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Jean-Yves Charbonnel. A remark on homogeneous affine varieties and related matters.. 2006. hal-00111188

HAL Id: hal-00111188
https://hal.science/hal-00111188v1

Preprint submitted on 3 Nov 2006

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hal-00111188, version 1 - 3 Nov 2006

A REMARK ON HOMOGENEOUS AFFINE VARIETIES AND RELATED MATTERS.

BY JEAN-YVES CHARBONNEL

ABSTRACT. — In this note we give an example of affine quotient G/H where G is an affine algebraic group over an algebraically closed field of characteristic 0 and H is a unipotent subgroup not contained in the unipotent radical of G. Some remarks about symmetric algebras of centralizers of nilpotent elements in simple Lie algebras, in particular cases, are added.

1. Introduction.

In this note $\mathbb{K}$ is an algebraically closed field of characteristic 0. Let G be an affine algebraic group over $\mathbb{K}$ and H a closed subgroup in G. We respectively denote by G_{u} and H_{u} the unipotent radicals of G and H. It is known [1](Proposition 3) that the homogeneous space G/H is affine if and only if it is so for the homogeneous space G/H_{u}. Moreover when H_{u} is contained in G_{u} then G/H is affine [1](Corollary 2). The question to know if the converse is true is an

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2000 Mathematics Subject Classification. — 14A10, 14R25, 17B20, 22E20, 22E46, 22E60.

Key words and phrases. — simple Lie algebra, nilpotent element, symmetric algebra, invariant, affine variety.

old question. But in this note we give an example which shows that it is not true in general. For that purpose we consider a simple Lie algebra $\mathfrak{g}$ of type F_4 over $\mathbb{K}$. In $\mathfrak{g}$ there exists a nilpotent element e whose centralizer $\mathfrak{g}(e)$ in $\mathfrak{g}$ has dimension 16 and its reductive factors are simple of dimension 3. By [3](Théorème 3.12), for any x in a non empty open subset of the dual of $\mathfrak{g}(e)$, its coadjoint orbit is closed. In particular it is an affine variety. We then show that for any x in a non empty open subset in the dual of $\mathfrak{g}(e)$, the stabilizer $\mathfrak{g}(e)(x)$ of x in $\mathfrak{g}(e)$ is not contained in the subset of nilpotent elements of the radical of $\mathfrak{g}(e)$ and any element of $\mathfrak{g}(e)(x)$ is nilpotent. Furthermore there is a natural question. Let $\mathfrak{g}$ be a semi-simple Lie algebra and e a nilpotent element. Does exist in the symmetric algebra of the centralizer of e in $\mathfrak{g}$ a semi-invariant which is not an invariant for the adjoint action? We give an example where it is not true and we check that in this case the index of its centralizer is equal to the rank of $\mathfrak{g}$. In the following sections, $\mathbb{K}$ is the ground field and we consider Zariski's topology. For an algebraic variety X and $\mathfrak{g}$ a Lie algebra acting on X we denote by $\mathfrak{g}(x)$ the stabilizer in $\mathfrak{g}$ of the element x in X.

2. Some remarks about invariants.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{K}$. We denote by $\mathfrak{g}^*$ its dual, $\mathrm{S}(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$, $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ the subalgebra of invariant elements in $\mathrm{S}(\mathfrak{g})$ for the adjoint action. We consider the coadjoint action of $\mathfrak{g}$ in $\mathfrak{g}^*$. For p in $\mathrm{S}(\mathfrak{g})$, the differential $\mathrm{d}p(x)$ of p at x is a linear form on $\mathfrak{g}^*$. So $\mathrm{d}p$ is a polynomial map from $\mathfrak{g}^*$ to $\mathfrak{g}$, that is to say $\mathrm{d}p$ is an element in $\mathrm{S}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$.

Lemma 2.1. — *Let p be an element in $\mathrm{S}(\mathfrak{g})$ and $\mathfrak{a}$ an ideal in $\mathfrak{g}$.*

i) *The element p is invariant for the adjoint action of $\mathfrak{a}$ in $\mathrm{S}(\mathfrak{g})$ if and only if $\mathrm{d}p(x)$ belongs to the stabilizer in $\mathfrak{g}$ of the restriction of x to $\mathfrak{a}$ for any x in a non empty open subset in $\mathfrak{g}^*$.*

ii) *The elements p is in $\mathrm{S}(\mathfrak{a})$ if and only if $\mathrm{d}p(x)$ is in $\mathfrak{a}$ for any x in a non empty open subset in $\mathfrak{g}^*$.*

Proof. — i) For any v in $\mathfrak{a}$ and any x in $\mathfrak{g}^*$, $[v,p](x)$ is equal to $\langle \mathrm{d}p(x), v.x\rangle$ where the coadjoint action of v on x is denoted by $v.x$. But by befinition of the coadjoint action, $\langle \mathrm{d}p(x), v.x\rangle$ is equal to $\langle [\mathrm{d}p(x), v], x\rangle$. Hence p is invariant for the adjoint action of $\mathfrak{a}$ if and only if $\mathrm{d}p(x)$ belongs to the stabilizer in $\mathfrak{g}$ of the restriction of x to $\mathfrak{a}$ for any x in $\mathfrak{g}^*$. So the statement is true because any non empty open subset in $\mathfrak{g}^*$ is everywhere dense in $\mathfrak{g}^*$.

ii) If p is in $\mathrm{S}(\mathfrak{a})$, $\mathrm{d}p$ is in $\mathrm{S}(\mathfrak{a}) \otimes_{\mathbb{C}} \mathfrak{a}$. Hence for any x in $\mathfrak{g}^*$, $\mathfrak{a}$ contains $\mathrm{d}p(x)$. Reciprocally if p is not in $\mathrm{S}(\mathfrak{a})$, there exists a basis $x_1, \ldots, x_n$ in $\mathfrak{g}$ such that x_1 is not in $\mathfrak{a}$ and $\frac{\partial p}{\partial x_1}$ is not equal to zero. So if x is not a zero of $\frac{\partial p}{\partial x_1}$ in $\mathfrak{g}^*$, $\mathrm{d}p(x)$ is not in $\mathfrak{a}$. □

As the intersection of two non empty open subsets in $\mathfrak{g}^*$ is non empty the following corollary is an easy consequence of lemma 2.1.

Corollary 2.2. — *Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Then $\mathfrak{a}$ does not contain $\mathfrak{g}(x)$ for any x in a non empty open subset in $\mathfrak{g}^*$ if and only if $\mathrm{S}(\mathfrak{a})$ does not contain $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.*

By definition the index $\mathrm{i}_{\mathfrak{g}}$ of $\mathfrak{g}$ is the smallest dimension of the stabilizers for the coadjoint action. By [5](Lemme 7), when $\mathfrak{g}$ is algebraic, $\mathrm{i}_{\mathfrak{g}}$ is the transcendence degree over $\mathbb{K}$ of the field of invariants for the adjoint action of $\mathfrak{g}$ in the fraction field of $\mathrm{S}(\mathfrak{g})$.

Proposition 2.3. — *We suppose that $\mathfrak{g}$ is algebraic. Then the following conditions are equivalent:*

1) *the field of invariants for the adjoint action of $\mathfrak{g}$ in the fraction field of $\mathrm{S}(\mathfrak{g})$ is the fraction field of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,*
2) *the subalgebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ contains $\mathrm{i}_{\mathfrak{g}}$ algebraically independent elements,*
3) *for any x in a non empty open subset in $\mathfrak{g}^*$, $\mathfrak{g}(x)$ is the image of the map $p \mapsto \mathrm{d}p(x)$.*

Proof. — We denote by K the fraction field of $\mathrm{S}(\mathfrak{g})$ and $K^{\mathfrak{g}}$ the field of invariants for the adjoint action of $\mathfrak{g}$ in K. We will prove the implications (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (2), (2) ⇒ (1).

