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SEMIGROUPS FOR GENERAL TRANSPORT EQUATIONS WITH ABSTRACT BOUNDARY CONDITIONS

L. ARLOTTI, J. BANASIAK & B. LODS

ABSTRACT. We investigate C_0 -semigroup generation properties of the Vlasov equation with general boundary conditions modeled by an abstract boundary operator H . For multiplicative boundary conditions we adapt techniques from [18] and in the case of conservative boundary conditions we show that there is an extension \mathcal{A} of the free streaming operator \mathcal{T}_H which generates a C_0 -semigroup $(V_H(t))_{t \geq 0}$ in L^1 . Furthermore, following the ideas of [6], we precisely describe its domain and provide necessary and sufficient conditions ensuring that $(V_H(t))_{t \geq 0}$ is stochastic.

CONTENTS

1. Introduction	1
2. Integration along the characteristics	3
2.1. Characteristic curves	3
2.2. Integration along characteristics	6
3. Trace operators and basic existence results	11
3.1. The maximal transport operator and trace results	11
3.2. Basic existence results	13
3.3. Transport equations with abstract boundary conditions	20
4. Multiplicative boundary conditions	22
5. Conservative boundary conditions	26
5.1. Characterization of $\mathcal{D}(\mathcal{A})$	27
5.2. Stochasticity of $(V_H(t))_{t \geq 0}$	29
Appendix A: Weak and strong definitions of the transport operator	32
References	43

1. INTRODUCTION

We are concerned in this paper with the well-posedness (in the sense of semigroup) of the general transport equation

$$(1.1a) \quad \partial_t f(\mathbf{x}, t) + \mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, t) = 0 \quad (\mathbf{x} \in \Omega, t > 0),$$

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supplemented by the abstract boundary condition

$$(1.1b) \quad f|_{\Gamma_-}(\mathbf{y}, t) = H(f|_{\Gamma_+})(\mathbf{y}, t), \quad (\mathbf{y} \in \Gamma_-, t > 0),$$

and the initial condition

$$(1.1c) \quad f(\mathbf{x}, 0) = f_0(\mathbf{x}), \quad (\mathbf{x} \in \Omega).$$

Here Ω is a smooth open subset of \mathbb{R}^N , Γ_{\pm} are suitable boundaries of the phase space and the boundary operator H is a linear *bounded* operator between trace spaces L^1_{\pm} corresponding to the boundaries Γ_{\pm} (see Section 2 for details). One of the novelty of our approach is that we assume \mathbb{R}^N to be endowed with a general positive Radon measure μ . The transport coefficient \mathcal{F} is a *time independent* vector field $\mathcal{F} : \Omega \rightarrow \mathbb{R}^N$ satisfying the following general assumptions:

Assumption 1. $\mathcal{F} : \Omega \rightarrow \mathbb{R}^N$ is Lipschitz-continuous with Lipschitz constant $\kappa > 0$, i.e.

$$|\mathcal{F}(\mathbf{x}_1) - \mathcal{F}(\mathbf{x}_2)| \leq \kappa |\mathbf{x}_1 - \mathbf{x}_2| \quad \text{for any } \mathbf{x}_1, \mathbf{x}_2 \in \Omega.$$

Assumption 2. The field \mathcal{F} is divergence-free with respect to μ in the sense that

$$\int_{\Omega} \mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x}) d\mu(\mathbf{x}) = 0$$

for any continuously differentiable function f with compact support on Ω .

A typical example of such a transport equation is the so-called Vlasov equation for which:

- i) The phase space Ω is given by the cylindrical domain $\Omega = \mathcal{D} \times \mathbb{R}^3 \subset \mathbb{R}^6$ where \mathcal{D} is a smooth open subset of \mathbb{R}^3 , referred to as the *position space*, while the so-called *velocity space* is here given by \mathbb{R}^3 . The measure μ is given by $d\mu(\mathbf{x}) = dx d\beta(v)$ where β is a suitable Radon measure on \mathbb{R}^3 (Lebesgue measure over \mathbb{R}^3 for continuous models or combination of Lebesgue measures over suitable spheres for the multigroup model).
- ii) For any $\mathbf{x} = (x, v) \in \mathcal{D} \times \mathbb{R}^3$,

$$(1.2) \quad \mathcal{F}(\mathbf{x}) = (v, \mathbf{F}(x, v)) \in \mathbb{R}^6$$

where $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$ is a time independent force field over $\mathcal{D} \times \mathbb{R}^3$ such that Assumptions 1 and 2 are fulfilled.

The existence of solution to the transport equation (1.1a) is a classical matter when considering the whole space $\Omega = \mathbb{R}^N$. In particular, the concept of renormalized solutions allows to consider irregular transport coefficient $\mathcal{F}(\cdot)$ (see [11] and the recent contributions [2, 17]) which is of particular relevance in fluid mechanics.

On the other hand, there are few results addressing the initial-boundary value problem (1.1), possibly due to difficulties created by the boundary conditions (1.1b). We mention here the seminal works by C. Bardos [9], and by R. Beals and V. Protopopescu [10] (see also [14]). Let us also mention more recent contributions [19] which also includes time-dependent transport coefficient, and [6, 18] dealing with the force-free ($\mathbf{F} \equiv 0$) Vlasov equation (1.2).

For $\mathbf{F} \neq 0$, the method of Beals and Protopopescu [10] provides the existence and a very precise description of a C_0 -semigroup governing (1.1) for $\|H\| < 1$ while, for nonnegative boundary operator H with $\|H\| = 1$, it ensures the existence of a C_0 -semigroup related to (1.1) without describing its generator. The method of [10] leaves totally open the multiplicative case $\|H\| > 1$.

We also mention that the existing theories introduce restrictive assumptions on the characteristics of the equation. For instance, fields with 'too many' periodic trajectories create serious difficulties. They are however covered in a natural way by the theory presented here.

On the other hand, in the force-free case $\mathbf{F} = 0$, the case of conservative boundary conditions $\|H\| = 1$ has been solved in [6], while the multiplicative case has been addressed in [18]. The results of [6, 18] are based upon a relatively simple representation, inspired by the fundamental work of [20], of the resolvent of the free-streaming operator \mathcal{T}_H (whose domain includes the boundary conditions (1.1b)) as a strongly convergent series.

The main objective of this work is to generalize the results of [6] and [18] to the general case $\mathbf{F} \neq 0$ and for a general Radon measure μ . Here again, the key ingredient is the derivation of a suitable representation of the resolvent of the free-streaming operator \mathcal{T}_H , see Theorem 3.6 and Proposition 3.10. We point out that the proof of Theorem 3.6 relies on a generalization of the result from [10, 14] which allows to compute integrals over Ω via integration along the integral curves of $\mathcal{F}(\cdot)$ coming from the boundary $\partial\Omega$, and which is free from some restrictive assumptions of *op. it.* In particular, we present a new proof of the Green formula which clarifies some points of the proofs in [10, 14].

The organization of the paper is as follows. In the following section (Section 2), we introduce the main tools used throughout the paper and present the aforementioned new results concerning *integration over the characteristic curves* of \mathcal{F} . Section 3 is dealing with various preliminary results. In particular, the question of the existence of trace results is addressed in Section 3.1 where a *totally new construction of the measures* over the incoming and outgoing parts Γ_{\pm} of $\partial\Omega$ is provided by Theorem 3.3. In Section 3.2, we establish some basic existence results, mainly pertaining to *stationary versions* of (1.1) and, as a consequence, we derive a new proof of *Green formula*, generalizing that of [10, 14] and Section 3.3 is concerned with the *setting of the problem* and with the representation of the resolvent of the free-streaming operator \mathcal{T}_H as a strongly convergent series. Note that the whole results of Section 3 clarify some obscure points of [10, 14] providing new proofs to some of the results of *op. cit.* In Section 4, we are dealing with the transport equation (1.1) for *multiplicative boundary conditions*, generalizing the results of [18]. Finally, in Section 5, we consider the delicate question of *conservative boundary conditions*. We employ a strategy already used in [6], borrowing some tools to the so-called additive perturbation theory of substochastic semigroups [8].

Notations: We shall adopt the following notations throughout this paper: for any unbounded operator A , the domain of A shall be denoted by $\mathcal{D}(A)$ whereas $\sigma_p(A)$, $\sigma_r(A)$, $\sigma_c(A)$ and $\varrho(A)$ will denote respectively the point spectrum, the residual spectrum, the continuous spectrum and the resolvent set of A . For $\lambda \in \varrho(A)$, the resolvent of A will be denoted by either $\mathcal{R}(\lambda, A)$ or $(\lambda - A)^{-1}$. For any Banach spaces X and Y , $\mathcal{B}(X, Y)$ denotes the space of bounded linear operators from X to Y whereas $\mathcal{B}(X) = \mathcal{B}(X, X)$.

2. INTEGRATION ALONG THE CHARACTERISTICS

2.1. Characteristic curves. Since \mathcal{F} is Lipschitz over Ω (with constant $\kappa > 0$), it is known from Kurzbraun's extension theorem [12, p. 201], that \mathcal{F} can be extended as a Lipschitz function (with

same Lipschitz constant $\kappa > 0$) over the whole space \mathbb{R}^N . We shall still denote this extension by \mathcal{F} . Note that, in general, the extended field \mathcal{F} is not divergence-free in \mathbb{R}^N . A crucial role in our study is played by the characteristic curves associated to the (extended) field \mathcal{F} . Precisely, for any $\mathbf{x} \in \mathbb{R}^N$ and $t \in \mathbb{R}$, consider the initial-value problem

$$(2.1) \quad \begin{cases} \frac{d}{ds} \mathbf{X}(s) = \mathcal{F}(\mathbf{X}(s)), & (s \in \mathbb{R}), \\ \mathbf{X}(t) = \mathbf{x}. \end{cases}$$

Since \mathcal{F} is Lipschitz-continuous on \mathbb{R}^N , Eq. (2.1) has a unique *global in time* solution and this allows to define the flow-mapping $\Theta : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^N$, such that, for $(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R}$, the mapping:

$$\mathbf{X}(\cdot) : s \in \mathbb{R} \mapsto \Theta(\mathbf{x}, t, s)$$

is the only solution of Eq. (2.1). Being concerned with solutions to the transport equation (1.1) in the region Ω , we have to introduce the definition of stay times of the characteristic curves in Ω :

Definition 2.1. *For any $\mathbf{x} \in \Omega$, define*

$$\tau_{\pm}(\mathbf{x}) = \inf\{s > 0; \Theta(\mathbf{x}, 0, \pm s) \notin \Omega\},$$

with the convention that $\inf \emptyset = \infty$, and set $\tau(\mathbf{x}) = \tau_+(\mathbf{x}) + \tau_-(\mathbf{x})$.

In other words, given $\mathbf{x} \in \Omega$, $(-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$ is the maximal interval for which $\Theta(\mathbf{x}, 0, s)$ lies in Ω for any $s \in I_{\mathbf{x}}$ and $\tau(\mathbf{x})$ is the length of the interval $I_{\mathbf{x}}$. Notice that $0 \leq \tau_{\pm}(\mathbf{x}) \leq \infty$. Thus, for any $t \in \mathbb{R}$, the function Θ is defined on the set

$$\left\{ (\mathbf{x}, t, s); \mathbf{x} \in \Omega, t \in \mathbb{R}, s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x})) \right\}.$$

Note that here we **do not** assume that the length of the interval $I_{\mathbf{x}} = (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$ is **finite**. In particular, $I_{\mathbf{x}} = \mathbb{R}$ for any stationary point \mathbf{x} of \mathcal{F} , i.e. $\mathcal{F}(\mathbf{x}) = 0$. If $\tau(\mathbf{x})$ is finite, then the function $\mathbf{X} : s \in I_{\mathbf{x}} \mapsto \Theta(\mathbf{x}, 0, s)$ is bounded since \mathcal{F} is Lipschitz-continuous. Moreover, still by virtue of the Lipschitz-continuity of \mathcal{F} , the only case when $\tau_{\pm}(\mathbf{x})$ is finite is when $\Theta(\mathbf{x}, 0, \pm s)$ reaches the boundary $\partial\Omega$ so that $\Theta(\mathbf{x}, 0, \pm\tau_{\pm}(\mathbf{x})) \in \partial\Omega$. We note that, since \mathcal{F} is Lipschitz around each point of $\partial\Omega$, the points of the set $\{\mathbf{y} \in \partial\Omega; \mathcal{F}(\mathbf{y}) = 0\}$ (introduced in [10, 14]) are equilibrium points of the extended field and cannot be reached in finite time.

Remark 2.2. *We emphasize that periodic trajectories which do not meet the boundaries have $\tau_{\pm} = \infty$ and thus are treated as infinite though geometrically they are bounded.*

We finally mention that it is not difficult to prove that the mappings $\tau_{\pm} : \Omega \rightarrow \mathbb{R}^+$ are lower semicontinuous and therefore measurable, see e.g., [8, p. 301]

The flow $\Theta(\mathbf{x}, t, s)$ defines, at each instant t , a mapping of the phase space Ω into \mathbb{R}^N . Through this mapping, to each point \mathbf{x} there corresponds the point $\mathbf{x}_{s,t} = \Theta(\mathbf{x}, t, s)$ reached at time s by a point which was in \mathbf{x} at the "initial" time t . This mapping is *one-to-one* and *measure-preserving* (Liouville's Theorem). More precisely, one can check easily that the flow Θ , defined on its maximal domain, has the following properties:

Proposition 2.3. *Let $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}$ be fixed. Then,*

- (i) $\Theta(\mathbf{x}, t, t) = \mathbf{x}$.
- (ii) $\Theta(\Theta(\mathbf{x}, t, s_1), s_1, s_2) = \Theta(\mathbf{x}, t, s_2)$, $\forall s_1, s_2 \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
- (iii) $\Theta(\mathbf{x}, t, s) = \Theta(\mathbf{x}, t - s, 0) = \Theta(\mathbf{x}, 0, s - t)$, $\forall s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
- (iv) $|\Theta(\mathbf{x}_1, t, s) - \Theta(\mathbf{x}_2, t, s)| \leq \exp(\kappa|t - s|)|\mathbf{x}_1 - \mathbf{x}_2|$ for any $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$, $s - t \in I_{\mathbf{x}_1} \cap I_{\mathbf{x}_2}$.
- (v) Let $t \in \mathbb{R}$. For any measurable set $A \subset \Omega$ such that $A_t := \{\Theta(\mathbf{x}, t, 0), \mathbf{x} \in A\} \subset \Omega$, one has $\mu(A) = \mu(A_t)$ (**Liouville's Theorem**).

Proof. We only prove Liouville's Theorem, the other four properties being classical (see [1]). Actually, Liouville's Theorem is a classical result if μ is absolutely continuous with respect to the Lebesgue measure. We have not been able to find a complete reference for general Radon measures and hence we provide it below. Since μ is regular, one may restrict ourselves to the case when the measurable set A is actually *compact*. Let us fix thus a compact set $K \subset \Omega$. We may assume without loss of generality that there exists $t_0 > 0$ such

$$\inf\{\mathbf{x} \in K; \tau_{\pm}(\mathbf{x})\} > t_0.$$

We define $K_t := \{\Theta(\mathbf{x}, 0, t), \mathbf{x} \in K\}$ for any $|t| \leq t_0$. From the continuity of the flow $\Theta(\cdot, 0, t)$, one sees that, for any $0 < \delta < t_0$ there exists a compact set $K_\delta \subset \Omega$ such that $K_t \subset K_\delta$, for all $|t| \leq \delta$. We set

$$\bar{\mu} = \chi_{K_\delta} \mu$$

so that $\bar{\mu}$ is a finite measure concentrated on a compact set of \mathbb{R}^N . We denote by $\bar{\mu}_t$ the image of $\bar{\mu}$ through the transformation $\mathbf{x} \mapsto \Theta(\mathbf{x}, t, 0)$, i.e.

$$\bar{\mu}_t = \Theta(\cdot, t, 0)_{\#} \bar{\mu},$$

where we adopt the notations of [3] for the push-forward measure. Since \mathcal{F} is Lipschitz, we have

$$\int_{\mathbb{R}^N} |\mathcal{F}(\mathbf{x})| d\bar{\mu}_t(\mathbf{x}) < \infty.$$

This proves that both \mathcal{F} and $\bar{\mu}_t$ are satisfying the assumptions of [3, Lemma 8.1.6, Prop. 8.1.7 & 8.1.8] so that $t \in (-\delta, \delta) \mapsto \bar{\mu}_t$ is the *unique distributional solution* to the measure-valued continuity equation

$$(2.2) \quad \partial_t \bar{\mu}_t + \nabla \cdot (\mathcal{F} \bar{\mu}_t) = 0, \quad \bar{\mu}_{t=0} = \bar{\mu},$$

in the sense that,

$$\int_{-\delta}^{\delta} dt \int_{\Omega} \left(\partial_t \varphi(\mathbf{x}, t) + \mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \varphi(\mathbf{x}, t) \right) d\bar{\mu}_t(\mathbf{x}) = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N \times (-\delta, \delta))$$

and

$$\lim_{t \rightarrow 0} \int_{\Omega} \psi(\mathbf{x}) d\bar{\mu}_t(\mathbf{x}) = \int_{\Omega} \psi(\mathbf{x}) d\bar{\mu}(\mathbf{x}), \quad \forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}^N).$$

Moreover, since \mathcal{F} is divergence free with respect to μ , it is easy to see that the constant map $t \in (-\delta, \delta) \mapsto \bar{\mu}$ is also a distributional solution of (2.2) so that

$$(2.3) \quad \bar{\mu}_t = \bar{\mu}, \quad \forall t \in (-\delta, \delta).$$

Since $\bar{\mu}_t(K) = \mu(K_t)$ and $\bar{\mu}(K) = \mu(K)$, we get $\mu(K_t) = \mu(K)$ for any $|t| < \delta$. Letting $\delta \rightarrow t_0$,

$$\mu(K_t) = \mu(K), \quad \forall t \in (-t_0, t_0)$$

and the proof is achieved. \square

An important consequence of (iii) above is that $\Theta(\mathbf{x}, 0, s) = \Theta(\mathbf{x}, -s, 0)$ for any $\mathbf{x} \in \Omega$, $0 \leq s \leq \tau_+(\mathbf{x})$. Therefore, from now on, to shorten notations we shall denote

$$\Phi(\mathbf{x}, t) = \Theta(\mathbf{x}, 0, t), \quad \forall t \in \mathbb{R},$$

so that $\Phi(\mathbf{x}, -t) = \Theta(\mathbf{x}, t, 0)$, $t \in \mathbb{R}$. We define the incoming and outgoing part of the boundary $\partial\Omega$ through the flow Φ :

Definition 2.4. *The incoming Γ_- and the outgoing Γ_+ parts of the boundary $\partial\Omega$ are defined by:*

$$(2.4) \quad \Gamma_{\pm} := \{ \mathbf{y} \in \partial\Omega; \exists \mathbf{x} \in \Omega, \tau_{\pm}(\mathbf{x}) < \infty \text{ and } \mathbf{y} = \Phi(\mathbf{x}, \pm\tau_{\pm}(\mathbf{x})) \}.$$

Properties of Φ and of τ_{\pm} imply that Γ_{\pm} are Borel sets. It is possible to extend the definition of τ_{\pm} to Γ_{\pm} as follows. If $\mathbf{x} \in \Gamma_-$ then we put $\tau_-(\mathbf{x}) = 0$ and denote $\tau_+(\mathbf{x})$ the length of the integral curve having \mathbf{x} as its left end-point; similarly if $\mathbf{x} \in \Gamma_+$ then we put $\tau_+(\mathbf{x}) = 0$ and denote $\tau_-(\mathbf{x})$ the length of the integral curve having \mathbf{x} as its right end-point. Note that with such a definition, it is not difficult to prove τ_{\pm} are measurable over $\Omega \cup \Gamma_- \cup \Gamma_+$.

