



Radar ranging and Doppler tracking in post-Einsteinian metric theories of gravity

Marc-Thierry Jaekel, Serge Reynaud

► To cite this version:

Marc-Thierry Jaekel, Serge Reynaud. Radar ranging and Doppler tracking in post-Einsteinian metric theories of gravity. *Classical and Quantum Gravity*, 2006, 23 (24), pp.7561. 10.1088/0264-9381/23/24/025 . hal-00110552

HAL Id: hal-00110552

<https://hal.science/hal-00110552>

Submitted on 30 Oct 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Radar ranging and Doppler tracking in post-Einsteinian metric theories of gravity

Marc-Thierry Jaekel¹ and Serge Reynaud²

¹*Laboratoire de Physique Théorique de l'Ecole Normale Supérieure,
CNRS, UPMC, 24 rue Lhomond, F75231 Paris Cedex 05*

²*Laboratoire Kastler Brossel, Université Pierre et Marie Curie,
case 74, CNRS, ENS, Campus Jussieu, F75252 Paris Cedex 05*

The study of post-Einsteinian metric extensions of general relativity (GR), which preserve the metric interpretation of gravity while considering metrics which may differ from that predicted by GR, is pushed one step further. We give a complete description of radar ranging and Doppler tracking in terms of the time delay affecting an electromagnetic signal travelling between the Earth and a remote probe. Results of previous publications concerning the Pioneer anomaly are corrected and an annually modulated anomaly is predicted besides the secular anomaly. Their correlation is shown to play an important role when extracting reliable information from Pioneer observations. The formalism developed here provides a basis for a quantitative analysis of the Pioneer data, in order to assess whether extended metric theories can be the appropriate description of gravity in the solar system.

I. INTRODUCTION

Experimental tests of gravity show a good agreement with General Relativity (GR) at all scales ranging from laboratory to the size of the solar system [1, 2, 3, 4]. However there exist a few anomalies which may be seen as challenging GR. Anomalies in the rotation curves of galaxies or in the relation between redshifts and luminosities can be accounted for by considering dark matter and dark energy but they can as well be thought of as consequences of modifications of GR at galactic or cosmological scales [5, 6].

The anomalous acceleration recorded on Pioneer 10/11 probes might point at some anomalous behaviour of gravity at a scale of the order of the size of the solar system [7, 8]. The observation of such an effect has stimulated a significant effort to find explanations in terms of systematic effects on board the spacecraft or in its environment but this effort has not met success up to now [9]. The Pioneer anomaly remains the subject of intensive investigation because of its potential implications in deep space navigation as well as fundamental physics [10, 11, 12, 13]. New missions have been proposed [14] and efforts have been made for recovering data associated with the whole duration of Pioneer 10/11 missions and submitting them to new analysis [15, 16].

The present paper follows up publications which have investigated whether or not metric extensions of GR had the capability to account for the Pioneer anomaly while remaining compatible with other gravity tests performed in the solar system. Such ‘post-Einsteinian’ extensions preserve the very core of GR with gravity identified with the metric tensor $g_{\mu\nu}$ and motions described by geodesics. In particular, the weak equivalence principle, one of the most accurately verified properties in physics, is preserved. However the extended metric may differ from its standard (GR) form so that observations may show deviations from standard expectations. An important point is that these extensions explore a broader family of metrics than in the usually considered PPN family [1], including in particular deviations in the outer solar system.

These extensions have been introduced in the context of a linearized treatment of gravitation fields [17, 18] and then discussed with non linearity taken into account [19]. We will show that the previous studies were only preliminary and that the more precise and detailed investigations presented in this paper change some of their conclusions. But the main result, namely that the post-Einsteinian extensions of GR show the capability to account for the Pioneer anomaly, will not be affected. Objections to this statement, contained in recent publications [20, 21], will be shown to miss their target.

The theoretical motivations for extensions of GR are rooted in its long confrontation with Quantum Field Theory. Their discussion, presented in [18, 19], is not repeated in the present paper. Here we will focus our attention on the phenomenological implications of these extensions, by testing the metric in the solar system through its confrontation with observations, particularly those associated with Doppler tracking of Pioneer 10/11 probes. These Pioneer data show an anomalous acceleration a_P directed towards the Sun with a roughly constant amplitude over a large range of heliocentric distances

$$a_P \sim 0.8 \text{ nm s}^{-2} \quad , \quad 20 \text{ AU} \leq r_P \leq 70 \text{ AU} \quad (1)$$

Note that the positive sign for an acceleration directed towards the Sun has been chosen to fit the convention of [8]. The numbers are given as indications which will allow us to discuss orders of magnitude later on. The symbol AU stands for the astronomical unit.

Besides this secular term, the recorded anomalous acceleration also shows diurnal and annual modulations [8]. As the secular one, these modulated anomalies could be the consequence of some not yet understood artefact. But the search for an artefact accounting for the secular anomaly is usually focused on systematic effects on board the spacecraft or in its local environment and it is clear that modulated anomalies can certainly not be explained in this manner, since nothing in the vicinity of Pioneer probes is expected to have diurnal or annual variations. This entails that secular and modulated anomalies can hardly be due to the same artefact.

The main result of the present paper will be that modulated anomalies are a natural prediction of post-Einsteinian extensions of GR. As a matter of fact, the Doppler observable not only depends on the motion of the Pioneer probe but also on the perturbation of electromagnetic propagation along the up- and down-links. As the paths followed by these links are themselves modulated by motions of the stations, the anomalous Doppler acceleration is expected to contain diurnal and annual modulations. The diurnal and annual anomalies have to be considered as further observables of great interest to be confronted to theoretical expectations. As these observables can be correlated with the more frequently discussed secular anomaly, this opens new perspectives for testing the metric in the solar system, even if the systematics associated with modulated and secular anomalies are likely not correlated to each other.

In the following, we will give a common description of the secular and modulated parts of the anomalous acceleration by introducing a representation of the Doppler tracking observables in terms of propagation time delays. The advantage of this representation will be to treat in a natural and consistent manner the influence of metric perturbations on probe motion on one hand, link propagation on the other hand. The benefit will appear clearly in the discussion of Doppler observables, deduced by differentiation of the so called radar ranges, that is to say time delays between emission and reception, as well as in the interpretation of observations. We have to stress at this point that Pioneer 10/11 missions were not equipped with range measurement capabilities, which is quite unfortunate. This indeed leads to ambiguities in the determination of ranges and will be shown to play an important role in the interpretation of the anomalies.

Basic definitions and relations between the various quantities will be written down in the context of the ‘post-Einsteinian’ extensions of GR in the next section (sect. 2). The delicate problem of taking into account motions of Earth and probe will then be addressed (sect. 3). Exact relations will be presented as well as analytical approximations accurate enough for the purpose of the present paper. We will use the fact that the deviation of the extensions from GR certainly remains small since most gravity tests are compatible with GR (sect. 4). Using these theoretical tools, we will study the Pioneer-like anomalies possibly arising in Doppler tracking of probes in the solar system (sect. 5). In order to discuss the relevant orders of magnitude, we will then make simplifying assumptions, considering the case of probes moving in the ecliptic plane and having nearly radial motions in the outer solar system. We will present theoretical expectations for the secular anomaly as well as for the modulated anomaly due to the motion of the Earth (sect. 6), taking into account the correlation arising between these two anomalies. We will then draw conclusions (sect. 7) from the results of this new analysis.

II. BASIC DEFINITIONS AND RELATIONS

As already discussed, the high accuracy of tests of the weak principle of equivalence allows us to focus our attention on metric extensions of GR. This does not mean that there are no violations of this principle but only that such violations are too small to account for the large Pioneer anomaly (of the order of one thousandth of the Newton acceleration at the place explored by Pioneer probes).

We also disregard the effects of rotation and non sphericity of the Sun which have an influence in the inner solar system but hardly in the outer one. Hence, we consider a static and isotropic metric representing space-time around a punctual and stationary source. This assumption notably simplifies the description with metric fields only depending on two functions g_{00} and g_{rr} of a single variable, the radius r ,

$$\begin{aligned} ds^2 &= g_{00}(r)c^2dt^2 + g_{rr}(r)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) \\ \partial_0 g_{\mu\nu} &= \partial_\theta g_{\mu\nu} = \partial_\varphi g_{\mu\nu} = 0 \end{aligned} \quad (2)$$

The metric has been written with Eddington isotropic coordinates recommended by the IAU convention [22, 23] to represent coordinates in the solar system [24]. Radii are defined from the gravity source ($r \equiv 0$ at Sun center), colatitude angles θ are defined with respect to the ecliptic plane ($\theta \equiv \pi/2$), and azimuth angles φ describe rotation within the ecliptic plane. For simplicity, we consider the Earth center of motion to have a uniform circular motion at frequency $\Omega_1 \equiv 2\pi \text{ yr}^{-1}$. We also disregard the problems associated with diurnal rotation of Earth and atmospheric perturbations. These simplifications lead to the drawback that diurnal modulations will not be modeled.

For the sake of precision, the following remarks have to be made with respect to the IAU convention. As already stated, we have first disregarded the effects of rotation and non sphericity of the Sun, since they have a small influence

on the observables studied thereafter. The idea is that differences between the values calculated with the standard metric of GR or with the modified metric (2), are only slightly affected by this simplification. The preliminary evaluation of anomalies performed in this manner will have to be confirmed by more complete calculations taking into account perturbations due to the structure and rotation of the Sun as well as to the presence of planets [8].