(1) ⇒ (2)

As the transcendence degree of $K^{\mathfrak{g}}$ over $\mathbb{K}$ is equal to $\mathrm{i}_{\mathfrak{g}}$, $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ contains $\mathrm{i}_{\mathfrak{g}}$ algebraically independent elements because $K^{\mathfrak{g}}$ is the fraction field of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ by hypothesis.

(2) $\Rightarrow$ (3)

Let $p_1, \ldots, p_{\mathrm{i}_{\mathfrak{g}}}$ be algebraically independent elements in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Then for any x in a non empty open subset in $\mathfrak{g}^*$ the elements $\mathrm{d}p_1(x), \ldots, \mathrm{d}p_{\mathrm{i}_{\mathfrak{g}}}(x)$ are linearly independent. But for any x in a non empty open subset in $\mathfrak{g}^*$, $\mathfrak{g}(x)$ has dimension $\mathrm{i}_{\mathfrak{g}}$. Hence by lemma 2.1, (i), $\mathfrak{g}(x)$ is the image of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ by the map $p \mapsto \mathrm{d}p(x)$.

(3) $\Rightarrow$ (2)

There exist x in $\mathfrak{g}^*$ and $p_1, \ldots, p_{\mathrm{i}_{\mathfrak{g}}}$ such that $\mathrm{d}p_1(x), \ldots, \mathrm{d}p_{\mathrm{i}_{\mathfrak{g}}}(x)$ are linearly independent. Hence $p_1, \ldots, p_{\mathrm{i}_{\mathfrak{g}}}$ are algebraically independent.

(2) $\Rightarrow$ (1)

As $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is integrally closed in $\mathrm{S}(\mathfrak{g})$, $K^{\mathfrak{g}}$ is the fraction field of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. □

3. About semi-invariants.

We consider $\mathfrak{g}$ as in 2 An element p in $\mathrm{S}(\mathfrak{g})$ is called semi-invariant if $[v, p]$ is colinear to p for any v in $\mathfrak{g}$. We denote by G be the adjoint algebraic group of $\mathfrak{g}$, G_{u} the unipotent radical of G, $\mathfrak{g}_0$ the intersection of the kernels of the weights of the semi-invariant elements in $\mathrm{S}(\mathfrak{g})$. For v in $\mathfrak{g}^*$, $G(v)$ denotes the stabilizer of v in G for the coadjoint action of G in $\mathfrak{g}^*$.

Lemma 3.1. — *For any x in a non empty open subset V in $\mathfrak{g}^*$, $\mathfrak{g}_0$ is the subset of elements v in $\mathfrak{g}$ such that $\mathrm{ad}\, v$ is in the Lie algebra of $G(x)[G, G]G_{\mathrm{u}}$. Moreover, for any x in V, $\mathfrak{g}_0$ contains the stabilizer in $\mathfrak{g}$ of the restriction of x to $\mathfrak{g}_0$.*

Proof. — By [5](Théorème 3.3), for any x in a non empty open subset V in $\mathfrak{g}^*$, $\mathfrak{g}_0$ is the subset of elements v in $\mathfrak{g}$ such that $\mathrm{ad}\, v$ is in the Lie algebra of $G(x)[G, G]G_{\mathrm{u}}$. Let x be in V, x_0 its restriction to $\mathfrak{g}_0$, $G(x_0)$ the stabilizer of x_0 in G. If T is a maximal torus contained in $G(x_0)$, the affine subspace $x + \mathfrak{g}_0^{\perp}$ is stable for the coadjoint action of T in $\mathfrak{g}^*$. So this action has a fixed point. But

any point in $\mathfrak{g}_0^\perp$ is fix for this action. Hence T is contained in $G(x)$ and G_0. As G/G_0 is a torus, $G(x_0)$ is contained in G_0 and $\mathfrak{g}_0$ contains the stabilizer of x_0 in $\mathfrak{g}$. □

Corollary 3.2. — *The subalgebra* $\mathrm{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ *is generated by the semi-invariant elements in* $\mathrm{S}(\mathfrak{g})$.

Proof. — Let $\mathfrak{A}$ be the subalgebra generated by semi-invariant elements in $\mathrm{S}(\mathfrak{g})$. By lemmas 2.1 and 3.1, any semi-invariant is contained in $\mathrm{S}(\mathfrak{g}_0)$ because it is invariant for the adjoint action of $\mathfrak{g}_0$. So $\mathrm{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ contains $\mathfrak{A}$. Any element in $\mathrm{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ is a sum of weight vectors for the action of G in $\mathrm{S}(\mathfrak{g}_0)$ because any element in $\mathrm{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$ is invariant by $[G,G]$ and G_{u}. So $\mathfrak{A}$ is equal to $\mathrm{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. □

Remark 3.3. — When $\mathfrak{g}$ is algebraic, $\mathrm{ad}\,\mathfrak{g}$ is the Lie algebra of G and the Lie algebra of G_{u} is the image by the adjoint representation of the subset $\mathfrak{g}_{\mathrm{u}}$ of elements v in $\mathfrak{g}$ such that $\mathrm{ad}\,v$ is in the Lie algebra of G_{u}. So for any x in a non empty open subset in $\mathfrak{g}^*$, $\mathfrak{g}_0$ is the sum of $\mathfrak{g}_{\mathrm{u}}$, $[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}(x)$.

We recall that K is the fraction field of $\mathrm{S}(\mathfrak{g})$ and $K^{\mathfrak{g}}$ is the field of invariants for the adjoint action of $\mathfrak{g}$ in K.

Lemma 3.4. — *The field* $K^{\mathfrak{g}}$ *is contained in the fraction field of the subalgebra* $\mathrm{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$.

Proof. — Let p be in $K^{\mathfrak{g}}$. As $\mathrm{S}(\mathfrak{g})$ is a unique factorization domain, p has a unique decomposition

$$a = a_1^{m_1} \cdots a_k^{m_k}$$

where $a_1,\ldots,a_p$ are prime elements in $\mathrm{S}(\mathfrak{g})$ and $m_1,\ldots,m_k$ are integers. But p is invariant for the adjoint action of $\mathfrak{g}$. So because of the unicity of the decomposition, $a_1,\ldots,a_p$ are semi-invariant elements in $\mathrm{S}(\mathfrak{g})$. Then by corollary 3.2, $a_1,\ldots,a_p$ are in $\mathrm{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. □

4. Some remarks about centralizers of nilpotent elements in a simple Lie algebra.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, G the adjoint group of $\mathfrak{g}$, e a nilpotent element in $\mathfrak{g}$. We identify $\mathfrak{g}$ and its dual by

the Killing form $\langle .,. \rangle$. By Jacobson-Morozov there exist elements h and f in $\mathfrak{g}$ such that e, h, f is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. The intersection $\mathfrak{l}$ of $\mathfrak{g}(e)$ and $\mathfrak{g}(f)$ is a reductive factor in $\mathfrak{g}(e)$. Moreover the restriction of $\langle .,. \rangle$ to $\mathfrak{g}(e) \times \mathfrak{g}(f)$ is non degenerate. So $\mathfrak{g}(f)$ is identified to the dual of $\mathfrak{g}(e)$.