The main aim of the present discussion is to represent Ω as a collection of characteristics running between points of Γ_- and Γ_+ so that the integral over Ω can be split into integrals over Γ_- (or Γ_+) and along the characteristics. However, we cannot do this in a precise way now since, in general, the sets Γ_+ and Γ_- do not provide a partition of $\partial\Omega$ since there still may be too many characteristics which extend to infinity on either side. Since we have not assumed Ω to be bounded, Γ_- or Γ_+ may be empty and also we may have characteristics running from $-\infty$ to $+\infty$ such as periodic ones. Thus, in general characteristics starting from Γ_- or ending at Γ_+ would not fill the whole Ω and, to proceed, we have to construct an auxiliary set by extending Ω into the time domain and use the approach of [10] which is explained below.

2.2. Integration along characteristics. For any $0 < T < \infty$, we define the domain

$$\Omega_T = \Omega \times (0, T)$$

and the measure $d\mu_T = d\mu \otimes dt$ on Ω_T . Consider the vector field over Ω_T :

$$Y = \frac{\partial}{\partial t} + \mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} = \mathcal{A}(\xi) \cdot \nabla_{\xi}$$

where $\mathcal{A}(\xi) = (\mathcal{F}(\mathbf{x}), 1)$ for any $\xi = (\mathbf{x}, t)$. We can define the characteristic curves of \mathcal{A} as the solution $\xi(s) = (\mathbf{X}(s), \theta(s))$ to the system $\frac{d}{ds}\xi(s) = \mathcal{A}(\xi(s))$, i.e.

$$\frac{d}{ds}\mathbf{X}(s) = \mathcal{F}(\mathbf{X}(s)), \quad \frac{d}{ds}\theta(s) = 1, \quad (s \in \mathbb{R}),$$

with

$$\mathbf{X}(0) = \mathbf{x}, \quad \theta(0) = t.$$

It is clear that the solution $\xi(s)$ to such a system is given by

$$\mathbf{X}(s) = \Phi(\mathbf{x}, s), \quad \theta(s) = s + t,$$

and we can define the flow of solution $\Psi(\xi, s) = (\Phi(\mathbf{x}, s), s+t)$ associated to \mathcal{A} and the existence times of the characteristic curves of Y are defined, for any $\xi = (\mathbf{x}, t) \in \Omega_T$, as

$$\ell_{\pm}(\xi) = \inf\{s > 0, (\Phi(\mathbf{x}, \pm s), \pm s + t) \notin \Omega_T\}.$$

The flow $\Psi(\cdot, \cdot)$ enjoys, *mutatis mutandis*, the properties listed in Proposition 2.3. Moreover, since \mathcal{A} is clearly Lipschitz continuous on $\overline{\Omega_T}$, no characteristic of Y can escape to infinity in finite time. In other words, all characteristic curves of Y now have finite lengths. Indeed, if $\Phi(\mathbf{x}, \pm s)$ does not reach $\partial\Omega$, then the characteristic curve $\Psi(\xi, \pm s)$ enters or leaves Ω_T through the bottom $\Omega \times \{0\}$, or through the top $\Omega \times \{T\}$ of it. Precisely, it is easy to verify that for $\xi = (\mathbf{x}, t) \in \Omega_T$ we have

$$\ell_+(\xi) = \tau_+(\mathbf{x}) \wedge (T - t) \quad \text{and} \quad \ell_-(\xi) = \tau_-(\mathbf{x}) \wedge t,$$

where \wedge denotes minimum. This clearly implies $\sup\{\ell_{\pm}(\xi) ; \xi \in \Omega_T\} \leq T$. Define now

$$\Sigma_{\pm, T} = \{\zeta \in \partial\Omega_T ; \exists \xi \in \Omega_T \text{ such that } \zeta = \Psi(\xi, \pm \ell_{\pm}(\xi))\}.$$

The definition of $\Sigma_{\pm, T}$ is analogous to Γ_{\pm} with the understanding that the characteristic curves now correspond to the vector field \mathcal{A} . In other words, $\Sigma_{-, T}$ (resp. $\Sigma_{+, T}$) is the subset of $\partial\Omega_T$ consisting of all left (resp. right) limits of characteristic curves of \mathcal{A} in Ω_T whereas Γ_- (resp. Γ_+) is the subset of $\partial\Omega$ consisting of all left (resp. right) limits of characteristic curves of \mathcal{F} in Ω . The main difference (and the interest of such a lifting to Ω_T) is the fact that *each characteristic curve of \mathcal{A} does reach the boundaries $\Sigma_{\pm, T}$ in finite time*. The above formulae allow us to extend functions ℓ_{\pm} to $\Sigma_{\pm, T}$ in the same way as we extended the functions τ_{\pm} to Γ_{\pm} . With these considerations, we can represent, up to a set of zero measure, the phase space Ω_T as

$$(2.5) \quad \begin{aligned} \Omega_T &= \{\Psi(\xi, s) ; \xi \in \Sigma_{-, T}, 0 < s < \ell_+(\xi)\} \\ &= \{\Psi(\xi, -s) ; \xi \in \Sigma_{+, T}, 0 < s < \ell_-(\xi)\}. \end{aligned}$$

With this realization one can prove the following:

Proposition 2.5. *Let $T > 0$ be fixed. There are unique positive Borel measures $d\nu_{\pm}$ on $\Sigma_{\pm, T}$ such that $d\mu_T = d\nu_+ \otimes ds = d\nu_- \otimes ds$.*

Proof. For any $\delta > 0$, define \mathcal{F}_{δ} as the set of all bounded Borel subsets E of $\Sigma_{-, T}$ such that $\ell_+(\xi) > \delta$ for any $\xi \in E$. Let us now fix $E \in \mathcal{F}_{\delta}$. For all $0 < \sigma \leq \delta$ put

$$E_{\sigma} = \{\Psi(\xi, s) ; \xi \in E, 0 < s \leq \sigma\}.$$

Clearly E_{σ} is a measurable subset of Ω_T . Define the mapping $h : \sigma \in (0, \delta] \mapsto h(\sigma) = \mu_T(E_{\sigma})$ with $h(0) = 0$. If σ_1 and σ_2 are two positive numbers such that $\sigma_1 + \sigma_2 \leq \delta$, then

$$E_{\sigma_1 + \sigma_2} \setminus E_{\sigma_1} = \{\Psi(\xi, s) ; \xi \in E, \sigma_1 < s \leq \sigma_1 + \sigma_2\} = \{\Psi(\eta, \sigma_1) ; \eta \in E_{\sigma_2}\}.$$

The properties of the flow Ψ (see Proposition 2.3) ensure that the mapping $\eta \mapsto \Psi(\eta, \sigma_1)$ is one-to-one and measure preserving, so that

$$\mu_T(E_{\sigma_1 + \sigma_2} \setminus E_{\sigma_1}) = \mu_T(E_{\sigma_2}) = h(\sigma_2).$$

Since $E_{\sigma_1+\sigma_2} = E_{\sigma_1} \cup (E_{\sigma_1+\sigma_2} \setminus E_{\sigma_1})$, we immediately obtain

$$(2.6) \quad h(\sigma_1 + \sigma_2) = h(\sigma_1) + h(\sigma_2) \quad \text{for any } \sigma_1, \sigma_2 > 0 \text{ with } \sigma_1 + \sigma_2 \leq \delta.$$

This is the well-known Cauchy equation, though defined only on an interval of the real line. It can be solved in a standard way using non negativity instead of continuity, yielding:

$$h(\sigma) = c_E \sigma \quad \text{for any } 0 < \sigma \leq \delta$$

where $c_E = h(\delta)/\delta$. We define $\nu_-(E) = c_E$. It is not difficult to see that, with the above procedure, the mapping $\nu_-(\cdot)$ defines a positive measure on the ring $\mathcal{F} = \bigcup_{\delta>0} \mathcal{F}_\delta$ of all the Borel subsets of $\Sigma_{-,T}$ on which the function $\ell_+(\xi)$ is bounded away from 0. Such a measure ν_- can be uniquely extended to the σ -algebra of the Borel subsets of $\Sigma_{-,T}$ (see e.g. [15, Theorem A, p. 54]). Consider now a Borel subset E of $\Sigma_{-,T}$ and a Borel subset I of \mathbb{R}^+ , such that for all $\xi \in E$ and $s \in I$ we have $0 < s < \ell_+(\xi)$. Then

$$E \times I = \{\Psi(\xi, s); \xi \in E, s \in I\} \subset \Omega_T.$$

Thanks to the definition of $\nu_-(\cdot)$, we can state that $\mu_T(E \times I) = \nu_-(E)\text{meas}(I)$ where $\text{meas}(I)$ denotes the Lebesgue measure of $I \subset \mathbb{R}$. This shows that $d\mu_T = d\nu_- \otimes ds$. Similarly we can define a measure ν_+ on $\Sigma_{+,T}$ and prove that $d\mu_T = d\nu_+ \otimes ds$. The uniqueness of the measures $d\nu_\pm$ is then obvious. \square

Remark 2.6. *Note that the above construction of the Borel measures $d\nu_\pm$ differs from that of [14, Lemmas XI.3.1 & 3.2], [10, Propositions 7 & 8] which merely apply when μ is absolutely continuous with respect to the Lebesgue measure. Our construction is much more general and can also be generalized to the case of a non-divergence force field \mathcal{F} [7].*

Next, by the cylindrical structure of Ω_T , and the representation of $\Sigma_{\pm,T}$ as

$$\Sigma_{-,T} = (\Gamma_- \times (0, T)) \cup \Omega \times \{0\} \quad \text{and} \quad \Sigma_{+,T} = (\Gamma_+ \times (0, T)) \cup \Omega \times \{T\},$$

the measures $d\nu_\pm$ over $\Gamma_\pm \times (0, T)$ can be written as $d\nu_\pm = d\mu_\pm \otimes dt$ where $d\mu_\pm$ are Borel measures on Γ_\pm . This leads to the following

Lemma 2.7. *There are unique positive Borel measures $d\mu_\pm$ on Γ_\pm such that, for any $f \in L^1(\Omega_T, d\mu_T)$*

$$(2.7) \quad \begin{aligned} \int_{\Omega_T} f(\mathbf{x}, t) d\mu_T(\mathbf{x}, t) &= \int_0^T dt \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y}) \wedge t} f(\Phi(\mathbf{y}, -s), t - s) ds \\ &\quad + \int_{\Omega} d\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x}) \wedge T} f(\Phi(\mathbf{x}, -s), T - s) ds, \end{aligned}$$

and

$$(2.8) \quad \begin{aligned} \int_{\Omega_T} f(\mathbf{x}, t) d\mu_T(\mathbf{x}, t) &= \int_0^T dt \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y}) \wedge (T-t)} f(\Phi(\mathbf{y}, s), t + s) ds \\ &\quad + \int_{\Omega} d\mu(\mathbf{x}) \int_0^{\tau_+(\mathbf{x}) \wedge T} f(\Phi(\mathbf{x}, s), s) ds. \end{aligned}$$

The above fundamental result allows to compute integral over the cylindrical phase-space Ω_T through integration along the characteristic curves. Let us now generalize it to the phase space Ω . Here the main difficulty stems from the fact that the characteristic curves of the vector field \mathcal{F} are no longer assumed to be of finite length. In order to extend Lemma 2.7 to possibly infinite existence times, first we prove the following:

Lemma 2.8. *Let $T > 0$ be fixed. Then, $\tau_+(\mathbf{x}) < T$ for any $\mathbf{x} \in \Omega$ if and only if $\tau_-(\mathbf{x}) < T$ for any $\mathbf{x} \in \Omega$.*

Proof. Assume that $T > \tau_+(\mathbf{x})$ for any $\mathbf{x} \in \Omega$ and that there is $\mathbf{z} \in \Omega$ such that $\tau_-(\mathbf{z}) \geq T$. One can assume without loss of generality that $\tau_-(\mathbf{z}) > T$. Indeed, if $\tau_-(\mathbf{z}) = T$, since Ω is open, the orbit passing through \mathbf{z} can be continued beyond ensuring the existence of $\mathbf{z}' \in \Omega$ with $\tau_-(\mathbf{z}') > T$. Now, if $\tau_-(\mathbf{z}) > T$, for any $T < t < \tau_-(\mathbf{z})$, $\mathbf{y} = \Phi(\mathbf{z}, -t) \in \Omega$ and $\Phi(\mathbf{y}, s) = \Phi(\mathbf{z}, t - s) \in \Omega$ for all $0 < s < t$. This leads to the contradiction that $\tau_+(\mathbf{z}) \geq t > T$. We proceed in the same way for the converse implication. \square

The above lemma allows to prove a representation formula for integral of the type $\int_{\Omega} f \, d\mu$ in terms of integrals over Γ_{\pm} . Hereafter, the support of a measurable function f defined on Ω is defined as $\text{Supp}f = \Omega \setminus \omega$ where ω is the maximal open subset of Ω on which f vanishes $d\mu$ -almost everywhere.

Proposition 2.9. *Let $f \in L^1(\Omega, d\mu)$. Assume that there exists $\tau_0 > 0$ such that $\tau_{\pm}(\mathbf{x}) < \tau_0$ for any $\mathbf{x} \in \text{Supp}(f)$. Then,*

$$(2.9) \quad \begin{aligned} \int_{\Omega} f(\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds \\ &= \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, s)) ds. \end{aligned}$$

Proof. For any $T > \tau_0$, define the domain $\Omega_T = \Omega \times (0, T)$. Since $T < \infty$, it is clear that $f \in L^1(\Omega_T, d\mu dt)$ and, by (2.7), we get

$$\begin{aligned} T \int_{\Omega} f(\mathbf{x}) d\mu(\mathbf{x}) &= \int_0^T dt \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{t \wedge \tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds + \\ &\quad \int_{\Omega} d\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x})} f(\Phi(\mathbf{x}, -s)) ds. \end{aligned}$$

Since the formula is valid for any $T > \tau_0$, differentiating with respect to T leads to the first assertion. The second assertion is proved in the same way by using formula (2.8). \square

To drop the finiteness assumption on $\tau_{\pm}(\mathbf{x})$, first we introduce the sets

$$\Omega_{\pm} = \{\mathbf{x} \in \Omega; \tau_{\pm}(\mathbf{x}) < \infty\}, \quad \Omega_{\pm\infty} = \{\mathbf{x} \in \Omega; \tau_{\pm}(\mathbf{x}) = \infty\},$$

and

$$\Gamma_{\pm\infty} = \{\mathbf{y} \in \Gamma_{\pm}; \tau_{\mp}(\mathbf{y}) = \infty\}.$$

One gets

Proposition 2.10. *Let $f \in L^1(\Omega, d\mu)$. Then*

$$(2.10) \quad \int_{\Omega_{\pm}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_{\pm}} d\mu_{\pm}(\mathbf{y}) \int_0^{\tau_{\mp}(\mathbf{y})} f(\Phi(\mathbf{y}, \mp s)) ds,$$

and

$$(2.11) \quad \int_{\Omega_{\pm} \cap \Omega_{\mp\infty}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_{\pm\infty}} d\mu_{\pm}(\mathbf{y}) \int_0^{\infty} f(\Phi(\mathbf{y}, \mp s)) ds.$$

Proof. Assume first $f \geq 0$. Let us fix $T > 0$. It is clear that $\mathbf{x} \in \Omega$ satisfies $\tau_+(\mathbf{x}) < T$ if and only if $\mathbf{x} = \Phi(\mathbf{y}, s, 0)$, with $\mathbf{y} \in \Gamma_+$ and $0 < s < T \wedge \tau_-(\mathbf{y})$. Then, by Proposition 2.9,

$$\int_{\{\tau_+(\mathbf{x}) < T\}} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{T \wedge \tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds.$$

Since $f \geq 0$, the inner integral is increasing with T and, using the monotone convergence theorem, we let $T \rightarrow \infty$ to get

$$\int_{\Omega_+} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f(\Phi(\mathbf{y}, -s)) ds$$

which coincides with (2.10). We proceed in the same way integration on Γ_- and get the second part of (2.10). Next we consider the set

$$\Delta = \{\mathbf{x} \in \Omega; \mathbf{x} = \Phi(\mathbf{y}, -s), \mathbf{y} \in \Omega_{+\infty}, 0 < s < T\}.$$

Proposition 2.9 asserts that

$$\int_{\Delta} f(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega_{+\infty}} d\mu_+(\mathbf{y}) \int_0^T f(\Phi(\mathbf{y}, -s)) ds.$$

Letting again $T \rightarrow \infty$, we get (2.11). We extend the results to arbitrary f by linearity. \square

Finally, with the following, we show that it is possible to transfer integrals over Γ_- to Γ_+ :

Proposition 2.11. *For any $\psi \in L^1(\Gamma_-, d\mu_-)$,*

$$(2.12) \quad \int_{\Gamma_- \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) d\mu_-(\mathbf{y}) = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} \psi(\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))) d\mu_+(\mathbf{z}).$$

Proof. For any $\epsilon > 0$, let f_{ϵ} be the function defined on $\Omega_+ \cap \Omega_-$ by

$$\psi_{\epsilon}(\mathbf{x}) = \begin{cases} \frac{\psi(\Phi(\mathbf{x}, -\tau_-(\mathbf{x})))}{\tau_+(\mathbf{x}) + \tau_-(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) > \epsilon, \\ 0 & \text{else.} \end{cases}$$

Since $\psi_{\epsilon} \in L^1(\Omega_+ \cap \Omega_-, d\mu)$, Eqs. (2.10) and (2.11) give

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} \psi_{\epsilon}(\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\mathbf{y}) \frac{ds}{\tau_+(\mathbf{y})} \\ &= \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) d\mu_-(\mathbf{y}). \end{aligned}$$

In the same way,

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} \psi_\epsilon(\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \psi(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) \frac{ds}{\tau_-(\mathbf{y})} \\ &= \int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} \psi(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) d\mu_-(\mathbf{y}), \end{aligned}$$

which leads to

$$\int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} \psi(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) d\mu_+(\mathbf{y}) = \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) d\mu_-(\mathbf{y})$$

for any $\epsilon > 0$. Passing to the limit as $\epsilon \rightarrow 0$ we get the conclusion. \square

3. TRACE OPERATORS AND BASIC EXISTENCE RESULTS

3.1. The maximal transport operator and trace results. The results of the previous section allow us to define the (maximal) transport \mathcal{T}_{\max} as the weak derivative along characteristic curves. To be precise, we define the set of test-functions \mathfrak{Y} as the collection of all bounded measurable functions ψ with compact support in Ω that, for any $\mathbf{x} \in \Omega$, the mapping

$$s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \mapsto \psi(\Phi(\mathbf{x}, s))$$

is continuously differentiable with

$$\left. \frac{d}{ds} \psi(\Phi(\cdot, s)) \right|_{s=0} \in L^1(\Omega, d\mu).$$

Then, we adopt the following definition:

Definition 3.1. We define the domain $\mathcal{D}(\mathcal{T}_{\max})$ of the transport operator \mathcal{T}_{\max} as the set of all $f \in L^1(\Omega, d\mu)$ for which there exists $g \in L^1(\Omega, d\mu)$ such that

$$\int_{\Omega} g(\mathbf{x}) \psi(\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega} f(\mathbf{x}) \left. \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \right|_{s=0} d\mu(\mathbf{x}), \quad \forall \psi \in \mathfrak{Y}.$$

In this case, $g =: -\mathcal{T}_{\max} f$.