Then, the IAU convention [22] explicitly refers to GR whereas we are here considering extensions of GR. We then have to face the implications for the definition of fundamental constants, primarily the velocity of light c . To this aim, we write the metric components as sums of standard GR expressions and small deviations

$$g_{\mu\nu} \equiv [g_{\mu\nu}]_{\text{st}} + \delta g_{\mu\nu} \quad , \quad |\delta g_{\mu\nu}| \ll 1 \quad (3)$$

and we convene that the deviations vanish at the radius of Earth orbit on which or in the vicinity of which the most accurate experiments are performed

$$\delta g_{\mu\nu}(r_1) \equiv 0 \quad , \quad r_1 \equiv 1 \text{ AU} \sim 150 \text{ Gm} \quad (4)$$

It thus remains to study the effects of variations with the radius r of the anomalous metric components δg_{00} and δg_{rr} . The standard metric is the GR solution written with Eddington isotropic coordinates

$$[g_{00}]_{\text{st}} = \left(\frac{1 - \frac{\kappa}{2r}}{1 + \frac{\kappa}{2r}} \right)^2 \quad , \quad [g_{rr}]_{\text{st}} = - \left(1 + \frac{\kappa}{2r} \right)^4 \quad (5)$$

The constant κ is related to the Schwarzschild radius (with G_N the Newton constant)

$$\kappa \equiv \frac{G_N M}{c^2} \sim 1.5 \text{ km} \sim 10^{-8} \text{ AU} \quad (6)$$

The dimensionless potential κ/r is small in the solar system, with a value $\sim 10^{-8}$ on Earth and even smaller values at the large radii explored by Pioneer probes.

The extensions of GR are often discussed within the PPN framework [1] where the metric (5) is expanded in terms of the Newton potential κ/r and Eddington parameters β and γ inserted in front of the terms of the expansion (with $\beta = \gamma = 1$ in GR)

$$g_{00} = 1 - 2\frac{\kappa}{r} + 2\beta\frac{\kappa^2}{r^2} + \dots \quad , \quad g_{rr} = -1 - 2\gamma\frac{\kappa}{r} - \dots \quad , \quad \text{PPN} \quad (7)$$

The PPN metric can be considered as a particular ‘post-Einsteinian’ extension of GR with anomalies showing specific dependences on the radius

$$\delta g_{00} \simeq 2(\beta - 1)\frac{\kappa^2}{r^2} \quad , \quad \delta(g_{00}g_{rr}) \simeq -2(\gamma - 1)\frac{\kappa}{r} \quad , \quad \text{PPN} \quad (8)$$

In this paper, we consider more general extensions of GR, which show significant deviations at long ranges (outer solar system) and not only short ones (inner solar system). With respect to the PPN metric (8), the more general extensions can be thought of as allowing for anomalies in the two sectors which may depend on scale.

Einstein curvatures corresponding to the extended metric have been studied in a detailed manner in [19]. In contrast to the standard expressions which vanish everywhere except on the gravity source, the anomalous curvatures are generally non null in space outside the source. This is already true for PPN extensions and, again, the general case corresponds to a more general r -dependence. This dependence can also be described in terms of running coupling constants which replace the Newton constant while depending on scale. We do not repeat these calculations [19] but recall that the natural metric extension of GR involves two running coupling constants which correspond to the sectors of traceless and traced tensors [25].

From the point of view of phenomenology, the two sectors are as well represented by the two functions $\delta g_{00}(r)$ and $\delta(g_{00}g_{rr})(r)$. The first sector represents an anomaly of the Newton potential which has to remain small to preserve the good agreement between GR and gravity tests performed on planetary orbits [2, 4, 26]. Meanwhile, the second sector represents an extension of PPN phenomenology with a scale dependent parameter γ . It opens an additional phenomenological freedom with respect to the mere modification of the Newton potential and this freedom opens the possibility to accomodate a Pioneer-like anomaly besides other gravity tests [17, 18, 19].

Recent publications force us to be more specific on the relation between the Pioneer anomaly and modifications of the Newton potential, *i.e.* anomalies in the first sector according to the terminology of the preceding paragraph. Interpreting the Pioneer anomaly in such a manner requires that δg_{00} varies roughly as r at the large radii explored by Pioneer probes. If this dependence also holds at smaller radii [8], or if the anomaly follows a simple Yukawa

law [4], one deduces that it cannot have escaped detection in the more constraining tests performed with martian probes [27, 28, 29]. Brownstein and Moffat have explored the possibility that the linear dependence holds at distances explored by Pioneer probes while being cut at the orbital radii of Mars [30]. Iorio and Giudice [20] as well as Tangen [21] have in contrast argued that the ephemeris of outer planets were accurate enough to discard the presence of the required linear dependence in the range of distances explored by the Pioneer probes. This argument has been contested by the authors of [30] so that the conflict remains to be settled.

Iorio and Giudice [20] and Tangen [21] have pushed their claim one step farther by restating their argument as an objection to the very possibility of accounting for the Pioneer anomaly in any viable metric theory of gravity. Even before entering the detailed developments to be presented in this paper, we can show that this claim is untenable just because it only considers metric anomalies in the first sector while disregarding those in the second sector. Later on in the paper, we will come back to the discussion of the compatibility of the metric modifications with observations performed in the solar system, often with an accuracy higher than that of Pioneer observations. This has to be done with care, accounting for the presence of the two sectors as well as for possible scale dependences. This question has already been discussed in [18, 19] for the cases of deflection experiments on electromagnetic sources passing behind the Sun [31, 32, 33, 34, 35]. We will see below that it has a particularly critical character for the ranging experiments which involve directly the Shapiro time delay [36].

III. RADAR RANGING AND DOPPLER TRACKING OBSERVABLES

In the present section, we introduce the time delay function, a two-point function the knowledge of which is equivalent to a characterization of the metric. We deduce from this function the radar ranging observables between the Earth and a probe in the solar system. We then analyze the case of Doppler tracking observables which are obtained from ranging ones through a time differentiation.

Starting from the static and isotropic metric (2) written in terms of Eddington isotropic coordinates, we define the time delay [37] as the time taken in this coordinate system by a light-like signal to propagate from a spatial position \mathbf{x}_1 to another one \mathbf{x}_2

$$\mathbf{x}_a \equiv r_a (\sin \theta_a \cos \varphi_a, \sin \theta_a \sin \varphi_a, \cos \theta_a) \quad , \quad a = 1, 2 \quad (9)$$

This defines a two-point function \mathcal{T} which depends on the positions \mathbf{x}_1 and \mathbf{x}_2 only through three real variables, which can be chosen as the two radii r_1 and r_2 and the angle ϕ between the two points as seen from the gravity source,

$$\cos \phi = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos (\varphi_2 - \varphi_1) \quad (10)$$

Stated differently, the time delay \mathcal{T} is a function of the triangle built on the emitter, the receiver and the gravity source.

The form of the time delay function for a static isotropic metric was obtained in [18, 19] by solving the Hamilton-Jacobi equation for a light ray

$$c\mathcal{T}(r_1, r_2, \phi) \equiv \int_{r_1}^{r_2} \frac{-\frac{g_{rr}}{g_{00}}(r)dr}{\sqrt{-\frac{g_{rr}}{g_{00}}(r) - \frac{\rho^2}{r^2}}} \quad , \quad \phi = \int_{r_1}^{r_2} \frac{\rho dr/r^2}{\sqrt{-\frac{g_{rr}}{g_{00}}(r) - \frac{\rho^2}{r^2}}} \quad (11)$$

These quantities are integrals over the light ray of integrands depending only on the conformally invariant ratio g_{00}/g_{rr} . The parameter ρ , hereafter called the impact parameter, is an implicit function of the variables r_1, r_2, ϕ determined by the second equation in (11). This definition fixes the relative sign of ρ and ϕ .

One can also deduce from the time delay function

$$c\mathcal{T} = \rho\phi + \int_{r_1}^{r_2} \mathcal{C}(r)dr \quad , \quad \mathcal{C}(r) \equiv \sqrt{-\frac{g_{rr}}{g_{00}}(r) - \frac{\rho^2}{r^2}} \quad (12)$$

Calculating the partial derivatives of \mathcal{T} , one indeed notices that cancellations appear in the angular derivative, due to the particular form (11) of the time delay function,

$$c\partial_{r_1}\mathcal{T} = -\mathcal{C}(r_1) \quad , \quad c\partial_{r_2}\mathcal{T} = \mathcal{C}(r_2) \quad , \quad c\partial_\phi\mathcal{T} = \rho \quad (13)$$

Second order derivatives of the time delay function \mathcal{T} may then be written

$$\begin{aligned} c\partial_\phi^2\mathcal{T} &= \partial_\phi\rho = \left(\int_{r_1}^{r_2} \frac{-\frac{g_{rr}}{g_{00}}(r)dr/r^2}{\mathcal{C}(r)^3} \right)^{-1} \\ c\partial_\phi\partial_{r_2}\mathcal{T} &= -\frac{\rho\partial_\phi\rho}{r_2^2\mathcal{C}(r_2)} = \partial_{r_2}\rho \quad , \quad c\partial_{r_2}^2\mathcal{T} = \partial_r\mathcal{C}(r_2) + \frac{\rho^2\partial_\phi\rho}{r_2^4\mathcal{C}(r_2)^2} \end{aligned} \quad (14)$$

Similar relations hold involving derivatives with respect to r_1 instead of r_2 . Note that non local expressions only come from angular derivatives, namely ρ and $\partial_\phi \rho$. It also turns out that the angular second derivative of the time delay function is positive (for $r_2 > r_1$).