4.1. — Let $\mathfrak{t}$ be a subspace in a Cartan subalgebra in $\mathfrak{l}$. We denote by $\mathfrak{a}$ the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. Then $\mathfrak{a}$ is a reductive Lie algebra in $\mathfrak{g}$ which contains e, h, f. Moreover the subspaces $\mathfrak{a}(f)$ and $[\mathfrak{t}, \mathfrak{g}(f)]$ are respectively orthogonal to $[\mathfrak{t}, \mathfrak{g}(e)]$ and $\mathfrak{a}(e)]$. We consider the coadjoint action $(v, x) \mapsto v.x$ of $\mathfrak{g}(e)$ in $\mathfrak{g}(f)$. Let $G(e)_0$ be the identity component of the centralizer of e in G. We denote by $(g, x) \mapsto g.x$ the coadjoint action of $G(e)_0$ on $\mathfrak{g}(f)$.

Lemma 4.1. — *Let τ be the map $(g, x) \mapsto g.x$ from $G(e)_0 \times \mathfrak{a}(f)$ to $\mathfrak{g}(f)$.*

i) *Let W_0 be the set of elements x in $\mathfrak{a}(f)$ such that the linear map $v \mapsto v.x$ from $[\mathfrak{t}, \mathfrak{g}(e)]$ to $\mathfrak{g}(f)$ is injective. Then W_0 is an open subset in $\mathfrak{a}(f)$. Moreover, for any x in W_0, the map $v \mapsto v.x$ is a linear isomorphism from $[\mathfrak{t}, \mathfrak{g}(e)]$ to $[\mathfrak{t}, \mathfrak{g}(f)]$ and $\mathfrak{g}(e)(x)$ is equal to $\mathfrak{a}(e)(x)$.*

ii) *For any x in W_0, the map τ is a submersion at $(\mathrm{id}_{\mathfrak{g}}, x)$.*

iii) *The restriction of τ to $G(e)_0 \times W_0$ is a smooth morphism from $G(e)_0 \times W_0$ onto an open subset in $\mathfrak{g}(f)$.*

iv) *When $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{l}$, for any simple factor $\mathfrak{a}_1$ in $\mathfrak{a}$, the component of e on $\mathfrak{a}_1$ is a distinguished nilpotent element in $\mathfrak{a}_1$.*

Proof. — i) Let x be in $\mathfrak{a}(f)$. For any v in $\mathfrak{g}(e)$ and t in $\mathfrak{t}$, we have

$$\langle w, [t, v].x \rangle = \langle [w, [t, v]], x \rangle = \langle [t, [w, v]], x \rangle = -\langle [w, v], [t, x] \rangle = 0$$

for any w in $\mathfrak{a}(e)$. Hence the image of $[\mathfrak{t}, \mathfrak{g}(e)]$ by the map $v \mapsto v.x$ is contained in $[\mathfrak{t}, \mathfrak{g}(f)]$. As $[\mathfrak{t}, \mathfrak{g}(e)]$ and $[\mathfrak{t}, \mathfrak{g}(f)]$ have the same dimension, W_0 is the set of points x in $\mathfrak{a}(f)$ where the map $v \mapsto v.x$ from $[\mathfrak{t}, \mathfrak{g}(e)]$ to $\mathfrak{g}(f)$ has maximal rank. So W_0 is an open subset in $\mathfrak{a}(f)$. As $\mathfrak{t}$ is contained in $\mathfrak{g}(e)(x)$ for any x in $\mathfrak{a}(f)$, $\mathfrak{g}(e)(x)$ is the direct sum of $\mathfrak{a}(e)(x)$ and its intersection with $[\mathfrak{t}, \mathfrak{g}(e)]$. Then for any x in W_0, $\mathfrak{g}(e)(x)$ is equal to $\mathfrak{a}(e)(x)$.

ii) Let x be in W_0. As the linear map tangent to τ at $(\mathrm{id}_{\mathfrak{g}}, x)$ is the map

$$\mathrm{ad}\,\mathfrak{g}(e) \times \mathfrak{a}(f) \to \mathfrak{g}(f) \ , \ (v, y) \mapsto v.x + y \ ,$$

τ is a submersion at $(\mathrm{id}_{\mathfrak{g}}, x)$ by (i).

iii) For any x in $\mathfrak{a}(f)$ the subset of elements g in $G(e)_0$ such that τ is a submersion at (g, x) is stable by left multiplication. So the restriction of τ to $G(e)_0 \times W_0$ is a smooth morphism from $G(e)_0 \times W_0$ onto an open subset in $\mathfrak{g}(f)$.

iv) Let us suppose that $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{l}$. Let $\mathfrak{a}_1$ be a simple factor in $\mathfrak{a}$ and e_1 the component of e on $\mathfrak{a}_1$. As $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{l}$, $\mathfrak{a}_1(e_1)$ has no semi-simple elements because $\mathfrak{a}_1(e_1)$ is the intersection of $\mathfrak{g}(e)$ and $\mathfrak{a}_1$. So e_1 is a distinguished nilpotent element. □

Corollary 4.2. — *The open subset W_0 is not empty if and only if for any x in a non empty open subset in $\mathfrak{g}(f)$ the orbit $G(e)_0.x$ contains an element whose stabilizer contains $\mathfrak{t}$. Moreover in this case $\mathfrak{g}(e)$ and $\mathfrak{a}(e)$ have the same index.*

Proof. — If W_0 is not empty then by lemma 4.1, (iii), for any x in a non empty open subset in $\mathfrak{g}(f)$, $\mathfrak{g}(e)(x)$ contains a subalgebra conjugated to $\mathfrak{t}$ by adjoint action because $\mathfrak{g}(e)(y)$ contains $\mathfrak{t}$ for any y in W_0. Reciprocally we suppose that there exists a non empty open subset U in $\mathfrak{g}(f)$ such that $G(e)_0.x$ contains an element whose stabilizer contains $\mathfrak{t}$. As we can suppose U invariant by $G(e)_0$ there exists an element x in $\mathfrak{a}(f)$ which is regular as a linear form on $\mathfrak{g}(e)$ and $\mathfrak{a}(e)$. In particular $\mathfrak{a}(e)(x)$ and $\mathfrak{g}(e)(x)$ are commutative subalgebra. Hence $\mathfrak{g}(e)(x)$ is contained in $\mathfrak{a}(e)(x)$ because it contains $\mathfrak{t}$. So the map $v \mapsto v.x$ from $[\mathfrak{t}, \mathfrak{g}(e)]$ to $\mathfrak{g}(f)$ is injective and W_0 contains x. Moreover for any v in $\mathfrak{a}(e)(x)$, x is orthogonal to $[v, [\mathfrak{t}, \mathfrak{g}(e)]]$ because it is contained in $[\mathfrak{t}, \mathfrak{g}(e)]$. So $\mathfrak{g}(e)(x)$ is equal to $\mathfrak{a}(e)(x)$ and $\mathfrak{a}(e)$ has the same index as $\mathfrak{g}(e)$. □

4.2. — For any λ in $\mathfrak{t}^*$, we respectively denote by E_λ and F_λ the weight subspaces of weight λ for the adjoint action of $\mathfrak{t}$ in $\mathfrak{g}(e)$ and $\mathfrak{g}(f)$.