Let us recall that, if f_1 and f_2 are two functions defined over Ω , we say that f_2 is a *representative* of f_1 if $\mu\{\mathbf{x} \in \Omega; f_1(\mathbf{x}) \neq f_2(\mathbf{x})\} = 0$, i.e. when $f_1(\mathbf{x}) = f_2(\mathbf{x})$ for μ -almost every $\mathbf{x} \in \Omega$. The following proposition can be adopted as an alternative definition of $(\mathcal{T}_{\max}, \mathcal{D}(\mathcal{T}_{\max}))$ (see Appendix A):

Proposition 3.2. The domain $\mathcal{D}(\mathcal{T}_{\max})$ is the set of all $f \in L^1(\Omega, d\mu)$ which admits a representative f^* such that:

(1) there exists $g \in L^1(\Omega, d\mu)$ such that for any $\mathbf{x} \in \Omega$ and any $-\tau_-(\mathbf{x}) < t_1 \leq t_2 < \tau_+(\mathbf{x})$

$$f^*(\Phi(\mathbf{x}, t_2)) - f^*(\Phi(\mathbf{x}, t_1)) = \int_{t_1}^{t_2} g(\Phi(\mathbf{x}, s)) ds;$$

(2) for μ -almost every $\mathbf{x} \in \Omega$, one has

$$\lim_{h \rightarrow 0} h^{-1} (f^*(\Phi(\mathbf{x}, h)) - f^*(\mathbf{x})) = g(\mathbf{x}).$$

In such a case, one defines $\mathcal{T}_{\max} f := -g \in L^1(\Omega, d\mu)$.

Note that, by virtue of (1), for any $\mathbf{x} \in \Omega$, the mapping $s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \mapsto f^*(\Phi(\mathbf{x}, s))$ is *absolutely continuous*. However, property (2) is not a consequence of (1) as an absolutely continuous function may fail to be differentiable at $s = 0$, which is the essence of (2). It turns out that $\mathcal{T}_{\max} f$ does not depend on the choice of the representative f^* . One can state the following trace result:

Proposition 3.3. *Let $f \in \mathcal{D}(\mathcal{T}_{\max})$. Then there is a representative f^* of f such that the limit*

$$\lim_{s \rightarrow 0^+} f^*(\Phi(\mathbf{y}, s))$$

exists for almost every $\mathbf{y} \in \Gamma_-$. Similarly, $\lim_{s \rightarrow 0^+} f^(\Phi(\mathbf{y}, -s))$ exists for almost every $\mathbf{y} \in \Gamma_+$.*

Proof. From the definition of $\mathcal{D}(\mathcal{T}_{\max})$ one has in particular $\mathcal{T}_{\max} f \in L^1(\Omega_-, d\mu)$. From Eq. (2.10), one sees then that, for almost every $\mathbf{y} \in \Gamma_-$, the mapping $t \in (0, \tau_+(\mathbf{y})) \mapsto f(\Phi(\mathbf{y}, t))$ is integrable over $(0, \tau_+(\mathbf{y}))$. We shall denote by \mathcal{Y}_- the set of such \mathbf{y} 's. Furthermore, using Proposition 3.2 (point (1)), there exists a representative f^* of f such that, for any $\mathbf{y} \in \mathcal{Y}_-$ and any $0 < s < s_0 < \tau_+(\mathbf{y})$, the following holds

$$f^*(\Phi(\mathbf{y}, s)) - f^*(\Phi(\mathbf{y}, s_0)) = - \int_s^{s_0} [\mathcal{T}_{\max} f](\Phi(\mathbf{y}, r)) dr.$$

It is clear that the limit $\lim_{s \rightarrow 0^+} f^*(\Phi(\mathbf{y}, s))$ exists and equals

$$f^*(\Phi(\mathbf{y}, s_0)) - \int_0^{s_0} [\mathcal{T}_{\max} f](\Phi(\mathbf{y}, r)) dr.$$

It is easy to check that this limit does not depend on s_0 . The existence of $\lim_{s \rightarrow 0^+} f^*(\Phi(\mathbf{y}, -s))$ for almost every $\mathbf{y} \in \Gamma_+$ follows by the same argument. \square

The above proposition allows to define the trace operators.

Definition 3.4. *For any $f \in \mathcal{D}(\mathcal{T}_{\max})$, define the traces $B^\pm f$ by*

$$B^+ f(\mathbf{y}) := \lim_{s \rightarrow 0^+} f^*(\Phi(\mathbf{y}, -s)) \quad \text{and} \quad B^- f(\mathbf{y}) := \lim_{s \rightarrow 0^+} f^*(\Phi(\mathbf{y}, s))$$

for any $\mathbf{y} \in \Gamma_\pm$ for which the limits exist, the representative f^ being provided by Proposition 3.3.*

Note that, as we saw in the proof of Proposition 3.3, for any $f \in \mathcal{D}(\mathcal{T}_{\max})$ and a.e. $\mathbf{z} \in \Gamma_+$,

$$B^+ f(\mathbf{z}) = f^*(\Phi(\mathbf{z}, -t)) - \int_0^t [\mathcal{T}_{\max} f](\Phi(\mathbf{z}, -s)) ds, \quad \forall t \in (0, \tau_-(\mathbf{z}))$$

where f^* is a suitable representative of f . A same formula holds for $B^- f$.

Lemma 2.7 provides the existence of Borel measures $d\mu_\pm$ on Γ_\pm which allow us to define the natural trace spaces associated to Problem (1.1), namely,

$$L_\pm^1 := L^1(\Gamma_\pm, d\mu_\pm).$$

We note, however, that for $f \in X$, the traces $\mathbf{y} \in \Gamma_{\pm} \mapsto \mathbf{B}^{\pm} f(\mathbf{y})$ not necessarily belong to L^1_{\pm} .

3.2. Basic existence results. Let \mathcal{T}_0 be the free streaming operator with *no re-entry boundary conditions*:

$$\mathcal{T}_0 \psi = \mathcal{T}_{\max} \psi \quad \text{for any } \psi \in \mathcal{D}(\mathcal{T}_0),$$

where the domain $\mathcal{D}(\mathcal{T}_0)$ is defined by

$$\mathcal{D}(\mathcal{T}_0) = \{\psi \in \mathcal{D}(\mathcal{T}_{\max}); \mathbf{B}^- \psi = 0\}.$$

We state the following generation result:

Theorem 3.5. *The operator $(\mathcal{T}_0, \mathcal{D}(\mathcal{T}_0))$ is the generator of a nonnegative C_0 -semigroup of contractions $(U_0(t))_{t \geq 0}$ in X given by*

$$U_0(t)f(\mathbf{x}) = f(\Phi(\mathbf{x}, -t))\chi_{\{t < \tau_-(\mathbf{x})\}}(\mathbf{x}), \quad (\mathbf{x} \in \Omega, f \in X),$$

where χ_A denotes the characteristic function of a set A .

Proof. The proof is divided into three steps:

• *Step 1.* Let us first check that the family of operators $(U_0(t))_{t \geq 0}$ is a nonnegative contractive C_0 -semigroup in X . Thanks to Proposition 2.3, we can prove that, for any $f \in X$ and any $t \geq 0$, the mapping $U_0(t)f : \Omega \rightarrow \mathbb{R}$ is measurable and the semigroup properties $U_0(0)f = f$ and $U_0(t)U_0(s)f = U_0(t+s)f$ ($t, s \geq 0$) hold. Let us now show that $\|U_0(t)f\|_X \leq \|f\|_X$. We have

$$\|U_0(t)f\|_X = \int_{\Omega_+} |U_0(t)f| d\mu + \int_{\Omega_- \cap \Omega_{+\infty}} |U_0(t)f| d\mu + \int_{\Omega_- \cap \Omega_{+\infty}} |U_0(t)f| d\mu.$$

Propositions 2.10 and 2.3 yield

$$\begin{aligned} \int_{\Omega_+} |U_0(t)f| d\mu &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |U_0(t)f(\Phi(\mathbf{y}, -s))| ds \\ &= \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\max(0, \tau_-(\mathbf{y})-t)} |f(\Phi(\mathbf{y}, -s-t))| ds \\ &\leq \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_t^{\max(t, \tau_-(\mathbf{y}))} |f(\Phi(\mathbf{y}, -r))| dr \leq \int_{\Omega_+} |f| d\mu. \end{aligned}$$

In the same way we obtain

$$\int_{\Omega_- \cap \Omega_{+\infty}} |U_0(t)f| d\mu = \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\infty} |U_0(t)f(\Phi(\mathbf{y}, s))| ds = \int_{\Omega_- \cap \Omega_{+\infty}} |f| d\mu,$$

and

$$\int_{\Omega_- \cap \Omega_{+\infty}} |U_0(t)f| d\mu = \int_{\Omega_- \cap \Omega_{+\infty}} |f| d\mu.$$

This proves contractivity of $U_0(t)$. Let us now show that $U_0(t)f$ is continuous, i.e.

$$\lim_{t \rightarrow 0} \|U_0(t)f - f\|_X = 0.$$

It is enough to show that this property holds for any $f \in \mathcal{C}_0^{\infty}(\Omega)$. In this case, $\lim_{t \rightarrow 0} U_0(t)f(\mathbf{x}) = f(\mathbf{x})$ for any $\mathbf{x} \in \Omega$. Moreover, $\sup_{\mathbf{x} \in \Omega} |U_0(t)f(\mathbf{x})| \leq \sup_{\mathbf{x} \in \Omega} |f(\mathbf{x})|$ and the support of $U_0(t)f$

is bounded, so that the Lebesgue dominated convergence theorem leads to the result. This proves that $(U_0(t))_{t \geq 0}$ is a C_0 -semigroup of contractions in X . Let \mathcal{A}_0 denote its generator.

• *Step 2.* To show that $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{T}_0)$ let $f \in \mathcal{D}(\mathcal{A}_0)$ be fixed. Then, for any $t > 0$, one has

$$(3.1) \quad U_0(t)f - f = \int_0^t U_0(s)\mathcal{A}_0f ds.$$

Let us fix $\delta > 0$ and set $\Omega_\delta = \{\mathbf{x} \in \Omega; \tau_-(\mathbf{x}) > \delta\}$. From (3.1), the mapping

$$u : t \in (0, \delta) \mapsto U_0(t)f \Big|_{\Omega_\delta} \in L^1(\Omega_\delta, d\mu)$$

is continuously differentiable with derivative

$$u'(t) = U_0(t)\mathcal{A}_0f \Big|_{\Omega_\delta}.$$

Since $u(t)(\mathbf{x}) = [U_0(t)f](\mathbf{x}) = f(\Phi(\mathbf{x}, -t))$ for any $0 < t < \delta$ and any $\mathbf{x} \in \Omega_\delta$, one deduces from [8, Theorem 2.40] (see also [16, Theorem 3.4.2]) that

- (i) for any $\mathbf{x} \in \Omega_\delta$, the function $t \in (0, \delta) \mapsto f(\Phi(\mathbf{x}, -t))$ is absolutely continuous,
- (ii) for almost every $(\mathbf{x}, t) \in \Omega_\delta \times (0, \delta)$, $\partial_t f(\Phi(\mathbf{x}, -t))$ exists and

$$\partial_t f(\Phi(\mathbf{x}, -t)) = \mathcal{A}_0f(\Phi(\mathbf{x}, -t)),$$

i.e.

$$\lim_{h \rightarrow 0} h^{-1} (f(\Phi(\mathbf{x}, -t+h)) - f(\Phi(\mathbf{x}, -t))) = \mathcal{A}_0f(\Phi(\mathbf{x}, -t)) \quad \text{for a.e. } (\mathbf{x}, t) \in \Omega_\delta \times (0, \delta).$$

In particular for any $0 < \epsilon < \delta$, there exists $0 < t_0 < \epsilon$ such that

$$\lim_{h \rightarrow 0} h^{-1} (f(\Phi(\mathbf{x}, -t_0+h)) - f(\Phi(\mathbf{x}, -t_0))) = \mathcal{A}_0f(\Phi(\mathbf{x}, -t_0)) \quad \text{for a.e. } \mathbf{x} \in \Omega_\delta.$$

Note that

$$\Omega_{\delta, \epsilon} := \{\mathbf{y} \in \Omega; \tau_-(\mathbf{y}) > \delta, \tau_+(\mathbf{y}) > \epsilon\} \subset \{\Phi(\mathbf{x}, -t_0); \mathbf{x} \in \Omega_\delta\}.$$

Therefore, setting

$$\mathcal{N}_\delta = \{\mathbf{x} \in \Omega_\delta; \lim_{h \rightarrow 0} h^{-1} (f(\Phi(\mathbf{x}, -t_0+h)) - f(\Phi(\mathbf{x}, -t_0))) \neq \mathcal{A}_0f(\Phi(\mathbf{x}, -t_0))\}$$

and $\mathcal{N}_{\delta, \epsilon} = \{\mathbf{y} \in \Omega_{\delta, \epsilon}; \lim_{h \rightarrow 0} h^{-1} (f(\Phi(\mathbf{y}, h)) - f(\mathbf{y})) \neq \mathcal{A}_0f(\mathbf{y})\}$ one sees that

$$\mathcal{N}_{\delta, \epsilon} \subset \Phi(\mathcal{N}_\delta, -t_0).$$

From Liouville's Theorem (Prop. 2.3), $\mu(\Phi(\mathcal{N}_\delta, -t_0)) = \mu(\mathcal{N}_\delta) = 0$, so that

$$\lim_{h \rightarrow 0} h^{-1} (f(\Phi(\mathbf{y}, h)) - f(\mathbf{y})) = \mathcal{A}_0f(\mathbf{y}), \quad \text{for } \mu\text{-almost every } \mathbf{y} \in \Omega_{\delta, \epsilon}.$$

Since $\delta > 0$ and $0 < \epsilon < \delta$ are arbitrary, one obtains that

$$\lim_{h \rightarrow 0} h^{-1} (f(\Phi(\mathbf{x}, h)) - f(\mathbf{x})) = \mathcal{A}_0f(\mathbf{x}), \quad \text{for } \mu\text{-almost every } \mathbf{x} \in \Omega.$$

Applying now [8, Theorem 2.40] to the *zero-th order time derivative* of $u(\cdot)$, one gets that, for any $\mathbf{x} \in \Omega$, the function $t \in (0, \tau_-(\mathbf{x})) \mapsto f(\Phi(\mathbf{x}, -t))$ is absolutely continuous with

$$f(\Phi(\mathbf{x}, -t_2)) - f(\Phi(\mathbf{x}, -t_1)) = \int_{t_1}^{t_2} [\mathcal{A}_0 f](\Phi(\mathbf{x}, -s)) ds, \quad \forall 0 < t_1 < t_2 < \tau_-(\mathbf{x}).$$

This proves that $f \in \mathcal{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max} f = \mathcal{A}_0 f$. To prove that $f \in \mathcal{D}(\mathcal{T}_0)$ it remains only to prove that $\mathbf{B}^- f = 0$. To do so, let $\lambda > 0$ and $g = (\lambda - \mathcal{A}_0)f$. Then,

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t) g(\Phi(\mathbf{x}, -t)) dt, \quad (\mathbf{x} \in \Omega).$$

Let $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$. Noting that $t = \tau_-(\Phi(\mathbf{y}, t))$, by Proposition 2.3 we obtain

$$\begin{aligned} f(\Phi(\mathbf{y}, t)) &= \int_0^t \exp(-\lambda s) g(\Phi(\Phi(\mathbf{y}, t), -s)) ds = \int_0^t \exp(-\lambda s) g(\Phi(\mathbf{y}, s - t)) ds \\ &= \int_0^t \exp(-\lambda(t - s)) g(\Phi(\mathbf{y}, s)) ds. \end{aligned}$$

Consequently, $\lim_{t \rightarrow 0^+} f(\Phi(\mathbf{y}, t)) = 0$ a.e. $\mathbf{y} \in \Gamma_-$, i.e. $\mathbf{B}^- f = 0$.

• *Step 3.* Let us show now the converse inclusion $\mathcal{D}(\mathcal{T}_0) \subset \mathcal{D}(\mathcal{A}_0)$. Let $f \in \mathcal{D}(\mathcal{T}_0)$. We may suppose, up to changing f on a set of zero measure, that $f = f^*$ where f^* is the representative of f provided by Prop. 3.2. Then, for any $\mathbf{x} \in \Omega$ and any $0 \leq t < \tau_-(\mathbf{x})$ one has

$$f(\Phi(\mathbf{x}, -t)) - f(\mathbf{x}) = \int_0^t [\mathcal{T}_{\max} f](\Phi(\mathbf{x}, -r)) dr$$

which, according to the explicit expression of $U_0(t)$, means that

$$(3.2) \quad U_0(t)f(\mathbf{x}) - f(\mathbf{x}) = \int_0^t U_0(r) \mathcal{T}_{\max} f(\mathbf{x}) dr$$

holds for any $\mathbf{x} \in \Omega$, and $t < \tau_-(\mathbf{x})$. Letting t converge towards $\tau_-(\mathbf{x})$ one obtains

$$f(\mathbf{x}) = - \int_0^{\tau_-(\mathbf{x})} [\mathcal{T}_{\max} f](\Phi(\mathbf{x}, -r)) dr.$$

In particular, Eq. (3.2) holds true for any $\mathbf{x} \in \Omega$ and any $t \geq \tau_-(\mathbf{x})$. Consequently, $f \in \mathcal{D}(\mathcal{A}_0)$ with $\mathcal{A}_0 f = \mathcal{T}_{\max} f$. \square

One can now state the following result.

Theorem 3.6. *Let $u \in L^1_-$ and $g \in X$ be given. Then the function*

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t) g(\Phi(\mathbf{x}, -t)) dt + \chi_{\{\tau_-(\mathbf{x}) < \infty\}} \exp(-\lambda \tau_-(\mathbf{x})) u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x})))$$

*is the **unique** solution $f \in \mathcal{D}(\mathcal{T}_{\max})$ of the boundary value problem:*

$$(3.3) \quad \begin{cases} (\lambda - \mathcal{T}_{\max})f = g \\ \mathbf{B}^- f = u \end{cases}$$

where $\lambda > 0$. Moreover, $B^+ f \in L^1_+$ and

$$(3.4) \quad \|B^+ f\|_{L^1_+} + \lambda \|f\|_X \leq \|u\|_{L^1_-} + \|g\|_X.$$

Furthermore, if $g \geq 0$ and $u \geq 0$, then (3.4) turns into equality.

