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A NEW APPROACH TO TRANSPORT EQUATIONS ASSOCIATED TO A REGULAR FIELD: TRACE RESULTS AND WELL-POSEDNESS.

L. ARLOTTI, J. BANASIAK & B. LODS

ABSTRACT. We generalize known results on transport equations associated to a Lipschitz field \mathcal{F} on some subspace of \mathbb{R}^N endowed with some general space measure μ . We provide a new definition of both the transport operator and the trace measures over the incoming and outgoing parts of $\partial\Omega$ generalizing known results from [9, 16]. We also prove the well-posedness of some suitable boundary-value transport problems and describe in full generality the generator of the transport semigroup with no-incoming boundary conditions.

1. INTRODUCTION

In this paper we present new methodological tools to investigate the well-posedness of the general transport equation

$$\partial_t f(\mathbf{x}, t) + \mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, t) = 0 \quad (\mathbf{x} \in \Omega, t > 0), \quad (1.1a)$$

supplemented by boundary condition

$$f|_{\Gamma_-}(\mathbf{y}, t) = \psi_-(\mathbf{y}, t), \quad (\mathbf{y} \in \Gamma_-, t > 0), \quad (1.1b)$$

and the initial condition

$$f(\mathbf{x}, 0) = f_0(\mathbf{x}), \quad (\mathbf{x} \in \Omega). \quad (1.1c)$$

Here Ω is a sufficiently smooth open subset of \mathbb{R}^N , Γ_{\pm} are suitable boundaries of the phase space and ψ_- is a given function of the trace space $L^1(\Gamma_-, d\mu_-)$ corresponding to the boundary Γ_- (see Section 2 for details).

The present paper is part of a series of papers on transport equations with general vector fields [5, 6] and introduce all the methodological tools that allow us not only to solve the initial-boundary problem (1.1) but also to treat in [6] the case of abstract boundary conditions relying the incoming and outgoing fluxes, generalizing the results of [9].

The main novelty of our approach is that we assume \mathbb{R}^N to be endowed with a general positive Radon measure μ . Here by a Radon measure we understand a Borel measure (or its completions, see [15, p. 332]) which is finite on compact sets. As we shall see it further on, taking into account such general Radon measure μ leads to a large amount of technical difficulties, in particular in the definition of trace spaces and in the derivation of Green's formula. Moreover, for such a measure μ , it is far from being trivial to identify the vector field $\mathcal{F} \cdot \nabla_x$ to the time derivative along the characteristic curves (as done in [9, Formulae (5.4) & (5.5), p.392]): the main difficulty stemming from the impossibility of applying classical convolution arguments (and the so-called Friedrich's lemma). We overcome this difficulty by introducing new mollification techniques along the characteristic curves. Let us explain in more details our general assumptions:

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AMS subject classifications (2000): 47D06, 47D05, 47N55, 35F05, 82C40.

1.1. General assumption and motivations. The transport coefficient \mathcal{F} is a *time independent* vector field $\mathcal{F} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ which is (globally) *Lipschitz-continuous* with Lipschitz constant $\kappa > 0$, i.e.

$$|\mathcal{F}(\mathbf{x}_1) - \mathcal{F}(\mathbf{x}_2)| \leq \kappa |\mathbf{x}_1 - \mathbf{x}_2| \quad \text{for any } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N. \quad (1.2)$$

Clearly, one can associate a flow $(T_t)_{t \in \mathbb{R}}$ to this field \mathcal{F} (with the notations of Section 2.1, $T_t = \Theta(\cdot, t, 0)$) and we make the following fundamental assumption (known as **Liouville's Theorem** whenever μ is the Lebesgue measure) on \mathcal{F} :

Assumption 1. *The measure μ is invariant under the flow $(T_t)_{t \in \mathbb{R}}$, i.e. $\mu(T_t A) = \mu(A)$ for any measurable subset $A \subset \mathbb{R}^N$ and any $t \in \mathbb{R}$.*

Remark 1.1. *Notice that, whenever μ is the Lebesgue measure over \mathbb{R}^N , it is well-known that Assumption 1 is equivalent to $\operatorname{div}(\mathcal{F}(\mathbf{x})) = 0$ for any $\mathbf{x} \in \mathbb{R}^N$. More generally, by virtue of [2, Remark 3 & Proposition 4], Assumption 1 holds for a general Borel measure μ provided the field \mathcal{F} is locally integrable with respect to μ and **divergence-free** with respect to μ in the sense that*

$$\int_{\mathbb{R}^N} \mathcal{F}(T_t(\mathbf{x})) \cdot \nabla_{\mathbf{x}} f(T_t(\mathbf{x})) d\mu(\mathbf{x}) = 0, \quad \forall t \in \mathbb{R}$$

for any infinitely differentiable function f with compact support.

A typical example of such a transport equation is the so-called Vlasov equation for which:

- i) The phase space Ω is given by the cylindrical domain $\Omega = \mathcal{D} \times \mathbb{R}^3 \subset \mathbb{R}^6$ where \mathcal{D} is a sufficiently smooth open subset of \mathbb{R}^3 , referred to as the *position space*, while the so-called *velocity space* is here given by \mathbb{R}^3 . The measure μ is given by $d\mu(\mathbf{x}) = dx d\beta(v)$ where β is a suitable Radon measure on \mathbb{R}^3 , e.g. Lebesgue measure over \mathbb{R}^3 for continuous models or combination of Lebesgue measures over suitable spheres for the multigroup model.
- ii) For any $\mathbf{x} = (x, v) \in \mathcal{D} \times \mathbb{R}^3$,

$$\mathcal{F}(\mathbf{x}) = (v, \mathbf{F}(x, v)) \in \mathbb{R}^6 \quad (1.3)$$

where $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$ is a time independent force field over $\mathcal{D} \times \mathbb{R}^3$ satisfying Assumption 1 and (1.2). The free transport case, investigated in [16, 4], corresponds to $\mathbf{F} = 0$.

The existence of solution to the transport equation (1.1a) is a classical matter when considering the whole space $\Omega = \mathbb{R}^N$. In particular, the concept of renormalized solutions allows to consider irregular transport coefficient $\mathcal{F}(\cdot)$ (see [10] and the recent contributions [2, 13]) which is of particular relevance in fluid mechanics.

On the other hand, there are few results addressing the initial-boundary value problem (1.1), possibly due to difficulties created by the boundary conditions (1.1b). We mention here the seminal works by C. Bardos [8], and by R. Beals and V. Protopopescu [9] (see also [11, 14]). Let us however mention that the results of [9, 11] introduce restrictive assumptions on the characteristics of the equation. For instance, fields with 'too many' periodic trajectories create serious difficulties. They are however covered in a natural way by the theory presented here, see Examples 2.5 & 2.6.

1.2. Presentation of the results. In this paper, we revisit and generalize the afore-mentioned results to the general case $\mathbf{F} \neq 0$ and for a general Radon measure μ . The latter, in particular, leads to numerous technical problems such as e.g. determination of suitable measures μ_{\pm} over the 'incoming' and 'outgoing' parts Γ_{\pm} of $\partial\Omega$. We provide here a general construction of these 'trace measures' generalizing, and making more precise, the results of [9, 11]. This construction allows us to establish Proposition 2.12 which allows to compute integrals over Ω via integration along

the integral curves of $\mathcal{F}(\cdot)$ coming from the boundary $\partial\Omega$, and which is free from some restrictive assumptions of *op. cit.* In particular, we present a new proof of the Green formula clarifying and removing gaps of the proofs in [9, 11]. Of course, the boundary condition (1.1b) we treat here is less general than the abstract ones investigated in [9, 11] but, as we already mentioned it, the tools we introduce here will allow us to generalize, in a subsequent paper [6], the results of the *op. cited* by dealing with abstract boundary conditions.

Another major difficulty, when dealing with a general Radon measure μ , is to provide a precise definition of the transport operator \mathcal{T}_{\max} associated to (1.1). It appears quite natural to define the transport operator \mathcal{T}_{\max} (with its maximal domain on $L^1(\Omega, d\mu)$) as a *weak directional derivative along the characteristic curves* in the L^1 -sense. However, it is not clear *a priori* that any function f for which the weak directional derivative exists in $L^1(\Omega, d\mu)$ (with appropriate and minimal class of test-functions) admits a trace over Γ_{\pm} . With the aim of proving such a trace result, we provide here a new characterization of the transport operator related to a *mild representation* of the solution to (1.1). Namely, we prove (Theorem 3.6) that the domain $\mathcal{D}(\mathcal{T}_{\max})$ (as defined in Section 3), is precisely the set of functions $f \in L^1(\Omega, d\mu)$ that admits a representative which is *absolutely continuous along almost any characteristic curve*.

Note that in the classical case when μ is the Lebesgue measure, such a representation is known to be true [10, Appendix]. Actually, in this case, one defines the domain $\mathcal{D}(\mathcal{T}_{\max})$ as the set of all $f \in L^1(\Omega, d\mu)$ for which the directional derivative $-\mathcal{F} \cdot \nabla f$ exists in the distributional sense and belongs to $L^1(\Omega, d\mu)$. Then, by convolution arguments, it is well-known that the set $\mathcal{C}_0^1(\Omega) \cap \mathcal{D}(\mathcal{T}_{\max})$ is dense in $\mathcal{D}(\mathcal{T}_{\max})$ for the graph norm $\|f\| = \|f\| + \|\mathcal{F} \cdot \nabla f\|$.

The question is much more delicate for a general Radon measure μ . Indeed, in such a case, the convolution argument used in the case of the Lebesgue measure does not apply anymore. Our strategy to prove the characterization of \mathcal{T}_{\max} is also based on a convolution argument but it uses *mollification technique along the characteristic curves* as developed in Section 3. Such a result shall allow us to obtain a rigorous derivation of Green's formula, clarifying some results of [9].

1.3. Plan of the paper. The organization of the paper is as follows. In Section 2 we introduce main tools used throughout the paper and present the aforementioned new results concerning *integration over the characteristic curves* of \mathcal{F} as well as a *new construction of the boundary measures* over the ‘incoming’ and ‘outgoing’ parts Γ_{\pm} of $\partial\Omega$ which generalizes and clarifies that of [9, 11]. In Section 3 we provide a construction of the maximal transport operator \mathcal{T}_{\max} . It is defined in a weak sense, through its action on suitably defined test functions. The fundamental result of this section shows that any function in the domain $\mathcal{D}(\mathcal{T}_{\max})$ admits a representation which is absolutely continuous along almost any characteristic which, in turn, allows for existence of its traces on the outgoing and incoming parts of the boundary. In Section 4 we apply the results of Section 3 to prove well-posedness of the time-dependent transport problem with no reentry boundary conditions associated with \mathcal{T}_{\max} . Moreover, we consider the corresponding stationary problem and, as a by-product, we recover a new proof of the *Green formula*.

2. INTEGRATION ALONG THE CHARACTERISTICS

2.1. Characteristic curves. A crucial role in our study is played by the characteristic curves associated to the field \mathcal{F} . Precisely, for any $\mathbf{x} \in \mathbb{R}^N$ and $t \in \mathbb{R}$, consider the initial-value problem

$$\begin{cases} \frac{d}{ds}\mathbf{X}(s) = \mathcal{F}(\mathbf{X}(s)), & (s \in \mathbb{R}); \\ \mathbf{X}(t) = \mathbf{x}. \end{cases} \quad (2.1)$$

Since \mathcal{F} is Lipschitz continuous on \mathbb{R}^N , Eq. (2.1) has a unique *global in time* solution and this allows to define the flow–mapping $\Theta : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^N$, such that, for $(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R}$, the mapping:

$$\mathbf{X}(\cdot) : s \in \mathbb{R} \mapsto \Theta(\mathbf{x}, t, s)$$

is the only solution of Eq. (2.1). Being concerned with solutions to the transport equation (1.1) in the region Ω , we have to introduce the definition of stay times of the characteristic curves in Ω :

Definition 2.1. For any $\mathbf{x} \in \Omega$, define $\tau_{\pm}(\mathbf{x}) = \inf\{s > 0; \Theta(\mathbf{x}, 0, \pm s) \notin \Omega\}$, with the convention that $\inf \emptyset = \infty$, and set $\tau(\mathbf{x}) = \tau_+(\mathbf{x}) + \tau_-(\mathbf{x})$.

In other words, given $\mathbf{x} \in \Omega$, $I_{\mathbf{x}} = (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$ is the maximal interval for which $\Theta(\mathbf{x}, 0, s)$ lies in Ω for any $s \in I_{\mathbf{x}}$ and $\tau(\mathbf{x})$ is the length of the interval $I_{\mathbf{x}}$. Notice that $0 \leq \tau_{\pm}(\mathbf{x}) \leq \infty$. Thus, the function Θ restricted to the set

$$\Lambda := \left\{ (\mathbf{x}, t, s); \mathbf{x} \in \Omega, t \in \mathbb{R}, s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x})) \right\}$$

is such that $\Theta(\Lambda) = \Omega$. Note that here we *do not* assume that the length of the interval $I_{\mathbf{x}} = (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$ is *finite*. In particular, $I_{\mathbf{x}} = \mathbb{R}$ for any stationary point \mathbf{x} of \mathcal{F} , i.e. $\mathcal{F}(\mathbf{x}) = 0$. If $\tau(\mathbf{x})$ is finite, then the function $\mathbf{X} : s \in I_{\mathbf{x}} \mapsto \Theta(\mathbf{x}, 0, s)$ is bounded since \mathcal{F} is Lipschitz continuous. Moreover, still by virtue of the Lipschitz continuity of \mathcal{F} , the only case when $\tau_{\pm}(\mathbf{x})$ is finite is when $\Theta(\mathbf{x}, 0, \pm s)$ reaches the boundary $\partial\Omega$ so that $\Theta(\mathbf{x}, 0, \pm\tau_{\pm}(\mathbf{x})) \in \partial\Omega$. We note that, since \mathcal{F} is Lipschitz around each point of $\partial\Omega$, the points of the set $\{\mathbf{y} \in \partial\Omega; \mathcal{F}(\mathbf{y}) = 0\}$ (introduced in [9, 11]) are equilibrium points of the \mathcal{F} and cannot be reached in finite time.

Remark 2.2. We emphasize that periodic trajectories which do not meet the boundaries have $\tau_{\pm} = \infty$ and thus are treated as infinite though geometrically they are bounded.

Finally we mention that it is not difficult to prove that the mappings $\tau_{\pm} : \Omega \rightarrow \mathbb{R}^+$ are lower semicontinuous and therefore measurable, see e.g., [7, p. 301]

The flow $\Theta(\mathbf{x}, t, s)$ defines, at each instant t , a mapping of the phase space Ω into \mathbb{R}^N . Through this mapping, to each point \mathbf{x} there corresponds the point $\mathbf{x}_{s,t} = \Theta(\mathbf{x}, t, s)$ reached at time s by the point which was at \mathbf{x} at the ‘initial’ time t . The flow Θ , restricted to Λ , has the properties:

Proposition 2.3. Let $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}$ be fixed. Then,

- (i) $\Theta(\mathbf{x}, t, t) = \mathbf{x}$.
- (ii) $\Theta(\Theta(\mathbf{x}, t, s_1), s_1, s_2) = \Theta(\mathbf{x}, t, s_2)$, $\forall s_1, s_2 \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
- (iii) $\Theta(\mathbf{x}, t, s) = \Theta(\mathbf{x}, t - s, 0) = \Theta(\mathbf{x}, 0, s - t)$, $\forall s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
- (iv) $|\Theta(\mathbf{x}_1, t, s) - \Theta(\mathbf{x}_2, t, s)| \leq \exp(\kappa|t - s|)|\mathbf{x}_1 - \mathbf{x}_2|$ for any $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$, $s - t \in I_{\mathbf{x}_1} \cap I_{\mathbf{x}_2}$.