We now study the observables which are used in deep space navigation [38] and make explicit their relation to the time delay function (11). We begin with radar ranging observables obtained by timing radio signals exchanged between stations on Earth and the deep space probe. An up-link radio signal is emitted from Earth at (t_1^-, \mathbf{x}_1^-) and received and sent back by the probe at (t_2, \mathbf{x}_2) with the down-link radio signal received on Earth at (t_1^+, \mathbf{x}_1^+) (positions identified with the center of motion of Earth, itself assumed to run uniformly on a circular orbit). Meanwhile the deep space probe is assumed to follow a geodesic trajectory in the outer solar system. As the gravity fields are very small there, this trajectory can be approximated with a good approximation as a Keplerian hyperbolic trajectory escaping the solar system [8].

The positions of the emission and reception events are connected by light cones, that is also by the following relations between event times and the time delay function,

$$t_2 - t_1^- = \mathcal{T}(\mathbf{x}_1^-, \mathbf{x}_2) \equiv \mathcal{T}_- \quad , \quad t_1^+ - t_2 = \mathcal{T}(\mathbf{x}_1^+, \mathbf{x}_2) \equiv \mathcal{T}_+ \quad (15)$$

For probes equipped with range measurement capabilities (which was not the case for Pioneer 10/11), the ranging observable may be defined as half the time elapsed on Earth from the emission time t_1^- to the reception time t_1^+

$$\Delta t \equiv \frac{t_1^+ - t_1^-}{2} = \frac{\mathcal{T}_- + \mathcal{T}_+}{2} \quad (16)$$

This quantity is not a proper gauge invariant quantity but it is directly related to such a quantity, the proper time $s_1^+ - s_1^-$ elapsed on Earth between the same two events, through a mere multiplication factor determined by the potentials created by Sun on Earth and the velocity of Earth on its orbit. This multiplication factor is constant in the simple context studied in this paper, and it can therefore be omitted. The ranging time is available for observers on Earth even if they don't have access to the transponding time t_2 defined on board the deep space probe. However, the transponding time can be deduced from the solution of equations of motion, at soon as the metric is known with a sufficient accuracy. We will see below how to deal with this complication.

For Pioneer 10/11 probes, the tracking technique was based on the measurement of the Doppler shift, a proper observable y defined from the ratio of cycle counting rates of reference clocks located at emission and reception stations [8, 38],

$$y \equiv \ln \frac{ds_1^+}{ds_1^-} = \ln \frac{dt_1^+}{dt_1^-} \quad (17)$$

Note that y has its definition gauge invariant when written in terms of proper times, but can as well be written in terms of coordinate times on Earth for the same reason as in the preceding paragraph. The same information can be encoded in a Doppler velocity

$$\frac{V}{c} \equiv \frac{d\Delta t}{dt} = \frac{dt_1^+ - dt_1^-}{dt_1^+ + dt_1^-} = \tanh \frac{y}{2} \quad , \quad y = \ln \frac{1 + V/c}{1 - V/c} \quad (18)$$

The difference between the observables V and y appears only at third order in the velocities and can be considered as a small term. This is due to the choice of the median observer time t to parametrize the data

$$t \equiv \frac{t_1^+ + t_1^-}{2} \quad , \quad dt_1^+ \equiv dt + d\Delta t \quad , \quad dt_1^- \equiv dt - d\Delta t \quad (19)$$

Should another time be used in its place, we would obtain second order corrections in the relation between V and y .

The following remarks are worth being kept in mind when using tracking data for obtaining knowledge on the motion of the deep space probe. Clearly the Doppler velocity is primarily correlated to the velocity of the deep space probe relatively to that of Earth, with relativistic as well as gravitational corrections fully accounted for in relation (18). But the velocity of the probe at some specific time is not known with a sufficient accuracy, unless informations extracted from Doppler data are used. It follows that the Pioneer gravity test is more appropriately discussed in terms of the Doppler acceleration A , which is just the time derivative of the Doppler velocity [8],

$$\frac{A}{c} \equiv \frac{dV}{cdt} = \frac{d^2 \Delta t}{dt^2} \quad (20)$$

This observable A gives a more direct access to the acceleration law of the probe to be compared with the theoretical expectation. Now the expected acceleration depends on the distance of the probe to the gravity source and the

latter also suffers an imprecise determination on probes for which ranging data are not available. These ‘ambiguity’ problems will be faced in the following, keeping in mind that relativistic and gravitational corrections can in principle be affected by the discussion.

In order to compare tracking observations with theory, we need to make more explicit the relation between the ranging observable Δt and the time delay function \mathcal{T} . To this aim, we note that, as the Earth has been assumed to have a circular motion, the time delay function \mathcal{T} reduces to a function of two variables r_2 and ϕ . For emission and reception events connected by light cones (15), these two variables may equivalently be replaced by the time parameters t_1 and t_2 which parametrize the Earth and probe trajectories respectively, at least once the metric and the trajectories are known. The history of ranging observations, *i.e.* of series of time triplets (see eqs.15), may then be conveniently represented under the form of a relation $\Delta t(t)$ with Δt and t defined by (16) and (19). The transponding time t_2 thus appears as a function of the median time t . These implicit relations may for instance be solved by approaching in an iterative manner the solution of the following equations

$$\begin{aligned} \mathcal{T}_- &= \mathcal{T}(\mathbf{x}_1(t - \Delta t), \mathbf{x}_2(t_2)) \quad , \quad \mathcal{T}_+ = \mathcal{T}(\mathbf{x}_1(t + \Delta t), \mathbf{x}_2(t_2)) \\ t_2 &= t + \frac{\mathcal{T}_- - \mathcal{T}_+}{2} \quad , \quad \Delta t = \frac{\mathcal{T}_+ + \mathcal{T}_-}{2} \end{aligned} \quad (21)$$

Derivatives of the function \mathcal{T} with respect to t_1 and t_2 are deduced from (13)

$$c\partial_1\mathcal{T} \equiv c\frac{\partial\mathcal{T}}{\partial t_1} = \rho\partial_1\phi \quad , \quad c\partial_2\mathcal{T} \equiv c\frac{\partial\mathcal{T}}{\partial t_2} = \mathcal{C}(r_2)\dot{r}_2 + \rho\partial_2\phi \quad (22)$$

The Doppler velocity and the derivative of transponding time are given by similar relations

$$\frac{c+V}{c-V} = \frac{1+\partial_2\mathcal{T}_+}{1-\partial_1\mathcal{T}_+} \frac{1+\partial_1\mathcal{T}_-}{1-\partial_2\mathcal{T}_-} \quad , \quad \frac{2}{\partial_t t_2} = \frac{1-\partial_2\mathcal{T}_-}{1+\partial_1\mathcal{T}_-} + \frac{1+\partial_2\mathcal{T}_+}{1-\partial_1\mathcal{T}_+} \quad (23)$$

IV. RANGING AND DOPPLER ANOMALIES

In order to bring the relations written in the preceding section to explicit formulas, we have to face rather complicated expressions, which can be dealt with in a numerical procedure but hardly in an analytical calculation. A simpler approach, extremely useful for a first discussion of the anomalies, is to use a first-order expansion of the observables in the metric perturbations [19]. The basic methods to be used in such an approach are presented in this section.

Let us first discuss the case of the time delay (11) with its standard form

$$[c\mathcal{T}]_{\text{st}}(r_1, r_2, \phi) = \int_{r_1}^{r_2} \frac{-[g_{rr}]_{\text{st}}(r)}{[g_{00}]_{\text{st}}} \frac{dr}{[\mathcal{C}(r)]_{\text{st}}} \quad (24)$$

At first order in the metric perturbation, the post-Einsteinian time delay may be written as the sum of this standard form and of an anomaly (see (11) and (12))

$$\begin{aligned} c\delta\mathcal{T}(r_1, r_2, \phi) &\equiv \{c\mathcal{T} - [c\mathcal{T}]_{\text{st}}\}(r_1, r_2, \phi) = \int_{r_1}^{r_2} \delta w(r) \frac{-[g_{rr}]_{\text{st}}(r)}{[g_{00}]_{\text{st}}} \frac{dr}{[\mathcal{C}(r)]_{\text{st}}} \\ \delta w &\equiv \delta \left\{ \ln \sqrt{\frac{-g_{rr}}{g_{00}}} \right\} = \frac{\delta g_{rr}}{2[g_{rr}]_{\text{st}}} - \frac{\delta g_{00}}{2[g_{00}]_{\text{st}}} \end{aligned} \quad (25)$$

We again notice that this variation is determined by the perturbation of the conformally invariant ratio g_{rr}/g_{00} . In order to compute the variation of the ranging time observable (15) induced by that of the time delay (25), we now introduce notations for anomalies of the ranging time \mathcal{T} and transponding time t_2

$$\delta\Delta t \equiv \Delta t - [\Delta t]_{\text{st}} \quad , \quad \delta t_2 \equiv t_2 - [t_2]_{\text{st}} \quad (26)$$

and write a first order equation for these anomalies by linearizing (21)

$$\begin{aligned} \frac{[\partial_1\mathcal{T}_+]_{\text{st}} + [\partial_1\mathcal{T}_-]_{\text{st}}}{2} \delta\Delta t + \left(1 + \frac{[\partial_2\mathcal{T}_+]_{\text{st}} - [\partial_2\mathcal{T}_-]_{\text{st}}}{2}\right) \delta t_2 &= -\frac{\delta\mathcal{T}_+ - \delta\mathcal{T}_-}{2} \\ \left(1 - \frac{[\partial_1\mathcal{T}_+]_{\text{st}} - [\partial_1\mathcal{T}_-]_{\text{st}}}{2}\right) \delta\Delta t - \frac{[\partial_2\mathcal{T}_+]_{\text{st}} + [\partial_2\mathcal{T}_-]_{\text{st}}}{2} \delta t_2 &= \frac{\delta\mathcal{T}_+ + \delta\mathcal{T}_-}{2} \end{aligned} \quad (27)$$