Proof. Let us write $f = f_1 + f_2$ with $f_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t) g(\Phi(\mathbf{x}, -t)) dt$, and

$$f_2(\mathbf{x}) = \chi_{\{\tau_-(\mathbf{x}) < \infty\}} \exp(-\lambda \tau_-(\mathbf{x})) u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x}))), \quad (\mathbf{x} \in \Omega).$$

According to Theorem 3.5, $f_1 = (\lambda - \mathcal{T}_0)^{-1} g$, i.e. $f_1 \in \mathcal{D}(\mathcal{T}_{\max})$ with $(\lambda - \mathcal{T}_{\max}) f_1 = g$ and $B^- f_1 = 0$. Therefore, to prove that f is a solution of (3.3) it suffices to check that $f_2 \in \mathcal{D}(\mathcal{T}_{\max})$, $(\lambda - \mathcal{T}_{\max}) f_2 = 0$ and $B^- f_2 = u$. It is easy to see that, for any $\mathbf{x} \in \Omega$ and any $s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$

$$(3.5) \quad f_2(\Phi(\mathbf{x}, s)) = \exp(-\lambda s) f_2(\mathbf{x}).$$

According to Prop. 3.2, it is then clear that

$$f_2 \in \mathcal{D}(\mathcal{T}_{\max}) \quad \text{with} \quad \mathcal{T}_{\max} f_2 = \lambda f_2.$$

Moreover, we deduces from (3.5) that

$$f_2(\Phi(\mathbf{y}, t)) = \exp(-\lambda t) u(\mathbf{y}), \quad (\mathbf{y} \in \Gamma_-, \quad 0 < t < \tau_+(\mathbf{y}))$$

from which we see that $B^- f_2 = u$. Consequently, f is a solution to (3.3). To prove that the solution is unique, it is sufficient to prove that the only solution $h \in \mathcal{D}(\mathcal{T}_{\max})$ to

$$(\lambda - \mathcal{T}_{\max}) h = 0, \quad B^- h = 0$$

is $h = 0$. This follows from the fact that such a solution h actually belongs to $\mathcal{D}(\mathcal{T}_0)$ while $\lambda \in \varrho(\mathcal{T}_0)$. Finally, it remains to prove (3.4). For simplicity, we still denote by f_2 its representative provided by Proposition 3.3. Using (3.5) and the fact that f_2 vanishes on $\Omega_{-\infty}$, we infer from (2.10) that

$$(3.6) \quad \begin{aligned} \lambda \int_{\Omega} |f_2| d\mu &= \lambda \int_{\Omega_-} |f_2| d\mu = \lambda \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} e^{-\lambda t} |u(\mathbf{y})| dt \\ &= \int_{\Gamma_-} |u(\mathbf{y})| \left(1 - e^{-\lambda \tau_+(\mathbf{y})}\right) d\mu_-(\mathbf{y}). \end{aligned}$$

Define $h : \mathbf{y} \in \Gamma_- \mapsto h(\mathbf{y}) = |u(\mathbf{y})| e^{-\lambda \tau_+(\mathbf{y})}$. It is clear that h vanishes on $\Gamma_{-\infty}$ and $h(\mathbf{y}) \leq |u(\mathbf{y})|$ for a.e. $\mathbf{y} \in \Gamma_-$. In particular, $h \in L^1_-$ and, according to (2.12),

$$\begin{aligned} \int_{\Gamma_-} h(\mathbf{y}) d\mu_-(\mathbf{y}) &= \int_{\Gamma_- \setminus \Gamma_{-\infty}} h(\mathbf{y}) d\mu_-(\mathbf{y}) = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} h(\Phi(\mathbf{z}, -\tau_-(\mathbf{z}))) d\mu_+(\mathbf{z}) \\ &= \int_{\Gamma_+ \setminus \Gamma_{+\infty}} e^{-\lambda \tau_-(\mathbf{z})} |u(\Phi(\mathbf{z}, -\tau_-(\mathbf{z})))| d\mu_+(\mathbf{z}) \\ &= \int_{\Gamma_+} |B^+ f_2(\mathbf{z})| d\mu_+(\mathbf{z}) = \|B^+ f_2\|_{L^1_+}. \end{aligned}$$

Combining this with (3.6) leads to

$$(3.7) \quad \lambda \|f_2\|_X + \|B^+ f_2\|_{L^1_+} = \|u\|_{L^1_-}.$$

Now, let us show that $B^+ f_1 \in L^1_+$ and $\|B^+ f_1\|_{L^1_+} + \lambda \|f\|_X \leq \|g\|_X$. For any $\mathbf{y} \in \Gamma_+$ and $0 < t < \tau_-(\mathbf{y})$, we see, as above, that

$$\begin{aligned} f_1(\Phi(\mathbf{y}, t, 0)) &= \int_0^{\tau_-(\mathbf{y})-t} \exp(-\lambda s) g(\Phi(\mathbf{y}, -s-t)) ds \\ &= \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t)) g(\Phi(\mathbf{y}, -s)) ds. \end{aligned}$$

This shows that $B^+ f_1(\mathbf{y}) = \lim_{t \rightarrow 0^+} f_1(\Phi(\mathbf{y}, -t)) = \int_0^{\tau_-(\mathbf{y})} \exp(-\lambda s) g(\Phi(\mathbf{y}, -s)) ds$. According to Proposition 2.10,

$$\int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |g(\Phi(\mathbf{y}, -s))| ds = \int_{\Omega_+} |g| d\mu$$

which, since $\exp(-\lambda(s-t)) |g(\Phi(\mathbf{y}, -s))| \leq |g(\Phi(\mathbf{y}, -s))|$, implies $B^+ f_1 \in L^1_+$. Let us now assume $g \geq 0$. Then $f_1 \geq 0$,

$$\lambda \|f_1\| = \lambda \int_{\Omega} f_1 d\mu = \lambda \int_{\Omega_+} f_1 d\mu + \lambda \int_{\Omega_- \cap \Omega_{+\infty}} f_1 d\mu + \lambda \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} f_1 d\mu.$$

Using similar arguments to those used in the study of f_2 , we have

$$\lambda \int_{\Omega_+} f_1 d\mu = \int_{\Gamma_+} d\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\Phi(\mathbf{y}, -t)) (1 - \exp(-\lambda t)) dt,$$

which, by Proposition 2.10, implies

$$\lambda \int_{\Omega_+} f_1 d\mu = \int_{\Omega_+} g d\mu - \int_{\Gamma_+} B^+ f_1 d\mu_+.$$

Similar argument shows that

$$\lambda \int_{\Omega_- \cap \Omega_{+\infty}} f_1 d\mu = \int_{\Omega_- \cap \Omega_{+\infty}} g d\mu,$$

while the equality

$$\lambda \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} f_1 d\mu = \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g d\mu,$$

follows since this case behaves as the whole space case. This shows that $\lambda \|f\|_X = \|g\|_X - \|B^+ f\|_{L^1_+}$ for $g \geq 0$. In general, defining

$$F_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda s) |g(\Phi(\mathbf{x}, -s))| ds, \quad (\mathbf{x} \in \Omega),$$

we obtain $\|B^+ f_1\|_{L^1_+} + \lambda \|f_1\|_X \leq \|B^+ F_1\|_{L^1_+} + \lambda \|F_1\|_X = \|g\|_X$, which combined with (3.7), gives (3.4). \square

Let us note that, with the notation of Theorem 3.6, we have

$$(3.8) \quad \int_{\Gamma_+} \mathbf{B}^+ f d\mu_+ + \lambda \int_{\Omega} f d\mu = \int_{\Gamma_-} u d\mu_- + \int_{\Omega} g d\mu.$$

Indeed, for nonnegative u and g , (3.4) turns out to be an identity which is precisely (3.8). Then, for arbitrary $u \in L^1_-$ and $g \in X$, we get (3.8) by splitting functions into positive and negative parts. This leads to the following generalization of Green's formula:

Proposition 3.7 (Green's formula). *Let $f \in \mathcal{D}(\mathcal{T}_{\max})$ be such that $\mathbf{B}^- f \in L^1_-$. Then $\mathbf{B}^+ f \in L^1_+$ and*

$$\int_{\Omega} \mathcal{T}_{\max} f d\mu = \int_{\Gamma_-} \mathbf{B}^- f d\mu_- - \int_{\Gamma_+} \mathbf{B}^+ f d\mu_+$$

Proof. For given $f \in \mathcal{D}(\mathcal{T}_{\max})$, we obtain the result by setting $u = \mathbf{B}^- f \in L^1_-$ and $g = (\lambda - \mathcal{T}_{\max})f \in X$ in Eq. (3.8). \square

Remark 3.8. *If $d\mu$ is the Lebesgue measure on \mathbb{R}^N , the above formula leads to a better understanding of the measures $d\mu_{\pm}$. Indeed, comparing it to the classical Green's formula (see e.g. [9]), one sees that the restriction of $d\mu_{\pm}$ on the set $\Sigma_{\pm} = \{\mathbf{y} \in \partial\Omega; \pm \mathcal{F}(\mathbf{y}) \cdot n(\mathbf{y}) > 0\}$ is equal to*

$$|\mathcal{F}(\mathbf{y}) \cdot n(\mathbf{y})| d\gamma(\mathbf{y})$$

where $d\gamma(\cdot)$ is the surface Lebesgue measure on $\partial\Omega$.

We conclude this section with a result similar to Theorem 3.6. Precisely, for boundary value problems with data given on Γ_+ , we have the following generalization of [6, Lemma 2.2]

Proposition 3.9. *Given $h \in L^1_+$, let*

$$f(\mathbf{x}) = \begin{cases} h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})) \frac{\tau_-(\mathbf{x}) e^{-\tau_+(\mathbf{x})}}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty, \\ h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})) e^{-\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) = \infty \text{ and } \tau_+(\mathbf{x}) < \infty, \\ 0 & \text{if } \tau_+(\mathbf{x}) = \infty. \end{cases}$$

Then, $f \in \mathcal{D}(\mathcal{T}_{\max})$, $\mathbf{B}^- f = 0$, and $\mathbf{B}^+ f = h$, with $\|f\|_X \leq \|h\|_{L^1_+}$ and $\|\mathcal{T}_0 f\|_X \leq \|h\|_{L^1_+}$.

Proof. Let us show that $f \in \mathcal{D}(\mathcal{T}_{\max})$ with

$$(3.9) \quad \mathcal{T}_{\max} f(\mathbf{x}) = \begin{cases} -h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})) e^{-\tau_+(\mathbf{x})} \frac{1 + \tau_-(\mathbf{x})}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty, \\ -h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})) e^{-\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) = \infty \text{ and } \tau_+(\mathbf{x}) < \infty, \\ 0 & \text{if } \tau_+(\mathbf{x}) = \infty. \end{cases}$$

Let

$$g(\mathbf{x}) = h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})) e^{-\tau_+(\mathbf{x})} \frac{1 + \tau_-(\mathbf{x})}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} \chi_{\{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty\}}.$$

Let $\mathbf{x} \in \Omega$ be fixed with $\tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty$. One has

$$f(\Phi(\mathbf{x}, t)) = h(\Phi(\mathbf{x}, \tau_+(\mathbf{x}))) \frac{(\tau_-(\mathbf{x}) + s) \exp(-\tau_+(\mathbf{x}) + s)}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})}$$

while

$$g(\Phi(\mathbf{x}, s)) = h(\Phi(\mathbf{x}, \tau_+(\mathbf{x}))) \exp(-\tau_+(\mathbf{x}) + s) \frac{1 + \tau_-(\mathbf{x}) + s}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})}, \quad \forall s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})).$$

Thus, for any given $-\tau_-(\mathbf{x}) < t_1 < t_2 < \tau_+(\mathbf{x})$,

$$\begin{aligned} \int_{t_1}^{t_2} g(\Phi(\mathbf{x}, s)) ds &= h(\Phi(\mathbf{x}, \tau_+(\mathbf{x}))) \frac{\exp(-\tau_+(\mathbf{x}))}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} \int_{t_1}^{t_2} \exp(s)(1 + \tau_-(\mathbf{x}) + s) ds \\ &= h(\Phi(\mathbf{x}, \tau_+(\mathbf{x}))) \frac{\exp(-\tau_+(\mathbf{x}))}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} (\tau_-(\mathbf{x}) + s) \exp(s) \Big|_{t_1}^{t_2}. \end{aligned}$$

It is then immediate to recognize that

$$\int_{t_1}^{t_2} g(\Phi(\mathbf{x}, s)) ds = f(\Phi(\mathbf{x}, t_2)) - f(\Phi(\mathbf{x}, t_1)).$$

Moreover, it is also easy to check that

$$\lim_{h \rightarrow 0} h^{-1} (f(\Phi(\mathbf{x}, h)) - f(\mathbf{x})) = g(\mathbf{x}), \quad \mu - \text{a.e. } \mathbf{x} \in \Omega, \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty.$$

One proceeds in the same way for $\mathbf{x} \in \Omega$ with $\tau_-(\mathbf{x}) = \infty$ and $\tau_+(\mathbf{x}) < \infty$ which proves that $f \in \mathcal{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max} f$ given by (3.9). Next we show $\|f\|_X \leq \|h\|_{L^1_+}$ and $\|\mathcal{T}_{\max} f\|_X \leq \|h\|_{L^1_+}$.

First we notice that

$$\int_{\Omega} |f(\mathbf{x})| d\mu(\mathbf{x}) = \int_{\Omega_+} |f(\mathbf{x})| d\mu(\mathbf{x}) = \int_{\Omega_+ \cap \Omega_-} |f(\mathbf{x})| d\mu(\mathbf{x}) + \int_{\Omega_+ \cap \Omega_{-\infty}} |f(\mathbf{x})| d\mu(\mathbf{x}),$$

since $f(\mathbf{x}) = 0$ whenever $\tau_+(\mathbf{x}) = \infty$. Now, according to the integration formula (2.10),

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-} |f(\mathbf{x})| d\mu(\mathbf{x}) &= \int_{\Omega_+ \cap \Omega_-} |h(\Phi(\mathbf{x}, \tau_+(\mathbf{x})))| \frac{\tau_-(\mathbf{x}) e^{-\tau_+(\mathbf{x})}}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} \\ &= \int_{\Gamma_+ \setminus \Gamma_{+\infty}} d\mu(\mathbf{z}) \int_0^{\tau_-(\mathbf{z})} \frac{|h(\mathbf{z})|}{\tau_-(\mathbf{z})} (\tau_-(\mathbf{z}) - s) e^{-s} ds = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} |h(\mathbf{z})| \frac{e^{-\tau_-(\mathbf{z})} + \tau_-(\mathbf{z}) - 1}{\tau_-(\mathbf{z})} d\mu(\mathbf{z}) \\ &\leq \int_{\Gamma_+ \setminus \Gamma_{+\infty}} |h(\mathbf{z})| d\mu(\mathbf{z}). \end{aligned}$$

In the same way, according to Eq. (2.11),

$$\int_{\Omega_+ \cap \Omega_{-\infty}} |f(\mathbf{x})| d\mu(\mathbf{x}) = \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{z}) \int_0^{\infty} |h(\mathbf{z})| e^{-s} ds = \int_{\Gamma_{+\infty}} |h(\mathbf{z})| d\mu_+(\mathbf{z}).$$

One obtains thus that

$$\|f\|_X = \int_{\Omega} |f(\mathbf{x})| d\mu(\mathbf{x}) \leq \int_{\Gamma_+ \setminus \Gamma_{+\infty}} |h(\mathbf{z})| d\mu(\mathbf{z}) + \int_{\Gamma_{+\infty}} |h(\mathbf{z})| d\mu(\mathbf{z}) = \|h\|_{L^1_+}.$$

One proceeds in the same way to show that $\|\mathcal{T}_{\max} f\|_X \leq \|h\|_{L^1_+}$. Since $f \in \mathcal{D}(\mathcal{T}_{\max})$, let us denote by f^* the representative of f provided by Proposition 3.3. Let $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$, we have

$$(3.10) \quad f^*(\Phi(\mathbf{y}, t)) = h(\Phi(\mathbf{y}, \tau_+(\mathbf{y}))) \frac{te^{t-\tau_+(\mathbf{y})}}{\tau_+(\mathbf{y})} \chi_{\{\tau_+(\mathbf{y}) < \infty\}}$$

and $B^- f(\mathbf{y}) = \lim_{t \rightarrow 0} f^*(\Phi(\mathbf{y}, t)) = 0$ for a.e. $\mathbf{y} \in \Gamma_-$. In the same way, given $\mathbf{z} \in \Gamma_+$ and $0 < t < \tau_+(\mathbf{z})$, we have

$$f^*(\Phi(\mathbf{z}, -t)) = h(\mathbf{z}) \left[\frac{(\tau_-(\mathbf{z}) - t)e^{-t}}{\tau_-(\mathbf{z})} \chi_{\{\tau_-(\mathbf{z}) < \infty\}} + e^{-t} \chi_{\{\tau_-(\mathbf{z}) = \infty\}} \right],$$

so that $B^+ f(\mathbf{z}) = h(\mathbf{z})$ for a.e. $\mathbf{z} \in \Gamma_+$. This ends the proof. \square

3.3. Transport equations with abstract boundary conditions. For any (linear) bounded boundary operator $H \in \mathcal{B}(L^1_+, L^1_-)$, define \mathcal{T}_H as

$$\mathcal{T}_H \psi = \mathcal{T}_{\max} \psi \quad \text{for any } \psi \in \mathcal{D}(\mathcal{T}_H),$$

where

$$\mathcal{D}(\mathcal{T}_H) = \{ \psi \in \mathcal{D}(\mathcal{T}_{\max}); B^+ \psi \in L^1_+, B^- \psi = HB^+ \psi \}.$$

For any $\lambda > 0$, we define the operators:

$$\left\{ \begin{array}{l} M_\lambda : L^1_- \longrightarrow L^1_+ \\ u \longmapsto [M_\lambda u](\mathbf{y}) = u(\Phi(\mathbf{y}, -\tau_-(\mathbf{y}))) \exp(-\lambda \tau_-(\mathbf{y})) \chi_{\{\tau_-(\mathbf{y}) < \infty\}}, \quad (\mathbf{y} \in \Gamma_+); \end{array} \right.$$

$$\left\{ \begin{array}{l} \Xi_\lambda : L^1_- \longrightarrow X \\ u \longmapsto [\Xi_\lambda u](\mathbf{x}) = u(\Phi(\mathbf{x}, -\tau_-(\mathbf{x}))) \exp(-\lambda \tau_-(\mathbf{x})) \chi_{\{\tau_-(\mathbf{x}) < \infty\}}, \quad (\mathbf{x} \in \Omega); \end{array} \right.$$

$$\left\{ \begin{array}{l} G_\lambda : X \longrightarrow L^1_+ \\ f \longmapsto [G_\lambda f](\mathbf{z}) = \int_0^{\tau_-(\mathbf{z})} f(\Phi(\mathbf{z}, -s)) \exp(-\lambda s) ds, \quad (\mathbf{z} \in \Gamma_+); \end{array} \right.$$

and

$$\left\{ \begin{array}{l} C_\lambda : X \longrightarrow X \\ f \longmapsto [C_\lambda f](\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} f(\Phi(\mathbf{x}, -s)) \exp(-\lambda s) ds, \quad (\mathbf{x} \in \Omega). \end{array} \right.$$

Thanks to Hölder's inequality, all these operators are bounded on their respective spaces. Note that Ξ_λ is a lifting operator which, to a given $u \in L^1_-$ associates a function $f = \Xi_\lambda u \in \mathcal{D}(\mathcal{T}_{\max})$ whose trace on Γ_- is exactly u (Theorem 3.6). The operator M_λ transfers functions defined on Γ_- to functions defined on Γ_+ and, using Theorem 3.5, it is easy to see that C_λ coincides with the resolvent of \mathcal{T}_0 , i.e. $C_\lambda f = (\lambda - \mathcal{T}_0)^{-1} f$ for any $f \in X$, $\lambda > 0$. In particular, $\text{Rank}(C_\lambda) \subset \mathcal{D}(\mathcal{T}_{\max})$. Moreover, still using Theorem 3.6, we see that $G_\lambda f = B^+ C_\lambda f$ for any $f \in X$ and $M_\lambda u = B^+ \Xi_\lambda u$ for any $u \in L^1_-$. Finally, we see that G_λ is surjective for any $\lambda > 0$. Indeed, according to Proposition 3.9, we have that for any $g \in L^1_+$, there is an $f \in \mathcal{D}(\mathcal{T}_{\max})$, such that $B^+ f = g$ and $B^- f = 0$. The latter property means that $f \in \mathcal{D}(\mathcal{T}_0)$ so that, for any $\lambda > 0$, there

is $\psi \in X$ such that $f = \mathcal{B}(\lambda, \mathcal{T}_0)\psi$. In this case, $g = \mathbf{B}^+ f = G_\lambda \psi$ and $\|\psi\|_X \leq \lambda \|f\|_X + \|\mathcal{T}_0 f\|_X \leq (1 + \lambda)\|g\|_X$. The above operators allow to solve the boundary-value problem

$$(3.11) \quad \begin{cases} (\lambda - \mathcal{T}_{\max})f = g \\ \mathbf{B}^- f = H\mathbf{B}^+ f \end{cases}$$

where $g \in X$ and $\lambda > 0$. Precisely, we have

Proposition 3.10. *Let $g \in X$ be given. Assume that for some $\lambda_0 > 0$ the series $\sum_{n=0}^{\infty} (M_{\lambda_0} H)^n G_{\lambda_0} g$ converges in L_+^1 . Then the function*

$$(3.12) \quad f = C_{\lambda_0} g + \Xi_{\lambda_0} H \left(\sum_{n=0}^{\infty} (M_{\lambda_0} H)^n G_{\lambda_0} g \right)$$

is a solution of $(\lambda_0 - \mathcal{T}_H)f = g$. If, moreover, $H \geq 0$, then the thesis is valid for all $\lambda \geq \lambda_0$.