An important consequence of (iii) above is that $\Theta(\mathbf{x}, 0, s) = \Theta(\mathbf{x}, -s, 0)$ for any $\mathbf{x} \in \Omega$, $0 \leq s \leq \tau_+(\mathbf{x})$. Therefore, from now on, to shorten notations we shall denote

$$\Phi(\mathbf{x}, t) = \Theta(\mathbf{x}, 0, t), \quad \forall t \in \mathbb{R},$$

so that $\Phi(\mathbf{x}, -t) = \Theta(\mathbf{x}, t, 0)$, $t \in \mathbb{R}$. We define the incoming and outgoing part of the boundary $\partial\Omega$ through the flow Φ :

Definition 2.4. The incoming Γ_- and the outgoing Γ_+ parts of the boundary $\partial\Omega$ are defined by:

$$\Gamma_{\pm} := \{\mathbf{y} \in \partial\Omega; \exists \mathbf{x} \in \Omega, \tau_{\pm}(\mathbf{x}) < \infty \text{ and } \mathbf{y} = \Phi(\mathbf{x}, \pm\tau_{\pm}(\mathbf{x}))\}. \quad (2.2)$$

Properties of Φ and of τ_{\pm} imply that Γ_{\pm} are Borel sets. It is possible to extend the definition of τ_{\pm} to Γ_{\pm} as follows. If $\mathbf{x} \in \Gamma_{-}$ then we put $\tau_{-}(\mathbf{x}) = 0$ and denote $\tau_{+}(\mathbf{x})$ the length of the integral curve having \mathbf{x} as its left end-point; similarly if $\mathbf{x} \in \Gamma_{+}$ then we put $\tau_{+}(\mathbf{x}) = 0$ and denote $\tau_{-}(\mathbf{x})$ the length of the integral curve having \mathbf{x} as its right endpoint. Note that this definition implies that τ_{\pm} are measurable over $\Omega \cup \Gamma_{-} \cup \Gamma_{+}$.

Let us illustrate the above definition of Γ_{\pm} by two simple 2D examples:

Example 2.5 (Harmonic oscillator in a rectangle). Let $\Omega = (-a, a) \times (-\xi, \xi)$ with $a, \xi > 0$ and let us consider the harmonic oscillator force field

$$\mathcal{F}(\mathbf{x}) = (v, -\omega^2 x), \quad \text{for any } \mathbf{x} = (x, v) \in \Omega \quad (2.3)$$

where $\omega > 0$. We take as μ the Lebesgue measure over \mathbb{R}^2 and, since \mathcal{F} is divergence-free, Assumption 1 is fulfilled. In this case, for any $\mathbf{x}_0 = (x_0, v_0) \in \Omega$, the solution $(x(t), v(t)) = \Phi(\mathbf{x}_0, t)$ to the characteristic equation $\frac{d}{dt}\mathbf{X}(t) = \mathcal{F}(\mathbf{X}(t))$, $\mathbf{X}(0) = \mathbf{x}_0$, given by

$$\Phi(\mathbf{x}_0, t) = \left(x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t); -x_0 \omega \sin(\omega t) + v_0 \cos(\omega t) \right),$$

is such that

$$\omega^2 x^2(t) + v^2(t) = \omega^2 x_0^2 + v_0^2, \quad t \in (-\tau_{-}(\mathbf{x}_0), \tau_{+}(\mathbf{x}_0))$$

which means that the integral curves associated to \mathcal{F} are *ellipses* centered at $(0, 0)$ and oriented in the counterclockwise direction. Now,

$$\partial\Omega = \left(\{-a\} \times [-\xi, \xi] \right) \cup \left(\{a\} \times [-\xi, \xi] \right) \cup \left([-a, a] \times \{-\xi\} \right) \cup \left([-a, a] \times \{\xi\} \right)$$

and it is easy to check that

$$\Gamma_{\pm} = \left(\{\pm a\} \times (-\xi, 0] \right) \cup \left(\{\mp a\} \times [0, \xi) \right) \cup \left([0, a] \times \{\pm \xi\} \right) \cup \left((-a, 0] \times \{\mp \xi\} \right).$$

Notice that $\Gamma_{+} \cap \Gamma_{-} = \{(a, 0), (0, \xi), (-a, 0), (0, -\xi)\}$ and

$$\partial\Omega \setminus (\Gamma_{+} \cup \Gamma_{-}) = \{(a, \xi), (a, -\xi), (-a, \xi), (-a, -\xi)\}$$

is a discrete set (of linear Lebesgue measure zero).

Example 2.6 (Harmonic oscillator in a stadium). Consider now the two-dimensional phase space (where \mathbb{R}^2 is still endowed with the Lebesgue measure μ):

$$\Omega = \{\mathbf{x} = (x, v) \in \mathbb{R}^2; x^2 + v^2 < 2 \text{ and } -1 < v < 1\}$$

and consider the harmonic oscillator force field \mathcal{F} given by (2.3) with $\omega = 1$ for simplicity. Then, the integral curves associated to \mathcal{F} are *circles* centered at $(0, 0)$ and oriented in the counterclockwise direction. In this case, one can see that

$$\Gamma_{\pm} = \{(x, -1); -1 < \pm x \leq 0\} \cup \{(x, 1); 0 \leq \pm x < 1\}.$$

In particular, one sees that $\partial\Omega \setminus (\Gamma_{+} \cup \Gamma_{-}) = \{(x, v) \in \mathbb{R}^2; x^2 + v^2 = 2; -1 \leq v \leq 1\}$ is a 'big' part of the boundary $\partial\Omega$ (with positive linear Lebesgue measure). Notice also that $\tau_{+}(\mathbf{x}) = +\infty$ for any $\mathbf{x} = (x, v)$ with $x^2 + v^2 < 1$.

The main aim of the present discussion is to represent Ω as a collection of characteristics running between points of Γ_{-} and Γ_{+} so that the integral over Ω can be split into integrals over Γ_{-} (or Γ_{+}) and along the characteristics. However, at present we cannot do this in a precise way since, in general, the sets Γ_{+} and Γ_{-} do not provide a partition of $\partial\Omega$ as there may be 'too

many' characteristics which extend to infinity on either side. Since we have not assumed Ω to be bounded, Γ_- or Γ_+ may be empty and also we may have characteristics running from $-\infty$ to $+\infty$ such as periodic ones. Thus, in general, characteristics starting from Γ_- or ending at Γ_+ would not fill the whole Ω and, to proceed, we have to construct an auxiliary set by extending Ω into the time domain and use the approach of [9] which is explained below.

2.2. Integration along characteristics. For any $0 < T < \infty$, we define the domain

$$\Omega_T = \Omega \times (0, T)$$

and the measure $d\mu_T = d\mu \otimes dt$ on Ω_T . Consider the vector field over Ω_T :

$$Y = \partial_t + \mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} = \mathcal{A}(\xi) \cdot \nabla_{\xi}$$

where $\mathcal{A}(\xi) = (\mathcal{F}(\mathbf{x}), 1)$ for any $\xi = (\mathbf{x}, t)$. We can define the characteristic curves of \mathcal{A} as the solution $\xi(s) = (\mathbf{X}(s), \theta(s))$ to the system $\frac{d}{ds}\xi(s) = \mathcal{A}(\xi(s))$, i.e.

$$\frac{d}{ds}\mathbf{X}(s) = \mathcal{F}(\mathbf{X}(s)), \quad \frac{d}{ds}\theta(s) = 1, \quad (s \in \mathbb{R}),$$

with

$$\mathbf{X}(0) = \mathbf{x}, \quad \theta(0) = t.$$

It is clear that the solution $\xi(s)$ to the above system is given by

$$\mathbf{X}(s) = \Phi(\mathbf{x}, s), \quad \theta(s) = s + t,$$

and we can define the flow of solution $\Psi(\xi, s) = (\Phi(\mathbf{x}, s), s+t)$ associated to \mathcal{A} and the existence times of the characteristic curves of Y are defined, for any $\xi = (\mathbf{x}, t) \in \Omega_T$, as

$$\ell_{\pm}(\xi) = \inf\{s > 0, (\Phi(\mathbf{x}, \pm s), \pm s + t) \notin \Omega_T\}.$$

The flow $\Psi(\cdot, \cdot)$ enjoys, *mutatis mutandis*, the properties listed in Proposition 2.3 and μ_T is invariant under Ψ . Moreover, since \mathcal{A} is clearly Lipschitz continuous on $\overline{\Omega_T}$, no characteristic of Y can escape to infinity in finite time. In other words, all characteristic curves of Y now have finite lengths. Indeed, if $\Phi(\mathbf{x}, \pm s)$ does not reach $\partial\Omega$, then the characteristic curve $\Psi(\xi, \pm s)$ enters or leaves Ω_T through the bottom $\Omega \times \{0\}$, or through the top $\Omega \times \{T\}$ of it. Precisely, it is easy to verify that for $\xi = (\mathbf{x}, t) \in \Omega_T$ we have

$$\ell_+(\xi) = \tau_+(\mathbf{x}) \wedge (T - t) \quad \text{and} \quad \ell_-(\xi) = \tau_-(\mathbf{x}) \wedge t,$$

where \wedge denotes minimum. This clearly implies $\sup\{\ell_{\pm}(\xi) ; \xi \in \Omega_T\} \leq T$. Define now

$$\Sigma_{\pm, T} = \{\zeta \in \partial\Omega_T ; \exists \xi \in \Omega_T \text{ such that } \zeta = \Psi(\xi, \pm \ell_{\pm}(\xi))\}.$$

The definition of $\Sigma_{\pm, T}$ is analogous to Γ_{\pm} with the understanding that now the characteristic curves correspond to the vector field \mathcal{A} . In other words, $\Sigma_{-, T}$ (resp. $\Sigma_{+, T}$) is the subset of $\partial\Omega_T$ consisting of all left (resp. right) limits of characteristic curves of \mathcal{A} in Ω_T whereas Γ_- (resp. Γ_+) is the subset of $\partial\Omega$ consisting of all left (resp. right) limits of characteristic curves of \mathcal{F} in Ω . The main difference (and the interest of such a lifting to Ω_T) is the fact that *each characteristic curve of \mathcal{A} does reach the boundaries $\Sigma_{\pm, T}$ in finite time*. The above formulae allow us to extend functions ℓ_{\pm} to $\Sigma_{\pm, T}$ in the same way as we extended the functions τ_{\pm} to Γ_{\pm} . With these considerations, we can represent, up to a set of zero measure, the phase space Ω_T as

$$\begin{aligned} \Omega_T &= \{\Psi(\xi, s) ; \xi \in \Sigma_{-, T}, 0 < s < \ell_+(\xi)\} \\ &= \{\Psi(\xi, -s) ; \xi \in \Sigma_{+, T}, 0 < s < \ell_-(\xi)\}. \end{aligned} \tag{2.4}$$

With this realization we can prove the following:

Proposition 2.7. *Let $T > 0$ be fixed. There are unique positive Borel measures $d\nu_{\pm}$ on $\Sigma_{\pm, T}$ such that $d\mu_T = d\nu_+ \otimes ds = d\nu_- \otimes ds$.*

Proof. For any $\delta > 0$, define \mathcal{E}_{δ} as the set of all bounded Borel subsets E of $\Sigma_{-, T}$ such that $\ell_+(\xi) > \delta$ for any $\xi \in E$. Let us now fix $E \in \mathcal{E}_{\delta}$. For all $0 < \sigma \leq \delta$ put

$$E_{\sigma} = \{\Psi(\xi, s); \xi \in E, 0 < s \leq \sigma\}.$$

Clearly E_{σ} is a measurable subset of Ω_T . Define the mapping $h : \sigma \in (0, \delta] \mapsto h(\sigma) = \mu_T(E_{\sigma})$ with $h(0) = 0$. If σ_1 and σ_2 are two positive numbers such that $\sigma_1 + \sigma_2 \leq \delta$, then

$$E_{\sigma_1 + \sigma_2} \setminus E_{\sigma_1} = \{\Psi(\xi, s); \xi \in E, \sigma_1 < s \leq \sigma_1 + \sigma_2\} = \{\Psi(\eta, \sigma_1); \eta \in E_{\sigma_2}\}.$$

The properties of the flow Ψ (see Proposition 2.3) ensure that the mapping $\eta \mapsto \Psi(\eta, \sigma_1)$ is one-to-one and measure preserving, so that

$$\mu_T(E_{\sigma_1 + \sigma_2} \setminus E_{\sigma_1}) = \mu_T(E_{\sigma_2}) = h(\sigma_2).$$

Since $E_{\sigma_1 + \sigma_2} = E_{\sigma_1} \cup (E_{\sigma_1 + \sigma_2} \setminus E_{\sigma_1})$, we immediately obtain

$$h(\sigma_1 + \sigma_2) = h(\sigma_1) + h(\sigma_2) \quad \text{for any } \sigma_1, \sigma_2 > 0 \text{ with } \sigma_1 + \sigma_2 \leq \delta. \quad (2.5)$$

This is the well-known Cauchy equation, though defined only on an interval of the real line. It can be solved in a standard way using non-negativity instead of continuity, yielding:

$$h(\sigma) = c_E \sigma \quad \text{for any } 0 < \sigma \leq \delta$$

where $c_E = h(\delta)/\delta$. We define $\nu_-(E) = c_E$. It is not difficult to see that, with the above procedure, the mapping $\nu_-(\cdot)$ defines a positive measure on the ring $\mathcal{E} = \bigcup_{\delta > 0} \mathcal{E}_{\delta}$ of all the Borel subsets of $\Sigma_{-, T}$ on which the function $\ell_+(\xi)$ is bounded away from 0. Such a measure ν_- can be uniquely extended to the σ -algebra of the Borel subsets of $\Sigma_{-, T}$ (see e.g. [12, Theorem A, p. 54]). Consider now a Borel subset E of $\Sigma_{-, T}$ and a Borel subset I of \mathbb{R}^+ , such that for all $\xi \in E$ and $s \in I$ we have $0 < s < \ell_+(\xi)$. Then

$$E \times I = \{\Psi(\xi, s); \xi \in E, s \in I\} \subset \Omega_T.$$

Thanks to the definition of $\nu_-(\cdot)$, we can state that $\mu_T(E \times I) = \nu_-(E)\text{meas}(I)$ where $\text{meas}(I)$ denotes the linear Lebesgue measure of $I \subset \mathbb{R}$. This shows that $d\mu_T = d\nu_- \otimes ds$. Similarly we can define a measure ν_+ on $\Sigma_{+, T}$ and prove that $d\mu_T = d\nu_+ \otimes ds$. The uniqueness of the measures $d\nu_{\pm}$ is then obvious. \square

Remark 2.8. *Note that the above construction of the Borel measures $d\nu_{\pm}$ differs from that of [11, Lemmas XI.3.1 & 3.2], [9, Propositions 7 & 8] which, moreover, only apply when μ is absolutely continuous with respect to the Lebesgue measure. Our construction is much more general and can also be generalized to the case of a non-divergence force field \mathcal{F} , [5].*

