equiv L_I\,(\psi = \pi/2, \varphi = 0)\, \kappa \sqrt{a} + o\left(\sqrt{a}\right) = L_I\,(\psi = \pi/2, \varphi = 0)\, \sigma_{yy}^{\infty} \sqrt{a} + o\left(\sqrt{a}\right)$ since the definition of the 'stress intensity factor κ of the notch' in the absence of the crack adopted here is such that it reduces to the uniform tensile stress σ_{yy} for $\psi = \pi/2$: see the Introduction. Comparison between these two expressions of $K_I(a)$ yields $L_I\,(\psi = \pi/2, \varphi = 0) \simeq 1.12\sqrt{\pi} \simeq 1.985$, which is in excellent agreement with the value of 1.99 found here (see Table 1).

The second special case is that where $\psi = \pi$ but φ is arbitrary (semi-infinite crack with a arbitrarily deviated extension). The comparison can be made only for an initial crack loaded in pure mode I; indeed this initial crack is considered here as the limit of a sharp notch for $\rightarrow \pi$, and any notch (with $\psi < \pi$) is always asymptotically loaded in pure mode I (see Section 2 of Part I). Leblond (1989) has shown that in this case

$$K_p(a) \equiv F_{p,I}(\varphi) K_I + O(\sqrt{a}) \quad (p = I, II), \tag{34}$$

where K_I denotes the mode I SIF of the initial crack, in the absence of the extension. This formula is of *universal value*, that is, it applies in all circumstances, whatever the far geometry of the body and whatever the loading. Furthermore, Amestoy and Leblond (1992) have provided formulae for the functions $F_{p,I}(\varphi)$ which allow for their evaluation with an accuracy better than 10^{-6}. Now for $\psi = \pi$, $\alpha = \frac{1}{2}$ so that, by Equation (2), $\beta = 0$. Hence Equations (1) and (4) read

$$K_p(a) \equiv L_p(\psi = \pi, \varphi)\kappa + o(a^0) = L_p(\psi = \pi, \varphi) K_I + o(a^0), \tag{35}$$

where use has been made of the fact that the 'stress intensity factor κ of the notch' reduces to the ordinary SIF K_I in the particular case where the notch becomes a crack ($\psi = \pi$) (see the Introduction). Comparison between formulae (34) and (35) shows that one must have

$$L_p(\psi = \pi, \varphi) \equiv F_{p,I}(\varphi)\,, \quad \forall \varphi \quad (p = I, II)\,. \tag{36}$$

Table 3. Comparison between the functions $L_p(\psi = \pi, \varphi)$ obtained here and the functions $F_{p,I}(\varphi)$ provided by Amestoy and Leblond (1992)

	$L_I(\psi = \pi, \varphi)$ (this work)	$F_{I,I}(\varphi)$ (Amestoy and Leblond, 1992)	$L_{II}(\psi = \pi, \varphi)$ (this work)	$F_{II,I}(\varphi)$ (Amestoy and Leblond, 1992)
$\varphi = 0°$	1	1	0	0
10°	0.989	0.989	0.086	0.086
20°	0.955	0.955	0.168	0.168
30°	0.902	0.902	0.240	0.240
40°	0.831	0.831	0.299	0.300
50°	0.748	0.748	0.343	0.343
60°	0.656	0.655	0.369	0.369
70°	0.559	0.559	0.378	0.378

Table 4. Comparison between the functions $L_p\,(\psi = \pi/2, \varphi)$ found here and provided by Fett and Munz (1997)

	$L_I(\psi = \pi/2, \varphi)$ (this work)	$L_I(\psi = \pi/2, \varphi)$ (Fett and Munz, 1997)	$L_{II}(\psi = \pi/2, \varphi)$ (this work)	$L_{II}(\psi = \pi/2, \varphi)$ (Fett and Munz, 1997)
$\varphi = 0°$	1.99	1.99	0	0
10°	1.95	1.94	0.19	0.19
20°	1.83	1.82	0.36	0.37
30°	1.65	1.65	0.48	0.52

Table 3 compares the functions $L_p(\psi = \pi, \varphi)$ obtained here with the functions $F_{p,I}(\varphi)$ resulting from the work of Amestoy and Leblond (1992). It can be seen that the discrepancy between the results is smaller than the numerical accuracy in the calculation of the functions $L_p(\psi = \pi, \varphi)$.

The third particular case corresponds to the situation where $\psi = \pi/2$ but φ is arbitrary (oblique edge crack originating from a regular, corner-free surface). The results of several authors (Wilson, 1969; Sha and Yang, 1985; Aliabadi, 1987) have recently been compiled and critically compared by Fett and Munz (1997). These authors provide the values of the functions $L_p\,(\psi = \pi/2, \varphi)$ in the form of tables (Nos. 52 and 53 in their book). (In fact, the SIFs are given for various crack lengths a, and one must extrapolate the results for $a \to 0$). Table 4 shows the comparison between the functions $L_p\,(\psi = \pi/2, \varphi)$ obtained here and those provided by these authors; the agreement is quite good.