Lemma 4.3. — *Let λ be in $\mathfrak{t}^*$. Then the subspaces E_λ, $E_{-\lambda}$, F_λ, $F_{-\lambda}$ have the same dimension. Moreover the eigenvalues of the restrictions of* $\mathrm{ad}\, h$ *to E_λ and $E_{-\lambda}$ are the same with the same multiplicities.*

Proof. — We denote by $\mathfrak{s}$ the subspace generated by e, h, f. It is well known that the restriction of $\langle .,. \rangle$ to $\mathfrak{g}(e) \times \mathfrak{g}(f)$ is non degenerate. But for μ in $\mathfrak{t}^*$, F_μ is orthogonal to E_λ if μ is different from $-\lambda$. So the restriction of $\langle .,. \rangle$ to $E_\lambda \times F_{-\lambda}$ is non degenerate because $\mathfrak{g}(e)$ and $\mathfrak{g}(f)$ are respectively the sum of the subspaces E_μ and F_μ where μ is in $\mathfrak{t}^*$. So the dimension of E_λ and $F_{-\lambda}$ are equal. As $\mathfrak{t}$ centralizes h, there exists a basis $w_1, \ldots, w_k$ in $F_{-\lambda}$ whose elements are weight vectors for h. For $i = 1, \ldots, k$, we denote by W_i the sub-$\mathfrak{s}$-module in $\mathfrak{g}$ generated by w_i. Then W_i is simple and the intersection of W_i and $\mathfrak{g}(e)$ has dimension 1. Let v_i be a non zero element in this intersection. As $\mathfrak{t}$ centralizes $\mathfrak{s}$, any element in W_i is a weight vector of weight $-\lambda$ for $\mathfrak{t}$. So $E_{-\lambda}$ contains $v_1, \ldots, v_k$. As $w_1, \ldots, w_k$ are linearly independent, the sum of subspaces $W_1, \ldots, W_k$ is direct. So the elements $v_1, \ldots, v_k$ are linearly independent. Then the dimension of $E_{-\lambda}$ is bigger than the dimension of E_λ. By the same reasons, $E_{-\lambda}$ and F_λ have the same dimension and it is smaller than the dimension of E_λ. Hence the subspaces E_λ, $E_{-\lambda}$, F_λ, $F_{-\lambda}$ have the same dimension. Moreover $v_1, \ldots, v_k$ is a basis in $E_{-\lambda}$ whose elements are eigenvectors for $\mathrm{ad}\, h$. So if d is an eigenvalue of the restriction of $\mathrm{ad}\, h$ to $E_{-\lambda}$ with multiplicity m, $-d$ is an eigenvalue of the restriction of $\mathrm{ad}\, h$ to $F_{-\lambda}$ with multiplicity m. From the duality between E_λ and $F_{-\lambda}$ we deduce that d is an eigenvalue of the restriction of $\mathrm{ad}\, h$ to $E_{-\lambda}$ with multiplicity m. □

Let λ be a non zero weight of the adjoint action of $\mathfrak{t}$ in $\mathfrak{g}(e)$. Let $v_1, \ldots, v_k$ be a basis in E_λ and $w_1, \ldots, w_k$ a basis in $E_{-\lambda}$.

Lemma 4.4. — *Let M_λ be the matrix*

$$M_\lambda = \begin{bmatrix} [v_1, w_1] & \cdots & [v_1, w_k] \\ \vdots & \ddots & \vdots \\ [v_k, w_1] & \cdots & [v_k, w_k] \end{bmatrix}$$

with coefficients in $\mathrm{S}(\mathfrak{g}(e))$ *and* δ_λ *its determinant.*

i) *The element* δ_λ *is in* $\mathrm{S}(\mathfrak{a}(e))$. *Moreover up to a multiplicative scalar* δ_λ *does not depend on the choice of the basis in* E_λ *and* $E_{-\lambda}$.

ii) *For any* x *in* $\mathfrak{a}(f)$, $\mathfrak{g}(e)(x)$ *is the sum of its intersections with the subspaces* E_μ *where* μ *is in* $\mathfrak{t}^*$.

iii) *Let* x *be in* $\mathfrak{a}(f)$. *Then the intersection of* E_λ *and* $\mathfrak{g}(e)(x)$ *is equal to* $\{0\}$ *if and only if* $\delta_\lambda(x)$ *is different from* 0.

Proof. — By definition, E_0 and F_0 are respectively equal to $\mathfrak{a}(e)$ and $\mathfrak{a}(f)$.

i) For v in E_λ and w in $E_{-\lambda}$, $[v,w]$ is in $\mathfrak{a}(e)$. So δ_λ is in $\mathrm{S}(\mathfrak{a}(e))$. Let α be a linear automorphism in E_λ, α its matrix in the basis $v_1,\dots,v_k$, $\alpha_{i,j}$ the coefficient of α on the i-rd line and j-rd column. From the equality:

$$\alpha(v_i) = \sum_{j=1}^{k} \alpha_{i,j} v_j$$

for $i = 1,\dots,k$, we deduce

$$\alpha M_\lambda = \begin{bmatrix} [\alpha(v_1), w_1] & \cdots & [\alpha(v_1), w_k] \\ \vdots & \ddots & \vdots \\ [\alpha(v_k), w_1] & \cdots & [\alpha(v_k), w_k] \end{bmatrix} .$$

Hence up to a multiplicative scalar δ_λ does not depend on the basis $v_1,\dots,v_k$. As it is the same for the basis in $E_{-\lambda}$, the proof is done.

ii) Let x be in $\mathfrak{a}(f)$. As $\mathfrak{a}(f)$ is orthogonal to E_μ for μ different from 0, $\mathfrak{t}$ is contained in $\mathfrak{g}(e)(x)$. Then $\mathfrak{g}(e)(x)$ is stable by the adjoint action of $\mathfrak{t}$ in $\mathfrak{g}(e)$. So $\mathfrak{g}(e)(x)$ is the sum of its intersections with the subspaces E_μ.

iii) Let v be in E_λ and $a_1,\dots,a_k$ its coordinates in the basis $v_1,\dots,v_k$. The element v is in $\mathfrak{g}(e)(x)$ if and only if $\langle x, [v, w_j]\rangle$ is equal to 0 for $j = 1,\dots,k$. This condition is equivalent to the equality:

$$\begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix} \begin{bmatrix} \langle x, [v_1, w_1]\rangle & \cdots & \langle x, [v_1, w_k]\rangle \\ \vdots & \ddots & \vdots \\ \langle x, [v_k, w_1]\rangle & \cdots & \langle x, [v_k, w_k]\rangle \end{bmatrix} = 0 .$$

So the intersection of E_λ and $\mathfrak{g}(e)(x)$ is equal to $\{0\}$ if and only if $\delta_\lambda(x)$ is different from 0. □

Let δ be the product of δ_λ where λ is a non zero weight of the adjoint action of $\mathfrak{t}$ in $\mathfrak{g}(e)$.

Corollary 4.5. — *The element δ in $\mathrm{S}(\mathfrak{a}(e))$ is different from 0 if and only if for any x in a non empty open subset in $\mathfrak{g}(f)$ the orbit $G(e)_0.x$ contains an element whose stabilizer contains $\mathfrak{t}$.*

Proof. — We consider the open subset W_0 in $\mathfrak{a}(f)$ introduced in lemma 4.1, (i). Then an element x in $\mathfrak{a}(f)$ is in W_0 if and only if $\delta(x)$ is different from 0. Hence the corollary is a consequence of corollary 4.2. □

We recall that an element x in $\mathfrak{g}(f)$ is called regular if the dimension of $\mathfrak{g}(e)(x)$ is minimal. Moreover the subset of regular elements in $\mathfrak{g}(f)$ is open.