Proof. Define $\mathcal{S}_{\lambda_0} g = \sum_{n=0}^{\infty} (M_{\lambda_0} H)^n G_{\lambda_0} g$. By assumption, $\mathcal{S}_{\lambda_0} g \in L_+^1$ so that $H\mathcal{S}_{\lambda_0} g \in L_-^1$. Then, as we have already seen, both $C_{\lambda_0} g$ and $\Xi_{\lambda_0} H\mathcal{S}_{\lambda_0} g$ belong to $\mathcal{D}(\mathcal{T}_{\max})$. This shows that $f = C_{\lambda_0} g + \Xi_{\lambda_0} H\mathcal{S}_{\lambda_0} g \in \mathcal{D}(\mathcal{T}_{\max})$. Furthermore,

$$\mathbf{B}^- f = \mathbf{B}^- C_{\lambda_0} g + \mathbf{B}^- \Xi_{\lambda_0} H\mathcal{S}_{\lambda_0} g = H\mathcal{S}_{\lambda_0} g$$

since $C_{\lambda_0} g \in \mathcal{D}(\mathcal{T}_0)$ and $\mathbf{B}^- \Xi_{\lambda_0} u = u$ for all $u \in L_-^1$. In a similar way,

$$\mathbf{B}^+ f = \mathbf{B}^+ C_{\lambda_0} g + \mathbf{B}^+ \Xi_{\lambda_0} H\mathcal{S}_{\lambda_0} g = G_{\lambda_0} g + M_{\lambda_0} H\mathcal{S}_{\lambda_0} g = \sum_{n=0}^{\infty} (M_{\lambda_0} H)^n G_{\lambda_0} g = \mathcal{S}_{\lambda_0} g,$$

so that $\mathbf{B}^- f = H\mathbf{B}^+ f$, i.e. $f \in \mathcal{D}(\mathcal{T}_H)$. Finally, Theorems 3.5 and 3.6 assert respectively that $(\lambda_0 - \mathcal{T}_{\max})C_{\lambda_0} g = g$ and $(\lambda_0 - \mathcal{T}_{\max})\Xi_{\lambda_0} u = 0$ for any $u \in L_+^1$ so that f solves (3.11) with $\lambda = \lambda_0$. The statement for $H \geq 0$ follows from the fact that M_λ and G_λ decrease with λ and thus the series in (3.12) converges for any $\lambda \geq \lambda_0$. \square

As a consequence, one gets the following generation result for contractive boundary operators already stated in [10, 14]:

Theorem 3.11. *Assume H to be strictly contractive, i.e. $\|H\|_{\mathcal{B}(L_+^1, L_-^1)} < 1$. Then \mathcal{T}_H generates a C_0 -semigroup of contractions $(U_H(t))_{t \geq 0}$ and the resolvent $(\lambda - \mathcal{T}_H)^{-1}$ is given by*

$$(3.13) \quad (\lambda - \mathcal{T}_H)^{-1} = C_\lambda + \Xi_\lambda H \left(\sum_{n=0}^{\infty} (M_\lambda H)^n G_\lambda \right) \quad \text{for any } \lambda > 0$$

where the series is convergent in $\mathcal{B}(X)$.

Proof. It is easy to see that $\|M_\lambda\| \leq 1$ for any $\lambda > 0$. In particular, $\|M_\lambda H\| < 1$ for any $\lambda > 0$ and the series $\sum_{n=0}^{\infty} (M_\lambda H)^n G_\lambda$ converges in $\mathcal{B}(X, L_+^1)$. Fix now $g \in X$ and let $f \in X$ be given by (3.12). Proposition 3.10 ensures that f is a solution of (3.11) while (3.4) implies that

$$\lambda \|f\|_X \leq \|g\|_X + \|\mathbf{B}^- f\|_{L_-^1} - \|\mathbf{B}^+ f\|_{L_+^1}.$$

Since $\|B^- f\|_{L_-^1} = \|HB^+ f\|_{L_-^1} < \|B^+ f\|_{L_+^1}$, we get $\lambda\|f\|_X < \|g\|_X$ or, equivalently,

$$\|(\lambda - \mathcal{T}_H)f\|_X \geq \lambda\|f\|_X.$$

Therefore \mathcal{T}_H is a densely defined dissipative operator (recall that $\mathcal{D}(\mathcal{T}_H)$ contains the set of compactly supported continuous functions) of X . Moreover, the range of $(\lambda - \mathcal{T}_H)$ is exactly X according to Proposition 3.10 so that the Lumer-Phillips Theorem leads to the generation result. \square

Remark 3.12. *Hadamard's criterion ensures that the series in (3.13) converges in $\mathcal{B}(X)$ for any $\lambda > 0$ and any boundary operator H such that $r_\sigma(M_\lambda H) < 1$.*

4. MULTIPLICATIVE BOUNDARY CONDITIONS

In this section, we consider the general case $\|H\|_{\mathcal{B}(L_+^1, L_-^1)} \geq 1$, and we provide, in the spirit of [18], a sufficient condition on H ensuring that \mathcal{T}_H generates a C_0 -semigroup in X . Let χ_ϵ denote the following multiplication operator in L_+^1 :

$$[\chi_\epsilon u](\mathbf{y}) = \begin{cases} u(\mathbf{y}) & \text{if } \tau_-(\mathbf{y}) \leq \epsilon, \\ 0 & \text{else,} \end{cases}$$

for any $u \in L_+^1$ and any $\epsilon > 0$. Our main result is the following

Theorem 4.1. *Let $H \in \mathcal{B}(L_+^1, L_-^1)$. If*

$$(4.1) \quad \limsup_{\epsilon \rightarrow 0} \|H\chi_\epsilon\|_{\mathcal{B}(L_+^1, L_-^1)} < 1,$$

then \mathcal{T}_H generates a C_0 -semigroup $(V_H(t))_{t \geq 0}$ in X such that

$$(4.2) \quad \|V_H(t)\| \leq \frac{\|H\|}{1 - \|H\chi_\epsilon\|} \exp \left[-\frac{t}{\epsilon} \log \left(\frac{1 - \|H\chi_\epsilon\|}{\|H\|} \right) \right] \quad (t \geq 0),$$

for any $\epsilon > 0$ such that $\|H\chi_\epsilon\| < 1$.

The strategy to prove this result is adapted from [18] and consists in performing a suitable change of the unknown function in (1.1) (similar to that used in [14, Chapter XIII]) so that the new unknown satisfies an equivalent evolution problem (4.4) but with a boundary operator which is contractive, provided the assumption (4.1) holds. More precisely, for any $\alpha < 0$, define the multiplication operator in L_+^1 :

$$\mathcal{M}_\alpha : L_+^1 \ni u \mapsto [\mathcal{M}_\alpha u](\mathbf{y}) = \exp\{\alpha(\tau_-(\mathbf{y}) \wedge k)\}u(\mathbf{y}) \in L_+^1,$$

where k is a positive real number to be fixed later. Let \mathcal{Z}_α be defined by

$$\mathcal{Z}_\alpha : X \ni f \mapsto [\mathcal{Z}_\alpha f](\mathbf{x}) = \exp\{\alpha(\tau_-(\mathbf{x}) \wedge k)\}f(\mathbf{x}) \in X.$$

Since $\mathcal{M}_\alpha \in \mathcal{B}(L_+^1)$, it is possible to define the free streaming operator $\mathcal{T}_{H\mathcal{M}_\alpha}$ associated to the boundary operator $H\mathcal{M}_\alpha \in \mathcal{B}(L_-^1, L_+^1)$ and the *absorption operator*

$$\mathcal{A}_{H,\alpha}\psi(\mathbf{x}) = \mathcal{T}_{H\mathcal{M}_\alpha}\psi(\mathbf{x}) - \alpha\chi_{\{\tau_-(\mathbf{x}) \leq k\}}\psi(\mathbf{x}), \quad \psi \in \mathcal{D}(\mathcal{A}_{H,\alpha}),$$

where

$$\mathcal{D}(\mathcal{A}_{H,\alpha}) = \mathcal{D}(\mathcal{T}_{H,\mathcal{M}_\alpha}) = \{\psi \in \mathcal{D}(\mathcal{T}_{\max}); \mathbf{B}^\pm \psi \in L^1_\pm, \mathbf{B}^- \psi = H\mathcal{M}_\alpha \mathbf{B}^+ \psi\}.$$

The unbounded operators \mathcal{T}_H and $\mathcal{A}_{H,\alpha}$ are related by the following lemma.

Lemma 4.2. *For any $0 < \alpha < 1$, $\mathcal{Z}_\alpha^{-1} \mathcal{D}(\mathcal{T}_H) = \mathcal{D}(\mathcal{A}_{H,\alpha})$ and $\mathcal{T}_H = \mathcal{Z}_\alpha \mathcal{A}_{H,\alpha} \mathcal{Z}_\alpha^{-1}$.*

Proof. Let $0 < \alpha < 1$ be fixed. One sees easily that \mathcal{Z}_α is a continuous bijection from X onto itself. Its inverse is given by

$$\mathcal{Z}_\alpha^{-1} : f \in X \mapsto \mathcal{Z}_\alpha^{-1} f(\mathbf{x}) = \exp\{-\alpha(\tau_-(\mathbf{x}) \wedge k)\} f(\mathbf{x}) \in X.$$

Note that $\mathcal{Z}_\alpha^{-1} \in \mathcal{B}(X)$ because $\sup\{\tau_-(\mathbf{x}) \wedge k; \mathbf{x} \in \Omega\} \leq k$. Now, let $f \in \mathcal{D}(\mathcal{T}_H)$ and $g = \mathcal{Z}_\alpha^{-1} f$. First we show that $g \in \mathcal{D}(\mathcal{T}_{\max})$. Setting $\Omega_k = \{\mathbf{x} \in \Omega; \tau_-(\mathbf{x}) \leq k\}$, define

$$u(\mathbf{x}) = \exp(-\tau_-(\mathbf{x})\alpha) (-\alpha f(\mathbf{x}) - \mathcal{T}_{\max} f(\mathbf{x})) \chi_{\Omega_k}.$$

One has, for any $\mathbf{x} \in \Omega_k$ and any $s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$:

$$u(\Phi(\mathbf{x}, s)) = \exp(-\alpha\tau_-(\mathbf{x})) \exp(-\alpha s) (-\alpha g(\Phi(\mathbf{x}, s)) - \mathcal{T}_{\max} g(\Phi(\mathbf{x}, s))).$$

Then, using the fact that, according to Prop. 3.2,

$$g(\Phi(\mathbf{x}, s)) = f(\mathbf{x}) - \int_0^s \mathcal{T}_{\max} f(\Phi(\mathbf{x}, r)) dr, \quad \forall \mathbf{x} \in \Omega_k, s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$$

since $g \in \mathcal{D}(\mathcal{T}_{\max})$, it is possible to show that, for any $\mathbf{x} \in \Omega_k$

$$\begin{aligned} \int_{s_1}^{s_2} u(\Phi(\mathbf{x}, s)) ds &= \exp(-\tau_-(\mathbf{x})\alpha) \left[\exp(-s_1\alpha) f(\Phi(\mathbf{x}, s_1)) - \exp(-s_2\alpha) f(\Phi(\mathbf{x}, s_2)) \right] \\ &= g(\Phi(\mathbf{x}, s_1)) - g(\Phi(\mathbf{x}, s_2)), \quad (-\tau_-(\mathbf{x}) < s_1 < s_2 < \tau_+(\mathbf{x})). \end{aligned}$$

Moreover, it is also clear that

$$\lim_{h \rightarrow 0} h^{-1} (g(\Phi(\mathbf{x}, h)) - g(\mathbf{x})) = u(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_k.$$

Since

$$g(\mathbf{x}) = e^{-\tau_-(\mathbf{x})\alpha} f(\mathbf{x}) \chi_{\{\tau_-(\mathbf{x}) \leq k\}} + e^{-k\alpha} f(\mathbf{x}) \chi_{\{\tau_-(\mathbf{x}) > k\}}$$

we obtain that $g \in \mathcal{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max} g(\mathbf{x}) = e^{-\alpha(\tau_-(\mathbf{x}) \wedge k)} (\alpha f(\mathbf{x}) \chi_{\{\tau_-(\mathbf{x}) \leq k\}} + \mathcal{T}_{\max} f(\mathbf{x}))$, i.e.

$$(4.3) \quad \mathcal{T}_{\max} \mathcal{Z}_\alpha^{-1} f = \mathcal{Z}_\alpha^{-1} (\alpha f \chi_{\{\tau_- \leq k\}} + \mathcal{T}_{\max} f).$$

Still denoting by f and g their corresponding representatives provided by Proposition 3.3, one sees that, since $\tau_-(\mathbf{y}) \wedge k = 0$ for any $\mathbf{y} \in \Gamma_-$,

$$\mathbf{B}^- g(\mathbf{y}) = \lim_{s \rightarrow 0^+} g(\Phi(\mathbf{y}, s)) = \lim_{s \rightarrow 0^+} \exp(-s\alpha) f(\Phi(\mathbf{y}, s)) = \mathbf{B}^- f(\mathbf{y}).$$

Now, for any $\mathbf{z} \in \Gamma_+$, using that $\tau_-(\Phi(\mathbf{z}, -s)) = \tau_-(\mathbf{z}) - s$ for any $0 < s < \tau_-(\mathbf{z})$, we have

$$\begin{aligned} \mathbf{B}^+ g(\mathbf{z}) &= \lim_{s \rightarrow 0^+} g(\Phi(\mathbf{z}, -s)) = \lim_{s \rightarrow 0^+} \exp(-\alpha((\tau_-(\mathbf{z}) \wedge k) - s)) f(\Phi(\mathbf{z}, s)) \\ &= \exp(-\alpha(\tau_-(\mathbf{z}) \wedge k)) \mathbf{B}^+ f(\mathbf{z}), \end{aligned}$$

i.e. $B^+g = \mathcal{M}_\alpha^{-1}B^+f$. Consequently $B^\pm g \in L^\pm_1$ and $B^-g = H\mathcal{M}_\alpha B^+g$. This proves that $g \in \mathcal{D}(\mathcal{A}_{H,\alpha})$ i.e. $\mathcal{Z}_\alpha^{-1}\mathcal{D}(\mathcal{T}_H) \subset \mathcal{D}(\mathcal{A}_{H,\alpha})$. The converse inclusion is proved similarly. Finally, for any $f \in \mathcal{D}(\mathcal{T}_H)$, (4.3) reads $\mathcal{T}_{H\mathcal{M}_\alpha}\mathcal{Z}_\alpha^{-1}f = \mathcal{Z}_\alpha^{-1}(\alpha\chi_{\{\tau_- \leq k\}}f + \mathcal{T}_Hf)$, i.e.

$$\mathcal{Z}_\alpha\mathcal{A}_{H,\alpha}\mathcal{Z}_\alpha^{-1}f = \mathcal{T}_Hf \quad \forall f \in \mathcal{D}(\mathcal{T}_H)$$

which completes the proof. \square

Remark 4.3. Note that the characteristic function $\chi_{\{\tau_- \leq k\}}$ in the definition of $\mathcal{A}_{H,\alpha}$ is missing in the force-free case studied in [18] but has to be considered if one wishes to take into account characteristic curves with infinite length.

The above lemma shows that the evolution problem

$$(4.4) \quad \begin{cases} \partial_t f(\mathbf{x}, t) + \mathcal{F}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} f(\mathbf{x}, t) + \alpha\chi_{\{\tau_-(\mathbf{x}) \leq k\}}f(\mathbf{x}, t) = 0 \\ B^-f = H\mathcal{M}_\alpha B^+f \\ f(\mathbf{x}, 0) = \exp\{-\alpha(\tau_-(\mathbf{x}) \wedge k)\}g_0(\mathbf{x}), \quad (\mathbf{x} \in \Omega) \end{cases}$$

is equivalent, by the change of variables, to problem (1.1). Consequently, to prove that \mathcal{T}_H is a generator of a C_0 -semigroup $(V_H(t))_{t \geq 0}$ in X , it suffices to show that $\mathcal{A}_{H,\alpha}$ generates a C_0 -semigroup $(V_{H,\alpha}(t))_{t \geq 0}$ in X (for some negative α). Moreover, by Theorem 3.11, it is enough to find a negative α such that $\|H\mathcal{M}_\alpha\| < 1$. We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. Define $\mathcal{Q} = \{\alpha < 0 ; \|H\mathcal{M}_\alpha\| < 1\}$. As explained above, Lemma 4.2 and Theorem 3.11 imply that if $\mathcal{Q} \neq \emptyset$, then \mathcal{T}_H generates a C_0 -semigroup $(V_H(t))_{t \geq 0}$ such that

$$(4.5) \quad V_H(t) = \mathcal{Z}_\alpha V_{H,\alpha}(t)\mathcal{Z}_\alpha^{-1} \quad (t \geq 0, \alpha \in \mathcal{Q}),$$

where $(V_{H,\alpha}(t))_{t \geq 0}$ is a C_0 -semigroup in X with generator $\mathcal{A}_{H,\alpha}$ ($\alpha \in \mathcal{Q}$). Using assumption (4.1), let us fix $\epsilon > 0$ so that $\|H\chi_\epsilon\| < 1$ and choose k to be larger than ϵ . Then, for any $0 < \alpha < 1$,

$$\|H\mathcal{M}_\alpha\| \leq \|H\chi_\epsilon\mathcal{M}_\alpha\| + \|H(I - \chi_\epsilon)\mathcal{M}_\alpha\| \leq \|H\chi_\epsilon\| + \|H\|(I - \chi_\epsilon)\mathcal{M}_\alpha\|.$$

Moreover

$$\begin{aligned} \|(I - \chi_\epsilon)\mathcal{M}_\alpha\| &= \sup \{ \exp[\alpha(\tau_-(\mathbf{y}) \wedge k)] ; \mathbf{y} \in \Gamma_+ \text{ and } (\tau_-(\mathbf{y}) \wedge k) \geq \epsilon \} \\ &\leq \exp(\epsilon\alpha) \quad (\alpha < 0). \end{aligned}$$

Consequently,

$$\|H\mathcal{M}_\alpha\| \leq \|H\chi_\epsilon\| + \|H\| \exp(\epsilon\alpha)$$

and $\alpha \in \mathcal{Q}$ provided

$$(4.6) \quad \epsilon\alpha < \log \left(\frac{1 - \|H\chi_\epsilon\|}{\|H\|} \right).$$

Therefore, $\mathcal{Q} \neq \emptyset$ and \mathcal{T}_H is a generator of a C_0 -semigroup $(V_H(t))_{t \geq 0}$ in X . On the other hand, since $\mathcal{A}_{H,\alpha}g = \mathcal{T}_{H\mathcal{M}_\alpha}g - \alpha\chi_{\{\tau_- \leq k\}}g$ for any $g \in \mathcal{D}(\mathcal{A}_{H,\alpha})$, and since $\mathcal{T}_{H\mathcal{M}_\alpha}$ generates a C_0 -semigroup of contractions, we see that

$$\|V_{H,\alpha}(t)\| \leq \exp(-t\alpha) \quad (t \geq 0, \alpha \in \mathcal{Q}).$$

Next, we see that

$$\|\mathcal{Z}_\alpha\| \leq 1 \quad \text{and} \quad \|\mathcal{Z}_\alpha^{-1}\| \leq \exp(-k\alpha), \quad (\alpha \in \mathcal{Q}),$$

hence (4.5) implies $\|V_H(t)\| \leq \exp(-(k+t)\alpha)$ for any $t \geq 0$ and any $\alpha \in \mathcal{Q}$. Noting that the set \mathcal{Q} is independent of k (actually it depends only on ϵ and H through (4.6)), we may let k go to ϵ so that

$$\|V_H(t)\| \leq \exp(-(\epsilon+t)\alpha) \quad (t \geq 0, \alpha \in \mathcal{Q}).$$

Now, for any fixed $\epsilon > 0$, optimizing the free parameter α in (4.6) we obtain (4.2). \square

The estimate (4.2) on $\|V_H(t)\|$ certainly is not optimal and can be improved for some geometries of the phase space. One such case is described in the corollary below.