Next, by the cylindrical structure of Ω_T , and the representation of $\Sigma_{\pm, T}$ as

$$\Sigma_{-, T} = (\Gamma_- \times (0, T)) \cup \Omega \times \{0\} \quad \text{and} \quad \Sigma_{+, T} = (\Gamma_+ \times (0, T)) \cup \Omega \times \{T\},$$

the measures $d\nu_{\pm}$ over $\Gamma_{\pm} \times (0, T)$ can be written as $d\nu_{\pm} = d\mu_{\pm} \otimes dt$, where $d\mu_{\pm}$ are Borel measures on Γ_{\pm} . This leads to the following

Lemma 2.9. *There are unique positive Borel measures $d\mu_{\pm}$ on Γ_{\pm} such that, for any $f \in L^1(\Omega_T, d\mu_T)$*

$$\begin{aligned} \int_{\Omega_T} f(\mathbf{x}, t) d\mu_T(\mathbf{x}, t) &= \int_0^T dt \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y}) \wedge t} f(\Phi(\mathbf{y}, -s), t-s) ds \\ &\quad + \int_{\Omega} d\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x}) \wedge T} f(\Phi(\mathbf{x}, -s), T-s) ds, \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \int_{\Omega_T} f(\mathbf{x}, t) d\mu_T(\mathbf{x}, t) &= \int_0^T dt \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y}) \wedge (T-t)} f(\Phi(\mathbf{y}, s), t+s) ds \\ &\quad + \int_{\Omega} d\mu(\mathbf{x}) \int_0^{\tau_+(\mathbf{x}) \wedge T} f(\Phi(\mathbf{x}, s), s) ds. \end{aligned} \quad (2.7)$$

The above fundamental result allows to compute integrals over the cylindrical phase-space Ω_T through integration along the characteristic curves. Let us now generalize it to the phase space Ω . Here the main difficulty stems from the fact that the characteristic curves of the vector field \mathcal{F} are no longer assumed to be of finite length. In order to extend Lemma 2.9 to possibly infinite existence times, first we prove the following:

Lemma 2.10. *Let $T > 0$ be fixed. Then, $\tau_+(\mathbf{x}) < T$ for any $\mathbf{x} \in \Omega$ if and only if $\tau_-(\mathbf{x}) < T$ for any $\mathbf{x} \in \Omega$.*

Proof. It is easy to see that $\tau_+(\mathbf{x}) < T$ for any $\mathbf{x} \in \Omega$ is equivalent to $\tau(\mathbf{x}) < T$ for any $\mathbf{x} \in \Omega$ and this is also equivalent to $\tau_-(\mathbf{x}) < T$ for any $\mathbf{x} \in \Omega$. \square

Hereafter, the support of a measurable function f defined on Ω is defined as $\text{Supp} f = \Omega \setminus \omega$ where ω is the maximal open subset of Ω on which f vanishes $d\mu$ -almost everywhere.

Proposition 2.11. *Let $f \in L^1(\Omega, d\mu)$. Assume that there exists $\tau_0 > 0$ such that $\tau_{\pm}(\mathbf{x}) < \tau_0$ for any $\mathbf{x} \in \text{Supp}(f)$. Then,*

$$\begin{aligned} \int_{\Omega} f(\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds \\ &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, s)) ds. \end{aligned} \quad (2.8)$$

Proof. For any $T > \tau_0$, define the domain $\Omega_T = \Omega \times (0, T)$. Since $T < \infty$, it is clear that $f \in L^1(\Omega_T, d\mu dt)$ and, by (2.6), we get

$$\begin{aligned} T \int_{\Omega} f(\mathbf{x}) d\mu(\mathbf{x}) &= \int_0^T dt \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{t \wedge \tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds + \\ &\quad \int_{\Omega} d\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x})} f(\Phi(\mathbf{x}, -s)) ds. \end{aligned}$$

Since the formula is valid for any $T > \tau_0$, differentiating with respect to T leads to the first assertion. The second assertion is proved in the same way by using formula (2.7). \square

To drop the finiteness assumption on $\tau_{\pm}(\mathbf{x})$, first we introduce the sets

$$\Omega_{\pm} = \{\mathbf{x} \in \Omega; \tau_{\pm}(\mathbf{x}) < \infty\}, \quad \Omega_{\pm\infty} = \{\mathbf{x} \in \Omega; \tau_{\pm}(\mathbf{x}) = \infty\},$$

and

$$\Gamma_{\pm\infty} = \{\mathbf{y} \in \Gamma_{\pm}; \tau_{\mp}(\mathbf{y}) = \infty\}.$$

Then

Proposition 2.12. *Given $f \in L^1(\Omega, d\mu)$, one has*

$$\int_{\Omega_{\pm}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_{\pm}} d\mu_{\pm}(\mathbf{y}) \int_0^{\tau_{\mp}(\mathbf{y})} f(\Phi(\mathbf{y}, \mp s)) ds, \quad (2.9)$$

and

$$\int_{\Omega_{\pm} \cap \Omega_{\mp\infty}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_{\pm\infty}} d\mu_{\pm}(\mathbf{y}) \int_0^{\infty} f(\Phi(\mathbf{y}, \mp s)) ds. \quad (2.10)$$

Proof. Assume first $f \geq 0$. Let us fix $T > 0$. It is clear that $\mathbf{x} \in \Omega$ satisfies $\tau_+(\mathbf{x}) < T$ if and only if $\mathbf{x} = \Phi(\mathbf{y}, -s)$, with $\mathbf{y} \in \Gamma_+$ and $0 < s < T \wedge \tau_-(\mathbf{y})$. Then, by Proposition 2.11,

$$\int_{\{\tau_+(\mathbf{x}) < T\}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{T \wedge \tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds.$$

Since $f \geq 0$, the inner integral is increasing with T and, using the monotone convergence theorem, we let $T \rightarrow \infty$ to get

$$\int_{\Omega_+} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds$$

which coincides with (2.9). We proceed in the same way with integration on Γ_- and get the second part of (2.9). Next we consider the set

$$\Delta = \{\mathbf{x} \in \Omega; \mathbf{x} = \Phi(\mathbf{y}, -s), \mathbf{y} \in \Omega_{+\infty}, 0 < s < T\}.$$

Proposition 2.11 asserts that

$$\int_{\Delta} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega_{+\infty}} d\mu_+(\mathbf{y}) \int_0^T f(\Phi(\mathbf{y}, -s)) ds.$$

Letting again $T \rightarrow \infty$, we get (2.10). We extend the results to arbitrary f by linearity. \square

Finally, with the following, we show that it is possible to transfer integrals over Γ_- to Γ_+ :

Proposition 2.13. *For any $\psi \in L^1(\Gamma_-, d\mu_-)$,*

$$\int_{\Gamma_- \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) d\mu_-(\mathbf{y}) = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} \psi(\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))) d\mu_+(\mathbf{z}). \quad (2.11)$$

Proof. For any $\epsilon > 0$, let f_{ϵ} be the function defined on $\Omega_+ \cap \Omega_-$ by

$$\psi_{\epsilon}(\mathbf{x}) = \begin{cases} \frac{\psi(\Phi(\mathbf{x}, -\tau_-(\mathbf{x})))}{\tau_+(\mathbf{x}) + \tau_-(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) > \epsilon, \\ 0 & \text{else.} \end{cases}$$

Since $\psi_\epsilon \in L^1(\Omega_+ \cap \Omega_-, d\mu)$, Eqs. (2.9) and (2.10) give

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} \psi_\epsilon(\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\mathbf{y}) \frac{ds}{\tau_+(\mathbf{y})} \\ &= \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) d\mu_-(\mathbf{y}). \end{aligned}$$

In the same way,

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} \psi_\epsilon(\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \psi(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) \frac{ds}{\tau_-(\mathbf{y})} \\ &= \int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} \psi(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) d\mu_-(\mathbf{y}), \end{aligned}$$

which leads to

$$\int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} \psi(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) d\mu_+(\mathbf{y}) = \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) d\mu_-(\mathbf{y})$$

for any $\epsilon > 0$. Passing to the limit as $\epsilon \rightarrow 0$ we get the conclusion. \square

We end this section with a technical result we shall need in the sequel (see Lemma 3.3):

Proposition 2.14. *Let K be a compact subset of Ω . Denote*

$$K_\pm := \{\mathbf{y} \in \Gamma_\pm; \exists t_0 \in \mathbb{R} \text{ such that } \Phi(\mathbf{y}, \pm t) \in K \text{ for any } t \geq t_0\}.$$

Then $\mu_\pm(K_\pm) = 0$.

Proof. Let K be a fixed compact subset of Ω . Applying Eq. (2.9) or (2.10) to the function $f(\mathbf{x}) = \chi_K(\mathbf{x})$, one has

$$\infty > \mu(K) \geq \int_{K_-} d\mu_-(\mathbf{y}) \int_0^\infty \chi_K(\Phi(\mathbf{y}, t)) dt. \quad (2.12)$$

By definition, if $\mathbf{y} \in K_-$, then for some $t_0 \in \mathbb{R}$, $\chi_K(\Phi(\mathbf{y}, t)) = 1$ for any $t \geq t_0$. Therefore,

$$\int_0^\infty \chi_K(\Phi(\mathbf{y}, t)) dt = \infty, \quad \forall \mathbf{y} \in K_-.$$

Inequality (2.12) implies that $\mu_-(K_-) = 0$. One proves the result for K_+ in the same way. \square

3. THE MAXIMAL TRANSPORT OPERATOR AND TRACE RESULTS

The results of the previous section allow us to define the (maximal) transport operator \mathcal{T}_{\max} as the weak derivative along the characteristic curves. To be precise, let us define the space of *test functions* \mathfrak{V} as follows:

Definition 3.1 (Test-functions). *Let \mathfrak{V} be the set of all measurable and bounded functions $\psi : \Omega \rightarrow \mathbb{R}$ with compact support in Ω and such that, for any $\mathbf{x} \in \Omega$, the mapping*

$$s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \mapsto \psi(\Phi(\mathbf{x}, s))$$

is continuously differentiable with

$$\mathbf{x} \in \Omega \mapsto \left. \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \right|_{s=0} \text{ measurable and bounded.} \quad (3.1)$$

Remark 3.2. Notice that the class of test-functions \mathfrak{Y} is not defined as a subset of $L^\infty(\Omega, d\mu)$; that is, we do not identify functions equal μ -almost everywhere. It is however a natural question to know whether two test-functions coinciding μ -almost everywhere are such that their derivatives (defined by (3.1)) do coincide μ -almost everywhere. We provide a positive answer to this question at the end of the paper (see Appendix).

An important property of test-functions is the following consequence of Proposition 2.14:

Lemma 3.3. Let $\psi \in \mathfrak{Y}$ be given. For μ_{\mp} -almost any $\mathbf{y} \in \Gamma_{\mp}$ there exists a sequence $(t_n^{\pm})_n$ (depending on \mathbf{y}) such that

$$\lim_{n \rightarrow \infty} t_n^{\pm} = \tau_{\pm}(\mathbf{y}) \quad \text{and} \quad \psi(\Phi(\mathbf{y}, \pm t_n^{\pm})) = 0 \quad \forall n \in \mathbb{N}.$$

Proof. Let $\psi \in \mathfrak{Y}$ be given and let $K = \text{Supp}(\psi)$. For any $\mathbf{y} \in \Gamma_-$ with $\tau_+(\mathbf{y}) < \infty$ one has $\Phi(\mathbf{y}, \tau_+(\mathbf{y})) \in \Gamma_+$ and, since K is compact in Ω , $\psi(\Phi(\mathbf{y}, \tau_+(\mathbf{y}))) = 0$ and the existence of a sequence $(t_n^+)_n$ converging to $\tau_+(\mathbf{y})$ with the above property is clear. Now, Proposition 2.14 applied to K shows that there exists a set $\Gamma'_- \subset \Gamma_-$ with $\mu_-(\Gamma \setminus \Gamma'_-) = 0$ and such that, for any $\mathbf{y} \in \Gamma'_-$, there is a sequence $(t_n^+)_n$ converging to ∞ such that $\Phi(\mathbf{y}, t_n) \notin K$ for any $n \in \mathbb{N}$. This proves the result. The statement for Γ_+ is proved in the same way. \square

In the next step we define the transport operator $(\mathcal{T}_{\max}, \mathcal{D}(\mathcal{T}_{\max}))$.

Definition 3.4 (Transport operator \mathcal{T}_{\max}). The domain of the maximal transport operator \mathcal{T}_{\max} is the set $\mathcal{D}(\mathcal{T}_{\max})$ of all $f \in L^1(\Omega, d\mu)$ for which there exists $g \in L^1(\Omega, d\mu)$ such that

$$\int_{\Omega} g(\mathbf{x})\psi(\mathbf{x})d\mu(\mathbf{x}) = \int_{\Omega} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}), \quad \forall \psi \in \mathfrak{Y}.$$

In this case, $g =: \mathcal{T}_{\max}f$.

Remark 3.5. Of course, in some weak sense, $\mathcal{T}_{\max}f = -\mathcal{F} \cdot \nabla f$. Precisely, for any $\varphi \in \mathcal{C}_0^1(\Omega)$, the following formula holds:

$$\int_{\Omega} (\mathcal{F}(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x})) f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega} \mathcal{T}_{\max}f(\mathbf{x}) \varphi(\mathbf{x}) d\mu(\mathbf{x}).$$

3.1. Fundamental representation formula: mild formulation. Recall that, if f_1 and f_2 are two functions defined over Ω , we say that f_2 is a *representative* of f_1 if $\mu\{\mathbf{x} \in \Omega; f_1(\mathbf{x}) \neq f_2(\mathbf{x})\} = 0$, i.e. when $f_1(\mathbf{x}) = f_2(\mathbf{x})$ for μ -almost every $\mathbf{x} \in \Omega$. The following fundamental result provides a characterization of the domain of $\mathcal{D}(\mathcal{T}_{\max})$:

Theorem 3.6. Let $f \in L^1(\Omega, \mu)$. The following are equivalent:

- (1) There exists $g \in L^1(\Omega, \mu)$ and a representative f^{\sharp} of f such that, for μ -almost every $\mathbf{x} \in \Omega$ and any $-\tau_-(\mathbf{x}) < t_1 \leq t_2 < \tau_+(\mathbf{x})$:

$$f^{\sharp}(\Phi(\mathbf{x}, t_1)) - f^{\sharp}(\Phi(\mathbf{x}, t_2)) = \int_{t_1}^{t_2} g(\Phi(\mathbf{x}, s)) ds. \quad (3.2)$$

- (2) $f \in \mathcal{D}(\mathcal{T}_{\max})$. In this case, $g = \mathcal{T}_{\max}f$.