The symbols appearing in (27) are defined according to

$$\begin{aligned}\delta\mathcal{T}_{\pm} &\equiv \delta\mathcal{T}(\mathbf{x}_1(t \pm [\Delta t]_{\text{st}}), \mathbf{x}_2([t_2]_{\text{st}})) \\ [\partial_a \mathcal{T}_{\pm}]_{\text{st}} &\equiv [\partial_a \mathcal{T}]_{\text{st}}(\mathbf{x}_1(t \pm [\Delta t]_{\text{st}}), \mathbf{x}_2([t_2]_{\text{st}})) \quad , \quad a \equiv 1, 2\end{aligned}\quad (28)$$

Solving these equations, one obtains the ranging and transponding time anomalies

$$\begin{aligned}\delta\Delta t &= \frac{1 - [V]_{\text{st}}/c}{2} \frac{\delta\mathcal{T}_+}{1 - [\partial_1 \mathcal{T}_+]_{\text{st}}} + \frac{1 + [V]_{\text{st}}/c}{2} \frac{\delta\mathcal{T}_-}{1 + [\partial_1 \mathcal{T}_-]_{\text{st}}} \\ \delta t_2 &= -\frac{1}{2} [\partial_t t_2]_{\text{st}} \left(\frac{\delta\mathcal{T}_+}{1 - [\partial_1 \mathcal{T}_+]_{\text{st}}} - \frac{\delta\mathcal{T}_-}{1 + [\partial_1 \mathcal{T}_-]_{\text{st}}} \right)\end{aligned}\quad (29)$$

The implicit equations (21) and other expressions written up to now will be approximated in the next section so that they are more easily used for explicit discussions of Pioneer-like anomalies. The approximation will be based on the fact that the Earth and probe velocities are much smaller than light velocity.

We now come to the fact that the anomalous time delay between emission and reception is not only affected by the perturbation of light propagation, but also by the perturbation of the probe trajectory. Precisely, the value of r_2 to be used in preceding calculations is not the same in extended theory as it would be in standard theory. This is characterized by differences $(\delta r_2, \delta\theta_2, \delta\varphi_2)$ of the coordinates of the probe. In order to discuss this point, we introduce a new notation for variations of any quantity f as sums of terms, the first one being calculated with endpoints fixed and the other ones associated with position differences,

$$\bar{\delta}f \equiv f(r_2, \phi) - [f]_{\text{st}}([r_2]_{\text{st}}, [\phi]_{\text{st}}) = \delta f + \delta r_2 \partial_r [f]_{\text{st}} + \delta\phi \partial_\phi [f]_{\text{st}} \quad (30)$$

The angular anomalies have been collected in $\delta\phi$ and no anomalies have been accounted for the position of Earth which is assumed to be known. The condition (4) of null metric anomalies at Earth ensures the consistency of this description with the conventions of metrology. Note that, in a first order expansion in the metric perturbation, all contributions to (30) except the first one may be calculated using the standard expression of f . For the time delay function in particular, equation (30) is read as

$$\bar{\delta}T \equiv \delta\mathcal{T}([r_2]_{\text{st}}, [\phi]_{\text{st}}) + [\mathcal{C}]_{\text{st}}([r_2]_{\text{st}}) \delta r_2 + [\rho]_{\text{st}} \delta\phi \quad (31)$$

The perturbation $\bar{\delta}T$ defined by (31) and evaluated at first order will be used in the following as a good approximation to the ranging time anomaly (29). Its evaluation still requires the solution of the equation of motion of the probe.

In particular, the position differences δr_2 and $\delta\phi$ have to be deduced from the geodesic equations written separately for the standard metric $[g]_{\text{st}}$ and the modified one $[g]_{\text{st}} + \delta g$. These geodesic equations have their usual form in a metric theory [39]

$$\frac{du^\mu}{ds} + \Gamma_{\nu\rho}^\mu u^\nu u^\rho \equiv 0 \quad (32)$$

with $\Gamma_{\nu\rho}^\mu$ the Christoffel symbols and u^μ the relativistic velocities

$$\Gamma_{\mu\nu}^\lambda \equiv \frac{g^{\lambda\rho}}{2} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad , \quad u^\mu \equiv \frac{dx^\mu}{ds} \quad , \quad g_{\mu\nu} u^\mu u^\nu = 1 \quad (33)$$

For the computations to be performed in the next section, it is worth noticing that the geodesic equations (32) may also be written as conservation laws for energy and angular momentum. The latter is a vector, and the conservation of its direction just means that the motion takes place in an orbital plane containing the Sun. This plane is characterized by two angles, the longitude of the ascending node Ω and the inclination of the orbit ι , which also give the spatial direction of the conserved angular momentum along the unit vector $(\sin \iota \sin \Omega, -\sin \iota \cos \Omega, \cos \iota)$.

In order to describe motion, we then introduce angular coordinates adapted to the orbital plane with $\tilde{\theta} = \frac{\pi}{2}$ on the orbit, and $\tilde{\varphi}$ measured in the orbital plane. The reduced energy e and angular momentum j are defined as the following conserved quantities

$$e \equiv g_{00} \frac{cdt}{ds} \quad , \quad j \equiv g_{rr} r^2 \frac{d\tilde{\varphi}}{ds} \quad (34)$$

These relations just give the velocity components u^0 and $u^{\tilde{\varphi}}$, with the third component $u^{\tilde{\theta}}$ vanishing and the fourth one u^r given by velocity normalization (33)

$$\frac{dr}{ds} \equiv u^r = U^r \quad , \quad (U^r)^2 \equiv \frac{1}{g_{00}g_{rr}} \left(g_{00} \left(1 - \frac{j^2}{g_{rr}r^2} \right) - e^2 \right) \quad (35)$$

U^r is a function of the variable r giving the radial velocity u^r and depending on the form of the metric and on the conserved quantities labelling the trajectory. Equations (34) can also be written in terms of functions U^0 and $U^{\tilde{\varphi}}$ of the variable r

$$\frac{cdt}{ds} = U^0 \equiv \frac{e}{g_{00}} \quad , \quad \frac{d\tilde{\varphi}}{ds} = U^{\tilde{\varphi}} \equiv \frac{j}{g_{rr}r^2} \quad (36)$$

Standard trajectories are obtained by integrating these equations, with the metric components having their standard expressions. At first order in the metric perturbation around GR, the probe trajectory is then obtained as the sum of standard and anomalous contributions. The latter are expressed in terms of variations of the functions U . In particular, the radial velocity and acceleration show the following anomalies

$$\begin{aligned} \frac{d\delta r}{ds} &= \delta \left(\frac{dr}{ds} \right) = \bar{\delta} U^r = \partial_r [U^r]_{\text{st}} \delta r + \delta U^r \\ \frac{d^2 \delta r}{ds^2} &= \delta \left(\frac{d^2 r}{ds^2} \right) = \bar{\delta} (U^r \partial_r U^r) = \frac{1}{2} \partial_r^2 \left\{ [U^r]_{\text{st}}^2 \right\} \delta r + \frac{1}{2} \partial_r \delta \left\{ (U^r)^2 \right\} \end{aligned} \quad (37)$$

For the explicit calculations to be performed in the next sections, we will write the solution for the distance variation as

$$\begin{aligned} \delta r_2 &= \delta r_* + [U^r(r_2)]_{\text{st}} \int_{r_*}^{r_2} \frac{\delta U^r}{[U^r]_{\text{st}}^2} dr \\ \frac{\delta U^r}{[U^r]_{\text{st}}} &= -\frac{\delta(g_{00}g_{rr})}{2[g_{00}g_{rr}]_{\text{st}}} \\ &\quad + \frac{1}{2[g_{00}g_{rr}(U^r)^2]_{\text{st}}} \left\{ \delta g_{00} - \frac{j^2}{r^2} \delta \left(\frac{g_{00}}{g_{rr}} \right) - \frac{2j\delta j}{r^2} \frac{[g_{00}]_{\text{st}}}{[g_{rr}]_{\text{st}}} - 2e\delta e \right\} \end{aligned} \quad (38)$$

The constant δr_* represents the initial radius variation at r_* between geodesics calculated for the extended and standard metrics. The angular variation is then written as

$$\begin{aligned} \delta \tilde{\varphi}_2 &= \delta \tilde{\varphi}_* + [U^{\tilde{\varphi}}(r_2)]_{\text{st}} \int_{r_*}^{r_2} \frac{\delta U^{\tilde{\varphi}}}{[U^{\tilde{\varphi}}]_{\text{st}}^2} dr + \int_{r_*}^{r_2} \left(\frac{\delta U^{\tilde{\varphi}}}{[U^{\tilde{\varphi}}]_{\text{st}}} - \frac{\delta U^r}{[U^r]_{\text{st}}} \right) \frac{[U^{\tilde{\varphi}}]_{\text{st}}}{[U^r]_{\text{st}}} dr \\ \frac{\delta U^{\tilde{\varphi}}}{[U^{\tilde{\varphi}}]_{\text{st}}} &= \frac{\delta j}{j} - \frac{\delta g_{rr}}{[g_{rr}]_{\text{st}}} \end{aligned} \quad (39)$$

The constant $\delta \tilde{\varphi}_*$ represents an initial angular difference between the geodesics.

When taken with the results of the preceding section, these equations provide us with an exact description of radar ranging and Doppler tracking, in the simplified context considered in the present paper and in a first order expansion in the metric deviation from its standard form.