The last special case is that where $\varphi = 0$ but ψ is arbitrary (crack lying on the bisecting line of an arbitrary notch). This case has again been considered by Fett and Munz (1997), on the basis of previous works of Gross (1970), Hasebe and Isida (1978) and Iida (1979). Fett and Munz' results are given in the form of an approximate but accurate formula for the function $L_I(\psi, \varphi = 0)$ ($L_{II}(\psi, \varphi = 0)$ being zero) (account being taken of a definition of the 'stress

Table 5. Comparison between the functions $L_p(\psi, \varphi = 0)$ found here and provided by Fett and Munz (1997)

	$L_I(\psi, \varphi = 0)$ (this work)	$L_I(\psi, \varphi = 0)$ (Fett and Munz, 1997)
$= 120°$	0.856	0.868
140°	0.829	0.811
160°	0.931	0.928
180°	1	1

intensity factor κ of the notch' differing from that adopted here). Table 5 compares our results with those of Fett and Munz; a good agreement can again be observed.

7. Conclusion: prediction of the kink angle of the crack extension

Let us finally consider the problem of determining the beginning of the propagation path of the crack, and more specifically the kink angle χ of the crack extension (see Fig. 1 of Part I). When this extension is still infinitesimal, the SIFs K_p^* $(p = I, II)$ at its tip are given by the *universal* formula (where Einstein's implicit summation convention over repeated indices is used)

$$K_p^* \equiv F_{p,q}(\chi) K_p(a) \quad (p, q = I, II) \tag{34'}$$

which was established by Leblond (1989) and is in fact a mere extension of formula (34) to the case where some mode II is initially present. The kink angle χ can then be deduced from Goldstein and Salganik's (1974) *principle of local symmetry*. This criterion is well-known and widely accepted and convincing arguments in its favour have been provided by Amestoy (1987). It stipulates that

$$K_{II}^* = F_{II,I}(\chi) K_I(a) + F_{II,II}(\chi) K_{II}(a) = 0 \,; \tag{37}$$

the numerical values of the functions $F_{p,q}(\chi)$ to be used when solving this equation with respect to χ can be deduced for instance from Amestoy and Leblond's (1992) approximate but highly accurate formulae for these functions. If a is small, formulae (1) and (4) for the SIFs $K_p(a)$ can be used and the preceding equation reduces to

$$F_{II,I}(\chi) L_I(\psi, \varphi) + F_{II,II}(\chi) L_{II}(\psi, \varphi) = 0 \,. \tag{37'}$$

This equation determines the kink angle χ *as a function of the sole angles* ψ *and* φ, *independently of the loading*. This feature, which may appear surprising at first sight, arises from the fact that as mentioned above, the notch is always asymptotically loaded in pure mode I, whatever the loading.

Precisely because of that property, it is intuitively obvious (and is in fact a rigorous consequence of Equation (37′)) that the angles φ and χ are of opposite signs, which means that *the direction of the crack is closer to that of the bisecting line of the notch after the kink than*

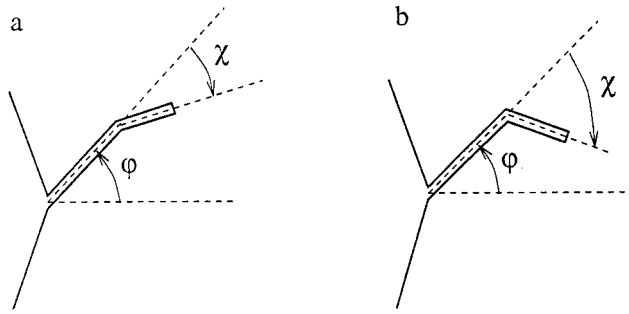

Figure 3. Figure 3. The two possible situations after the kink: $|\chi| < \varphi$ (a); $|\chi| > \varphi$ (b).

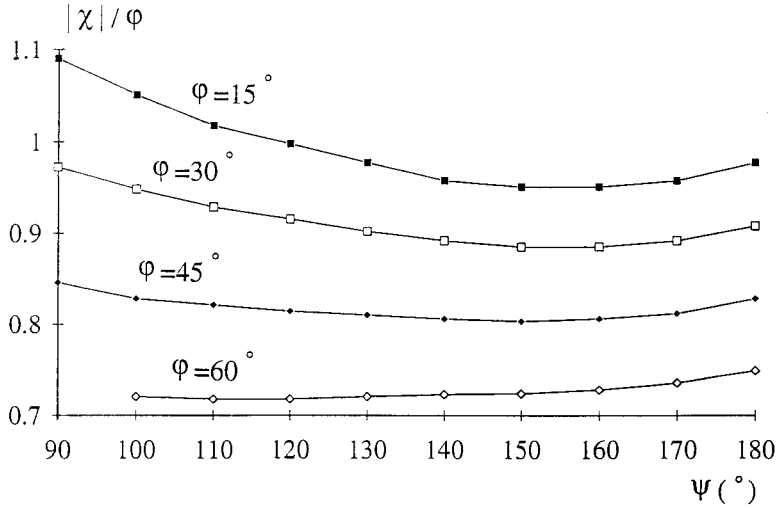

Figure 4. Figure 4. The ratio $|\chi|/\varphi$ as a function of the angle ψ, for various values of the angle φ.