The proof of the theorem is made of several steps. The difficult part of the proof is the implication (2) \implies (1). It is carried out through several technical lemmas based upon *mollification along the characteristic curves* (recall that, whenever μ is not absolutely continuous with respect to the Lebesgue measure, no global convolution argument is available). Let us make precise what

this is all about. Consider a sequence $(\varrho_n)_n$ of one dimensional mollifiers supported in $[0, 1]$, i.e. for any $n \in \mathbb{N}$, $\varrho_n \in \mathcal{C}_0^\infty(\mathbb{R})$, $\varrho_n(s) = 0$ if $s \notin [0, 1/n]$, $\varrho_n(s) \geq 0$ and $\int_0^{1/n} \varrho_n(s) ds = 1$. Then, for any $f \in L^1(\Omega, d\mu)$, define the (extended) mollification:

$$\varrho_n \diamond f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \varrho_n(s) f(\Phi(\mathbf{x}, -s)) ds.$$

As we shall see later, such a definition corresponds precisely to a time convolution over any characteristic curves (see e.g. (3.4)). Note that, with such a definition, it is not clear *a priori* that $\varrho_n \diamond f$ defines a measurable function, finite almost everywhere. It is proved in the following that actually such a function is integrable.

Lemma 3.7. *Given $f \in L^1(\Omega, d\mu)$, $\varrho_n \diamond f \in L^1(\Omega, d\mu)$ for any $n \in \mathbb{N}$. Moreover,*

$$\|\varrho_n \diamond f\| \leq \|f\|, \quad \forall f \in L^1(\Omega, d\mu), n \in \mathbb{N}. \quad (3.3)$$

Proof. One considers, for a given $f \in L^1(\Omega, d\mu)$, the extension of f by zero outside Ω :

$$\bar{f}(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \quad \bar{f}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbb{R}^N \setminus \Omega.$$

Then $\bar{f} \in L^1(\mathbb{R}^N, d\mu)$. Let us consider the transformation:

$$\Upsilon : (\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \Upsilon(\mathbf{x}, s) = (\Phi(\mathbf{x}, -s), -s) \in \mathbb{R}^N \times \mathbb{R}.$$

As a homeomorphism, Υ is measure preserving for pure Borel measures. It is also measure preserving for completions of Borel measures (such as a Lebesgue measure) since it is measure-preserving on Borel sets and the completion of a measure is obtained by adding to the Borel σ -algebra all sets contained in a measure-zero Borel sets, see [12, Theorem 13.B, p. 55]. Then, according to [12, Theorem 39.B, p. 162], the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \bar{f}(\Phi(\mathbf{x}, -s))$$

is measurable as the composition of Υ with the measurable function $(\mathbf{x}, s) \mapsto \bar{f}(\mathbf{x})$. Define now $\Lambda = \{(\mathbf{x}, s); \mathbf{x} \in \Omega, 0 < s < \tau_-(\mathbf{x})\}$, Λ is a measurable subset of $\mathbb{R}^N \times \mathbb{R}$. Therefore, the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \bar{f}(\Phi(\mathbf{x}, -s)) \chi_\Lambda(\mathbf{x}, s) \varrho_n(s)$$

is measurable. Since ϱ_n is compactly supported, it is also integrable over $\mathbb{R}^N \times \mathbb{R}$ and, according to Fubini's Theorem

$$[\varrho_n \diamond f](\mathbf{x}) := \int_{\mathbb{R}} \bar{f}(\Phi(\mathbf{x}, -s)) \chi_\Lambda(\mathbf{x}, s) \varrho_n(s) ds = \int_0^{\tau_-(\mathbf{x})} \varrho_n(s) f(\Phi(\mathbf{x}, -s)) ds$$

is finite for almost every $\mathbf{x} \in \Omega$ and the associated application $\varrho_n \diamond f$ is integrable.

Let us prove now (3.3). Since $|\varrho_n \diamond f| \leq \varrho_n \diamond |f|$, to show that $\varrho_n \diamond f \in L^1(\Omega, d\mu)$, it suffices to deal with a *nonnegative function* $f \in L^1(\Omega, d\mu)$. One sees easily that, for any $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$,

$$(\varrho_n \diamond f)(\Phi(\mathbf{y}, t)) = \int_0^t \varrho_n(s) f(\Phi(\mathbf{y}, t-s)) ds = \int_0^t \varrho_n(t-s) f(\Phi(\mathbf{y}, s)) ds. \quad (3.4)$$

Thus,

$$\begin{aligned} \int_0^{\tau_+(\mathbf{y})} [\varrho_n \diamond f](\Phi(\mathbf{y}, t)) dt &= \int_0^{\tau_+(\mathbf{y})} ds \int_s^{\tau_+(\mathbf{y})} \varrho_n(s) f(\Phi(\mathbf{y}, t-s)) dt \\ &= \int_0^{\tau_+(\mathbf{y}) \wedge 1/n} \varrho_n(s) ds \int_0^{\tau_+(\mathbf{y})-s} f(\Phi(\mathbf{y}, r)) dr. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\leq \int_0^{\tau_+(\mathbf{y})} [\varrho_n \diamond f](\Phi(\mathbf{y}, t)) dt \leq \int_0^{1/n} \varrho_n(s) ds \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r)) dr \\ &= \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r)) dr, \quad \forall \mathbf{y} \in \Gamma_-, n \in \mathbb{N} \end{aligned}$$

so that

$$\int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} [\varrho_n \diamond f](\Phi(\mathbf{y}, t)) dt \leq \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r)) dr.$$

This proves, thanks to Proposition 2.12, that

$$\int_{\Omega_-} [\varrho_n \diamond f] d\mu \leq \int_{\Omega_-} f d\mu. \quad (3.5)$$

Now, in the same way:

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} [\varrho_n \diamond f](\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^\infty [\varrho_n \diamond f](\Phi(\mathbf{y}, -t)) dt \\ &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^\infty dt \int_0^\infty \varrho_n(s) f(\Phi(\mathbf{y}, -s-t)) ds \\ &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^\infty dt \int_t^\infty \varrho_n(r-t) f(\Phi(\mathbf{y}, -r)) dr. \end{aligned}$$

so that

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} [\varrho_n \diamond f](\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^\infty f(\Phi(\mathbf{y}, -r)) dr \int_0^r \varrho_n(r-t) dt \\ &\leq \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^\infty f(\Phi(\mathbf{y}, -r)) dr \end{aligned}$$

i.e.

$$\int_{\Omega_+ \cap \Omega_-} \varrho_n \diamond f(\mathbf{x}) d\mu(\mathbf{x}) \leq \int_{\Omega_+ \cap \Omega_-} f(\mathbf{x}) d\mu(\mathbf{x}). \quad (3.6)$$

Finally

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} [\varrho_n \diamond f](\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Omega_+ \cap \Omega_-} d\mu(\mathbf{x}) \int_0^\infty \varrho_n(s) f(\Phi(\mathbf{x}, -s)) ds \\ &= \int_0^\infty \varrho_n(s) ds \int_{\Omega_+ \cap \Omega_-} f(\Phi(\mathbf{x}, -s)) d\mu(\mathbf{x}). \end{aligned}$$

Now, from Assumption 1, for any $s \geq 0$,

$$\int_{\Omega_+ \cap \Omega_-} f(\Phi(\mathbf{x}, -s)) d\mu(\mathbf{x}) = \int_{\Omega_+ \cap \Omega_-} f(\mathbf{x}) d\mu(\mathbf{x}),$$

so that

$$\int_{\Omega_{+\infty} \cap \Omega_{-\infty}} [\varrho_n \diamond f](\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{x}) d\mu(\mathbf{x}). \quad (3.7)$$

Combining (3.5), (3.6) and (3.7), one finally gets $\|\varrho_n \diamond f\| \leq \|f\|$. \square

As it is the case for classical convolution, the family $(\varrho_n \diamond f)_n$ approximates f in L^1 -norm:

Proposition 3.8. *Given $f \in L^1(\Omega, d\mu)$,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} |(\varrho_n \diamond f)(\mathbf{x}) - f(\mathbf{x})| d\mu(\mathbf{x}) = 0. \quad (3.8)$$

Proof. According to (3.3) and from the density of $\mathcal{C}_0(\Omega)$ in $L^1(\Omega, d\mu)$, it suffices to prove the result for any f continuous over Ω and compactly supported. Splitting f into positive and negative parts, $f = f^+ - f^-$, one can also assume f to be nonnegative. From the continuity of both f and $\Phi(\cdot, \cdot)$, one has

$$\mathcal{K}_n := \text{Supp}(\varrho_n \diamond f) = \overline{\left\{ \mathbf{x} \in \Omega, \exists s_0 \in \text{Supp}(\varrho_n) \text{ such that } \Phi(\mathbf{x}, -s_0) \in \text{Supp}(f) \right\}}.$$

Moreover, it is easily seen that $\mathcal{K}_{n+1} \subset \mathcal{K}_n$ for any $n \geq 1$. Finally, it is clear that

$$\mathcal{K}_1 \subset \{ \mathbf{x} \in \overline{\Omega}; \exists \mathbf{y} \in \text{Supp}(f) \text{ with } |\mathbf{x} - \mathbf{y}| \leq d \}$$

where $d = \sup\{|\Phi(\mathbf{x}, s) - \mathbf{x}|; 0 \leq s \leq 1; \mathbf{x} \in \text{Supp}(f)\} < \infty$. Therefore, \mathcal{K}_1 is compact. Set now

$$\mathcal{O}_n := \mathcal{K}_n \cup \text{Supp}(f) \quad \text{and} \quad \mathcal{O}_n^- = \{ \mathbf{x} \in \mathcal{O}_n; \tau_-(\mathbf{x}) < 1/n \}.$$

Noticing that $\mu(\mathcal{O}_1)$ is finite, one can see easily that $\lim_n \mu(\mathcal{O}_n^-) = 0$. Since $\sup_{\mathbf{x} \in \Omega} |\varrho_n \diamond f(\mathbf{x})| \leq \sup_{\mathbf{x} \in \Omega} |f(\mathbf{x})|$, for any $\varepsilon > 0$, there exists $n_0 \geq 1$ such that

$$\int_{\mathcal{O}_n^-} |f(\mathbf{x})| d\mu(\mathbf{x}) \leq \varepsilon, \quad \text{and} \quad \int_{\mathcal{O}_n^-} |\varrho_n \diamond f(\mathbf{x})| d\mu(\mathbf{x}) \leq \varepsilon \quad \forall n \geq n_0.$$

Now, noticing that $\text{Supp}(\varrho_n \diamond f - f) \subset \mathcal{O}_n$, one has for any $n \geq n_0$,

$$\int_{\Omega} |\varrho_n \diamond f - f| d\mu = \int_{\mathcal{O}_n} |\varrho_n \diamond f - f| \leq 2\varepsilon + \int_{\mathcal{O}_n \setminus \mathcal{O}_n^-} |\varrho_n \diamond f - f| d\mu.$$

For any $\mathbf{x} \in \mathcal{O}_n \setminus \mathcal{O}_n^-$, since ϱ is supported in $[0, 1/n]$, one has

$$\begin{aligned} [\varrho_n \diamond f](\mathbf{x}) - f(\mathbf{x}) &= \int_0^{1/n} \varrho_n(s) f(\Phi(\mathbf{x}, -s)) ds - f(\mathbf{x}) \\ &= \int_0^{1/n} \varrho_n(s) (f(\Phi(\mathbf{x}, -s)) - f(\mathbf{x})) ds. \end{aligned}$$

Note that, thanks to Gronwall's lemma,

$$|\Phi(\mathbf{x}, -s) - \mathbf{x}| \leq \frac{L}{\kappa} (\exp(\kappa s) - 1) \leq \frac{L}{\kappa} (\exp(\kappa/n) - 1), \quad \forall \mathbf{x} \in \mathcal{O}_1, s \in (0, 1/n)$$

where $L = \sup\{|\mathcal{F}(\mathbf{x})|, \mathbf{x} \in \mathcal{O}_1\}$. Since f is uniformly continuous on \mathcal{O}_1 , it follows that

$$\lim_{n \rightarrow \infty} \sup \left\{ |f(\Phi(\mathbf{x}, -s)) - f(\mathbf{x})|; \mathbf{x} \in \mathcal{O}_1, s \in (0, 1/n) \right\} = 0$$

from which we deduce that there exists some $n_1 \geq 0$, such that $|\varrho_n \diamond f(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon$ for any $\mathbf{x} \in \mathcal{O}_n \setminus \mathcal{O}_n^-$ and any $n \geq n_1$. One obtains then, for any $n \geq n_1$,

$$\int_{\Omega} |\varrho_n \diamond f - f| d\mu \leq 2\varepsilon + \varepsilon\mu(\mathcal{O}_n \setminus \mathcal{O}_n^-) \leq 2\varepsilon + \varepsilon\mu(\mathcal{O}_1)$$

which proves the result. \square

We saw that, for a given $f \in L^1(\Omega, d\mu)$, $\varrho_n \diamond f$ is also integrable ($n \in \mathbb{N}$). Actually, we shall see that $\varrho_n \diamond f$ is even more regular than f :

Lemma 3.9. *Given $f \in L^1(\Omega, d\mu)$, set $f_n = \varrho_n \diamond f$, $n \in \mathbb{N}$. Then, $f_n \in \mathcal{D}(\mathcal{T}_{\max})$ with*

$$[\mathcal{T}_{\max} f_n](\mathbf{x}) = - \int_0^{\tau_-(\mathbf{x})} \varrho'_n(s) f(\Phi(\mathbf{x}, -s)) ds, \quad \mathbf{x} \in \Omega.$$

Proof. Set $g_n(\mathbf{x}) = - \int_0^{\tau_-(\mathbf{x})} \varrho'_n(s) f(\Phi(\mathbf{x}, -s)) ds$, $\mathbf{x} \in \Omega$. It is easy to see that $g_n \in L^1(\Omega, d\mu)$. Now, given $\psi \in \mathfrak{Y}$, let us consider the quantity

$$I = \int_{\Omega} f_n(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}).$$

One has to prove that $I = \int_{\Omega} g_n(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x})$. We split the above integral over Ω into three integrals I_- , I_+ and I_{∞} over Ω_- , $\Omega_{-\infty} \cap \Omega_+$ and $\Omega_{+\infty} \cap \Omega_{-\infty}$ respectively. Recall that, for any $\mathbf{x} \in \Omega_-$, there is some $\mathbf{y} \in \Gamma_-$ and some $t \in (0, \tau_+(\mathbf{y}))$ such that $\mathbf{x} = \Phi(\mathbf{y}, t)$. In such a case

$$\frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} = \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)). \quad (3.9)$$

Then, according to Prop. 2.12 and Eq. (3.4):

$$\begin{aligned} I_- &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f_n(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \\ &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \int_0^t \varrho_n(t-s) f(\Phi(\mathbf{y}, s)) ds \\ &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, s)) ds \int_s^{\tau_+(\mathbf{y})} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) \varrho_n(t-s) dt. \end{aligned} \quad (3.10)$$

Let us now investigate more carefully this last integral. Let $\mathbf{y} \in \Gamma_-$ be fixed. If $\tau_+(\mathbf{y}) < \infty$ then, since ψ is compactly supported, we have $\psi(\Phi(\mathbf{y}, \tau_+(\mathbf{y}))) = 0$ and integration by part (together with $\varrho_n(0) = 0$) leads to

$$\int_s^{\tau_+(\mathbf{y})} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) \varrho_n(t-s) dt = - \int_s^{\tau_+(\mathbf{y})} \varrho'_n(t-s) \psi(\Phi(\mathbf{y}, t)) dt.$$