V. PIONEER LIKE ANOMALIES

Implicit equations written in the preceding section have to be solved in an iterative manner, which is well adapted to a numerical procedure but not easily performed in an analytical work. In order to be able to present qualitative but explicit discussions of Pioneer-like anomalies, we now introduce approximated forms of these equations. We will also consider Pioneer-like probes with high excentricity orbits, so that it will be possible to neglect angular terms. For simplicity, we also consider that the probe moves in the ecliptic plane, *i.e.* that the inclination of the orbital plane vanishes ($\iota = 0$).

The main argument pleading for these approximations is the fact that the Earth velocity is much smaller than light velocity $\Omega_1 r_1 / c \simeq 10^{-4}$. This entails that the change of time delay function due to motion of Earth during the time of flight of the signal is small. Furthermore, the parity of equations (21) and (29) leads to corrections induced by Earth motion appearing only at second order in Earth velocity ($(\Omega_1 r_1 / c)^2 \simeq 10^{-8}$). Hence, the modifications of the ranging time anomalies (29) due to Earth motion may be ignored in a first discussion. Note that this statement applies to the anomalous part (29) of the ranging time and to the effect of motion during the ranging time only. It does neither hold for the standard ranging time which suffers an appreciable effect due to the Earth motion [8], nor for the effect of Earth motion on a longer term. The latter effect is evaluated below and found to play a significant role in the interpretation of the anomaly.

As already discussed, the anomalous time delay is also affected by the perturbation of the probe trajectory. This is taken into account by using equation (31) which now gives the true anomaly of the ranging observable

$$c\bar{\delta}\Delta t(t) \simeq c\bar{\delta}T(t) = c\delta\mathcal{T}(r_1, [r_2]_{\text{st}}, [\phi]_{\text{st}}) + [\mathcal{C}]_{\text{st}} ([r_2]_{\text{st}})\delta r_2 + [\rho]_{\text{st}} \delta\phi \quad (40)$$

The anomaly of Doppler velocity observable is then obtained from (18) and (25)

$$\begin{aligned} \bar{\delta}V &\simeq [\dot{r}_2]_{\text{st}} \left\{ \frac{-[g_{rr}]_{\text{st}} \delta w}{[g_{00}]_{\text{st}} [\mathcal{C}(r_2)]_{\text{st}}} + [\partial_r \mathcal{C}(r_2)]_{\text{st}} \delta r_2 \right\} + [\mathcal{C}(r_2)]_{\text{st}} \delta \dot{r}_2 \\ &+ [\rho]_{\text{st}} \delta \dot{\phi} + [\dot{\rho}]_{\text{st}} \frac{\bar{\delta}\rho}{[\partial_\phi \rho]_{\text{st}}} \end{aligned} \quad (41)$$

where we have introduced shorthand notations $\dot{r}_2 \equiv \frac{dr_2}{dt}$, $\dot{\phi} \equiv \frac{d\phi}{dt}$ and $\delta \dot{r}_2 \equiv \frac{d\delta r_2}{dt}$, $\delta \dot{\phi} \equiv \frac{d\delta \phi}{dt}$. The time derivative of the impact parameter appearing in (41) is given by

$$\dot{\rho} = \left(-\frac{\dot{r}_2 \rho}{r_2^2 \mathcal{C}(r_2)} + \dot{\phi} \right) \partial_\phi \rho \quad (42)$$

All contributions to the Doppler shift anomaly (41) are local except some contributions to the impact parameter anomaly $\bar{\delta}\rho$

$$\frac{\bar{\delta}\rho}{[\partial_\phi \rho]_{\text{st}}} = [\rho]_{\text{st}} \left(\int_{r_1}^{r_2} \frac{-[g_{rr}]_{\text{st}} \delta w}{[g_{00}]_{\text{st}} [\mathcal{C}]_{\text{st}}^{\frac{3}{2}} r^2} - \frac{\delta r_2}{[r_2]_{\text{st}}^2 [\mathcal{C}]_{\text{st}}} \right) + \delta\phi \quad (43)$$

Note that explicit dependences of \mathcal{C} on r_2 are omitted from now on.

The anomaly of Doppler acceleration observable is obtained similarly

$$\begin{aligned} \bar{\delta}A &= [\ddot{r}_2]_{\text{st}} \frac{-[g_{rr}]_{\text{st}} \delta w}{[g_{00}]_{\text{st}} [\mathcal{C}]_{\text{st}}} + [\dot{r}_2]_{\text{st}}^2 \partial_r \left(\frac{-[g_{rr}]_{\text{st}} \delta w}{[g_{00}]_{\text{st}} [\mathcal{C}]_{\text{st}}} \right) \\ &+ \frac{d}{dt} ([\dot{r}_2]_{\text{st}} [\partial_r \mathcal{C}]_{\text{st}} \delta r_2) + \frac{d[\mathcal{C}]_{\text{st}}}{dt} \delta \dot{r}_2 + [\mathcal{C}]_{\text{st}} \delta \ddot{r}_2 \\ &+ \frac{[\rho]_{\text{st}} [\dot{\rho}]_{\text{st}} [\dot{r}_2]_{\text{st}}}{[r_2]_{\text{st}}^2} \left(\frac{-[g_{rr}]_{\text{st}} \delta w}{[g_{00}]_{\text{st}} [\mathcal{C}]_{\text{st}}^3} \right) \\ &+ \frac{d}{dt} \left\{ [\rho]_{\text{st}} \delta \dot{\phi} + \frac{[\dot{\rho}]_{\text{st}}}{[\partial_\phi \rho]_{\text{st}}} \bar{\delta}\rho \right\} \end{aligned} \quad (44)$$

with shorthand notations $\ddot{r}_2 \equiv \frac{d\dot{r}_2}{dt}$ and $\delta \ddot{r}_2 \equiv \frac{d\delta \dot{r}_2}{dt}$. The first and second line in (44) contain all the terms which are not modulated by the annual motion of Earth. In particular, they contain the secular contribution to the anomalous Doppler acceleration which was calculated in [18, 19]. Note the relative signs between terms in first and second lines of (44), which correct an error made in [18, 19]. The third line contains modulated terms depending locally on anomalies of the probe trajectory and shown below to give negligible contributions to Pioneer-like anomalies. The fourth line includes terms which depend on the non local anomaly $\bar{\delta}\rho$. These important terms (see below) were ignored in [18, 19]. They vary with the Earth motion around the Sun and determine annual modulations of the anomaly.

The Pioneer anomaly has been recorded on deep space probes with high excentricity orbits and, therefore, nearly radial motions. For the sake of simplicity, we neglect from now on all the terms proportional to angular velocities or angular accelerations. As proper time relativistic corrections depend on the probe velocity squared \dot{r}_2^2 , it is also possible to use the simplification $ds \simeq cdt$. In this context, one deduces from (35) the radial acceleration read as the sum of standard and anomalous contributions

$$\begin{aligned} [\ddot{r}_2]_{\text{st}} &\simeq \frac{c^2}{2} \frac{\partial_r \ln [g_{00}]_{\text{st}}}{[g_{rr}]_{\text{st}}} - \frac{[\dot{r}_2]_{\text{st}}^2}{2} \partial_r \ln [g_{00} g_{rr}]_{\text{st}} \\ \delta \ddot{r}_2 &\simeq \frac{c^2}{2} \frac{\bar{\delta}(\partial_r g_{00})}{[g_{00} g_{rr}]_{\text{st}}} - [\ddot{r}_2]_{\text{st}} \frac{\bar{\delta}(g_{00} g_{rr})}{[g_{00} g_{rr}]_{\text{st}}} \\ &\quad - \frac{[\dot{r}_2]_{\text{st}}^2}{2} \frac{\bar{\delta}(\partial_r (g_{00} g_{rr}))}{[g_{00} g_{rr}]_{\text{st}}} - \frac{\partial_r [g_{00} g_{rr}]_{\text{st}}}{[g_{00} g_{rr}]_{\text{st}}} [\dot{r}_2]_{\text{st}} \delta \dot{r}_2 \end{aligned} \quad (45)$$

We have disregarded the effect of planets as gravity sources, which is justified once again by the fact that we focus our attention on the anomalies (this effect has to be taken into account in the data analysis process [8]). Equation (35) is integrated to obtain the anomaly expressed on the position or velocity of the probe

$$\begin{aligned}\delta\dot{r}_2 &\simeq \frac{c^2\bar{\delta}g_{00}/2 - c^2[e]_{\text{st}}\delta e}{[g_{00}g_{rr}]_{\text{st}}[\dot{r}_2]_{\text{st}}} - \frac{\bar{\delta}(g_{00}g_{rr})}{2[g_{00}g_{rr}]_{\text{st}}}[\dot{r}_2]_{\text{st}} \\ \delta r_2 &\simeq [\dot{r}_2]_{\text{st}} \left\{ \delta\tau + \int_{r_1}^{r_2} \frac{c^2\delta g_{00} - 2c^2[e]_{\text{st}}\delta e - [\dot{r}_2]_{\text{st}}^2\delta(g_{00}g_{rr})}{2[\dot{r}_2]_{\text{st}}^3[g_{00}g_{rr}]_{\text{st}}} dr \right\}\end{aligned}\quad (46)$$

These relations are just particular cases of the more general results obtained in the end of the preceding section. They have been written in terms of a difference $\delta\tau$ of epoch, that is also the time of passage at the initial radius r_* . For convenience, this initial radius has been pushed back to r_1 .