Corollary 4.6. — *If δ is equal to 0, then for any regular element x in $\mathfrak{g}(f)$, $\mathfrak{g}(e)(x)$ does not contain a conjugate of $\mathfrak{t}$ by the adjoint group of $\mathfrak{g}(e)$.*

Proof. — Let us suppose that there exists a regular element x in $\mathfrak{g}(f)$ such that $\mathfrak{g}(e)(x)$ contains a conjugate of $\mathfrak{t}$ by the adjoint group of $\mathfrak{g}(e)$. As the subset of regular elements in $\mathfrak{g}(f)$ is stable for the action of the adjoint group of $\mathfrak{g}(e)$ we can suppose that $\mathfrak{g}(e)(x)$ contains $\mathfrak{t}$. As x is regular, $\mathfrak{g}(e)(x)$ is commutative and contained in $\mathfrak{a}(e)$. Then by lemma 4.4, (iii), $\delta_\lambda(x)$ is not equal to 0 for any non zero weight λ of the adjoint action of $\mathfrak{t}$ in $\mathfrak{g}(e)$. □

We denote by $\mathfrak{g}(e)_{\mathrm{u}}$ the subset of nilpotent elements in the radical of $\mathfrak{g}(e)$.

Corollary 4.7. — *Let d be the biggest eigenvalue of $\mathrm{ad}\,h$. We suppose that the kernel of $\mathrm{ad}\,h - d$ has dimension smaller than 3 and does not centralize $\mathfrak{t}$. Moreover if the kernel of $\mathrm{ad}\,h - d$ has dimension 3, $\mathfrak{t}$ is contained in the center of $\mathfrak{l}$.*

i) *For any regular element x in $\mathfrak{g}(f)$, $\mathfrak{g}(e)(x)$ does not contain a conjugate of $\mathfrak{t}$ by the adjoint representation.*

ii) *If* $\mathfrak{l}$ *has rank* 1 *then for any* x *in a non empty open subset in* $\mathfrak{g}(f)$*, the elements of* $\mathfrak{g}(e)(x)$ *are nilpotent.*

iii) *If* $\mathfrak{t}$ *is the center of* $\mathfrak{l}$*, then the symmetric algebra* $\mathrm{S}(\mathfrak{g}(e))$ *of* $\mathfrak{g}(e)$ *contains semi-invariant elements which are not in* $\mathrm{S}(\mathfrak{g}(e))^{\mathfrak{g}(e)}$*.*

Proof. — i) As d is the biggest eigenvalue of $\mathrm{ad}\,h$, the kernel of $\mathrm{ad}\,h - d$ is contained in $\mathfrak{g}(e)$. By hypothesis and lemma 4.3, there exists a non zero linear form λ on $\mathfrak{t}$ such that λ and $-\lambda$ are weights of the adjoint action of $\mathfrak{t}$ in the kernel $\mathrm{ad}\,h - d$. Let w_1 be a non zero element in E_λ such that $[h, w_1]$ is equal to dw_1. As d is the biggest eigenvalue of the restriction of $\mathrm{ad}\,h$ to $\mathfrak{g}(e)_{\mathrm{u}}$, w_1 centralizes $\mathfrak{g}(e)_{\mathrm{u}}$ because any eigenvalue of the restriction of $\mathrm{ad}\,h$ to $\mathfrak{g}(e)_{\mathrm{u}}$ is strictly positive. If there exists a non zero element v in the intersection of $E_{-\lambda}$ and $\mathfrak{l}$, the dimension of the kernel of $\mathrm{ad}\,h - d$ is equal to 2 and $[v, w_1]$ is an element in the kernel of $\mathrm{ad}\,h - d$ which centralizes $\mathfrak{t}$. So w_1 centralizes $E_{-\lambda}$ and δ_λ is equal to 0. Hence by corollary 4.6, for any regular element x in $\mathfrak{g}(f)$, $\mathfrak{g}(e)(x)$ does not contain a conjugate of $\mathfrak{t}$ by the adjoint group of $\mathfrak{g}(e)$.

ii) We suppose that $\mathfrak{l}$ has rank 1. Then for any non zero semi-simple element v in $\mathfrak{g}(e)$ the line containing v is conjugate to $\mathfrak{t}$ by the adjoint action. So by (i), for any x in a non empty open subset the elements of $\mathfrak{g}(e)(x)$ are nilpotent.

iii) We suppose that $\mathfrak{t}$ is the center of $\mathfrak{l}$. Then by (i), for any x in a non empty open subset, $\mathfrak{g}(e)$ strictly contains the sum of the subspaces $\mathfrak{g}(e)(x)$, $[\mathfrak{l}, \mathfrak{l}]$, $\mathfrak{g}(e)_{\mathrm{u}}$. Hence by corollary 3.2, $\mathrm{S}(\mathfrak{g}(e))$ contains semi-invariant elements which are not invariant. □

Lemma 4.8. — *Let* d *be the biggest eigenvalue of* $\mathrm{ad}\,h$*. We suppose that the following conditions are satisfied:*

1) *the kernel of* $\mathrm{ad}\,h - d$ *has dimension smaller than* 3 *and does not centralize* $\mathfrak{t}$*,*
2) *if the kernel of* $\mathrm{ad}\,h - d$ *has dimension* 3 *then* $\mathfrak{t}$ *is contained in the center of* $\mathfrak{l}$*,*
3) *if* μ *is a non zero weight of the adjoint action of* $\mathfrak{t}$ *in* $\mathfrak{g}(e)$*,* δ_μ *is not equal to* 0*,*

4) *if λ is a non zero weight of the adjoint action of $\mathfrak{t}$ in the kernel of* $\mathrm{ad}\, h - d$*, there exists a principal minor in the matrix M_λ which is not equal to* 0.

Then the index of $\mathfrak{a}(e)$ is bigger than the index of $\mathfrak{g}(e)$.

Proof. — For i in $\{-1, +1\}$ we denote by $\mathfrak{k}_{i\lambda}$ the intersection of $E_{i\lambda}$ and the kernel of $\mathrm{ad}\, h - d$. Let $\mathfrak{k}$ be the sum of $\mathfrak{k}_\lambda$ and $\mathfrak{k}_{-\lambda}$. Then by condition (1) and lemma 4.1, $\mathfrak{k}$ has dimension 2. Moreover by condition (2), $\mathfrak{k}$ is an ideal in $\mathfrak{g}(e)$. Let $\mathfrak{q}$ be the quotient of $\mathfrak{g}(e)$ by $\mathfrak{k}$. Then the restriction to $\mathfrak{a}(e)$ of the canonical morphism from $\mathfrak{g}(e)$ to $\mathfrak{q}$ is injective. So we can identify $\mathfrak{a}(e)$ with its image in $\mathfrak{q}$. Then $\mathfrak{a}(e)$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{q}$. For any weight μ of the adjoint action of $\mathfrak{t}$ in $\mathfrak{q}$ we denote by E'_μ the weight subspace of weight μ. Then E'_μ is the image of E_μ by the canonical morphism from $\mathfrak{g}(e)$ to $\mathfrak{q}$. When μ is not equal to λ or $-\lambda$, E_μ and E'_μ have the same dimension. Otherwise, E'_λ and $E'_{-\lambda}$ have dimension $\dim E_\lambda - 1$. Moreover by condition (4) if $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ are basis in E'_λ and $E'_{-\lambda}$, we have

$$\det \begin{bmatrix} [v_1, w_1] & \cdots & [v_1, w_k] \\ \vdots & \ddots & \vdots \\ [v_k, w_1] & \cdots & [v_k, w_k] \end{bmatrix} \neq 0 \ ,$$

because the intersection of E_λ and $\mathfrak{k}$ centralizes $E_{-\lambda}$. Then by corollaries 4.5 and 4.2, $\mathfrak{a}(e)$ and $\mathfrak{q}$ have the same index. Let x be an element in $\mathfrak{a}(f)$ which is a linear regular form on $\mathfrak{a}(e)$ and $\mathfrak{q}$. Then $\mathfrak{g}(e)(x)$ is equal to $\mathfrak{a}(e)(x) + \mathfrak{k}$. In particular the dimension of $\mathfrak{g}(e)(x)$ is $\mathrm{i}_{\mathfrak{a}(e)} + 2$ and $\mathfrak{g}(e)(x)$ is not commutative because it contains $\mathfrak{t}$ and $\mathfrak{k}$. Hence x is not a regular linear form on $\mathfrak{g}(e)$ and $\mathrm{i}_{\mathfrak{g}(e)}$ is smaller than $\mathrm{i}_{\mathfrak{a}(e)}$. □

We finish this section by giving an example of a simple Lie algebra and a nilpotent element in it for which the symmetric algebra of its centralizer contains a semi-invariant which is not invariant.