If now $\tau_+(\mathbf{y}) > \infty$, then, since ϱ_n is supported in $[0, 1/n]$, one has

$$\begin{aligned} \int_s^{\tau_+(\mathbf{y})} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) \varrho_n(t-s) dt &= \int_s^{s+1/n} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) \varrho_n(t-s) dt \\ &= - \int_s^{\tau_+(\mathbf{y})} \varrho'_n(t-s) \psi(\Phi(\mathbf{y}, t)) dt \end{aligned}$$

Finally, we obtain,

$$\begin{aligned} I_- &= - \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, s)) ds \int_s^{\tau_+(\mathbf{y})} \psi(\Phi(\mathbf{y}, t)) \varrho'_n(t-s) dt \\ &= - \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\Phi(\mathbf{y}, t)) dt \int_0^t \varrho'_n(s) f(\Phi(\mathbf{y}, t-s)) ds. \end{aligned}$$

Using again Prop. 2.12, we finally get

$$I_- = \int_{\Omega_-} g_n(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}).$$

One proves in the same way that

$$I_+ = \int_{\Omega_+ \cap \Omega_{-\infty}} f_n(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega_+ \cap \Omega_{-\infty}} g_n(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}).$$

It remains to consider $I_\infty = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f_n(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x})$. One has

$$\begin{aligned} I_\infty &= \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) \int_0^\infty \varrho_n(t) f(\Phi(\mathbf{x}, -t)) dt \\ &= \int_0^\infty \varrho_n(t) dt \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} f(\Phi(\mathbf{x}, -t)) d\mu(\mathbf{x}). \end{aligned}$$

For any $\mathbf{x} \in \Omega_{+\infty} \cap \Omega_{-\infty}$ and any $t \geq 0$, setting $\mathbf{y} = \Phi(\mathbf{x}, -t)$, one has $\mathbf{y} \in \Omega_{-\infty} \cap \Omega_{+\infty}$ and $\frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} = \frac{d}{dt} \psi(\Phi(\mathbf{y}, t))$ from which Liouville's Theorem (Assumption 1) yields

$$\int_{\Omega_{+\infty} \cap \Omega_{-\infty}} \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} f(\Phi(\mathbf{x}, -t)) d\mu(\mathbf{x}) = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) f(\mathbf{y}) d\mu(\mathbf{y}).$$

Therefore,

$$\begin{aligned} I_\infty &= \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{y}) d\mu(\mathbf{y}) \int_0^\infty \varrho_n(t) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \\ &= - \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{y}) d\mu(\mathbf{y}) \int_0^\infty \varrho'_n(t) \psi(\Phi(\mathbf{y}, t)) dt \\ &= - \int_0^\infty \varrho'_n(t) dt \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{y}) \psi(\Phi(\mathbf{y}, t)) d\mu(\mathbf{y}). \end{aligned}$$

Arguing as above, one can "turn back" to the \mathbf{x} variable to get

$$\int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{y}) \psi(\Phi(\mathbf{y}, t)) d\mu(\mathbf{y}) = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\Phi(\mathbf{x}, -t)) \psi(\mathbf{x}) d\mu(\mathbf{x}),$$

i.e.

$$I_\infty = - \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} \psi(\mathbf{x}) d\mu(\mathbf{x}) \int_0^\infty \varrho'_n(t) f(\Phi(\mathbf{x}, -t)) dt = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} \psi(\mathbf{x}) g_n(\mathbf{x}) d\mu(\mathbf{x})$$

and the Lemma is proven. \square

Remark 3.10. Notice that Proposition 3.8 together with Lemma 3.9 prove that $\mathcal{D}(\mathcal{T}_{\max})$ is a dense subset of $L^1(\Omega, d\mu)$.

Now, whenever $f \in \mathcal{D}(\mathcal{T}_{\max})$, one has the following more precise result:

Proposition 3.11. *If $f \in \mathcal{D}(\mathcal{T}_{\max})$, then*

$$[\mathcal{T}_{\max}(\varrho_n \diamond f)](\mathbf{x}) = [\varrho_n \diamond \mathcal{T}_{\max}f](\mathbf{x}), \quad (\mathbf{x} \in \Omega, n \in \mathbb{N}). \quad (3.11)$$

Before proving this result, we need the following very simple lemma:

Lemma 3.12. *For any $\psi \in \mathfrak{Y}$ and any $n \in \mathbb{N}$, define*

$$\chi_n(\mathbf{x}) = \int_0^{\tau_+(\mathbf{x})} \varrho_n(s) \psi(\Phi(\mathbf{x}, s)) ds, \quad \mathbf{x} \in \Omega.$$

Then, χ_n belongs to \mathfrak{Y} .

Proof. Since τ_+ is measurable and ϱ_n is compactly supported, it is easy to see that χ_n is measurable and bounded over Ω . Now, for any $\mathbf{x} \in \Omega$, and any $t \in (\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$, one has

$$\chi_n(\Phi(\mathbf{x}, t)) = \int_t^{\tau_+(\mathbf{x})} \varrho_n(s-t) \psi(\Phi(\mathbf{x}, s)) ds.$$

It is clear then from the properties of ϱ_n that the mapping $t \in (\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \mapsto \chi_n(\Phi(\mathbf{x}, t))$ is continuously differentiable with

$$\frac{d}{dt} \chi_n(\Phi(\mathbf{x}, t)) = - \int_t^{\tau_+(\mathbf{x})} \varrho_n'(s-t) \psi(\Phi(\mathbf{x}, s)) ds = \int_t^{\tau_+(\mathbf{x})} \varrho_n(s-t) \frac{d}{ds} [\psi(\Phi(\mathbf{x}, s))] ds. \quad (3.12)$$

In particular, for $t = 0$, one gets

$$\left. \frac{d}{dt} \chi_n(\Phi(\mathbf{x}, t)) \right|_{t=0} = - \int_0^{\tau_+(\mathbf{x})} \varrho_n'(s) \psi(\Phi(\mathbf{x}, s)) ds.$$

Since ϱ_n' is compactly supported and $\psi \in \mathfrak{Y}$, the application $\mathbf{x} \in \Omega \mapsto \left. \frac{d}{dt} \chi_n(\Phi(\mathbf{x}, t)) \right|_{t=0}$ is measurable and bounded. \square

PROOF OF PROPOSITION 3.11. We use the notations of Lemma 3.9. Since $\varrho_n \diamond \mathcal{T}_{\max}f \in L^1(\Omega, d\mu)$, it suffices to show that

$$\int_{\Omega} f_n(\mathbf{x}) \left. \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \right|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega} \psi(\mathbf{x}) [\varrho_n \diamond \mathcal{T}_{\max}f](\mathbf{x}) d\mu(\mathbf{x}), \quad \forall \psi \in \mathfrak{Y}.$$

Here again, we shall deal separately with the integrals over Ω_- , $\Omega_+ \cap \Omega_{-\infty}$ and $\Omega_{+\infty} \cap \Omega_{-\infty}$. Let χ_n be defined as in Lemma 3.12, as we already saw it (see (3.12)), for any $\mathbf{y} \in \Gamma_-$, and any $0 < s < \tau_+(\mathbf{y})$, $\frac{d}{ds} \chi_n(\Phi(\mathbf{y}, s)) = \int_s^{\tau_+(\mathbf{y})} \varrho_n(t-s) \frac{d}{dt} [\psi(\Phi(\mathbf{y}, t))] dt$. Consequently, according to (3.10),

$$\begin{aligned} \int_{\Omega_-} f_n(\mathbf{x}) \left. \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \right|_{s=0} d\mu(\mathbf{x}) &= \int_{\Gamma_-} d\mu(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r)) \frac{d}{dr} \chi_n(\Phi(\mathbf{y}, r)) dr \\ &= \int_{\Omega_-} f(\mathbf{x}) \left. \frac{d}{ds} \chi_n(\Phi(\mathbf{x}, s)) \right|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega_-} \chi_n(\mathbf{x}) [\mathcal{T}_{\max}f](\mathbf{x}) d\mu(\mathbf{x}) \end{aligned}$$

where, for the two last identities, we used (3.9) and the fact that $\chi_n \in \mathfrak{Y}$. Now, using Prop. 2.12

$$\begin{aligned} \int_{\Omega_-} \chi_n(\mathbf{x})[\mathcal{T}_{\max}f](\mathbf{x})d\mu(\mathbf{x}) &= \int_{\Omega_-} [\mathcal{T}_{\max}f](\mathbf{x})d\mu(\mathbf{x}) \int_0^{\tau_+(\mathbf{x})} \varrho_n(r)\psi(\Phi(\mathbf{x}, r))dr \\ &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\Phi(\mathbf{y}, s))ds \int_0^s \varrho_n(s-t)[\mathcal{T}_{\max}f](\Phi(\mathbf{y}, t))dt. \end{aligned}$$

Therefore, Eq. (3.4) leads to

$$\begin{aligned} \int_{\Omega_-} \chi_n(\mathbf{x})[\mathcal{T}_{\max}f](\mathbf{x})d\mu(\mathbf{x}) &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\Phi(\mathbf{y}, s))[\varrho_n \diamond \mathcal{T}_{\max}f](\Phi(\mathbf{y}, s))ds \\ &= \int_{\Omega_-} \psi(\mathbf{x}) [\varrho_n \diamond \mathcal{T}_{\max}f](\mathbf{x})d\mu(\mathbf{x}). \end{aligned}$$

The integrals over $\Omega_+ \cap \Omega_{-\infty}$ and $\Omega_{-\infty} \cap \Omega_{+\infty}$ are evaluated in the same way. \square

We are in position to prove the following

Proposition 3.13. *Let $f \in L^1(\Omega, d\mu)$ and $f_n = \varrho_n \diamond f$, $n \in \mathbb{N}$. Then, for μ_- -a. e. $\mathbf{y} \in \Gamma_-$,*

$$f_n(\Phi(\mathbf{y}, s)) - f_n(\Phi(\mathbf{y}, t)) = \int_s^t [\mathcal{T}_{\max}f_n](\Phi(\mathbf{y}, r))dr \quad \forall 0 < s < t < \tau_+(\mathbf{y}). \quad (3.13)$$

In the same way, for almost every $\mathbf{z} \in \Gamma_+$,

$$f_n(\Phi(\mathbf{z}, -s)) - f_n(\Phi(\mathbf{z}, -t)) = \int_s^t \mathcal{T}_{\max}f_n(\Phi(\mathbf{z}, -r))dr, \quad \forall 0 < s < t < \tau_-(\mathbf{z}).$$

Proof. We focus only on (3.13), the second assertion following the same lines. Since $f \in L^1(\Omega_-, d\mu)$, Proposition 2.12 implies that the integral $\int_0^{\tau_+(\mathbf{y})} |f(\Phi(\mathbf{y}, r))|dr$ exists and is finite for μ_- -almost every $\mathbf{y} \in \Gamma_-$. Therefore, for μ_- -almost every $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$, the quantities $\int_0^t \varrho_n(t-s)f(\Phi(\mathbf{y}, s))ds$ and $\int_0^t \varrho_n'(t-s)f(\Phi(\mathbf{y}, s))ds$ are well-defined and finite. Moreover, thanks to Eq. (3.4) Lemma 3.9, they are respectively equal to $f_n(\Phi(\mathbf{y}, t))$ and $[\mathcal{T}_{\max}f_n](\Phi(\mathbf{y}, t))$. In particular, the mapping $t \in (0, \tau_+(\mathbf{y})) \mapsto [\mathcal{T}_{\max}f_n](\Phi(\mathbf{y}, t)) \in \mathbb{R}$ is continuous. It is then easy to see that, for any $0 < s < t < \tau_+(\mathbf{y})$

$$\begin{aligned} \int_s^t [\mathcal{T}_{\max}f_n](\Phi(\mathbf{y}, r))dr &= - \int_s^t dr \int_0^r \varrho_n'(r-u)f(\Phi(\mathbf{y}, u))du \\ &= - \int_0^s f(\Phi(\mathbf{y}, u))du \int_s^t \varrho_n'(r-u)dr - \int_s^t f(\Phi(\mathbf{y}, u))du \int_u^t \varrho_n'(r-u)dr \\ &= - \int_0^t f(\Phi(\mathbf{y}, u))\varrho_n(t-u)du + \int_0^s f(\Phi(\mathbf{y}, u))\varrho_n(s-u)du, \end{aligned}$$

which is nothing but (3.13). \square

As a consequence, one gets the following result :

Proposition 3.14. *For any $f \in \mathcal{D}(\mathcal{T}_{\max})$, there exists some functions $\tilde{f}_{\pm} \in L^1(\Omega_{\pm}, d\mu)$ such that $\tilde{f}_{\pm}(\mathbf{x}) = f(\mathbf{x})$ for μ -almost every $\mathbf{x} \in \Omega_{\pm}$ and, for μ_- -almost every $\mathbf{y} \in \Gamma_-$:*

$$\tilde{f}_-(\Phi(\mathbf{y}, s)) - \tilde{f}_-(\Phi(\mathbf{y}, t)) = \int_s^t [\mathcal{T}_{\max}f](\Phi(\mathbf{y}, r))dr \quad \forall 0 < s < t < \tau_+(\mathbf{y}), \quad (3.14)$$

while, for μ_+ -almost every $\mathbf{z} \in \Gamma_+$:

$$\tilde{f}_+(\Phi(\mathbf{z}, -s)) - \tilde{f}_+(\Phi(\mathbf{z}, -t)) = \int_s^t [\mathcal{T}_{\max} f](\Phi(\mathbf{z}, -r)) dr \quad \forall 0 < s < t < \tau_-(\mathbf{z}).$$

Proof. Define, for any $n \geq 1$, $f_n = \varrho_n \diamond f$, so that, from Propositions 3.11 and 3.8, $\lim_{n \rightarrow \infty} \|f_n - f\| + \|\mathcal{T}_{\max} f_n - \mathcal{T}_{\max} f\| = 0$. In particular,

$$\lim_{n \rightarrow \infty} \int_{\Omega_-} |f_n(\mathbf{x}) - f(\mathbf{x})| + |[\mathcal{T}_{\max} f_n](\mathbf{x}) - [\mathcal{T}_{\max} f](\mathbf{x})| d\mu(\mathbf{x}) = 0.$$

Then Eq. (2.9) yields

$$\begin{aligned} & \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |f_n(\Phi(\mathbf{y}, s)) - f(\Phi(\mathbf{y}, s))| ds \\ & \quad + \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |[\mathcal{T}_{\max} f_n](\Phi(\mathbf{y}, s)) - [\mathcal{T}_{\max} f](\Phi(\mathbf{y}, s))| ds \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

since $\mathcal{T}_{\max} f$ and $\mathcal{T}_{\max} f_n$ both belong to $L^1(\Omega, d\mu)$. Consequently, for almost every $\mathbf{y} \in \Gamma_-$ (up to a subsequence, still denoted by f_n) we get

$$\begin{cases} f_n(\Phi(\mathbf{y}, \cdot)) \longrightarrow f(\Phi(\mathbf{y}, \cdot)) \\ [\mathcal{T}_{\max} f_n](\Phi(\mathbf{y}, \cdot)) \longrightarrow [\mathcal{T}_{\max} f](\Phi(\mathbf{y}, \cdot)) \end{cases} \quad \text{in } L^1((0, \tau_+(\mathbf{y})), ds)$$

as $n \rightarrow \infty$. Let us fix $\mathbf{y} \in \Gamma_-$ for which this holds. Passing again to a subsequence, we may assume that $f_n(\Phi(\mathbf{y}, s))$ converges (pointwise) to $f(\Phi(\mathbf{y}, s))$ for almost every $s \in (0, \tau_+(\mathbf{y}))$. Let us fix such a s_0 . Then,

$$f_n(\Phi(\mathbf{y}, s_0)) - f_n(\Phi(\mathbf{y}, s)) = \int_{s_0}^s [\mathcal{T}_{\max} f_n](\Phi(\mathbf{y}, r)) dr \quad \forall s \in (0, \tau_+(\mathbf{y})).$$

Now, the right-hand-side has a limit as $n \rightarrow \infty$ so that the first term on the left-hand side also must converge as $n \rightarrow \infty$. Thus, for any $s \in (0, \tau_+(\mathbf{y}))$, the limit

$$\lim_{n \rightarrow \infty} f_n(\Phi(\mathbf{y}, s)) = \tilde{f}_-(\Phi(\mathbf{y}, s))$$

exists and, for any $0 < s < \tau_+(\mathbf{y})$

$$\tilde{f}_-(\Phi(\mathbf{y}, s)) = \tilde{f}_-(\Phi(\mathbf{y}, s_0)) - \int_{s_0}^s [\mathcal{T}_{\max} f](\Phi(\mathbf{y}, r)) dr.$$

It is easy to check then that $\tilde{f}_-(\mathbf{x}) = f(\mathbf{x})$ for almost every $\mathbf{x} \in \Omega_-$. The same arguments lead to the existence of \tilde{f}_+ . \square

The above result shows that the mild formulation of Theorem 3.6 is fulfilled for any $\mathbf{x} \in \Omega_- \cup \Omega_+$. It remains to deal with $\Omega_\infty := \Omega_{-\infty} \cap \Omega_{+\infty}$.