The Doppler acceleration anomaly (44) may then be expressed in terms of the radial velocity $[\dot{r}_2]_{\text{st}}$ and acceleration $[\ddot{r}_2]_{\text{st}}$

$$\begin{aligned}\bar{\delta}A &= \frac{c^2[\mathcal{C}]_{\text{st}}}{2} \frac{\bar{\delta}(\partial_r g_{00})}{[g_{00}g_{rr}]_{\text{st}}} \\ &- [\ddot{r}_2]_{\text{st}} \left\{ \frac{[\mathcal{C}]_{\text{st}}\bar{\delta}(g_{00}g_{rr})}{[g_{00}g_{rr}]_{\text{st}}} + \frac{\delta(g_{00}g_{rr})}{2[g_{00}]_{\text{st}}^2[\mathcal{C}]_{\text{st}}} - \frac{[g_{rr}]_{\text{st}}\delta g_{00}}{[g_{00}]_{\text{st}}^2[\mathcal{C}]_{\text{st}}} \right\} \\ &- [\dot{r}_2]_{\text{st}}^2 \left\{ \frac{[\mathcal{C}]_{\text{st}}}{2} \left(\frac{\bar{\delta}(\partial_r(g_{00}g_{rr}))}{[g_{00}g_{rr}]_{\text{st}}} - \frac{\partial_r[g_{00}g_{rr}]_{\text{st}}\bar{\delta}(g_{00}g_{rr})}{[g_{00}g_{rr}]_{\text{st}}^2} \right) \right. \\ &\quad \left. + \partial_r \left(\frac{\delta(g_{00}g_{rr})}{2[g_{00}^2\mathcal{C}]_{\text{st}}} \right) + \frac{1}{2[\dot{r}_2]_{\text{st}}} \frac{d[\mathcal{C}]_{\text{st}}}{dt} \frac{\bar{\delta}(g_{00}g_{rr})}{[g_{00}g_{rr}]_{\text{st}}} - \partial_r \left(\frac{[g_{rr}]_{\text{st}}\delta g_{00}}{[g_{00}^2\mathcal{C}]_{\text{st}}} \right) \right\} \\ &+ \left\{ \frac{1}{[\dot{r}_2]_{\text{st}}} \frac{d[\mathcal{C}]_{\text{st}}}{dt} - [\mathcal{C}]_{\text{st}} \frac{\partial_r[g_{00}g_{rr}]_{\text{st}}}{[g_{00}g_{rr}]_{\text{st}}} \right\} \frac{c^2\bar{\delta}g_{00} - 2c^2[e]_{\text{st}}\delta e}{2[g_{00}]_{\text{st}}[g_{rr}]_{\text{st}}} \\ &- \frac{[\rho]_{\text{st}}[\dot{\rho}]_{\text{st}}[\dot{r}_2]_{\text{st}}}{[r_2]_{\text{st}}^2[g_{00}]_{\text{st}}^2[\mathcal{C}]_{\text{st}}^3} \left\{ \frac{\delta(g_{00}g_{rr})}{2} - [g_{rr}]_{\text{st}}\delta g_{00} \right\} \\ &+ \frac{d}{dt} \left\{ [\dot{r}_2]_{\text{st}}[\partial_r\mathcal{C}]_{\text{st}}\delta r_2 + [\rho]_{\text{st}}\delta\dot{\phi} + \frac{[\dot{\rho}]_{\text{st}}}{[\partial_\phi\rho]_{\text{st}}}\bar{\delta}\rho \right\}\end{aligned}\quad (47)$$

We now evaluate this expression by using orders of magnitude known for the different quantities. We first know that the metric components are close to 1 for g_{00} , to -1 for g_{rr} , the differences being of the order of the Newton potential κ/r . The latter has a value $\sim 10^{-8}$ on Earth orbit and values 20 to 70 times smaller at the distances explored by Pioneer probes. The square of velocity divided by light velocity has the same value $\sim 10^{-8}$ for Earth, due to the virial theorem, and it is roughly 6 times smaller for Pioneer probes (velocity ~ 12 km/s ~ 0.4 times that of Earth). As we study deep space probes at large heliocentric distances $r_2 \gg r_1$, we also use the fact that the parameter $[\rho]_{\text{st}}$ is at least 20 times smaller than $[r_2]_{\text{st}}$, so that terms scaling as $[\rho]_{\text{st}}^2/[r_2]_{\text{st}}^2$ are at least 400 times smaller than unity. For the same reason, terms proportional to angular velocity anomaly $\delta\dot{\phi}$ are found to have a negligible effect. After these remarks, expression (47) is simplified to the following dominant contributions

$$\begin{aligned}\bar{\delta}A &\simeq \bar{\delta}A_{\text{sec}} + \bar{\delta}A_{\text{ann}} \\ \bar{\delta}A_{\text{sec}} &\simeq -\frac{c^2}{2}\partial_r(\delta g_{00}) + [\ddot{r}_2]_{\text{st}} \left\{ \frac{\delta(g_{00}g_{rr})}{2} - \delta g_{00} \right\} - \frac{c^2}{2}\partial_r^2[g_{00}]_{\text{st}}\delta r_2 \\ \bar{\delta}A_{\text{ann}} &\simeq \frac{d}{dt} \left\{ [\dot{\phi}]_{\text{st}}\bar{\delta}\rho \right\}\end{aligned}\quad (48)$$

The term $\bar{\delta}A_{\text{sec}}$ contains secular contributions proportional to metric anomalies in the first and second sectors as well as to the range ambiguity

$$\delta r_2 \simeq [\dot{r}_2]_{\text{st}} \left\{ \delta\tau - \int_{r_1}^{r_2} \frac{c^2\delta g_{00} - 2c^2[e]_{\text{st}}\delta e - [\dot{r}_2]_{\text{st}}^2\delta(g_{00}g_{rr})}{2[\dot{r}_2]_{\text{st}}^3} dr \right\}\quad (49)$$

The last term $\bar{\delta}A_{\text{ann}}$ is a modulated contribution which is proportional to the anomaly of the impact parameter

$$\bar{\delta}\rho \simeq -[\partial_\phi\rho]_{\text{st}}[\rho]_{\text{st}} \left\{ \int_{r_1}^{r_2} \left(\frac{\delta(g_{00}g_{rr})}{2} + \delta g_{00} \right) \frac{dr}{r^2} + \frac{\delta r_2}{[r_2]_{\text{st}}^2} \right\}\quad (50)$$

At this point, it is worth emphasizing the differences between the expression (48) obtained for the anomalous acceleration δA and the result previously obtained in [17, 18].

The secular anomaly arising from the second sector replaces the result obtained in preliminary calculations [17, 18] which was spoiled by a calculation error. The previous result was linear in the gravity fields and proportional to the kinetic energy of the probe. The new expression appears at the second order in gravity fields, since it is proportional on one hand to the anomalous potential $\delta(g_{00}g_{rr})$ at the position of the probe and on another hand to the standard probe acceleration $[\ddot{r}_2]_{\text{st}}$, that is also the gradient of the standard Newton potential. It however remains of first order with respect to the metric anomaly, and this property has been used in the derivation. Note that the change of form of this term has no consequence on the comparison of the anomalies recorded on Pioneer 10 and 11 since the two probes had nearly equal velocities and anomalous accelerations [8]. But it will affect several conclusions to be drawn in the next section.

The previous results [17, 18] were also preliminary for the following reasons. First, the secular anomaly is corrected by a term proportional to the range ambiguity, because the position of the probe is not known directly (no range measurement capabilities on Pioneer probes), and this important fact was not discussed previously. The range ambiguity (49) contains contributions proportional to anomalies as well as trajectory mismodeling, *i.e.* modifications of the standard acceleration observable due to changes $\delta\tau$ and δe of the constants of motion. Then, the range ambiguity also affects the evaluation of the annually modulated anomaly, probably the most striking new feature of the expression (48). Expectations for the annual and secular anomalies are thus correlated, a property which will turn out to be of uttermost importance in the next section. Finally, the comparison of these expectations may open a road to a genuine test of the ‘post-Einsteinian’ phenomenological framework, as discussed below.

VI. DISCUSSION OF ORDERS OF MAGNITUDE

Obviously, the interest of this new road depends in a critical manner on the orders of magnitude of the various terms, to be discussed in the present section. This discussion is heavily dependent on the stringent constraints put on possible metric anomalies by the gravity tests already performed in the solar system. In order to write down the relevant arguments, we introduce potentials Φ_N and Φ_P in the two sectors as in [18, 19]

$$g_{00} \simeq 1 + 2\Phi_N \quad , \quad -g_{00}g_{rr} \simeq 1 + 2\Phi_P \quad (51)$$

We also use the simplest form of the standard impact parameter $[\rho]_{\text{st}}$ which is modulated by Earth rotation

$$[\rho]_{\text{st}} \simeq -r_1 \sin(\Omega_1(t - t_{\text{conj}})) \quad (52)$$

The time t_{conj} corresponds to a conjunction (closest approach) of Earth and deep space probe. The simple expression (52) is sufficient for the purpose of the present section.

In this context, the secular and modulated anomalies in (48) are reduced to

$$\begin{aligned} \bar{\delta}A_{\text{sec}} &\simeq -c^2 \partial_r \delta\Phi_N(r_2) + a_1 \frac{r_1^2}{r_2^2} \left\{ \frac{2\delta r_2}{r_2} + \delta\Phi_P(r_2) + 2\delta\Phi_N(r_2) \right\} \\ \bar{\delta}A_{\text{ann}} &\simeq -\mathcal{A} \cos(2\Omega_1(t - t_{\text{conj}})) - \frac{d\mathcal{A}}{2\Omega_1 dt} \sin(2\Omega_1(t - t_{\text{conj}})) \\ \mathcal{A} &\equiv a_1 \frac{r_1}{r_2} \left\{ \frac{\delta r_2}{r_2} - r_2 \int_{r_1}^{r_2} (\delta\Phi_P - 2\delta\Phi_N) \frac{dr}{r^2} \right\} \\ a_1 &\equiv \frac{c^2 \kappa}{r_1^2} = \frac{G_N M}{r_1^2} = \Omega_1^2 r_1 \simeq 6 \times 10^{-3} \text{ m s}^{-2} \end{aligned} \quad (53)$$

These expressions of the secular and modulated anomalies can be considered as the key predictions of the post-Einsteinian framework presented in this paper.