Let $\mathfrak{g}$ be a simple Lie algebra of type E_7. By the tables in [2], $\mathfrak{g}$ contains a nilpotent element e such that $\mathfrak{g}(e)$ has dimension 35. Considering e, h, f as above, $\mathfrak{l}$ is a direct product of a torus $\mathfrak{t}$ of dimension 1 by a simple Lie algebra of dimension 3. The biggest

eigenvalue of $\mathrm{ad}\,h$ is 6 and its multiplicity is 3. Furthermore the kernel of $\mathrm{ad}\,h - 6$ does not centralize $\mathfrak{t}$. Then by corollary 4.7, (iii), the symmetric algebra of $\mathfrak{g}(e)$ contains semi-invariant elements which are not invariant. The centralizer $\mathfrak{a}$ of $\mathfrak{t}$ has dimension 27, $\mathfrak{t}$ is the center of $\mathfrak{a}$, $\mathfrak{a}(e)$ has dimension 9. Then $[\mathfrak{a}, \mathfrak{a}]$ is isomorphic to the direct product of $\mathfrak{sl}_4$, $\mathfrak{sl}_3$, $\mathfrak{sl}_2$. So the index of $\mathfrak{a}(e)$ is 7. There exists an element t in $\mathrm{ad}\,\mathfrak{t}$ whose eigenvalues are integers. The strictly positive values of these integers are $1, 2, 3, 4$ and their respective multiplicities are $6, 4, 2, 1$. Moreover the eigenvalues of the restriction of t to the kernel of $\mathrm{ad}\,h - 6$ are $-2, 0, 2$. As $\mathfrak{t}$ has dimension 1, we get elements $\delta_1, \delta_2, \delta_3, \delta_4$ in $\mathrm{S}(\mathfrak{a}(e))$. For i not equal to 2, δ_i is different from 0. Moreover in the matrix whose determinant is δ_2 there is a principal minor which is not equal to 0. Hence by lemma 4.4, the index of $\mathfrak{g}(e)$ is smaller than 7. So by Vinberg's result, the index of $\mathfrak{g}(e)$ is 7.

5. An example of affine quotient.

Let $\mathfrak{g}$ be a simple Lie algebra of type F_4 and G its adjoint group. Following the tables in [2], there exists a nilpotent element e in $\mathfrak{g}$ such that $\mathfrak{g}(e)$ has dimension 16 and the reductive factors of $\mathfrak{g}(e)$ are simple of dimension 3. Then we have the proposition:

Proposition 5.1. — *Let $\mathfrak{g}(e)^*$ be the dual of $\mathfrak{g}(e)$. For any x in a non empty open subset in $\mathfrak{g}(e)^*$, the coadjoint orbit of x is closed in $\mathfrak{g}(e)^*$, the elements of the stabilizer $\mathfrak{g}(e)(x)$ of x in $\mathfrak{g}(e)$ are nilpotent and $\mathfrak{g}(e)(x)$ is not contained in the subset of nilpotent elements in the radical of $\mathfrak{g}(e)$.*

Denoting by G^e the adjoint group of $\mathfrak{g}(e)$, for any x in a non empty open subset in $\mathfrak{g}(e)^*$, the coadjoint orbit $G^e.x$ is an affine variety and the identity component H of the stabilizer of x in G^e is a unipotent subgroup, not contained in the unipotent radical of G^e. So we get an example of affine quotient G^e/H where H is a unipotent subgroup in G^e not contained in the unipotent radical of G^e.

As above we consider elements h and f in $\mathfrak{g}$ such that e, h, f is an $\mathfrak{sl}_2$-triple. We identify $\mathfrak{g}(f)$ and $\mathfrak{g}(e)^*$ by the Killing form. The intersection $\mathfrak{l}$ of $\mathfrak{g}(e)$ and $\mathfrak{g}(f)$ is a simple Lie algebra of dimension 3 and the subset $\mathfrak{g}(e)_{\mathrm{u}}$ of nilpotent elements in the radical of $\mathfrak{g}(e)$ is an ideal of dimension 13. As $\mathfrak{g}(e)$ is a semi-direct product of $\mathfrak{l}$ and $\mathfrak{g}(e)_{\mathrm{u}}$, $\mathfrak{g}(e)$ is a unimodular Lie algebra. So by [3](Théorème 3.12), for any x in a non empty open subset in $\mathfrak{g}(f)$ the coadjoint orbit of x is closed in $\mathfrak{g}(f)$. Using computation by logicial Gap4 we find a basis $x_1, \ldots, x_{16}$ in $\mathfrak{g}(e)$ whose image by $\mathrm{ad}\, h$ is the following sequence

$$x_1, 0, x_3, 2x_4, 2x_5, 3x_6, x_7, 4x_8, 3x_9, 4x_{10}, 5x_{11}, 4x_{12}, 5x_{13}, x_{14}, 0, 0\ .$$

Moreover e is equal to $x_4 + x_5$. Then x_2, x_{15}, x_{16} is a basis in $\mathfrak{l}$ and $x_1, x_3, \ldots, x_{14}$ is a basis in $\mathfrak{g}(e)_{\mathrm{u}}$. The biggest eigenvalue of $\mathrm{ad}\, h$ is equal to five and its multiplicity is equal to 2. So by corollary 4.6, for any x in a non empty open subset in $\mathfrak{g}(f)$, the elements of $\mathfrak{g}(e)(x)$ are nilpotent. The element p in $\mathrm{S}(\mathfrak{g}(e))$

$$p = -9x_{13}x_{12}x_{14} + 9x_{11}x_8x_7 + 3/2x_{13}x_4x_6 - 3{x_{11}}^2x_2 + 3{x_{13}}^2x_{15} + 3/4x_4{x_{10}}^2$$
$$-9/4x_5{x_{10}}^2 - 3x_1x_{13}x_{10} + 3x_1x_{12}x_{11} - 3x_3x_{13}x_8 + 3x_3x_{11}x_{10} - 3x_4x_{12}x_8$$
$$+ 9x_5x_{12}x_8 + 3/2x_{11}x_4x_9 + 3x_{11}x_{13}x_{16}\ ,$$

is invariant for the adjoint action. We remark that p is not contained in $\mathrm{S}(\mathfrak{g}(e)_{\mathrm{u}})$. So by corollary 2.2, for any x in a non empty open subset in $\mathfrak{g}(f)$, $\mathfrak{g}(e)(x)$ is not contained in $\mathfrak{g}(e)_{\mathrm{u}}$. As the intersection of a finitely many non empty open subsets in $\mathfrak{g}(f)$ is non empty, for any x in a non empty open subset in $\mathfrak{g}(f)$ the adjoint orbit of x is closed in $\mathfrak{g}(f)$, $\mathfrak{g}(e)(x)$ is not contained in $\mathfrak{g}(e)_{\mathrm{u}}$ and its elements are nilpotent. In appendix a Maple program is given to compute 4 algebraic independent elements $p(1), p(2), p(3), p(4)$ in $\mathrm{S}(\mathfrak{g}(e))^{\mathfrak{g}(e)}$. The element p above is equal to $p(3)$. As a consequence of lemma 4.8, the index of $\mathfrak{g}(e)$ is equal to 4. As there are sufficiently many variables with degree 1 in the polynomials $p(1), p(2), p(3), p(4)$, it is easy to see that the generic fiber of the morphism τ whose comorphism is the canonical injection from $\mathbb{C}[p(1), p(2), p(3), p(4)]$ to $\mathrm{S}(\mathfrak{g}(e))$, is irreducible. Hence the subfield generated by $p(1), p(2), p(3), p(4)$ is the subfield of invariants for the