Proposition 3.15. *Let $f \in \mathcal{D}(\mathcal{T}_{\max})$. Then, there exists a set $\mathcal{O} \subset \Omega_\infty$ with $\mu(\mathcal{O}) = 0$ and a function \tilde{f} defined on $\{\mathbf{z} = \Phi(\mathbf{x}, t), \mathbf{x} \in \Omega_\infty \setminus \mathcal{O}, t \in \mathbb{R}\}$ such that $f(\mathbf{x}) = \tilde{f}(\mathbf{x})$ μ -almost every $\mathbf{x} \in \Omega_\infty$ and*

$$\tilde{f}(\Phi(\mathbf{x}, s)) - \tilde{f}(\Phi(\mathbf{x}, t)) = \int_s^t [\mathcal{T}_{\max} f](\Phi(\mathbf{x}, r)) dr, \quad \forall \mathbf{x} \in \Omega_\infty \setminus \mathcal{O}, s < t.$$

Proof. Since $(\mathbf{x}, t) \mapsto (\mathbf{z}, t) = (\Phi(\mathbf{x}, t), t)$ is a measurable and measure preserving mapping from $\Omega_\infty \times \mathbb{R}$ onto itself, Propositions 3.8 and 3.11 give

$$\lim_{n \rightarrow \infty} \int_{\Omega_\infty} d\mu(\mathbf{x}) \int_{I_k} |f_n(\Phi(\mathbf{x}, t)) - f(\Phi(\mathbf{x}, t))| dt = 0 \quad (3.15)$$

$$\lim_{n \rightarrow \infty} \int_{\Omega_\infty} d\mu(\mathbf{x}) \int_{I_k} |\mathcal{T}_{\max} f_n(\Phi(\mathbf{x}, t)) - \mathcal{T}_{\max} f(\Phi(\mathbf{x}, t))| dt = 0, \quad (3.16)$$

for any $I_k = [-k, k]$, $k \in \mathbb{N}$. This shows, in particular, that there is (a maximal) $\mathcal{E} \subset \Omega_\infty$ with $\mu(\mathcal{E}) = 0$ such that, for almost every $\mathbf{x} \in \Omega_\infty \setminus \mathcal{E}$ and any bounded interval $I \subset \mathbb{R}$:

$$\int_I |f(\Phi(\mathbf{x}, t))| dt + \int_I |[\mathcal{T}_{\max} f](\Phi(\mathbf{x}, t))| dt < \infty$$

and we can argue as in Proposition 3.13 that

$$f_n(\Phi(\mathbf{x}, s)) - f_n(\mathbf{x}) = - \int_0^s \mathcal{T}_{\max} f_n(\Phi(\mathbf{x}, r)) dr, \quad \forall s \in \mathbb{R}.$$

Proposition 3.8 yields the existence of a subsequence $(f_{n_p})_p$ and a μ -null set A_0 with $\mathcal{E} \subset A_0 \subset \Omega_\infty$ such that

$$\lim_{p \rightarrow \infty} f_{n_p}(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_\infty \setminus A_0.$$

Now, for any $k \in \mathbb{N}$,

$$\lim_{p \rightarrow \infty} \int_{\Omega_\infty} d\mu(\mathbf{x}) \int_{I_k} |\mathcal{T}_{\max} f_{n_p}(\Phi(\mathbf{x}, t)) - \mathcal{T}_{\max} f(\Phi(\mathbf{x}, t))| dt = 0$$

so that, there is a subsequence (depending on k) and a μ -null set A_k with $A_0 \subset A_k \subset \Omega_\infty$ such that

$$\lim_{p(k) \rightarrow \infty} \int_{I_k} |\mathcal{T}_{\max} f_{n_{p(k)}}(\Phi(\mathbf{x}, t)) - \mathcal{T}_{\max} f(\Phi(\mathbf{x}, t))| dt = 0, \quad \forall \mathbf{x} \in \Omega_\infty \setminus A_k.$$

Let $\mathbf{x} \in \Omega_\infty \setminus A_k$ and $|s| < k$ be fixed. From

$$f_{n_{p(k)}}(\Phi(\mathbf{x}, s)) - f_{n_{p(k)}}(\mathbf{x}) = - \int_0^s \mathcal{T}_{\max} f_{n_{p(k)}}(\Phi(\mathbf{x}, r)) dr$$

we deduce that the limit $\lim_{p(k) \rightarrow \infty} f_{n_{p(k)}}(\Phi(\mathbf{x}, s))$ exists and is equal to

$$\lim_{p(k) \rightarrow \infty} f_{n_{p(k)}}(\Phi(\mathbf{x}, s)) = f(\mathbf{x}) - \int_0^s \mathcal{T}_{\max} f(\Phi(\mathbf{x}, r)) dr.$$

We define then \tilde{f} by

$$\tilde{f}(\Phi(\mathbf{x}, s)) = \lim_{p(k) \rightarrow \infty} f_{n_{p(k)}}(\Phi(\mathbf{x}, s)), \quad \mathbf{x} \in \Omega_\infty \setminus A_k, |s| < k$$

and defining $\mathcal{O} = \bigcup_{k \geq 1} A_k$, we get the result. \square

Before the proof of Theorem 3.6, we have to establish existence of the trace on Γ_- .

Proposition 3.16. *Let f satisfies condition (1) of Theorem 3.6. Then*

$$\lim_{t \rightarrow 0^+} f^\sharp(\Phi(\mathbf{y}, t))$$

exists for almost every $\mathbf{y} \in \Gamma_-$. Similarly, $\lim_{t \rightarrow 0^+} f^\sharp(\Phi(\mathbf{y}, -t))$ exists for almost every $\mathbf{y} \in \Gamma_+$.

Proof. First we note that there is $\tilde{\Omega}_- \subset \Omega_-$ with $\mu(\Omega_- \setminus \tilde{\Omega}_-) = 0$ such that (3.2) is valid any $\mathbf{x} \in \tilde{\Omega}_-$. Let $\tilde{\Gamma}_- = \{\mathbf{y} \in \Gamma_-; \mathbf{y} = \Phi(\mathbf{x}, \tau_-(\mathbf{x})), \mathbf{x} \in \tilde{\Omega}_-\}$. It is easy to see that $\mu_-(\Gamma_- \setminus \tilde{\Gamma}_-) = 0$. Indeed, otherwise, by (2.9), there would be a subset of Ω_- of positive μ -measure, not intersecting $\tilde{\Omega}_-$, which would contradict (3.2). Consequently, any $\mathbf{x} \in \tilde{\Omega}_-$ can be written as $\mathbf{x} = \Phi(\mathbf{y}, \tau_-(\mathbf{y}))$, $\mathbf{y} \in \tilde{\Gamma}_-$ and (3.2) can be recast as

$$f^\sharp(\Phi(\mathbf{y}, t)) - f^\sharp(\Phi(\mathbf{y}, t_0)) = \int_t^{t_0} g(\Phi(\mathbf{y}, s)) ds. \quad (3.17)$$

for almost any $\mathbf{y} \in \Gamma_-$, where $0 < t \leq t_0 < \tau_+(\mathbf{y})$. Using again (2.9), $s \mapsto g(\Phi(\mathbf{y}, s))$ is integrable on $(0, \tau_+(\mathbf{y}))$ for almost any $\mathbf{y} \in \Gamma_-$. Consequently, for almost every $\mathbf{y} \in \Gamma_-$ we can pass to the limit in (3.17) with $t \rightarrow 0$; it is easy to check that this limit does not depend on t_0 . The existence of $\lim_{t \rightarrow 0^+} f^\sharp(\Phi(\mathbf{y}, -t))$ for a. e. $\mathbf{y} \in \Gamma_+$ follows by the same argument. \square

The above proposition allows to define the trace operators.

Definition 3.17. For any $f \in \mathcal{D}(\mathcal{T}_{\max})$, define the traces $B^\pm f$ by

$$B^+ f(\mathbf{y}) := \lim_{t \rightarrow 0^+} f^\sharp(\Phi(\mathbf{y}, -t)) \quad \text{and} \quad B^- f(\mathbf{y}) := \lim_{t \rightarrow 0^+} f^\sharp(\Phi(\mathbf{y}, t))$$

for any $\mathbf{y} \in \Gamma_\pm$ for which the limits exist, where f^\sharp is a suitable representative of f .

PROOF OF THEOREM 3.6. To prove that (2) \implies (1), given $f \in \mathcal{D}(\mathcal{T}_{\max})$, set

$$f^\sharp(\mathbf{x}) = \begin{cases} \tilde{f}_-(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_-, \\ \tilde{f}_+(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_+ \cap \Omega_{-\infty}, \\ \tilde{f}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_{-\infty} \cap \Omega_{+\infty}, \end{cases}$$

where \tilde{f}_\pm are given by Proposition 3.14 while \tilde{f} is provided by Prop. 3.15. Then, it is clear that for any $\mathbf{x} \in \Omega$ and any $-\tau_-(\mathbf{x}) < t_1 \leq t_2 < \tau_+(\mathbf{x})$

$$f^\sharp(\Phi(\mathbf{x}, t_1)) - f^\sharp(\Phi(\mathbf{x}, t_2)) = \int_{t_1}^{t_2} [\mathcal{T}_{\max} f](\Phi(\mathbf{x}, s)) ds$$

and (3.2) is proven.

Let us now prove that (1) \implies (2). Let us fix $\psi \in \mathfrak{Y}$, one has

$$\begin{aligned} \int_{\Omega_-} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \\ &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f^\sharp(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt. \end{aligned}$$

Notice that since both $\int_{\Omega_-} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x})$ and $\int_{\Omega_-} \psi(\mathbf{x}) g(\mathbf{x}) d\mu(\mathbf{x})$ exist, Proposition 2.12 and Fubini's Theorem, the integrals

$$\int_0^{\tau_+(\mathbf{y})} f^\sharp(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \quad \text{and} \quad \int_0^{\tau_+(\mathbf{y})} g(\Phi(\mathbf{y}, t)) \psi(\Phi(\mathbf{y}, t)) dt$$

are well-defined for μ_- -almost every $\mathbf{y} \in \Gamma_-$. Let us prove that these two integrals coincide for almost-every $\mathbf{y} \in \Gamma_-$. According to Lemma 3.3, for almost every $\mathbf{y} \in \Gamma_-$, there is a sequence

$(t_n)_n$ (depending on \mathbf{y}) such that $\psi(\Phi(\mathbf{y}, t_n)) = 0$ and $t_n \rightarrow \tau_+(\mathbf{y})$. Thus,

$$\int_0^{\tau_+(\mathbf{y})} f^\sharp(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt = \lim_{n \rightarrow \infty} \int_0^{t_n} f^\sharp(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt$$

and

$$\int_0^{\tau_+(\mathbf{y})} g(\Phi(\mathbf{y}, t)) \psi(\Phi(\mathbf{y}, t)) dt = \lim_{n \rightarrow \infty} \int_0^{t_n} g(\Phi(\mathbf{y}, t)) \psi(\Phi(\mathbf{y}, t)) dt.$$

Further, for almost every $\mathbf{y} \in \Gamma_-$, according to (3.2),

$$f^\sharp(\Phi(\mathbf{y}, t)) = B^- f(\mathbf{y}) - \int_0^t g(\Phi(\mathbf{y}, r)) dr, \quad \forall t \in (0, \tau_+(\mathbf{y})).$$

Integration by parts, using the fact that $\psi(\Phi(\mathbf{y}, 0)) = \psi(\Phi(\mathbf{y}, t_n)) = 0$ for any n , leads to

$$\int_0^{t_n} f^\sharp(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt = \int_0^{t_n} g(\Phi(\mathbf{y}, t)) \psi(\Phi(\mathbf{y}, t)) dt.$$

Consequently, for μ_- almost every $\mathbf{y} \in \Gamma_-$:

$$\int_0^{\tau_+(\mathbf{y})} f^\sharp(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt = \int_0^{\tau_+(\mathbf{y})} g(\Phi(\mathbf{y}, t)) \psi(\Phi(\mathbf{y}, t)) dt. \quad (3.18)$$

Finally, we get

$$\begin{aligned} \int_{\Omega_-} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} g(\Phi(\mathbf{y}, t)) \psi(\Phi(\mathbf{y}, t)) dt \\ &= \int_{\Omega_-} g(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}). \end{aligned} \quad (3.19)$$

Using now parametrization over Γ_+ , we prove in the same way that

$$\int_{\Omega_+ \cap \Omega_{-\infty}} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega_+ \cap \Omega_{-\infty}} g(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}). \quad (3.20)$$

It remains now to evaluate $A := \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x})$. According to Assumption 1

$$A = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f^\sharp(\Phi(\mathbf{x}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{x}, t)) d\mu(\mathbf{x}), \quad \forall t \in \mathbb{R}.$$

Let us integrate the above identity over $(0, 1)$, so that

$$A = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} d\mu(\mathbf{x}) \int_0^1 f^\sharp(\Phi(\mathbf{x}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{x}, t)) dt.$$

Let us fix $\mathbf{x} \in \Omega_{-\infty} \cap \Omega_{+\infty}$. For any $t \in (0, 1)$, one has $f^\sharp(\Phi(\mathbf{x}, t)) = f^\sharp(\mathbf{x}) - \int_0^t g(\Phi(\mathbf{x}, s)) ds$ and integration by parts yields

$$\begin{aligned} \int_0^1 f^\sharp(\Phi(\mathbf{x}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{x}, t)) dt &= \int_0^1 \psi(\Phi(\mathbf{x}, t)) g(\Phi(\mathbf{x}, t)) dt - \psi(\mathbf{x}) f^\sharp(\mathbf{x}) \\ &\quad + \psi(\Phi(\mathbf{x}, 1)) \left(f^\sharp(\mathbf{x}) - \int_0^1 g(\Phi(\mathbf{x}, s)) ds \right) \\ &= \int_0^1 \psi(\Phi(\mathbf{x}, t)) g(\Phi(\mathbf{x}, t)) dt + \psi(\Phi(\mathbf{x}, 1)) f^\sharp(\Phi(\mathbf{x}, 1)) - \psi(\mathbf{x}) f^\sharp(\mathbf{x}) \end{aligned}$$

where we used again (3.2). Integrating over $\Omega_{-\infty} \cap \Omega_{+\infty}$ we see from Liouville's Theorem (Assumption 1) that

$$\int_{\Omega_{-\infty} \cap \Omega_{+\infty}} \psi(\Phi(\mathbf{x}, 1)) f^\sharp(\Phi(\mathbf{x}, 1)) d\mu(\mathbf{x}) = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} \psi(\mathbf{x}) f^\sharp(\mathbf{x}) d\mu(\mathbf{x}),$$

i.e.

$$A = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} d\mu(\mathbf{x}) \int_0^1 \psi(\Phi(\mathbf{x}, t)) g(\Phi(\mathbf{x}, t)) dt$$

which, thanks to Liouville's Theorem, is nothing but

$$\int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}). \quad (3.21)$$

Combining (3.19), (3.20) and (3.21), we obtain

$$\int_{\Omega} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega} g(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}), \quad \forall \psi \in \mathfrak{D}$$

which exactly means that $f \in \mathcal{D}(\mathcal{T}_{\max})$ with $g = \mathcal{T}_{\max}$ and the proof is complete. \square

Corollary 3.18. *Traces $B^\pm f$ on Γ_\pm can be defined for any $f \in \mathcal{D}(\mathcal{T}_{\max})$. For μ_- -almost any $\mathbf{y} \in \Gamma_-$ we have*

$$B^- f(\mathbf{y}) = f^\sharp(\Phi(\mathbf{y}, t)) + \int_0^t [\mathcal{T}_{\max} f](\Phi(\mathbf{y}, s)) ds, \quad \forall t \in (0, \tau_+(\mathbf{y})),$$

where f^\sharp is a suitable representative of f . An analogous formula holds for $B^+ f$.