The secular anomaly $\bar{\delta}A_{\text{sec}}$ contains a first term describing the effect of a Newton law modification and a second one gathering the contributions of the range ambiguity and of the potentials. The annual anomaly $\bar{\delta}A_{\text{ann}}$ is determined by an amplitude \mathcal{A} which contains contributions of the range ambiguity and of the two anomalous potentials. This amplitude is not annually modulated but suffers a secular change during the probe’s journey. Considering that \mathcal{A} varies slowly over a year, the modulation in (53) is essentially at twice the orbital frequency. The observed behaviour [8] has a richer structure but its discussion must take into account the following remarks. First the constants of motion, and therefore \mathcal{A} , are modified at maneuvers which are occurring twice a year on the average. Then, the mismodeling contributions on angles φ_2 and θ_2 , not studied in detail here, have periods Ω_1 and $2\Omega_1$, as soon as a non

null inclination angle ι is accounted for. These remarks qualitatively explain why the observed annual modulation may be more complicated than the simple expression (53).

The maneuvers are not frequent for Pioneer 10/11 probes, which is one of the main causes of their excellent navigational accuracy [8]. They interrupt the free geodesic segments by changing the values of the constants of motion. This essentially amounts to a change of the local velocity with no change of the position if the maneuver can be modeled as nearly instantaneous. The detailed description of the maneuvers is one of the most delicate parts of the data analysis process and it certainly goes much farther than the purpose of the present paper. In the following paragraphs, we focus our attention on the main features which can be qualitatively expected rather than on quantitative results. To this aim, we can go along with the simplified expression (53).

The hierarchy of magnitudes appearing in (53), $a_1 r_1^2 / r_2^2$ in $\bar{\delta}A_{\text{sec}}$ and $a_1 r_1 / r_2$ in $\bar{\delta}A_{\text{ann}}$, plays a central role in this discussion. This is the reason why we have used in (53) the acceleration a_1 of Earth on its orbit as the natural scale for measuring anomalous Doppler accelerations, except for the first term which is the direct effect of a Newton law modification. We briefly discuss this term now, before embarking on a more complete analysis of the terms induced by the range ambiguity and the second potential and putting into evidence the correlation between the secular and modulated anomalies.

Should the Pioneer anomaly be explained by an anomaly in the first sector, a linear dependence of the potential $\delta\Phi_N$ would be needed to reproduce the fact that the anomaly has a roughly constant value (1) over a large range of heliocentric distances r_P

$$c^2 \partial_r \delta\Phi_N \simeq a_P \quad , \quad 20 \text{ AU} \leq r_P \leq 70 \text{ AU} \quad (54)$$

The simplest way to modelize the anomaly would thus correspond to a potential varying linearly with r and vanishing at Earth orbit to fit the convention (4)

$$\delta\Phi_N \simeq \frac{r - r_1}{\ell_P} \quad (55)$$

We have introduced a length ℓ_P characteristic of the Pioneer anomaly

$$\ell_P^{-1} \equiv \frac{a_P}{c^2} \simeq 0.8 \times 10^{-26} \text{ m}^{-1} \simeq 1.2 \times 10^{-15} \text{ AU}^{-1} \quad (56)$$

Should this model effectively describe the metric in the vicinity of Earth and Mars, its effects could not have escaped detection in the very accurate tests performed with martian probes such as Viking [27]. Numbers shedding light on this point are given in [8] (see also [4] where similar conclusions are obtained for a modification of the Newton potential having the form of a Yukawa potential). The effect of the perturbation (55) on planets would produce a change of their orbital radius. The order of magnitude (56) would lead to range variations of $\sim 50\text{km}$ and $\sim 100\text{km}$ respectively at smallest and largest distances. Meanwhile, the Viking data constrain these measurements to agree with standard expectations at a level of $\sim 100\text{m}$ and $\sim 150\text{m}$ respectively. These numbers are different enough to eliminate the simple model (55) with the coefficient ℓ_P chosen to fit the Pioneer anomaly. Note that the effect of the Shapiro time delay in the range evaluation, which should in principle have been taken into account in the discussion, has here a negligible influence [40].

As already discussed, these results do not prove that the Pioneer anomaly cannot be reproduced in a metric theory. First, there is the possibility that the linear dependence needed to reproduce (54) at distances explored by Pioneer probes is cut at the orbital radii of planets on which the strongest constraints are obtained [30]. As it was already discussed, it then remains to decide whether or not the ephemeris of the outer planets are accurate enough to forbid the presence of the linear dependence (54) in the range of distances explored by the Pioneer probes [20, 21]. This point remains to be settled [30]. In any case, there is another possibility, namely that the Pioneer anomaly is induced by the second anomalous potential $\delta\Phi_P$ rather than the first one $\delta\Phi_N$. We now consider these terms which are still here even if there is no anomaly at all in the first sector ($\delta\Phi_N = 0$). More thorough studies will have to be performed later on to study the correlated effects of anomalies in the two sectors.

We now focus our attention on the secular anomaly $\bar{\delta}A_{\text{sec}}$ and the annual amplitude \mathcal{A} which are determined in (53) by the range ambiguity and the second potential

$$\bar{\delta}A_{\text{sec}} \simeq a_1 \frac{r_1^2}{r_2^2} \left\{ \frac{2\delta r_2}{r_2} + \delta\Phi_P(r_2) \right\} , \quad \mathcal{A} \simeq a_1 \frac{r_1}{r_2} \left\{ \frac{\delta r_2}{r_2} - r_2 \int_{r_1}^{r_2} \frac{\delta\Phi_P}{r^2} dr \right\} \quad (57)$$

With the same assumption $\delta\Phi_N = 0$, the range ambiguity is given by

$$\delta r_2 \simeq [\dot{r}_2]_{\text{st}} \left\{ \delta\tau + \int_{r_1}^{r_2} \frac{\delta\varepsilon}{[\dot{r}]_{\text{st}}^3} dr - \int_{r_1}^{r_2} \frac{\delta\Phi_P}{[\dot{r}]_{\text{st}}} dr \right\} \quad (58)$$

We have introduced the non relativistic reduced energy ε which allows us to express the standard velocity $[\dot{r}]_{\text{st}}$ as a function of distance, using (35),

$$\varepsilon \equiv \frac{c^2 (e^2 - 1)}{2} \quad , \quad [\dot{r}]_{\text{st}} = \sqrt{2 \left(\varepsilon + \frac{c^2 \kappa}{r} \right)} \quad (59)$$

The secular anomaly and annual amplitude appear in (57) as different linear superpositions of the reduced range ambiguity $\delta r_2/r_2$ and of the second potential $\delta\Phi_P$. The term in front of the annual amplitude turns out to be larger than the term in front of the secular anomaly by a factor r_2/r_1 . This could appear to be contradictory with the fact that the annual anomaly is only a fraction of the secular one [8] in the data, but this is not the case. As a matter of fact, the data analysis process described in [8] is based on the *a priori* assumption that there is no annually modulated anomaly in the physical signal of interest. As explained above, this assumption is valid for anomalies induced by $\delta\Phi_N$ but not for anomalies induced by $\delta\Phi_P$. In the context of this assumption, the choice of the best trajectory fitting the data tends to produce a null or quasi null value for the annual anomaly. This corresponds to a choice of the trajectory leading to a perfect or nearly perfect compensation of the two contributions to \mathcal{A} in (57), in conformity with the assumed absence of annual anomalies.

In the context of the present paper where annual anomaly of the Doppler acceleration is a natural expectation, the observations reported in [8] have a different significance. They mean that the unknown epoch $\delta\tau$ characterizing the motion of the probe has been fixed so that the two contributions to \mathcal{A} compensate each other at some well chosen radius r_2

$$|\mathcal{A}| \ll a_1 \frac{r_1}{r_2} \quad , \quad \frac{\delta r_2}{r_2} \simeq r_2 \int_{r_1}^{r_2} \frac{\delta\Phi_P}{r^2} dr \quad (60)$$

As a consequence of already presented arguments, this compensation has to be effective within a fraction of the order of or even smaller than r_1/r_2 . Now, this compensation cannot remain perfect over a long period of time. The first reason for that is the different dependence on r_2 of the two terms to be compensated by each other. The second reason is due to the maneuvers which change the constants of motion and thus affect the compensation. A precise estimation of the annual anomaly thus requires a complete solution of the motion including a detailed description of the maneuvers. As this task is outside the scope of the present paper, we will not be able to conclude whether or not the annual modulations reported in [8] are effectively accounted for in a quantitative manner by the effect of the second potential.