Corollary 4.4. *Assume that $\inf\{\tau_-(\mathbf{y}); \mathbf{y} \in \Gamma_+\} = \ell_0 > 0$. Then, for any boundary operator $H \in \mathcal{B}(L_+^1, L_-^1)$, \mathcal{T}_H generates a C_0 -semigroup $(V_H(t))_{t \geq 0}$ in X such that*

$$\|V_H(t)\| \leq \max\{1, \|H\|\} \exp(\max\{0, \log \|H\|\}t/\ell_0) \quad (t \geq 0).$$

Proof. According to Theorem 3.11, it suffices to prove the result for $\|H\| \geq 1$. Noting that

$$\|H\chi_\epsilon\| = \begin{cases} 0 & \text{if } 0 < \epsilon < \ell_0 \\ \|H\| & \text{if } \epsilon \geq \ell_0, \end{cases}$$

we immediately see that

$$\inf \left\{ \frac{1}{\epsilon} \log \left(\frac{1 - \|H\chi_\epsilon\|}{\|H\|} \right); \|H\chi_\epsilon\| < 1 \right\} = -\frac{\log \|H\|}{\ell_0}.$$

The proof becomes now a straightforward application of Theorem 4.1. \square

This corollary shows that if Ω is a phase space in which the lengths of characteristic curves are bounded away from 0, then the general transport equation (1.1) is **well-posed for any bounded boundary operator** $H \in \mathcal{B}(L_+^1, L_-^1)$. Let us illustrate the above result with a few examples.

Example 4.5. In the first example we consider the force-free Vlasov equation in a slab of thickness $2a$, ($a > 0$). In such a case, $\Omega = \{\mathbf{x} = (x, v) \in \mathbb{R}^2; -a < x < a, -1 < v < 1\}$ and $\mathcal{F}(\mathbf{x}) = (v, 0)$. It is not difficult to see (see for instance [18, Section 4.1]) that the above Corollary applies in this case since $\inf\{\tau_-(\mathbf{y}); \mathbf{y} \in \Gamma_+\} = 2a > 0$.

Consider now an example of the Vlasov equation with a non trivial force term for which Corollary 4.4 still applies.

Example 4.6. Let us consider the following two-dimensional phase space:

$$\Omega = \{\mathbf{x} = (x, y) \in \mathbb{R}^2; x^2 + y^2 < 2 \text{ and } -1 < y < 1\}$$

with the field $\mathcal{F}(\mathbf{x}) = (-y, x)$ for any $\mathbf{x} = (x, y) \in \Omega$. In such a case, the characteristic curves are *circular*, namely

$$\Phi(\mathbf{x}, s) = (x \cos s - y \sin s, x \sin s + y \cos s), \quad \mathbf{x} = (x, y), \quad s \in \mathbb{R}.$$

In particular, for any $\mathbf{x} = (x, y) \in \Omega$ such that $x^2 + y^2 < 1$, one has $\tau_{\pm}(\mathbf{x}) = \infty$. Moreover,

$$\Gamma_{\pm} = \{(x, -1); -1 < \pm x < 0\} \cup \{(x, 1); 0 < \pm x < 1\}.$$

In this case, one can easily check that $\inf\{\tau_{-}(\mathbf{y}); \mathbf{y} \in \Gamma_{+}\} = \pi/2$.

5. CONSERVATIVE BOUNDARY CONDITIONS

In this section we consider the case of conservative boundary conditions. Note that such boundary conditions arise naturally in the study of gas dynamics [14] and are typically associated to a boundary operator H such that

$$\|H\psi\| = \|\psi\|, \quad \text{for any } \psi \in L_{+}^1, \psi \geq 0.$$

Theorem 4.1 does not apply to such boundary operators since

$$\|H\chi_{\epsilon}\| = \|\chi_{\epsilon}\| = 1 \quad \text{for any } \epsilon > 0.$$

Therefore to deal with the generation properties of the operator \mathcal{T}_H , we shall proceed in a different way adapting techniques used in [6] in the force-free case. From now on, we adopt the following assumptions, which are more specific than the condition above.

Assumption 3.

- (a) *The boundary operator $H \in \mathcal{B}(L_{+}^1, L_{-}^1)$ is **positive**.*
- (b) $\|H\| = 1$.
- (c) *If $f \in L_{+}^1$ is non-negative and $Hf = 0$, then $f = 0$.*

Under these hypotheses we can prove the existence result given in Theorem 5.1. This result, with different proof, can be found in [10]. A less general version of it has been obtained also in [6, Theorem 2.8]. The proof for a general force field \mathcal{F} , which we present below for self-consistency of the paper, is the same as in [6] since it only uses the series representation of the resolvent of \mathcal{T}_H and the generation result for contractive boundary operators (Theorem 3.11).

For any $0 < r < 1$, let $(V_r(t))_{t \geq 0}$ be the C_0 -semigroup of X generated by \mathcal{T}_{rH} (whose existence is given by Theorem 3.11).

Theorem 5.1. *Let H satisfy Assumption (3). Then, for any $t \geq 0$ and any $f \in X$ the limit $V_H(t)f = \lim_{r \nearrow 1} V_r(t)f$ exists in X and defines a substochastic semigroup $(V_H(t))_{t \geq 0}$. If $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the generator of $(V_H(t))_{t \geq 0}$, then its resolvent is given by*

$$(5.1) \quad (\lambda - \mathcal{A})^{-1}f = C_{\lambda}f + \sum_{n=0}^{\infty} \Xi_{\lambda}H(M_{\lambda}H)^n G_{\lambda}f \quad \text{for any } f \in X, \lambda > 0,$$

where the series converges in X .

Proof. According to Theorem 3.11, for any $0 < r < 1$ and any fixed $\lambda > 0$, the resolvent of \mathcal{T}_{rH} is given by

$$(\lambda - \mathcal{T}_{rH})^{-1} = C_{\lambda} + \sum_{n=0}^{\infty} r^{n+1} B_{\lambda}H(M_{\lambda}H)^n G_{\lambda}$$

with $\sup_{0 < r < 1} \|(\lambda - \mathcal{T}_{rH})^{-1}\| \leq \lambda^{-1}$. Then, for any $f \geq 0$, the function $r \in (0, 1) \mapsto (\lambda - \mathcal{T}_{rH})^{-1}f$ is non-negative and non-decreasing so that the following limit exists

$$\mathcal{R}(\lambda)f = \lim_{r \nearrow 1} (\lambda - \mathcal{T}_{rH})^{-1}f = C_\lambda f + \sum_{n=0}^{\infty} \Xi_\lambda H(M_\lambda H)^n G_\lambda f$$

where the series converges absolutely because of the monotone convergence theorem. It is easy to check that $\|\mathcal{R}(\lambda)f\| \leq \lambda^{-1}\|f\|$. Now, for $f = f^+ - f^-$, we define $\mathcal{R}(\lambda)f = \mathcal{R}(\lambda)f^+ - \mathcal{R}(\lambda)f^-$ so that $\mathcal{R}(\lambda)$ is a linear and bounded operator in X with $\|\mathcal{R}(\lambda)\| \leq \lambda^{-1}$. Furthermore, the range of $\mathcal{R}(\lambda)$ is dense in X since it contains the $\mathcal{C}_0^\infty(\Omega)$. Indeed, if $f \in \mathcal{C}_0^\infty(\Omega)$, then $(\lambda - \mathcal{T}_{rH})f = (\lambda - \mathcal{T}_H)f = g$ is independent of $0 < r < 1$, so that $(\lambda - \mathcal{T}_{rH})^{-1}g = f \rightarrow \mathcal{R}(\lambda)g$ as $r \nearrow 1$. Now, thanks to Trotter-Kato Theorem, there exists an operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ which generates a C_0 -semigroup $(V_H(t))_{t \geq 0}$ in X and such that $\mathcal{R}(\lambda) = (\lambda - \mathcal{A})^{-1}$ for any $\lambda > 0$ and $V_H(t)f = \lim_{r \nearrow 1} V_r(t)f$, for any $t \geq 0$. \square

Remark 5.2. We note that the expression (5.1) implies that $(V_H(t))_{t \geq 0}$ does not depend on the choice of the approximating sequence of semigroups $(V_r(t))_{t \geq 0}$. Indeed, for any sequence of non-negative boundary operators $(H_n)_n \subset \mathcal{B}(L_+^1, L_-^1)$ with $H_n f \nearrow H f$ as $n \rightarrow \infty$ for any nonnegative $f \in L_+^1$, one can check that $V_{H_n}(t)$ converges strongly to $V_H(t)$.

Remark 5.3. Note that, in contrast to what happens in the force-free case [6, Theorem 2.8] we cannot say at this moment that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is an extension of $(\mathcal{T}_H, \mathcal{D}(\mathcal{T}_H))$. This, however, will become clear by Theorem 5.7.

Remark 5.4. Note that, arguing as in [6, Corollary 2.10], we can show that, for any $\lambda > 0$, the series $\sum_{n=0}^{\infty} \Xi_\lambda H(M_\lambda H)^n$ defines a bounded linear operator from L_+^1 to X whose norm is less than (or equal to) $(\lambda + 1)/\lambda$.

5.1. Characterization of $\mathcal{D}(\mathcal{A})$. In this section we characterize the domain of \mathcal{A} by adapting the extensions techniques used in [6, Section 3] in the force-free case. Such extension techniques are similar to those introduced in [5] in a different context (see also [8]). Precisely, let us denote by E_- the set of all measurable functions defined on $(\Gamma_-, d\mu_-)$ taking values in the extended set of reals $\mathbb{R} \cup \{\pm\infty\}$. It is clear that $L_-^1 \subset E_-$. In the sequel we shall denote $\Xi := \Xi_1 \in \mathcal{B}(L_-^1, X)$. Through Ξ , we define the set $F_- \subset E_-$ as follows: $f \in F_-$ if and only if for any non-negative and non-decreasing sequence $(f_n)_n \subset L_-^1$, satisfying $\sup_n f_n = |f|$ we have $\sup_n \Xi f_n \in X$. Such a sequence will be called a Ξ -approximating sequence of f .

Definition 5.5. For any $f \in F_-$, $f \geq 0$, we define $\Pi f := \sup_n \Xi f_n \in X$, for any Ξ -approximating sequence $(f_n)_n$ of f . If $f = f_+ - f_-$, we define Πf as $\Pi f = \Pi f_+ - \Pi f_-$.

Note that, from [6, Lemma 3.1], the operator Π is well-defined from F_- to X in the sense that the value of Π does not depend on the choice of the Ξ -approximating sequence of f .

In the same way, we define the set $E_+ \supset L_+^1$ to be the set of all extended real-valued measurable functions defined on $(\Gamma_+, d\mu_+)$. Now, through the boundary operator H , we construct a subset F_+ of E_+ as the set of all functions $\psi \in E_+$ such that $\sup_n H\psi_n \in F_-$ for any non-negative and nondecreasing sequence $(\psi_n)_n$ of L_+^1 such that $\sup_n \psi_n = |\psi|$. Such a sequence will be called an H -approximating sequence of ψ . We have the following definition

Definition 5.6. For any $\psi \in F_+$, $\psi \geq 0$, define $H\psi := \sup_n H\psi_n \in F_-$, for any H -approximating sequence (ψ_n) of ψ . If $\psi = \psi_+ - \psi_-$, we define $H\psi$ as $H\psi = H\psi_+ - H\psi_-$.

Here again, the above operator is well-defined by virtue of [6, Lemma 3.4]. We are now in position to precisely describe the domain of \mathcal{A} .

Theorem 5.7. Let Assumption 3 be satisfied. Then $\varphi \in \mathcal{D}(\mathcal{A})$ if and only if

- (1) $\varphi \in \mathcal{D}(\mathcal{T}_{\max})$, $B^\pm \varphi \in F_\pm$,
- (2) $B^- \varphi = HB^+ \varphi$
- (3) $\lim_{n \rightarrow \infty} \|\Pi H(MH)^n B^+ \varphi\|_X = 0$.

Moreover, for any $\varphi \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}\varphi = \mathcal{T}_{\max}\varphi$.

Proof. We refer the reader to [6, Theorem 3.6] for the proof of the above Theorem. Actually, the main ingredient of the proof is the representation formula (3.13) whereas the explicit expressions of the operators M_λ , Ξ_λ , C_λ and G_λ do not play any role in the proof. Note that, though the range of M is $E_+ \subsetneq F_+$, it can be check that, for any φ satisfying 1) and 2), the sequence $(\Pi H(MH)^n B^+ \varphi)_n$ is well-defined. \square

An important consequence of the above characterization is that it explains the link between the domains of \mathcal{T}_H and that of \mathcal{A} .

Proposition 5.8. Let $\varphi \in \mathcal{D}(\mathcal{A})$ be such that $\varphi|_{\Gamma_\pm} \in L_\pm^1$. Then $\varphi \in \mathcal{D}(\mathcal{T}_H)$, i.e. $\varphi|_{\Gamma_-} = H(\varphi|_{\Gamma_+})$. More precisely, $\varphi \in \mathcal{D}(\mathcal{T}_H)$ if and only if $\varphi \in \mathcal{D}(\mathcal{A})$ and the series $\sum_{n=0}^{\infty} (M_1 H)^n G_1 f$ is convergent in L_+^1 where $f = (1 - \mathcal{A})\varphi$. In particular, $\mathcal{A} = \mathcal{T}_H$ if and only if $\sum_{n=0}^{\infty} (M_1 H)^n G_1 f$ converges in L_+^1 for any $f \in X$.

Proof. Let $\varphi \in \mathcal{D}(\mathcal{A})$. According to Theorem 5.7, $B^- \varphi = HB^+ \varphi$ which, since $B^- \varphi \in L_-^1$, reads $\varphi|_{\Gamma_-} = H(\varphi|_{\Gamma_+})$. Since $\varphi \in \mathcal{D}(\mathcal{T}_{\max})$, it is then clear that $\varphi \in \mathcal{D}(\mathcal{T}_H)$.

Assume now that $\varphi \in \mathcal{D}(\mathcal{T}_H)$. As above, $\varphi \in \mathcal{D}(\mathcal{A})$ and $\varphi|_{\Gamma_\pm} \in L_\pm^1$. Let $f = (1 - \mathcal{A})\varphi$ and let $\psi_n = \sum_{k=0}^{n+1} (M_1 H)^k G_1 f$ for $n \geq 0$. Assume for a while that $f \geq 0$. We can show that

$$\sup_n \psi_n = B^+ \varphi \in L_+^1$$

which implies the convergence of the series $\sum_{k=0}^{\infty} (M_1 H)^k G_1 f$ in L_+^1 , which extends for arbitrary f by linearity. Conversely, let $\varphi \in \mathcal{D}(\mathcal{A})$ and $f = (1 - \mathcal{A})\varphi$. If $\sum_{k=0}^{\infty} (M_1 H)^k G_1 f$ converges in L_+^1 , then we get $B^+ \varphi \in L_+^1$ in the same way and, from the first part, $\varphi \in \mathcal{D}(\mathcal{T}_H)$. \square

The above result shows that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is an extension of $(\mathcal{T}_H, \mathcal{D}(\mathcal{T}_H))$. Moreover, if \mathcal{T}_H does not generate a C_0 -semigroup in X , then the set

$$\mathcal{D}(\mathcal{A}) \setminus \mathcal{D}(\mathcal{T}_H) = \{f \in \mathcal{D}(\mathcal{A}); f|_{\Gamma_\pm} \notin L_\pm^1\} \neq \emptyset,$$

and if \mathcal{T}_H is not closed, then there exists $\varphi \in \overline{\mathcal{D}(\mathcal{T}_H)}$ such that $\varphi|_{\Gamma_\pm} \notin L_\pm^1$. The main scope of the following section is to determine the necessary and sufficient condition on H ensuring the stochasticity of $(V_H(t))_{t \geq 0}$.

5.2. **Stochasticity of $(V_H(t))_{t \geq 0}$.** In this section, we assume that, besides **Assumption 3**, H satisfies conservativeness assumption mentioned at the beginning of this section, i.e.

$$(5.2) \quad \|H\psi\|_{L^1_-} = \|\psi\|_{L^1_+} \quad \text{for any } \psi \in L^1_+, \psi \geq 0.$$

In such a case, one expects the semigroup $(V_H(t))_{t \geq 0}$ to be *stochastic*, that is,

$$(5.3) \quad \int_{\Omega} V_H(t)f \, d\mu = \int_{\Omega} f \, d\mu \quad (f \in X).$$

Indeed, a consequence of Green's formula (Proposition 3.7) is that

$$\int_{\Omega} \mathcal{T}_H f \, d\mu = 0 \quad \text{for any } f \in \mathcal{D}(\mathcal{T}_H).$$

Since $\frac{d}{dt}V_H(t)f = \mathcal{A}V_H(t)f$ for any $t \geq 0$ and any $f \in \mathcal{D}(\mathcal{A})$, (5.3) should be true at least when $\mathcal{A} = \overline{\mathcal{T}_H}$ (see [20]). In this section we give necessary and sufficient conditions ensuring (5.3) to hold. For any $f \in X$, $f \geq 0$, we define

$$(5.4) \quad \beta(f) := \lim_{n \rightarrow \infty} \int_{\Gamma_+} (M_1 H)^n G_1 f(\mathbf{y}) \, d\mu_+(\mathbf{y}) \geq 0.$$

This limit exists since $\|M_1 H\| \leq 1$ so that the the right-hand-side of (5.4) is a decreasing numerical sequence. For arbitrary $f \in X$, $\beta(f)$ is defined by linearity. We have

Theorem 5.9. *The C_0 -semigroup $(V_H(t))_{t \geq 0}$ is **stochastic** in X if and only if $\beta(f) = 0$ for any $f \in X$.*

Proof. Let us fix $f \in X$, $f \geq 0$ and let $\varphi = (1 - \mathcal{A})^{-1}f$. For any $n \geq 1$, define

$$\varphi_n = \mathcal{R}(1, \mathcal{T}_0)f + \sum_{k=0}^n \Xi H (M_1 H)^k G_1 f = \mathcal{R}(1, \mathcal{T}_0)f + \Xi H \sum_{k=0}^n (M_1 H)^k G_1 f.$$

According to (5.1), we have $\varphi_n \rightarrow \varphi$ in X and $\varphi_n \in \mathcal{D}(\mathcal{T}_{\max})$ with $\mathcal{T}_{\max}\varphi_n + f = \varphi_n$ for any $n \geq 1$. Now, set

$$u_n := \sum_{k=0}^n H (M_1 H)^k G_1 f \in L^1_-, \quad \text{and} \quad \psi_n = \sum_{k=0}^{n+1} (M_1 H)^k G_1 f \in L^1_+.$$

Then it is clear that $\psi_n = B^+ \varphi_n$ and $\varphi_n = \mathcal{R}(1, \mathcal{T}_0)f + \Xi u_n$. Consequently, Green's formula (Proposition 3.7) yields

$$(5.5) \quad \int_{\Omega} \varphi_n \, d\mu = \int_{\Omega} f \, d\mu + \int_{\Gamma_-} u_n \, d\mu_- - \int_{\Gamma_+} \psi_n \, d\mu_+.$$

Since $u_n = H\psi_{n-1}$ and $\psi_n \geq 0$, (5.2) yields

$$\int_{\Omega} \varphi_n \, d\mu = \int_{\Omega} f \, d\mu + \int_{\Gamma_+} (\psi_{n-1} - \psi_n) \, d\mu_+.$$

Now, using that $\psi_n - \psi_{n-1} = (M_1 H)^{n+1} G_1 f$ and passing to the limit as $n \rightarrow \infty$, we obtain

$$(5.6) \quad \int_{\Omega} \varphi \, d\mu = \int_{\Omega} f \, d\mu - \beta(f).$$

Consequently, $\beta(f) = 0$ if and only if $\|(1 - \mathcal{A})^{-1} f\|_X = \|f\|_X$. Now, it is easy to see that the stochasticity of $(V_H(t))_{t \geq 0}$ is equivalent to the property that $\|(1 - \mathcal{A})^{-1} f\|_X = \|f\|_X$ for any nonnegative $f \in X$. \square

Remark 5.10. Note that, as in [6], for any $f \in X$ and $\varphi = (1 - \mathcal{A})^{-1} f$:

$$\beta(f) = \int_{\Omega} \mathcal{A}\varphi(\mathbf{x}) \, d\mu(\mathbf{x}).$$

Remark 5.11. Since G_1 is surjective according to Proposition 3.9, we have $\beta(f) = 0$ for any $f \in X$ if and only if $\|(M_1 H)^n g\|_{L^1_+} \rightarrow 0$ for any $g \in L^1_+$.