Lemma 2.9 provides the existence of Borel measures $d\mu_\pm$ on Γ_\pm , which allow us to define the natural trace spaces associated to Problem (1.1), namely,

$$L_\pm^1 := L^1(\Gamma_\pm, d\mu_\pm).$$

However, the traces $B^\pm f$, $f \in \mathcal{D}(\mathcal{T}_{\max})$, not necessarily belong to L_\pm^1 .

4. WELL-POSEDNESS FOR INITIAL AND BOUNDARY-VALUE PROBLEMS

4.1. Absorption semigroup. From now on, we will denote $X = L^1(\Omega, d\mu)$ endowed with its natural norm $\|\cdot\|_X$. Let \mathcal{T}_0 be the free streaming operator with *no re-entry boundary conditions*:

$$\mathcal{T}_0 \psi = \mathcal{T}_{\max} \psi, \quad \text{for any } \psi \in \mathcal{D}(\mathcal{T}_0),$$

where the domain $\mathcal{D}(\mathcal{T}_0)$ is defined by

$$\mathcal{D}(\mathcal{T}_0) = \{\psi \in \mathcal{D}(\mathcal{T}_{\max}); B^- \psi = 0\}.$$

We state the following generation result:

Theorem 4.1. *The operator $(\mathcal{T}_0, \mathcal{D}(\mathcal{T}_0))$ is the generator of a nonnegative C_0 -semigroup of contractions $(U_0(t))_{t \geq 0}$ in $L^1(\Omega, d\mu)$ given by*

$$U_0(t) f(\mathbf{x}) = f(\Phi(\mathbf{x}, -t)) \chi_{\{t < \tau_-(\mathbf{x})\}}(\mathbf{x}), \quad (\mathbf{x} \in \Omega, f \in X),$$

where χ_A denotes the characteristic function of a set A .

Proof. The proof is divided into three steps:

• *Step 1.* Let us first check that the family of operators $(U_0(t))_{t \geq 0}$ is a nonnegative contractive C_0 -semigroup in X . Thanks to Proposition 2.3, we can prove that, for any $f \in X$ and any $t \geq 0$, the mapping $U_0(t)f : \Omega \rightarrow \mathbb{R}$ is measurable and the semigroup properties $U_0(0)f = f$ and $U_0(t)U_0(s)f = U_0(t+s)f$ ($t, s \geq 0$) hold. Let us now show that $\|U_0(t)f\|_X \leq \|f\|_X$. We have

$$\|U_0(t)f\|_X = \int_{\Omega_+} |U_0(t)f| d\mu + \int_{\Omega_- \cap \Omega_{+\infty}} |U_0(t)f| d\mu + \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} |U_0(t)f| d\mu.$$

Propositions 2.12 and 2.3 yield

$$\begin{aligned} \int_{\Omega_+} |U_0(t)f| d\mu &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |U_0(t)f(\Phi(\mathbf{y}, -s))| ds \\ &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\max(0, \tau_-(\mathbf{y})-t)} |f(\Phi(\mathbf{y}, -s-t))| ds \\ &\leq \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_t^{\max(t, \tau_-(\mathbf{y}))} |f(\Phi(\mathbf{y}, -r))| dr \leq \int_{\Omega_+} |f| d\mu. \end{aligned}$$

In the same way we obtain

$$\int_{\Omega_- \cap \Omega_{+\infty}} |U_0(t)f| d\mu = \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^\infty |U_0(t)f(\Phi(\mathbf{y}, s))| ds = \int_{\Omega_- \cap \Omega_{+\infty}} |f| d\mu,$$

and

$$\int_{\Omega_{-\infty} \cap \Omega_{+\infty}} |U_0(t)f| d\mu = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} |f| d\mu.$$

This proves contractivity of $U_0(t)$. Let us now show that $U_0(t)f$ is continuous, i.e.

$$\lim_{t \rightarrow 0} \|U_0(t)f - f\|_X = 0.$$

It is enough to show that this property holds for any $f \in \mathcal{C}_0(\Omega)$. In this case, $\lim_{t \rightarrow 0} U_0(t)f(\mathbf{x}) = f(\mathbf{x})$ for any $\mathbf{x} \in \Omega$. Moreover, $\sup_{\mathbf{x} \in \Omega} |U_0(t)f(\mathbf{x})| \leq \sup_{\mathbf{x} \in \Omega} |f(\mathbf{x})|$ and the support of $U_0(t)f$ is bounded, so that the Lebesgue dominated convergence theorem leads to the result. This proves that $(U_0(t))_{t \geq 0}$ is a C_0 -semigroup of contractions in X . Let \mathcal{A}_0 denote its generator.

• *Step 2.* To show that $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{T}_0)$, fix $f \in \mathcal{D}(\mathcal{A}_0)$, $\lambda > 0$ and $g = (\lambda - \mathcal{A}_0)f$. Then,

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t) g(\Phi(\mathbf{x}, -t)) dt, \quad (\mathbf{x} \in \Omega). \quad (4.1)$$

To prove that $f \in \mathcal{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max}f = \mathcal{A}_0f$, it suffices to prove that

$$\int_{\Omega} (\lambda f(\mathbf{x}) - g(\mathbf{x})) \psi(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}), \quad \forall \psi \in \mathfrak{Y}.$$

Let us fix $\psi \in \mathfrak{Y}$, set $\varphi(\mathbf{x}) := \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0}$ and write

$$\begin{aligned} \int_{\Omega} f(\mathbf{x}) \varphi(\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Omega_+} f(\mathbf{x}) \varphi(\mathbf{x}) d\mu(\mathbf{x}) + \int_{\Omega_{+\infty} \cap \Omega_-} f(\mathbf{x}) \varphi(\mathbf{x}) d\mu(\mathbf{x}) \\ &\quad + \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{x}) \varphi(\mathbf{x}) d\mu(\mathbf{x}) = I_1 + I_2 + I_3. \end{aligned}$$

We first deal with I_1 . For any $\mathbf{y} \in \Gamma_+$ and $t \in (0, \tau_-(\mathbf{y}))$ we have $\varphi(\Phi(\mathbf{y}, -t)) = -\frac{d}{dt}\psi(\Phi(\mathbf{y}, -t))$ and $f(\Phi(\mathbf{y}, -t)) = \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t))g(\Phi(\mathbf{y}, -s))ds$. Then, by Proposition 2.12,

$$\begin{aligned} I_1 &= - \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \frac{d}{dt}\psi(\Phi(\mathbf{y}, -t))dt \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t))g(\Phi(\mathbf{y}, -s))ds \\ &= - \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\Phi(\mathbf{y}, -s))ds \int_0^s \exp(-\lambda(s-t))\frac{d}{dt}(\psi(\Phi(\mathbf{y}, -t))) dt \\ &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\Phi(\mathbf{y}, -s)) \times \\ &\quad \times \left\{ \lambda \int_0^s \exp(-\lambda(s-t))\psi(\Phi(\mathbf{y}, -t))dt - \psi(\Phi(\mathbf{y}, -s)) \right\} ds \end{aligned}$$

where we used the fact that $\psi(\Phi(\mathbf{y}, 0)) = 0$ for any $\mathbf{y} \in \Gamma_+$ since ψ is compactly supported. Thus

$$\begin{aligned} I_1 &= \lambda \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \psi(\Phi(\mathbf{y}, -t))dt \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t))g(\Phi(\mathbf{y}, -s))ds \\ &\quad - \int_{\Gamma_-} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\Phi(\mathbf{y}, -s))\psi(\Phi(\mathbf{y}, -s))ds \\ &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \psi(\Phi(\mathbf{y}, -t))(\lambda f(\Phi(\mathbf{y}, -t)) - g(\Phi(\mathbf{y}, -t)))dt. \end{aligned}$$

Using again Proposition 2.12, we obtain

$$I_1 = \int_{\Omega_+} (\lambda f(\mathbf{x}) - g(\mathbf{x})) \psi(\mathbf{x})d\mu(\mathbf{x}). \quad (4.2)$$

Arguing in a similar way, we prove that

$$I_2 = - \int_{\Omega_- \cap \Omega_{+\infty}} (\lambda f(\mathbf{x}) - g(\mathbf{x})) \psi(\mathbf{x})d\mu(\mathbf{x}). \quad (4.3)$$

Finally, since

$$f(\mathbf{x}) = \int_0^\infty \exp(-\lambda t)g(\Phi(\mathbf{x}, -t)) dt \quad \text{for any } \mathbf{x} \in \Omega_{-\infty} \cap \Omega_{+\infty},$$

one has

$$\begin{aligned} I_3 &= \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} \varphi(\mathbf{x})d\mu(\mathbf{x}) \int_0^\infty \exp(-\lambda t)g(\Phi(\mathbf{x}, -t))dt \\ &= \int_0^\infty \exp(-\lambda t)dt \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} \varphi(\mathbf{x})g(\Phi(\mathbf{x}, -t))d\mu(\mathbf{x}). \end{aligned}$$

Now, Assumption 1 asserts that

$$\int_{\Omega_{-\infty} \cap \Omega_{+\infty}} \varphi(\mathbf{x})g(\Phi(\mathbf{x}, -t))d\mu(\mathbf{x}) = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g(\mathbf{x})\varphi(\Phi(\mathbf{x}, t))d\mu(\mathbf{x}), \quad \forall t \geq 0,$$

and, since $\varphi(\Phi(\mathbf{x}, t)) = \frac{d}{dt}\psi(\Phi(\mathbf{x}, t))$, finally

$$\begin{aligned} I_3 &= \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g(\mathbf{x}) d\mu(\mathbf{x}) \int_0^\infty \exp(-\lambda t) \frac{d}{dt} (\psi(\Phi(\mathbf{x}, t))) dt \\ &= - \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}) + \lambda \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g(\mathbf{x}) d\mu(\mathbf{x}) \int_0^\infty \exp(-\lambda t) \psi(\Phi(\mathbf{x}, t)) dt. \end{aligned}$$

Using again Assumption 1, this finally gives

$$I_3 = - \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} (g(\mathbf{x}) - \lambda f(\mathbf{x})) \psi(\mathbf{x}) d\mu(\mathbf{x}). \quad (4.4)$$

Combining (4.2)–(4.4) leads to

$$\int_{\Omega} f(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) = - \int_{\Omega} (g(\mathbf{x}) - \lambda f(\mathbf{x})) \psi(\mathbf{x}) d\mu(\mathbf{x})$$

which proves that $f \in \mathcal{D}(\mathcal{T}_{\max})$ and $(\lambda - \mathcal{T}_{\max})f = g$. Next, for $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$ we write $t = \tau_-(\Phi(\mathbf{y}, t))$ and, by Proposition 2.3 and (4.1), we obtain

$$f(\Phi(\mathbf{y}, t)) = \int_0^t \exp(-\lambda(t-s)) g(\Phi(\mathbf{y}, s)) ds. \quad (4.5)$$

Consequently, $\lim_{t \rightarrow 0^+} f(\Phi(\mathbf{y}, t)) = 0$ a.e. $\mathbf{y} \in \Gamma_-$, i.e. $B^-f = 0$ so that $f \in \mathcal{D}(\mathcal{T}_0)$ and $\mathcal{A}_0 f = \mathcal{T}_0 f = \lambda f - g$.

• *Step 3.* Now let us show the converse inclusion $\mathcal{D}(\mathcal{T}_0) \subset \mathcal{D}(\mathcal{A}_0)$. Let $f \in \mathcal{D}(\mathcal{T}_0)$. Changing possibly f on a set of zero measure, we may write $f = f^\sharp$, where f^\sharp is the representative of f given by Theorem 3.6. Then, for any $\mathbf{x} \in \Omega$ and any $0 \leq t < \tau_-(\mathbf{x})$

$$f(\Phi(\mathbf{x}, -t)) - f(\mathbf{x}) = \int_0^t [\mathcal{T}_{\max} f](\Phi(\mathbf{x}, -r)) dr$$

which, according to the explicit expression of $U_0(t)$, means that

$$U_0(t)f(\mathbf{x}) - f(\mathbf{x}) = \int_0^t U_0(r) \mathcal{T}_{\max} f(\mathbf{x}) dr \quad (4.6)$$

for any $\mathbf{x} \in \Omega$ and $t < \tau_-(\mathbf{x})$. Letting t converge towards $\tau_-(\mathbf{x})$ we obtain

$$f(\mathbf{x}) = - \int_0^{\tau_-(\mathbf{x})} [\mathcal{T}_{\max} f](\Phi(\mathbf{x}, -r)) dr.$$

In particular, Eq. (4.6) holds true for any $\mathbf{x} \in \Omega$ and any $t \geq \tau_-(\mathbf{x})$. Arguing exactly as in [16, p. 38], the pointwise identity (4.6) represents the X -integral, i.e. $U_0(t)f - f = \int_0^t U_0(r) \mathcal{T}_{\max} f dr$ in $L^1(\Omega, d\mu)$. Consequently, $f \in \mathcal{D}(\mathcal{A}_0)$ with $\mathcal{A}_0 f = \mathcal{T}_{\max} f$. \square

4.2. Green's formula. The above result allows us to treat more general boundary-value problem:

Theorem 4.2. *Let $u \in L^1_-$ and $g \in X$ be given. Then the function*

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t) g(\Phi(\mathbf{x}, -t)) dt + \chi_{\{\tau_-(\mathbf{x}) < \infty\}} \exp(-\lambda \tau_-(\mathbf{x})) u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x})))$$

is a **unique** solution $f \in \mathcal{D}(\mathcal{T}_{\max})$ of the boundary value problem:

$$\begin{cases} (\lambda - \mathcal{T}_{\max})f = g, \\ B^-f = u, \end{cases} \quad (4.7)$$

where $\lambda > 0$. Moreover, $\mathbf{B}^+ f \in L^1_+$ and

$$\|\mathbf{B}^+ f\|_{L^1_+} + \lambda \|f\|_X \leq \|u\|_{L^1_-} + \|g\|_X, \quad (4.8)$$

with equality sign if $g \geq 0$ and $u \geq 0$.