before it. However, two possibilities remain (Fig. 3): φ being for instance taken as positive, either (a) $|\chi| < \varphi$, which means that the distance between the crack tip and the bisecting line of the notch still increases after the kink, although less markedly than before it; or (b) $|\chi| > \varphi$, so that this distance decreases after the kink.

Figure 4 shows the ratio $|\chi|/\varphi$ (φ being taken as positive) as a function of the angle ψ for various values of the angle φ. It can be observed that *in most cases, possibility (a) prevails: the distance from the crack tip to the bisecting line still increases after the kink.* However, possibility (b) seems to occur for values of ψ and φ close to $\pi/2$ and 0, respectively; then this distance decreases after the kink. It may nevertheless be remarked that the ratio $|\chi|/\varphi$ is close to unity then, and that conditions for good convergence of the series (32) are not well fulfilled, since ψ is close to $\pi/2$. Hence the numerical accuracy may not be sufficient to warrant that the ratio $|\chi|/\varphi$ is really greater than 1. Anyway, since it is close to unity then, and since in addition φ is small, the decrease of the distance from the crack tip to the bisecting line is marginal: the crack extension almost follows the direction of the bisecting line.

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References

Aliabadi, M.H., Rooke, D.P. and Cartwright, D.J. (1987): Mixed-mode Bueckner weight functions using boundary element analysis. *International Journal of Fracture* **34**, 131–147.

Amestoy, M. (1987): *Propagation de fissures en élasticité plane*. Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris, France.

Amestoy, M., Bui, H.D. and Dang Van, K. (1979): Déviation infinitésimale d'une fissure dans une direction arbitraire. *Comptes-Rendus de l'Académie des Sciences de Paris* **289B**, 99–102.

Amestoy, M. and Leblond, J.-B. (1992): Crack paths in plane situations – II. Detailed form of the expansion of the stress intensity factors. *International Journal of Solids and Structures* **29**, 465–501.

Andersson, H. (1969): Stress intensity factors at the tips of a star-shaped contour in an infinite tensile sheet. *Journal of the Mechanics and Physics of Solids* **17**, 405–414.

Bilby, B.A. and Cardew, G.E. (1975): The crack with a kinked tip. *International Journal of Fracture* **11**, 708–712.

Fett, T. and Munz, D. (1997): *Stress Intensity Factors and Weight Functions*, Computational Mechanics Publications, Southampton, UK.

Goldstein, R.V. and Salganik, R.L. (1974): Brittle fracture of solids with arbitrary cracks. *International Journal of Fracture* **10**, 507–523.

Gross, B. (1970): *Some Plane Problem Elastostatic Solutions for Plates Having a V-Notch*, Ph.D. Thesis, Case Western Reserve University, USA.

Hasebe, N. and Iida, J. (1978): A crack originating from a triangular notch on rim of a semi-infinite plate. *Engineering Fracture Mechanics* **10**, 773–782.

Isida, M. (1979): Tension of a half-plane containing array cracks, branched cracks and cracks emanating from sharp notches. *Transactions of the Japan Society of Mechanical Engineers* **45**, 306–317.

Leblond, J.-B. (1989): Crack paths in two-dimensional situations – I. General form of the expansion of the stress intensity factors. *International Journal of Solids and Structures* **25**, 1311–1325.

Mouchrif, S.-E. (1994): *Trajets de propagation de fissures en mécanique linéaire tridimensionnelle de la rupture fragile*. Thèse de Doctorat, Université Pierre et Marie Curie, Paris, France.

Muskhelishvili, N.I. (1952): *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Leiden, the Netherlands.

Sha, G.T and Yang, C.T. (1985): Weight function calculations for mixed-mode fracture problems with the virtual crack extension technique. *Engineering Fracture Mechanics* **21**, 1119–1149.

Wilson, W.K. (1969): *On Combined Mode Fracture Mechanics*, Research Report 69-1E7 FMECH-R1, Westinghouse Research Laboratories, Pittsburgh, USA.

Wu, C.H. (1978): Elasticity problems of a slender Z-crack. *Journal of Elasticity* **8**, 183–205.