Proposition 5.12. Assume that H is conservative. Then, the following are equivalent:

- 1) $(V_H(t))_{t \geq 0}$ is stochastic;
- 2) $\mathcal{A} = \overline{\mathcal{T}_H}$;
- 3) $\int_{\Omega} \mathcal{A}\varphi = 0$ for any $\varphi \in \mathcal{D}(\mathcal{A})$.

Proof. The equivalence between 1) and 3) is nothing but (5.6). Let us prove the implication 1) \Rightarrow 2). Take $\varphi \in \mathcal{D}(\mathcal{A})$, the implication is proven if we are able to construct a sequence $(\varphi_n)_n \subset \mathcal{D}(\mathcal{T}_H)$ such that

$$(5.7) \quad \begin{cases} \varphi_n \rightarrow \varphi & (n \rightarrow \infty) \\ (1 - \mathcal{T}_H)\varphi_n \rightarrow f = (1 - \mathcal{A})\varphi & \text{in } X. \end{cases}$$

For any $n \in \mathbb{N}$, define $g_n = (M_1 H)^n G_1 f \in L^1_+$. Then, using Proposition 3.9, for any $n \geq 1$, there exists $\psi_n \in \mathcal{D}(\mathcal{T}_{\max})$ such that $B^- \psi_n = 0$ and $B^+ \psi_n = g_n$ with $\|\psi_n\|_X \leq \|g_n\|_{L^1_+}$ and $\|\mathcal{T}_0 \psi_n\|_X \leq \|g_n\|_{L^1_+}$. As in [6, Proposition 4.4], we can define

$$\varphi_n = \mathcal{R}(1, \mathcal{T}_0) f + \sum_{k=0}^{n-1} \Xi H (M_1 H)^k G_1 f - \psi_n$$

and show that $\varphi_n \in \mathcal{D}(\mathcal{T}_H)$. Since $(V_H(t))_{t \geq 0}$ is assumed to be stochastic, from Theorem 5.9 we infer that $\|g_n\|_{L^1_+} \rightarrow 0$ as $n \rightarrow \infty$ so that $\psi_n \rightarrow 0$ and $\mathcal{T}_0 \psi_n \rightarrow 0$. Then it is easy to see that $(\varphi_n)_n$ satisfies (5.7). This proves that 1) \Rightarrow 2). Finally we explained the idea underlying the converse implication 2) \Rightarrow 1) at the beginning of this subsection (see the considerations after formula (5.3)). We refer to [6, Proposition 2.11] for a detailed proof using both Green's formula and a density argument. \square

Now we discuss spectral properties $M_\lambda H$ which ensure stochasticity of $(V_H(t))_{t \geq 0}$. The proof of the following can be seen as a simple adaptation of that of [6, Theorem 4.5], where the explicit expressions of the various operators Ξ_λ , M_λ , G_λ do not play any role but the main idea goes back to [13] (see also [8, Theorem 4.3]).

- Theorem 5.13.** 1) For any $\lambda > 0$, $1 \notin \sigma_p(M_\lambda H)$;
 2) $1 \in \varrho(M_\lambda H)$ for some/all $\lambda > 0$ if and only if $\mathcal{A} = \overline{\mathcal{T}_H}$;
 3) $1 \in \sigma_c(M_\lambda H)$ for some/all $\lambda > 0$ if and only if $\mathcal{A} = \overline{\mathcal{T}_H} \neq \mathcal{T}_H$.
 4) $1 \in \sigma_r(M_\lambda H)$ for some/all $\lambda > 0$ if and only if $\mathcal{A} \not\supseteq \overline{\mathcal{T}_H}$.

Proof. 1) The fact that 1 cannot belong to the point spectrum of $M_\lambda H$ ($\lambda > 0$) is a simple consequence of Assumption 3 (c) and of the inclusion $\{\lambda > 0\} \subset \varrho(A)$.

2) If there exists $\lambda > 0$ such that $1 \in \varrho(M_\lambda H)$, then, since the series (3.13) converges in the norm topology to $(\lambda - \mathcal{T}_H)^{-1}$, we have $\mathcal{A} = \overline{\mathcal{T}_H}$. Conversely, assume that $\mathcal{A} = \overline{\mathcal{T}_H}$. Then, for any $f \in X$, the series $\sum_{n=0}^{\infty} (M_\lambda H)^n G_\lambda f$ converges in L_+^1 according to Proposition 5.8. Now, since G_λ is surjective, the series $\sum_{n=0}^{\infty} (M_\lambda H)^n g$ converges in L_+^1 for any $g \in L_+^1$. Denoting by $\mathcal{R}(\lambda)g$ the limit, we see from the Banach-Steinhaus Theorem that $\mathcal{R}(\lambda) \in \mathfrak{B}(L_+^1)$ and that $\mathcal{R}(\lambda)(1 - M_\lambda H) = (1 - M_\lambda H)\mathcal{R}(\lambda)$, which proves that $1 \in \varrho(M_\lambda H)$.

3) Let $\lambda > 0$ be such that $1 \in \sigma_c(M_\lambda H)$. Then

$$(5.8) \quad \overline{(1 - M_\lambda H)L_+^1} = L_+^1.$$

Let $\varphi \in \mathcal{D}(\mathcal{A})$ be given and let $f \in X$ be such that $\varphi = (\lambda - \mathcal{A})^{-1}f$. Since $G_\lambda f \in L_+^1$, there is a sequence $(\phi_n)_n \subset L_+^1$ such that $\|\phi_n - M_\lambda H \phi_n - G_\lambda f\|_{L_+^1} \rightarrow 0$. Now, define $g_n = \phi_n - M_\lambda H \phi_n - G_\lambda f$. According to Proposition 3.9, there exists $f_n \in \mathcal{D}(\mathcal{T}_{\max})$ such that $B^+ f_n = g_n$ and $B^- f_n = 0$. Moreover, $f_n \rightarrow 0$ and $\mathcal{T}_0 f_n \rightarrow 0$. Now, setting

$$\varphi_n = f_n + \mathcal{R}(\lambda, \mathcal{T}_0)f + \Xi_\lambda H \phi_n$$

we see that $(\varphi_n)_n \subset \mathcal{D}(\mathcal{T}_H)$ and $(\lambda - \mathcal{T}_H)\varphi_n \rightarrow f$. Furthermore

$$\Xi_\lambda H \phi_n = \sum_{k=0}^{\infty} \Xi_\lambda H (M_\lambda H)^k (\phi_n - M_\lambda H \phi_n) = \sum_{k=0}^{\infty} \Xi_\lambda H (M_\lambda H)^k (g_n + G_\lambda f),$$

where both above series are convergent by Remark 5.4. Using again Remark 5.4, we see that $\sum_{k=0}^{\infty} \Xi_\lambda H (M_\lambda H)^k g_n \rightarrow 0$ so that $\varphi_n \rightarrow \varphi$ and this proves that $\mathcal{A} = \overline{\mathcal{T}_H}$.

Conversely, assume $\mathcal{A} = \overline{\mathcal{T}_H} \neq \mathcal{T}_H$ and let $g \in L_+^1$. Define $g_n = \sum_{k=0}^{n-1} (M_1 H)^k g$. Then, $g_n \in L_+^1$ and, clearly $(1 - M_1 H)g_n = g - (M_1 H)^n g$. Since $\beta(f) = 0$, according to Remark 5.11, one has $\|(1 - M_1 H)g_n - g\|_{L_+^1} \rightarrow 0$ so that (5.8) holds. Since $\mathcal{A} \neq \mathcal{T}_H$, one has $1 \in \sigma(M_1 H) \setminus \sigma_p(M_1 H)$ which proves that $1 \in \sigma_c(M_1 H)$.

4) The last assertions is now clear since all the possibilities have been exhausted. \square

As in [6, Corollary 4.6], we provide here a useful criterion (see [6, Section 5] for several application in the force-free case).

Corollary 5.14. $(V_H(t))_{t \geq 0}$ is stochastic if and only if $1 \notin \sigma_p((M_\lambda H)^*)$ for any $\lambda > 0$. Moreover, if $(V_H(t))_{t \geq 0}$ is not stochastic, then there exists a **non-negative** $\gamma \in (L_+^1)^*$, $\gamma \neq 0$, such that $\gamma = (M_\lambda H)^* \gamma$.

APPENDIX A: WEAK AND STRONG DEFINITIONS OF THE TRANSPORT OPERATOR

We prove here Proposition 3.2 that shows the equivalence between the definition of the transport operator \mathcal{T}_{\max} as weak derivative along the characteristic curves and the strong derivative. Precisely, let us define the operator \mathbf{T} in $L^1(\Omega, d\mu)$ whose domain $\mathcal{D}(\mathbf{T})$ is the set of all $f \in L^1(\Omega, d\mu)$ which admits a representative f^* such that:

- (1) there exists $g \in L^1(\Omega, d\mu)$ such that for any $\mathbf{x} \in \Omega$ and any $-\tau_-(\mathbf{x}) < t_1 \leq t_2 < \tau_+(\mathbf{x})$

$$f^*(\Phi(\mathbf{x}, t_2)) - f^*(\Phi(\mathbf{x}, t_1)) = \int_{t_1}^{t_2} g(\Phi(\mathbf{x}, s)) ds;$$

- (2) for μ -almost every $\mathbf{x} \in \Omega$, one has

$$\lim_{h \rightarrow 0} h^{-1} (f^*(\Phi(\mathbf{x}, h)) - f^*(\mathbf{x})) = g(\mathbf{x}).$$

In such a case, one defines $\mathbf{T}f := -g \in L^1(\Omega, d\mu)$. We prove the following result

Proposition A.1. One has $\mathcal{D}(\mathbf{T}) = \mathcal{D}(\mathcal{T}_{\max})$ and $\mathbf{T}f = \mathcal{T}_{\max}f$, for any $f \in \mathcal{D}(\mathbf{T})$.

The fact that the weak derivative operator \mathcal{T}_{\max} is an extension of \mathbf{T} is a classical matter. The proof of the converse property is carried out through several technical lemmas based upon *mollification along the characteristic curves*. Recall that, whenever μ is not absolutely continuous with respect to the Lebesgue measure, no global convolution argument is available.

Let us make precise what this is all about. Consider a sequence $(\varrho_n)_n$ of one dimensional mollifiers supported in $[0, 1]$, i.e. for any $n \in \mathbb{N}$, $\varrho_n \in \mathcal{C}_0^\infty(\mathbb{R})$, $\varrho_n(s) = 0$ if $s \notin [0, 1/n]$, $\varrho_n(s) \geq 0$ and $\int_0^{1/n} \varrho_n(s) ds = 1$. Then, for any $f \in L^1(\Omega, d\mu)$, define the mollification:

$$\varrho_n \diamond f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \varrho_n(s) f(\Phi(\mathbf{x}, -s)) ds.$$

Note that, with such a definition, it is not clear *a priori* that $\varrho_n \diamond f$ defines a measurable function, finite almost everywhere. It is proved in the following that actually such a function is integrable.

Lemma A.1. Given $f \in L^1(\Omega, d\mu)$, $\varrho_n \diamond f \in L^1(\Omega, d\mu)$ for any $n \in \mathbb{N}$. Moreover,

$$(A.1) \quad \|\varrho_n \diamond f\| \leq \|f\|, \quad \forall f \in L^1(\Omega, d\mu), \quad n \in \mathbb{N}.$$

Proof. One considers, for a given $f \in L^1(\Omega, d\mu)$, the extension of f by zero outside Ω :

$$\bar{f}(\mathbf{x}) = f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \quad \bar{f}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \mathbb{R}^N \setminus \Omega.$$

Then $\bar{f} \in L^1(\mathbb{R}^N, d\mu)$. Let us consider the measurable transformation:

$$\Upsilon : (\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \Upsilon(\mathbf{x}, s) = (\Phi(\mathbf{x}, -s), -s) \in \mathbb{R}^N \times \mathbb{R}.$$

Note that Υ is not necessarily measure-preserving with respect to $d\mu \otimes ds$ because the extension of the field \mathcal{F} to the whole space is not necessarily divergence-free. According to [15, Theorem B, p. 162], the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \bar{f}(\Phi(\mathbf{x}, -s))$$

is measurable as the composition of Υ with the measurable function $(\mathbf{x}, s) \mapsto \bar{f}(\mathbf{x})$. Define now $\Lambda = \{(\mathbf{x}, s); \mathbf{x} \in \Omega, 0 < s < \tau_-(\mathbf{x})\}$, Λ is a measurable subset of $\mathbb{R}^N \times \mathbb{R}$. Therefore, the mapping

$$(\mathbf{x}, s) \in \mathbb{R}^N \times \mathbb{R} \mapsto \bar{f}(\Phi(\mathbf{x}, -s))\chi_\Lambda(\mathbf{x}, s)\varrho_n(s)$$

is measurable. Since ϱ_n is compactly supported, it is also integrable over $\mathbb{R}^N \times \mathbb{R}$ and, according to Fubini's Theorem

$$[\varrho_n \diamond f](\mathbf{x}) := \int_{\mathbb{R}} \bar{f}(\Phi(\mathbf{x}, -s))\chi_\Lambda(\mathbf{x}, s)\varrho_n(s)ds = \int_0^{\tau_-(\mathbf{x})} \varrho_n(s)f(\Phi(\mathbf{x}, -s))ds$$

is finite for almost every $\mathbf{x} \in \Omega$ and the associated application $\varrho_n \diamond f$ is integrable.

Let us prove now (A.1). Since $|\varrho_n \diamond f| \leq \varrho_n \diamond |f|$, to show that $\varrho_n \diamond f \in L^1(\Omega, d\mu)$, it suffices to deal with a *nonnegative function* $f \in L^1(\Omega, d\mu)$. One sees easily that, for any $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$,

$$(A.2) \quad (\varrho_n \diamond f)(\Phi(\mathbf{y}, t)) = \int_0^t \varrho_n(s)f(\Phi(\mathbf{y}, t-s))ds = \int_0^t \varrho_n(t-s)f(\Phi(\mathbf{y}, s))ds.$$

Thus,

$$\begin{aligned} \int_0^{\tau_+(\mathbf{y})} [\varrho_n \diamond f](\Phi(\mathbf{y}, t))dt &= \int_0^{\tau_+(\mathbf{y})} ds \int_s^{\tau_+(\mathbf{y})} \varrho_n(s)f(\Phi(\mathbf{y}, t-s))dt \\ &= \int_0^{\tau_+(\mathbf{y}) \wedge 1/n} \varrho_n(s)ds \int_0^{\tau_+(\mathbf{y})-s} f(\Phi(\mathbf{y}, r))dr. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 \leq \int_0^{\tau_+(\mathbf{y})} [\varrho_n \diamond f](\Phi(\mathbf{y}, t))dt &\leq \int_0^{1/n} \varrho_n(s)ds \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r))dr \\ &= \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r))dr, \quad \forall \mathbf{y} \in \Gamma_-, n \in \mathbb{N} \end{aligned}$$

so that

$$\int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} [\varrho_n \diamond f](\Phi(\mathbf{y}, t))dt \leq \int_{\Gamma_-} d\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r))dr.$$

This proves, thanks to Proposition 2.10, that

$$(A.3) \quad \int_{\Omega_-} [\varrho_n \diamond f]d\mu \leq \int_{\Omega_-} fd\mu.$$

Now, in the same way:

$$\begin{aligned} \int_{\Omega_+ \cap \Omega_-^\infty} [\varrho_n \diamond f](\mathbf{x})d\mu(\mathbf{x}) &= \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^\infty [\varrho_n \diamond f](\Phi(\mathbf{y}, -t))dt \\ &= \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^\infty dt \int_t^\infty \varrho_n(r-t)f(\Phi(\mathbf{y}, -r))dr \end{aligned}$$

so that

$$(A.4) \quad \begin{aligned} \int_{\Omega_+ \cap \Omega_-} [\varrho_n \diamond f](\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^\infty f(\Phi(\mathbf{y}, -r)) dr \int_0^r \varrho_n(r-t) dt \\ &\leq \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^\infty f(\Phi(\mathbf{y}, -r)) dr = \int_{\Omega_+ \cap \Omega_-} f(\mathbf{x}) d\mu(\mathbf{x}). \end{aligned}$$

Finally

$$\begin{aligned} \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} [\varrho_n \diamond f](\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} d\mu(\mathbf{x}) \int_0^\infty \varrho_n(s) f(\Phi(\mathbf{x}, -s)) ds \\ &= \int_0^\infty \varrho_n(s) ds \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\Phi(\mathbf{x}, -s)) d\mu(\mathbf{x}). \end{aligned}$$

Now, from Liouville's Theorem, for any $s \geq 0$,

$$\int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\Phi(\mathbf{x}, -s)) d\mu(\mathbf{x}) = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{x}) d\mu(\mathbf{x}),$$

so that

$$(A.5) \quad \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} [\varrho_n \diamond f](\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega_{+\infty} \cap \Omega_{-\infty}} f(\mathbf{x}) d\mu(\mathbf{x}).$$

Combining (A.3), (A.4) and (A.5), one finally gets $\|\varrho_n \diamond f\| \leq \|f\|$. \square

As it is the case for classical convolution, the family $(\varrho_n \diamond f)_n$ approximates f in L^1 -norm:

Proposition A.2. Given $f \in L^1(\Omega, d\mu)$,

$$(A.6) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |(\varrho_n \diamond f)(\mathbf{x}) - f(\mathbf{x})| d\mu(\mathbf{x}) = 0.$$