Proof. Let us write $f = f_1 + f_2$ with $f_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t) g(\Phi(\mathbf{x}, -t)) dt$, and

$$f_2(\mathbf{x}) = \chi_{\{\tau_-(\mathbf{x}) < \infty\}} \exp(-\lambda \tau_-(\mathbf{x})) u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x}))), \quad \mathbf{x} \in \Omega.$$

According to Theorem 4.1, $f_1 = (\lambda - \mathcal{T}_0)^{-1} g$, i.e. $f_1 \in \mathcal{D}(\mathcal{T}_{\max})$ with $(\lambda - \mathcal{T}_{\max})f_1 = g$ and $\mathbf{B}^- f_1 = 0$. Therefore, to prove that f is a solution of (4.7) it suffices to check that $f_2 \in \mathcal{D}(\mathcal{T}_{\max})$, $(\lambda - \mathcal{T}_{\max})f_2 = 0$ and $\mathbf{B}^- f_2 = u$. It is easy to see that $f_2 \in L^1(\Omega, d\mu)$ (see also (4.10)). To prove that $f_2 \in \mathcal{D}(\mathcal{T}_{\max})$ one argues as in the proof of Theorem 4.1. Precisely, let $\psi \in \mathfrak{V}$, noticing that f_2 vanishes outside Ω_- , one has thanks to (4.9)

$$\begin{aligned} \int_{\Omega} f_2(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f_2(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \\ &= \int_{\Gamma_-} u(\mathbf{y}) d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \exp(-\lambda t) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt. \end{aligned}$$

For almost every $\mathbf{y} \in \Gamma_-$, we compute the integral over $(0, \tau_+(\mathbf{y}))$ by parts, which yields $f_2 \in \mathcal{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max} f_2 = \lambda f_2$. Also,

$$f_2(\Phi(\mathbf{y}, t)) = \exp(-\lambda t) u(\mathbf{y}), \quad \mathbf{y} \in \Gamma_-, \quad 0 < t < \tau_+(\mathbf{y}) \quad (4.9)$$

from which we see that $\mathbf{B}^- f_2 = u$.

Consequently, f is a solution to (4.7). To prove that the solution is unique, it is sufficient to prove that the only solution $h \in \mathcal{D}(\mathcal{T}_{\max})$ to $(\lambda - \mathcal{T}_{\max})h = 0$, $\mathbf{B}^- h = 0$, is $h = 0$. This follows from the fact that such a solution h actually belongs to $\mathcal{D}(\mathcal{T}_0)$ if $\lambda \in \rho(\mathcal{T}_0)$. Finally, it remains to prove (4.8). For simplicity, we denote the representative of f_i , $i = 1, 2$, defined in Proposition 3.16, with the same letter. Using (4.9) and the fact that f_2 vanishes on $\Omega_{-\infty}$, from (2.9) we get

$$\begin{aligned} \lambda \int_{\Omega} |f_2| d\mu &= \lambda \int_{\Omega_-} |f_2| d\mu = \lambda \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} e^{-\lambda t} |u(\mathbf{y})| dt \\ &= \int_{\Gamma_-} |u(\mathbf{y})| (1 - e^{-\lambda \tau_+(\mathbf{y})}) d\mu_-(\mathbf{y}). \end{aligned} \quad (4.10)$$

Define $h : \mathbf{y} \in \Gamma_- \mapsto h(\mathbf{y}) = |u(\mathbf{y})| e^{-\lambda \tau_+(\mathbf{y})}$. It is clear that h vanishes on $\Gamma_{-\infty}$ and $h(\mathbf{y}) \leq |u(\mathbf{y})|$ for a.e. $\mathbf{y} \in \Gamma_-$. In particular, $h \in L^1_-$ and, according to (2.11),

$$\begin{aligned} \int_{\Gamma_-} h(\mathbf{y}) d\mu_-(\mathbf{y}) &= \int_{\Gamma_- \setminus \Gamma_{-\infty}} h(\mathbf{y}) d\mu_-(\mathbf{y}) = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} h(\Phi(z, -\tau_-(z))) d\mu_+(z) \\ &= \int_{\Gamma_+ \setminus \Gamma_{+\infty}} e^{-\lambda \tau_-(z)} |u(\Phi(z, -\tau_-(z)))| d\mu_+(z) = \int_{\Gamma_+} |\mathbf{B}^+ f_2(z)| d\mu_+(z) = \|\mathbf{B}^+ f_2\|_{L^1_+}. \end{aligned}$$

Combining this with (4.10) leads to

$$\lambda \|f_2\|_X + \|\mathbf{B}^+ f_2\|_{L^1_+} = \|u\|_{L^1_-}. \quad (4.11)$$

Now, let us show that $\mathbf{B}^+ f_1 \in L^1_+$ and $\|\mathbf{B}^+ f_1\|_{L^1_+} + \lambda \|f_1\|_X \leq \|g\|_X$. For any $\mathbf{y} \in \Gamma_+$ and $0 < t < \tau_-(\mathbf{y})$, we see, as above, that $f_1(\Phi(\mathbf{y}, -t)) = \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t)) g(\Phi(\mathbf{y}, -s)) ds$.

This shows that

$$\mathbb{B}^+ f_1(\mathbf{y}) = \lim_{t \rightarrow 0^+} f_1(\Phi(\mathbf{y}, -t)) = \int_0^{\tau_-(\mathbf{y})} \exp(-\lambda s) g(\Phi(\mathbf{y}, -s)) ds.$$

According to Proposition 2.12,

$$\int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |g(\Phi(\mathbf{y}, -s))| ds = \int_{\Omega_+} |g| d\mu$$

which, since $\exp(-\lambda(s-t))|g(\Phi(\mathbf{y}, -s))| \leq |g(\Phi(\mathbf{y}, -s))|$, implies $\mathbb{B}^+ f_1 \in L^1_+$. Let us now assume $g \geq 0$. Then $f_1 \geq 0$ and hence

$$\lambda \|f_1\| = \lambda \int_{\Omega} f_1 d\mu = \lambda \int_{\Omega_+} f_1 d\mu + \lambda \int_{\Omega_- \cap \Omega_{+\infty}} f_1 d\mu + \lambda \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} f_1 d\mu.$$

Using similar arguments to those used in the study of f_2 , we have

$$\lambda \int_{\Omega_+} f_1 d\mu = \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\Phi(\mathbf{y}, -t)) (1 - \exp(-\lambda t)) dt,$$

which, by Proposition 2.12, implies $\lambda \int_{\Omega_+} f_1 d\mu = \int_{\Omega_+} g d\mu - \int_{\Gamma_+} \mathbb{B}^+ f_1 d\mu_+$. Similar argument shows that $\lambda \int_{\Omega_- \cap \Omega_{+\infty}} f_1 d\mu = \int_{\Omega_- \cap \Omega_{+\infty}} g d\mu$, while the equality

$$\lambda \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} f_1 d\mu = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g d\mu,$$

is a direct consequence of the invariance of μ with respect to $\Phi(\cdot, t)$. This shows that $\lambda \|f\|_X = \|g\|_X - \|\mathbb{B}^+ f\|_{L^1_+}$ for $g \geq 0$. In general, defining

$$F_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda s) |g(\Phi(\mathbf{x}, -s))| ds, \quad \mathbf{x} \in \Omega,$$

we obtain $\|\mathbb{B}^+ f_1\|_{L^1_+} + \lambda \|f_1\|_X \leq \|\mathbb{B}^+ F_1\|_{L^1_+} + \lambda \|F_1\|_X = \|g\|_X$ which, combined with (4.11), gives (4.8). \square

Remark 4.3. Notice that, in order to get the existence and uniqueness of the solution f to (4.7), it is not necessary for u to belong to $L^1(\Gamma_-, d\mu_-)$. Indeed, we only have to make sure that $f_2 \in L^1(\Omega, d\mu)$, i.e., from (4.10), $\int_{\Gamma_-} |u(\mathbf{y})| (1 - e^{-\lambda\tau_+(\mathbf{y})}) d\mu_-(\mathbf{y}) < \infty$. Of course, to get (4.8), the assumption $u \in L^1(\Gamma_-, d\mu_-)$ is necessary.

Let us note that, with the notation of Theorem 4.2, we have

$$\int_{\Gamma_+} \mathbb{B}^+ f d\mu_+ + \lambda \int_{\Omega} f d\mu = \int_{\Gamma_-} u d\mu_- + \int_{\Omega} g d\mu. \quad (4.12)$$

Indeed, for nonnegative u and g , (4.8) turns out to be precisely (4.12). Then, for arbitrary $u \in L^1_-$ and $g \in X$, we get (4.12) by splitting functions into positive and negative parts. This leads to the following generalization of Green's formula:

Proposition 4.4 (Green's formula). Let $f \in \mathcal{D}(\mathcal{T}_{\max})$ satisfies $\mathbb{B}^- f \in L^1_-$. Then $\mathbb{B}^+ f \in L^1_+$ and

$$\int_{\Omega} \mathcal{T}_{\max} f d\mu = \int_{\Gamma_-} \mathbb{B}^- f d\mu_- - \int_{\Gamma_+} \mathbb{B}^+ f d\mu_+$$

Proof. For given $f \in \mathcal{D}(\mathcal{T}_{\max})$, we obtain the result by setting $u = B^- f \in L^1_-$ and $g = (\lambda - \mathcal{T}_{\max})f \in X$ in Eq. (4.12). \square

Remark 4.5. If $d\mu$ is the Lebesgue measure on \mathbb{R}^N , the above formula leads to a better understanding of the measures $d\mu_{\pm}$. Indeed, comparing it to the classical Green's formula (see e.g. [8]), we see that the restriction of $d\mu_{\pm}$ on the set $\Sigma_{\pm} = \{\mathbf{y} \in \partial\Omega; \pm \mathcal{F}(\mathbf{y}) \cdot n(\mathbf{y}) > 0\}$ equals

$$|\mathcal{F}(\mathbf{y}) \cdot n(\mathbf{y})| d\gamma(\mathbf{y}),$$

where $d\gamma(\cdot)$ is the surface Lebesgue measure on $\partial\Omega$.

APPENDIX: ABOUT THE CLASS OF TEST-FUNCTIONS

We answer in this Appendix a natural question concerning the definition of the class of test-functions \mathfrak{J} . Precisely, we prove that two test-functions equal μ -almost everywhere are such that their derivatives (in the sense of (3.1)) also coincide μ -almost everywhere. To prove our claim, it clearly suffices to prove that, given $\psi \in \mathfrak{J}$ such that $\psi(\mathbf{x}) = 0$ for μ -a. e. $\mathbf{x} \in \Omega$, one has $\varphi(\mathbf{x}) = 0$ for μ -a. e. $\mathbf{x} \in \Omega$ where $\varphi(\mathbf{x}) = \left. \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \right|_{s=0}$. Let

$$E := \{\mathbf{x} \in \Omega; \psi(\mathbf{x}) = 0 \text{ and } \varphi(\mathbf{x}) \neq 0\}.$$

It is clear that E is measurable and that one has to prove that $\mu(E) = 0$. It is no loss of generality to assume that E is bounded. We observe that for any $\mathbf{x} \in E$, there exists $\delta_{\mathbf{x}} > 0$ such that

$$\psi(\Phi(\mathbf{x}, t)) \neq 0, \quad \forall 0 < |t| < \delta_{\mathbf{x}}. \quad (\text{A.1})$$

Let us split E as follows

$$E = (E \cap \Omega_-) \cup (E \cap \Omega_+ \cap \Omega_{-\infty}) \cup (E \cap \Omega_{+\infty} \cap \Omega_{-\infty}) := E_- \cup E_+ \cup E_{\infty}$$

and prove that $\mu(E_-) = \mu(E_+) = \mu(E_{\infty}) = 0$.

- (1) First consider E_- . Since $\psi(\mathbf{x}) = 0$ for μ -a. e. $\mathbf{x} \in \Omega_-$ and using the fact that any $\mathbf{x} \in \Omega_-$ can be written as $\mathbf{x} = \Phi(\mathbf{y}, t)$ for some $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$, we observe that, for μ_- a. e. $\mathbf{y} \in \Gamma_-$, $\psi(\Phi(\mathbf{y}, t)) = 0$ for almost every (in the sense of the Lebesgue measure in \mathbb{R}) $0 < t < \tau_+(\mathbf{y})$. For such a $\mathbf{y} \in \Gamma_-$, continuous differentiability of $t \mapsto \psi(\Phi(\mathbf{y}, t))$ implies $\psi(\Phi(\mathbf{y}, t)) = 0$ for any $0 < t < \tau_+(\mathbf{y})$. This means, according to (A.1) that, for μ_- -a. e. $\mathbf{y} \in \Gamma_-$, $\Phi(\mathbf{y}, t) \notin E$ for any $0 < t < \tau_+(\mathbf{y})$. Since

$$\mu(E \cap \Omega_-) = \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \chi_E(\Phi(\mathbf{y}, t)) dt$$

we see that $\mu(E_-) = 0$.

- (2) In the same way, using Γ_+ instead of Γ_- , we show that $\mu(E \cap \Omega_+ \cap \Omega_{-\infty}) = 0$.
- (3) It remains to prove that $\mu(E_{\infty}) = 0$. In accordance with (A.1), we define for, any $n \in \mathbb{N}$,

$$E_n := \left\{ \mathbf{x} \in E_{\infty}; \delta_{\mathbf{x}} \geq 1/n \right\} = \left\{ \mathbf{x} \in E_{\infty}; \psi(\Phi(\mathbf{x}, t)) \neq 0, \quad \forall 0 < |t| < 1/n \right\}.$$

According to Assumption 1, it is easy to see that $\mu(E_n) = 0$ for any $n \in \mathbb{N}$ since $\psi(\mathbf{x}) = 0$ for μ -a.e. $\mathbf{x} \in \Omega$. Moreover, $E_1 \subset E_2 \subset \dots \subset E_n \subset E_{n+1} \subset \dots$, and

$$\bigcap_{n=1}^{\infty} (E_{\infty} \setminus E_n) = \emptyset.$$

Since we assumed $\mu(E) < \infty$, we have $\mu(E_\infty \setminus E_1) < \infty$ and $\lim_{n \rightarrow \infty} \mu(E_\infty \setminus E_n) = 0$. Writing $E_\infty = E_n \cup (E_\infty \setminus E_n)$, we see that $\mu(E_\infty) = 0$.

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