adjoint action of $\mathfrak{g}$ in the fraction field of $\mathrm{S}(\mathfrak{g}(e))$. The main point is that the nullvariety of $p(1), p(2), p(3), p(4)$ in $\mathfrak{g}(e)^*$ has codimension 4. Then we deduce that τ is open and surjective. Hence for any q in $\mathrm{S}(\mathfrak{g}(e))^{\mathfrak{g}(e)}$ the morphism whose comorphism is the canonical injection from $\mathbb{C}[p(1), p(2), p(3), p(4)]$ to $\mathbb{C}[p(1), p(2), p(3), p(4)][q]$ is surjective and quasi finite. So by main Zariski's theorem, this morphism is an isomorphism. Hence $\mathrm{S}(\mathfrak{g}(e))^{\mathfrak{g}(e)}$ is a polynomial algebra generated by $p(1), p(2), p(3), p(4)$. Moreover $\mathrm{S}(\mathfrak{g}(e))$ is a faithfully flat extension of $\mathrm{S}(\mathfrak{g}(e))^{\mathfrak{g}(e)}$. The same method proves that for any distinguished nilpotent element x of an exceptional simple Lie algebra $\mathfrak{e}$, the algebra $\mathrm{S}(\mathfrak{e}(x))^{\mathfrak{e}(x)}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{e}(x))$ is a faithfully flat extension of $\mathrm{S}(\mathfrak{e}(x))^{\mathfrak{e}(x)}$.

Appendix A
Appendix.