Despite this deficiency, the description just given of the annual anomaly nevertheless leads to a quantitative estimation of the secular anomaly. Equation (60) indeed fixes the otherwise unknown range ambiguity $\delta r_2/r_2$, so that $\bar{\delta}A_{\text{sec}}$ can be rewritten

$$\bar{\delta}A_{\text{sec}} \simeq a_1 r_1^2 \left\{ \frac{2}{r_2} \int_{r_1}^{r_2} \chi_P(r) dr + \chi_P(r_2) \right\} \quad , \quad \delta\Phi_P(r) \equiv \chi_P(r) r^2 \quad (61)$$

A roughly constant anomaly is produced when χ_P is constant, *i.e.* when $\delta\Phi_P(r)$ is quadratic in r , in the range of Pioneer distances. Identifying the expression $\bar{\delta}A_{\text{sec}}$ to the observed Pioneer anomaly a_P fixes the value of the constant

$$\bar{\delta}A_{\text{sec}} \simeq -a_P \quad \rightarrow \quad \chi_P \simeq -\frac{1}{3\kappa\ell_P} \simeq -4 \times 10^{-8} \text{ AU}^{-2} \quad (62)$$

This value is 3 times smaller from what would have been obtained without accounting for the range ambiguity. Note that χ_P can take different values outside the range 20-70 AU of Pioneer distances and that it is not even forced to be exactly constant in this range. In fact, we know that χ_P has to vanish at Earth radius in order to obey the convention (4). We also show in the next paragraph that it may have to be smaller than (62) between Earth and Mars in order to be compatible with planetary observations.

To this aim, we consider a simple model with the potential obtained as the sum of linear and quadratic terms vanishing at Earth orbit to fit (4)

$$\delta\Phi_P \simeq -\frac{(r - r_1)^2 + \mu_P(r - r_1)}{3\kappa\ell_P} \quad (63)$$

The quadratic coefficient has been fixed according to (62). The further characteristic length μ_P has been introduced to represent the radial derivative of the metric anomaly at Earth orbit. This linear term has to be small enough to be dominated by the quadratic one at distances explored by Pioneer probes ($\mu_P \ll r_P$). Now the metric perturbation

(63) has also an influence on the already discussed range measurements on martian probes. As a consequence of the Shapiro effect, a range variation is found with a value of the order of $\sim 700\text{m}$, in conflict with Viking data. This conflict may be cured by cutting off the simple dependence (63) at the orbital radii of Earth and Mars. It is easily checked out that this does not affect significantly the predictions made for the Pioneer probes which are at much larger distances.

VII. CONCLUDING REMARKS

As stated in the introduction, the present paper follows up publications [17, 18, 19] devoted to the study of post-Einsteinian metric extensions of GR. A more complete theory of radar ranging and Doppler tracking has been given in terms of the time delay function, allowing us to discuss the annual anomaly besides the secular one. A mistake made in previous publications in the evaluation of the secular anomaly has been corrected and important new results have been obtained. In particular, the annual anomaly has been found to be correlated with the secular anomaly through terms arising from ambiguities in the position of the probe. This correlation, in principle available in the data, has to be scrutinized in order to extract reliable information from Pioneer observations.

The following qualitative statements summarize the results of the present paper. As the secular anomaly, the annual anomaly is a natural consequence of the presence of a second potential. This has to be contrasted with the first potential which does not produce a significant effect along the links [40]. Then, the annual anomaly produced by propagation along the up- and down-links can be compensated near an arbitrary point by an appropriate choice of the trajectory of the probe. In fact this compensation is an output of any best fit procedure based on the a priori assumption that there is no annual anomaly. However, the compensation cannot remain exact when the probe moves as the two compensating terms have different dependences on the heliocentric distance. It follows that the annual anomaly reappears, either after some free evolution or after the next maneuver. This qualitative behaviour is reminiscent of the observations of annual anomalies which were reported in [8]. This situation certainly pleads for pushing this study and comparing the theoretical expectations with Pioneer data. It is only after a quantitative comparison, taking into account all the details known to be important for data analysis [8], that it will be possible to decide whether or not the post-Einsteinian phenomenological framework does fit the observations.

It is clear after these remarks that some of the conclusions of our previous papers have to be amended: the secular anomaly turns out to be proportional to the standard acceleration and to the second potential. The corrected expression is quadratic, and no longer linear, in the gravity fields, with one contribution standard and the second one anomalous. Identifying the expectation with the observed Pioneer anomaly now points to a second potential with a quadratic dependence on the radius. This corresponds to a constant curvature with an unexpectedly large value in the outer solar system (eq. (62) of the present paper). This quadratic dependence may have to be cut off at distances exceeding the size of the solar system as well as in the inner solar system (in order to pass Shapiro tests on martian probes). Note that the discussion of the preceding section was mainly focused on the change of the Shapiro delay, due to the anomaly on Φ_P . The modification of the orbital radii, which could in principle play a role, has been ignored because it was expected to have a negligible influence. The evaluation of correlated effects of anomalies in the two sectors will be necessary in order to be able to reach definitive conclusions.

These conclusions constitute motivations for new experiments in the solar system. Clearly, experiments with ranging capabilities will offer qualitatively better perspectives than Pioneer observations which were performed without such capabilities. Missions going to the borders of the solar system [14] will either prove or disprove the existence of the anomaly at such long distances. Comparison with the theoretical expectation of the present paper will give an answer to the question whether such an anomaly may have a metric origin, with the metric possibly departing from the GR prediction. This idea could also be tested on a shorter time scale by adding specially designed instruments on planetary probes going to Mars, Jupiter, or Saturn, the reduction of the explored heliocentric distance being compensated by a potentially large improvement of the measurement accuracy.

-
- [1] Will C M 1993 *Theory and Experiment in Gravitational Physics* (Cambridge: Cambridge U. P.)
Will C M 2001 *Living Rev. Rel.* **4** 4
 - [2] Fischbach E and Talmadge C 1998 *The Search for Non Newtonian Gravity* (Berlin: Springer)
 - [3] Adelberger E G, Heckel B R and Nelson A E 2003 *Ann. Rev. Nucl. Part. Sci.* **53** 77
 - [4] Jaekel M-T and Reynaud S 2005 *Int. J. Mod. Phys. A* **20** 2294 and references therein
 - [5] Lue A, Scoccimari R and Starkman G 2004 *Phys. Rev. D* **69** 044005
 - [6] Carroll S M *et al* 2004 *Phys. Rev. D* **70** 043528
 - [7] Anderson J D *et al* 1998 *Phys. Rev. Lett.* **81** 2858

- [8] Anderson J D *et al* 2002 *Phys. Rev.* **D 65** 082004
- [9] Anderson J D *et al* 2003 *Mod. Phys. Lett.* **A17** 875
- [10] Nieto M M and Turyshev S G 2004 *Class. Quantum Grav.* **21** 4005
- [11] Turyshev S G, Nieto M M and Anderson J D 2004 *35th COSPAR Scientific Assembly Preprint* gr-qc/0409117
- [12] Bertolami O and Paramos J 2004 *Class. Quantum Grav.* **21** 3309
see also *Preprint* astro-ph/0408216 and *Preprint* gr-qc/0411020
- [13] Lämmerzahl C, Preuss O and Dittus H 2006 *Preprint* gr-qc/0604052
- [14] Dittus H *et al* 2006 in *Trends in Space Science and Cosmic Vision 2020* to be published (ESA), also in *Preprint* gr-qc/0506139
- [15] Nieto M M and Anderson J D 2005 *Preprint* gr-qc/0507052
- [16] Turyshev S G, Toth V T, Kellogg L R *et al* 2006 *Int. J. Mod. Phys.* **D15** 1
see also *Preprint* gr-qc/0603016
- [17] Jaekel M T and Reynaud S 2005 *Mod. Phys. Lett.* **A20** 1047
- [18] Jaekel M T and Reynaud S 2005 *Class. Quantum Grav.* **22** 2135
- [19] Jaekel M T and Reynaud S 2006 *Class. Quantum Grav.* **23** 777
- [20] Iorio L and Giudice G 2006 *Preprint* gr-qc/0601055
- [21] Tangen K 2006 *Preprint* gr-qc/0602089
- [22] Guinot B *et al* 1997 *Metrologia* **34** 261
- [23] Petit G and Wolf P 2005 *Metrologia* **42** S138
- [24] An equivalent representation of the same spacetime is given in [19] in terms of Schwarzschild coordinates
- [25] Jaekel M T and Reynaud S 1995 *Annalen der Physik* **4** 68
- [26] Talmadge C *et al* 1998 *Phys. Rev. Lett.* **61** 1159
- [27] Reasenberg R D, Shapiro I I, MacNeil P E *et al* 1979 *Astrophys. J. Lett.* **234** L219
- [28] Hellings R W *et al* , 1983 *Phys. Rev. Lett.* **51** 1609
- [29] Anderson J D *et al* 1996 *Astrophys. J.* **459** 365
- [30] Brownstein J R and Moffat J W 2006 *Preprint* gr-qc/0511026, to appear in *Class. Quantum Grav.*
- [31] Iess L *et al* , 1999 *Class. Quantum Grav.* **16** 1487
- [32] Bertotti B, Iess L and Tortora P 2003 *Nature* **425** 374
- [33] Shapiro S S *et al* 2004 *Phys. Rev. Lett.* **92** 121101
- [34] Turyshev S G *et al* 2006 in *Trends in Space Science and Cosmic Vision 2020* to be published (ESA, 2005), also in *Preprint* gr-qc/0506104
- [35] Vecchiato A, Lattanzi M G and Bucciarelli B 2003 *Astron. Astrophys.* **399** 337
- [36] Shapiro I I 1999 *Rev. Mod. Phys.* **71** S41
- [37] Le Poncin-Lafitte C, Linet B and Teyssandier P 2004 *Class. Quantum Grav.* **21** 4463
- [38] Moyer T D 2003 *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation* , (John Wiley & Sons, Inc., 2003); see also the JPL website <http://descanso.jpl.nasa.gov/>
- [39] Landau L D and Lifschitz E M 1975 *The classical theory of fields* (New York: Butterworth)
- [40] Moffat J W 2006 *Preprint* gr-qc/0605141