In this appendix we give the Maple program which computes four algebraically independent elements in $\mathrm{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ when $\mathfrak{g}$ is simple of type F_4 and e is a nilpotent element such that $\mathfrak{g}(e)$ has dimension 16 and its reductive factors are simple of dimension 3. In this program $x_1, \ldots, x_d$ is a basis of $\mathfrak{g}(e)$ and $[x_i, x_j]$ is equal to $L(i, j)$. Moreover for $i = 1, \ldots, 16$, $y(i)$ is equal to $[h, x_i]$ and the center $\mathfrak{z}$ of $\mathfrak{g}(e)$ has dimension 1.

```
# The procedures k and f are elements in the kernel of
  the matrix A.

 with(linalg):

  L := proc(i,j)
    if i=1 and j=2 then return 2*x[3]:
  elif i=1 and j=3 then return 3*x[5]:
  elif i=1 and j=4 then return 0:
  elif i=1 and j=5 then return 0:
  elif i=1 and j=6 then return 2*x[8]:
  elif i=1 and j=7 then return 0:
```

```
elif i=1 and j=8 then return 0:
elif i=1 and j=9 then return -2*x[10]:
elif i=1 and j=10 then return 2*x[11]:
elif i=1 and j=11 then return 0:
elif i=1 and j=12 then return x[13]:
elif i=1 and j=13 then return 0:
elif i=1 and j=14 then return 0:
elif i=1 and j=15 then return 3*x[14]:
elif i=1 and j=16 then return x[1]:
elif i=2 and j=3 then return -3*x[7]:
elif i=2 and j=4 then return 0:
elif i=2 and j=5 then return 0:
elif i=2 and j=6 then return x[9]:
elif i=2 and j=7 then return 0:
elif i=2 and j=8 then return x[10]:
elif i=2 and j=9 then return 0:
elif i=2 and j=10 then return 2*x[12]:
elif i=2 and j=11 then return -x[13]:
elif i=2 and j=12 then return 0:
elif i=2 and j=13 then return 0:
elif i=2 and j=14 then return -x[1]:
elif i=2 and j=15 then return x[16]:
elif i=2 and j=16 then return -2*x[2]:
elif i=3 and j=4 then return 0:
elif i=3 and j=5 then return 0:
elif i=3 and j=6 then return -2*x[10]:
elif i=3 and j=7 then return 0:
elif i=3 and j=8 then return x[11]:
elif i=3 and j=9 then return 2*x[12]:
elif i=3 and j=10 then return 2*x[13]:
elif i=3 and j=11 then return 0:
elif i=3 and j=12 then return 0:
elif i=3 and j=13 then return 0:
elif i=3 and j=14 then return 0:
elif i=3 and j=15 then return 2*x[1]:
elif i=3 and j=16 then return -x[3]:
```

```
elif i=4 and j=5 then return 0:
elif i=4 and j=6 then return 2*x[11]:
elif i=4 and j=7 then return 0:
elif i=4 and j=8 then return 0:
elif i=4 and j=9 then return -2*x[13]:
elif i=4 and j=10 then return 0:
elif i=4 and j=11 then return 0:
elif i=4 and j=12 then return 0:
elif i=4 and j=13 then return 0:
elif i=4 and j=14 then return 0:
elif i=4 and j=15 then return 0:
elif i=4 and j=16 then return 0:
elif i=5 and j=6 then return -2*x[11]:
elif i=5 and j=7 then return 0:
elif i=5 and j=8 then return 0:
elif i=5 and j=9 then return 2*x[13]:
elif i=5 and j=10 then return 0:
elif i=5 and j=11 then return 0:
elif i=5 and j=12 then return 0:
elif i=5 and j=13 then return 0:
elif i=5 and j=14 then return 0:
elif i=5 and j=15 then return 0:
elif i=5 and j=16 then return 0:
elif i=6 and j=7 then return -2*x[12]:
elif i=6 and j=8 then return 0:
elif i=6 and j=9 then return 0:
elif i=6 and j=10 then return 0:
elif i=6 and j=11 then return 0:
elif i=6 and j=12 then return 0:
elif i=6 and j=13 then return 0:
elif i=6 and j=14 then return 0:
elif i=6 and j=15 then return 0:
elif i=6 and j=16 then return x[6]:
elif i=7 and j=8 then return x[13]:
elif i=7 and j=9 then return 0:
elif i=7 and j=10 then return 0:
```

```
elif i=7 and j=11 then return 0:
elif i=7 and j=12 then return 0:
elif i=7 and j=13 then return 0:
elif i=7 and j=14 then return x[5]:
elif i=7 and j=15 then return x[3]:
elif i=7 and j=16 then return -3*x[7]:
elif i=8 and j=9 then return 0:
elif i=8 and j=10 then return 0:
elif i=8 and j=11 then return 0:
elif i=8 and j=12 then return 0:
elif i=8 and j=13 then return 0:
elif i=8 and j=14 then return 0:
elif i=8 and j=15 then return 0:
elif i=8 and j=16 then return 2*x[8]:
elif i=9 and j=10 then return 0:
elif i=9 and j=11 then return 0:
elif i=9 and j=12 then return 0:
elif i=9 and j=13 then return 0:
elif i=9 and j=14 then return -2*x[8]:
elif i=9 and j=15 then return -x[6]:
elif i=9 and j=16 then return -x[9]:
elif i=10 and j=11 then return 0:
elif i=10 and j=12 then return 0:
elif i=10 and j=13 then return 0:
elif i=10 and j=14 then return 0:
elif i=10 and j=15 then return -2*x[8]:
elif i=10 and j=16 then return 0:
elif i=11 and j=12 then return 0:
elif i=11 and j=13 then return 0:
elif i=11 and j=14 then return 0:
elif i=11 and j=15 then return 0:
elif i=11 and j=16 then return x[11]:
elif i=12 and j=13 then return 0:
elif i=12 and j=14 then return -x[11]:
elif i=12 and j=15 then return -x[10]:
elif i=12 and j=16 then return -2*x[12]:
```

```
 elif i=13 and j=14 then return 0:
 elif i=13 and j=15 then return x[11]:
 elif i=13 and j=16 then return -x[13]:
 elif i=14 and j=15 then return 0:
 elif i=14 and j=16 then return 3*x[14]:
 elif i=15 and j=16 then return 2*x[15]:
 fi:
 if i=j then return 0 fi:
 if j<i then return -L(j,i) fi:
 end:

 y := proc(i)
  if i=1 then return x[1]:
elif i=2 then return 0*x[2]:
elif i=3 then return 1*x[3]:
elif i=4 then return 2*x[4]:
elif i=5 then return 2*x[5]:
elif i=6 then return 3*x[6]:
elif i=7 then return 1*x[7]:
elif i=8 then return 4*x[8]:
elif i=9 then return 3*x[9]:
elif i=10 then return 4*x[10]:
elif i=11 then return 5*x[11]:
elif i=12 then return 4*x[12]:
elif i=13 then return 5*x[13]:
elif i=14 then return x[14]:
elif i=15 then return 0*x[15]:
elif i=16 then return 0*x[16]:
fi: end:

 E := diag(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1):

 A := matrix(16,16):
 for i from 1 to 16 do
 for j from 1 to 16 do
 A[i,j] := L(j,i):
```

```
  od: od:

  f := proc(i)

  if i=1 then return matrix(1,16,[0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0]):
  elif i=2 then return
matrix(1,16,[0,0,0,0,0,x[13],0,-2*x[12],x[11],x[10],x[9],-2*x[8],x[6],
0,0,0]):
  elif i=3 then return
matrix(1,16,[-x[11]*x[13]^2,0,x[13]*x[11]^2,x[13]^2*x[8]+
x[13]*x[11]*x[10]+x[12]*x[11]^2,0,0,-x[11]^3,-x[13]^2*x[5],
0,-x[11]*x[13]*x[5],-2*x[11]*x[12]*x[5]-3*x[11]^2*x[7]-
x[10]*x[13]*x[5]-x[13]^2*x[1]+2*x[13]*x[11]*x[3],-x[11]^2*x[5],
-x[11]*x[5]*x[10]+3*x[13]^2*x[14]+x[11]^2*x[3]-
2*x[13]*x[11]*x[1]-2*x[5]*x[13]*x[8],x[13]^3,0,0]):
  elif i=4 then return
matrix(1,16,[-(x[13]*x[10]+2*x[12]*x[11])*x[13]^2,-x[13]^2*x[11]^2,
-(x[13]^2*x[8]-3*x[12]*x[11]^2-x[13]*x[11]*x[10])*x[13],
3*x[12]^2*x[11]^2+1/2*x[13]^3*x[6]+1/2*x[13]^2*x[9]*x[11]-
x[12]*x[13]^2*x[8]+x[13]^2*x[10]^2+3*x[13]*x[10]*x[12]*x[11],
0,0,3*x[11]*(x[13]^2*x[8]-x[12]*x[11]^2),-x[13]^2*(x[13]*x[3]-
3*x[11]*x[7]),0,1/2*x[13]*(-2*x[13]^2*x[1]+2*x[13]*x[11]*x[3]-
6*x[11]*x[12]*x[5]-3*x[10]*x[13]*x[5]),-3*x[10]*x[13]*x[5]*x[12]+
6*x[13]*x[11]*x[3]*x[12]+3*x[13]^2*x[7]*x[8]+x[10]*x[13]^2*x[3]-
9*x[11]^2*x[7]*x[12]+x[13]^3*x[16]-
2*x[13]^2*x[11]*x[2]-6*x[11]*x[12]^2*x[5]-2*x[13]^2*x[12]*x[1],
-3*x[13]^3*x[14]+x[13]^2*x[11]*x[1]+3*x[5]*x[13]^2*x[8]-
3*x[11]^2*x[5]*x[12],-3*x[11]*x[12]*x[10]*x[5]-
x[10]*x[13]^2*x[1]+6*x[13]^2*x[14]*x[12]+
2*x[13]^3*x[15]+3*x[11]^2*x[3]*x[12]-6*x[12]*x[13]*x[11]*x[1]-
6*x[13]*x[8]*x[5]*x[12]-x[3]*x[13]^2*x[8]+x[11]*x[13]^2*x[16],0,x[13]^4,
x[11]*x[13]^3]):
 elif i=5 then return
simplify(multiply(matadd(f(4),multiply(f(3),-3*x[12]*E)),1/x[13]^2*E)):

    fi: end:
```

```
   k :=  proc(i)
    if i=1 then return f(1):
  elif i=2 then return f(2):
  elif i=3 then return
simplify(multiply(matadd(f(4),multiply(f(3),-3*x[12]*E)),1/x[13]^2*E)):
  elif i=4 then return f(3):
  fi: end:

   r := proc(i)
     if i=1 then return 1/2*x[4]:
   elif i=2 then return 3*x[12]*x[8]-3/4*x[10]^2:
   elif i=3 then return -x[8]*x[13]^2-x[13]*x[11]*x[10]-x[12]*x[11]^2:
  fi: end:

   M := proc(i)
if i=1 then return
matadd(matadd(k(3),multiply(k(2),r(1)*E)),multiply(k(1),r(2)*E)):
elif i=2 then return matadd(k(4),multiply(k(1),r(3)*E)):
fi: end:

    p := proc(i)

    if i=1 then return add(k(1)[1,j]*x[j],j=1..16):
   elif i=2 then return add(k(2)[1,j]*x[j],j=1..16):
   elif i=3 then return add(M(1)[1,j]*x[j],j=1..16):
   elif i=4 then return add(M(2)[1,j]*x[j],j=1..16):
    fi: end:

     P := proc(i,j::posint)
if i=1 and j<17 then return simplify(add(diff(p(1),x[l])*L(j,l),l=1..16)):
elif i=2 and j<17 then return simplify(add(diff(p(2),x[l])*L(j,l),l=1..16)):
elif i=3 and j<17 then return simplify(add(diff(p(3),x[l])*L(j,l),l=1..16)):
elif i=4 and j<17 then return simplify(add(diff(p(4),x[l])*L(j,l),l=1..16)):
  fi: end:
```

We add the Gap4 program to compute the bracket L(i,j) which is equal to Bg[i]*Bg[j] in the Gap4 program.

```
L := SimpleLieAlgebra("F",4,Rationals);;
R := RootSystem(L);;
P := PositiveRoots(R);;
x := PositiveRootVectors(R);;
y := NegativeRootVectors(R);;
e := x[11]+x[12]+x[13];;
IsNilpotentElement(L,e);;
if true then FindSl2(L,e);;
Bs := BasisVectors(Basis(FindSl2(L,e)));;
F := function(i)
return (i)*((Bs[1]*Bs[3])*Bs[1]);;
end;;

numbers := [1..20];;
for i in numbers do
  if F(i) = (2)*e then
    f := (i)*Bs[3];;
    fi;;
    od;;

h := e*f;;

g := LieCentralizer(L,Subspace(L,[e]));;
Bg := BasisVectors(Basis(g));;
z := LieCentre(g);;
Bz := BasisVectors(Basis(z));;
fi;;
```

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