Proof. According to (A.1) and from the density of $\mathcal{C}_0(\Omega)$ in $L^1(\Omega, d\mu)$, it suffices to prove the result for any f continuous over Ω and compactly supported. Splitting f into positive and negative parts, $f = f^+ - f^-$, one can also assume f to be nonnegative. From the continuity of both f and $\Phi(\cdot, \cdot)$, one has

$$\mathcal{K}_n := \text{Supp}(\varrho_n \diamond f) = \overline{\Delta_n}$$

where $\Delta_n = \{\mathbf{x} \in \Omega, \exists s_0 \in (0, 1/n) \text{ such that } \Phi(\mathbf{x}, -s_0) \in \text{Supp}(f)\}$. Moreover, it is easily seen that $\mathcal{K}_{n+1} \subset \mathcal{K}_n$ for any $n \geq 1$. Finally, it is clear that

$$\mathcal{K}_1 \subset \{\mathbf{x} \in \overline{\Omega}; \exists \mathbf{y} \in \text{Supp}(f) \text{ with } |\mathbf{x} - \mathbf{y}| \leq d\}$$

where $d = \sup\{|\Phi(\mathbf{x}, s) - \mathbf{x}|; 0 \leq s \leq 1; \mathbf{x} \in \text{Supp}(f)\} < \infty$. Therefore, \mathcal{K}_1 is compact. Set now

$$\mathcal{O}_n := \mathcal{K}_n \cup \text{Supp}(f) \quad \text{and} \quad \mathcal{O}_n^- = \{\mathbf{x} \in \mathcal{O}_n; \tau_-(\mathbf{x}) < 1/n\}.$$

Noticing that $\mu(\mathcal{O}_1)$ is finite, one can see easily that $\lim_n \mu(\mathcal{O}_n^-) = 0$. Since $\sup_{\mathbf{x} \in \Omega} |\varrho_n \diamond f(\mathbf{x})| \leq \sup_{\mathbf{x} \in \Omega} |f(\mathbf{x})|$, for any $\epsilon > 0$, there exists $n_0 \geq 1$ such that

$$\int_{\mathcal{O}_n^-} |f(\mathbf{x})| d\mu(\mathbf{x}) \leq \epsilon, \quad \text{and} \quad \int_{\mathcal{O}_n^-} |\varrho_n \diamond f(\mathbf{x})| d\mu(\mathbf{x}) \leq \epsilon \quad \forall n \geq n_0.$$

Now, noticing that $\text{Supp}(\varrho_n \diamond f - f) \subset \mathcal{O}_n$, one has for any $n \geq n_0$,

$$\int_{\Omega} |\varrho_n \diamond f - f| d\mu = \int_{\mathcal{O}_n} |\varrho_n \diamond f - f| d\mu \leq 2\varepsilon + \int_{\mathcal{O}_n \setminus \mathcal{O}_n^-} |\varrho_n \diamond f - f| d\mu.$$

For any $\mathbf{x} \in \mathcal{O}_n \setminus \mathcal{O}_n^-$, since ϱ is supported in $[0, 1/n]$, one has

$$[\varrho_n \diamond f](\mathbf{x}) - f(\mathbf{x}) = \int_0^{1/n} \varrho_n(s) f(\Phi(\mathbf{x}, -s)) ds - f(\mathbf{x}) = \int_0^{1/n} \varrho_n(s) (f(\Phi(\mathbf{x}, -s)) - f(\mathbf{x})) ds.$$

Note that, thanks to Gronwall's lemma,

$$|\Phi(\mathbf{x}, -s) - \mathbf{x}| \leq \frac{L}{\kappa} (\exp(\kappa s) - 1) \leq \frac{L}{\kappa} (\exp(\kappa/n) - 1), \quad \forall \mathbf{x} \in \mathcal{O}_1, s \in (0, 1/n)$$

where $L = \sup\{|\mathcal{F}(\mathbf{x})|, \mathbf{x} \in \mathcal{O}_1\}$. Since f is uniformly continuous on \mathcal{O}_1 , it follows that

$$\lim_{n \rightarrow \infty} \sup \left\{ |f(\Phi(\mathbf{x}, -s)) - f(\mathbf{x})|; \mathbf{x} \in \mathcal{O}_1, s \in (0, 1/n) \right\} = 0$$

from which we deduces that there exists some $n_1 \geq 0$, such that $|\varrho_n \diamond f(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon$ for any $\mathbf{x} \in \mathcal{O}_n \setminus \mathcal{O}_n^-$ and any $n \geq n_1$. One obtains then, for any $n \geq n_1$,

$$\int_{\Omega} |\varrho_n \diamond f - f| d\mu \leq 2\varepsilon + \varepsilon \mu(\mathcal{O}_n \setminus \mathcal{O}_n^-) \leq 2\varepsilon + \varepsilon \mu(\mathcal{O}_1)$$

which proves the result. \square

The previous result asserts that, for a given $f \in L^1(\Omega, d\mu)$, $\varrho_n \diamond f$ is also integrable ($n \in \mathbb{N}$). Actually, we shall see that $\varrho_n \diamond f$ is even more regular than f since, for almost every $\mathbf{x} \in \Omega$, the mapping $s \in (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \mapsto [\varrho_n \diamond f](\Phi(\mathbf{x}, s))$ is absolutely continuous. Precisely, let us fix $\delta > 0$, and define

$$\Omega_\delta = \left\{ \mathbf{x} \in \Omega; \min(\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) > \delta \right\}.$$

Approximating f by a continuous function with compact support on Ω , it is not difficult to prove that the mapping $t \in (-\delta, \delta) \mapsto f(\Phi(\cdot, t)) \in L^1(\Omega_\delta, d\mu)$ is continuous, i.e.

$$\lim_{h \rightarrow 0} \int_{\Omega_\delta} |f(\Phi(\mathbf{x}, t+h)) - f(\Phi(\mathbf{x}, t))| d\mu(x) = 0,$$

for any $f \in L^1(\Omega, d\mu)$. Now, since $\varrho_n \diamond f \in L^1(\Omega, d\mu)$, one also has

$$\lim_{h \rightarrow 0} \int_{\Omega_\delta} \left| [\varrho_n \diamond f](\Phi(\mathbf{x}, t+h)) - [\varrho_n \diamond f](\Phi(\mathbf{x}, t)) \right| d\mu(x) = 0.$$

On the other hand, one has

Lemma A.2. Given $f \in L^1(\Omega, d\mu)$, the mapping

$$t \in (-\delta, \delta) \mapsto [\varrho_n \diamond f](\Phi(\cdot, t)) \in L^1(\Omega_\delta)$$

is continuously differentiable with, for any $|t| < \delta$,

$$(A.7) \quad \lim_{h \rightarrow 0} h^{-1} \left([\varrho_n \diamond f](\Phi(\mathbf{x}, t+h)) - [\varrho_n \diamond f](\Phi(\mathbf{x}, t)) \right) = \int_{-\tau_-(\mathbf{x})}^t \varrho_n'(t-s) f(\Phi(\mathbf{x}, s)) ds$$

where the convergence holds in $L^1(\Omega_\delta, d\mu)$ norm.

Proof. For any $\mathbf{x} \in \Omega_\delta$ and any $|t| < \delta$, one has

$$[\varrho_n \diamond f](\Phi(\mathbf{x}, t)) = \int_{-\tau_-(\mathbf{x})}^t \varrho_n(t-r) f(\Phi(\mathbf{x}, r)) dr.$$

Consequently, for any real h such that $|t+h| < \delta$, one can define

$$S_t(\mathbf{x}, h) = h^{-1} \left([\varrho_n \diamond f](\Phi(\mathbf{x}, t+h)) - [\varrho_n \diamond f](\Phi(\mathbf{x}, t)) \right) - \int_{-\tau_-(\mathbf{x})}^t \varrho'_n(t-s) f(\Phi(\mathbf{x}, s)) ds.$$

Assuming for a while $h > 0$ leads

$$\begin{aligned} S_t(\mathbf{x}, h) &= S_{t,1}(\mathbf{x}, h) + S_{t,2}(\mathbf{x}, h) := h^{-1} \int_t^{t+h} \varrho_n(t+h-s) f(\Phi(\mathbf{x}, s)) ds \\ &\quad + \int_{-\tau_-(\mathbf{x})}^t \left[\frac{\varrho_n(t+h-s) - \varrho_n(t-s)}{h} - \varrho'_n(t-s) \right] f(\Phi(\mathbf{x}, s)) ds. \end{aligned}$$

Now, since $\Phi(\mathbf{x}, s)$ lies in Ω as long as $s \in (t, t+h)$, one obtains easily the estimate

$$\int_{\Omega_\delta} |S_{t,1}(\mathbf{x}, h)| d\mu(\mathbf{x}) \leq \frac{\|f\|}{h} \int_t^{t+h} \varrho_n(t+h-s) ds \longrightarrow 0 \quad \text{as } h \rightarrow 0.$$

On the other hand, for any $\mathbf{x} \in \Omega_{n,\delta} := \{\mathbf{x} \in \Omega_\delta; \tau_-(\mathbf{x}) > -t + 1/n\}$,

$$S_{t,2}(\mathbf{x}, h) = \int_{t-1/n}^t \left[\frac{\varrho_n(t+h-s) - \varrho_n(t-s)}{h} - \varrho'_n(t-s) \right] f(\Phi(\mathbf{x}, s)) ds$$

since $\varrho_n(\cdot)$ is supported in $(0, 1/n)$ and both $t-s$ and $t+h-s$ are larger than $1/n$ if $s \in (-\tau_-(\mathbf{x}), t-1/n)$. Then, it is easy to see that,

$$\int_{\Omega_{n,\delta}} |S_{t,2}(\mathbf{x}, h)| d\mu(\mathbf{x}) \leq \|f\| \int_{t-1/n}^t \left| \frac{\varrho_n(t+h-s) - \varrho_n(t-s)}{h} - \varrho'_n(t-s) \right| ds$$

which goes to 0 as $h \rightarrow 0$. Let us now investigate the remainder integral over $\Omega_\delta \setminus \Omega_{n,\delta}$. Note that, for any $\mathbf{x} \in \Omega_\delta \setminus \Omega_{n,\delta}$, one has $\tau_-(\mathbf{x}) \leq -t + 1/n$ and one can write $\mathbf{x} = \Phi(\mathbf{y}, r)$, for some $\mathbf{y} \in \Gamma_-$, with $\tau_+(\mathbf{y}) \geq 2\delta$, and some $r = \tau_-(\mathbf{x}) \geq \delta$. Moreover, $\tau_+(\mathbf{x}) = \tau_+(\mathbf{y}) - r \geq \delta$. Setting $\Gamma_{-,\delta} = \{\mathbf{y} \in \Gamma_-; \tau_+(\mathbf{y}) \geq 2\delta\}$, then

$$\int_{\Omega_\delta \setminus \Omega_{n,\delta}} |S_{t,2}(\mathbf{x}, h)| d\mu(\mathbf{x}) \leq \int_{\Gamma_{-,\delta}} d\mu_-(\mathbf{y}) \int_\delta^{(\delta+1/n) \wedge (\tau_+(\mathbf{y}) - \delta)} |S_{t,2}(\Phi(\mathbf{y}, r), h)| dr.$$

Setting $\vartheta_n(t, h, r, s) = |h^{-1}(\varrho_n(t + h + r - s) - \varrho_n(t + r - s)) - \varrho_n(t + r - s)|$, for any $t, r, s, h \geq 0$, one gets

$$\begin{aligned} & \int_{\Omega_\delta \setminus \Omega_{n,\delta}} |S_{t,2}(\mathbf{x}, h)| d\mu(\mathbf{x}) \\ & \leq \int_{\Gamma_{-, \delta}} d\mu_{-}(\mathbf{y}) \int_{\delta}^{(\delta+1/n) \wedge (\tau_+(\mathbf{y}) - \delta)} dr \int_0^{t+r} |f(\Phi(\mathbf{y}, s))| \vartheta_n(t, h, r, s) ds \\ & \leq \int_{\Gamma_{-, \delta}} d\mu_{-}(\mathbf{y}) \int_0^{t+\delta} |f(\Phi(\mathbf{y}, s))| ds \int_{\delta}^{(\delta+1/n) \wedge (\tau_+(\mathbf{y}) - \delta)} \vartheta_n(t, h, r, s) dr \\ & + \int_{\Gamma_{-, \delta}} d\mu_{-}(\mathbf{y}) \int_{t+\delta}^{(t+\delta+1/n) \wedge (t+\tau_+(\mathbf{y}) - \delta)} |f(\Phi(\mathbf{y}, s))| ds \int_{s-t}^{(\delta+1/n) \wedge (\tau_+(\mathbf{y}) - \delta)} \vartheta_n(t, h, r, s) dr. \end{aligned}$$

Given $\varepsilon > 0$, it is not difficult to see, using the uniform continuity of ϱ'_n over $(0, 1/n)$, that

$$\sup_{0 \leq t+r-s \leq 1/n} \vartheta_n(t, h, r, s) \leq \varepsilon \quad \text{for } h \text{ sufficiently small.}$$

For such an h one checks easily that

$$\int_{\Omega_\delta \setminus \Omega_{n,\delta}} |S_{t,2}(\mathbf{x}, h)| d\mu(\mathbf{x}) \leq \varepsilon/n \int_{\Gamma_{-}} d\mu_{-}(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} |f(\Phi(\mathbf{y}, s))| ds \leq \varepsilon \|f\|$$

which achieves to prove that

$$\int_{\Omega_\delta} |S_t(\mathbf{x}, h)| d\mu(\mathbf{x}) \longrightarrow 0 \quad \text{as } h \rightarrow 0.$$

One proceeds in the same way for negative h and this proves the Lemma. \square

Now, since the application

$$u_n : t \in (-\delta, \delta) \longmapsto [\varrho_n \diamond f](\Phi(\cdot, t)) \in L^1(\Omega_\delta)$$

is continuously differentiable, according to [8, Theorem 2.40],

- (i) for any $\mathbf{x} \in \Omega_\delta$, the function $t \in (-\delta, \delta) \mapsto \varrho_n \diamond f(\Phi(\mathbf{x}, t))$ is absolutely continuous,
- (ii) for almost every $(\mathbf{x}, t) \in \Omega_\delta \times (-\delta, \delta)$, $\partial_t[\varrho_n \diamond f](\Phi(\mathbf{x}, t))$ exists and

$$\partial_t[\varrho_n \diamond f](\Phi(\mathbf{x}, t)) = \int_{-\tau_-(\mathbf{x})}^t \varrho'_n(t-s) f(\Phi(\mathbf{x}, s)) ds$$

This allows to prove the following:

Proposition A.3. For any $f \in L^1(\Omega, d\mu)$, $\varrho_n \diamond f \in \mathcal{D}(\mathbf{T})$ with

$$\mathbf{T}[\varrho_n \diamond f](\mathbf{x}) = - \int_0^{\tau_-(\mathbf{x})} \varrho'_n(s) f(\Phi(\mathbf{x}, -s)) ds, \quad \text{a.e. } \mathbf{x} \in \Omega.$$

Proof. Set

$$g_n(\mathbf{x}) = \int_{-\tau_-(\mathbf{x})}^0 \varrho'_n(-s) f(\Phi(\mathbf{x}, s)) ds$$

for any $\mathbf{x} \in \Omega$. Again, one checks as in Lemma A.1 that $g_n(\cdot)$ belongs to $L^1(\Omega)$. Note also that, given $\delta > 0$, $g_n(\Phi(\mathbf{x}, t)) = \int_{-\tau_-(\mathbf{x})}^t \varrho_n(t-s) f(\Phi(\mathbf{x}, s)) ds$, for any $|t| < \delta$ and any $\mathbf{x} \in \Omega_\delta$. Let us prove as in Theorem 2.4 that, from points (i) and (ii) here above, $\varrho_n \diamond f \in \mathcal{D}(\mathbf{T})$ with $\mathbf{T}(\varrho_n \diamond f)(\mathbf{x}) = -g_n(\mathbf{x})$. The only problem is to prove that the conclusion of point (ii) holds for $t = 0$ and almost every $\mathbf{x} \in \Omega_\delta$ (recall that, *a priori* point (ii) holds for almost every $\mathbf{x} \in \Omega$ and almost every $t \in (-\delta, \delta)$). Actually, for any $0 < \epsilon < \delta$, there exists $|t_0| < \epsilon$ such that

$$\lim_{h \rightarrow 0} h^{-1} \left([\varrho_n \diamond f](\Phi(\mathbf{x}, t_0 + h)) - [\varrho_n \diamond f](\Phi(\mathbf{x}, t_0)) \right) = g_n(\Phi(\mathbf{x}, t_0)) \quad \text{for a.e. } \mathbf{x} \in \Omega_\delta.$$

Setting

$$\mathcal{N}_\delta = \left\{ \mathbf{x} \in \Omega_\delta; \lim_{h \rightarrow 0} h^{-1} \left([\varrho_n \diamond f](\Phi(\mathbf{x}, t_0 + h)) - [\varrho_n \diamond f](\Phi(\mathbf{x}, t_0)) \right) \neq g_n(\Phi(\mathbf{x}, t_0)) \right\}$$

and

$$\mathcal{M} = \left\{ \mathbf{y} \in \Omega_{\delta+\epsilon}; \lim_{h \rightarrow 0} h^{-1} \left([\varrho_n \diamond f](\Phi(\mathbf{y}, h)) - [\varrho_n \diamond f](\mathbf{y}) \right) \neq g_n(\mathbf{y}) \right\}$$

and noticing that $\Omega_{\delta+\epsilon} \subset \{\Phi(\mathbf{x}, t_0); \mathbf{x} \in \Omega_\delta\}$ we see that

$$\mathcal{M} \subset \Phi(\mathcal{N}_\delta, t_0).$$

According to Liouville's Theorem (Prop. 2.3), $\mu(\Phi(\mathcal{N}_\delta, t_0)) = \mu(\mathcal{N}_\delta) = 0$, so that

$$\lim_{h \rightarrow 0} h^{-1} \left([\varrho_n \diamond f](\Phi(\mathbf{y}, h)) - [\varrho_n \diamond f](\mathbf{y}) \right) = g_n(\mathbf{y}), \quad \text{for } \mu\text{-almost every } \mathbf{y} \in \Omega_{\delta+\epsilon}.$$

Since $\delta > 0$ and $0 < \epsilon < \delta$ are arbitrary, we get

$$\lim_{h \rightarrow 0} h^{-1} \left([\varrho_n \diamond f](\Phi(\mathbf{x}, h)) - [\varrho_n \diamond f](\mathbf{x}) \right) = g_n(\mathbf{x}), \quad \text{for } \mu\text{-almost every } \mathbf{x} \in \Omega.$$

Applying now [8, Theorem 2.40] to the *zero-th order time derivative* of $u_n(\cdot)$, we obtain that, for almost any $\mathbf{z} \in \Omega_\delta$, the function $t \in (-\delta, \delta) \mapsto \varrho_n \diamond f(\Phi(\mathbf{z}, t))$ is absolutely continuous with

$$[\varrho_n \diamond f](\Phi(\mathbf{z}, t_2)) - [\varrho_n \diamond f](\Phi(\mathbf{z}, t_1)) = \int_{t_1}^{t_2} g_n(\Phi(\mathbf{z}, s)) ds, \quad \forall -\delta < t_1 < t_2 < \delta.$$

In order to show that such an identity holds for any $\mathbf{x} \in \Omega$ and any $-\tau_-(\mathbf{x}) < t_1 < t_2 < \tau_+(\mathbf{x})$, one proceeds as follows. Let us first fix $\mathbf{x} \in \Omega_\delta$ and set $\mathbf{y} = \Phi(\mathbf{x}, -\delta/2)$, then $\tau_-(\mathbf{y}) > \delta/2$ and $\tau_+(\mathbf{y}) > 3\delta/2$. Then, applying the above identity to \mathbf{y} and $s_i = t_i + \delta/2$, $i = 1, 2$, one has

$$[\varrho_n \diamond f](\Phi(\mathbf{y}, s_2)) - [\varrho_n \diamond f](\Phi(\mathbf{y}, s_1)) = \int_{s_1}^{s_2} g_n(\Phi(\mathbf{y}, s)) ds, \quad \forall -\delta/2 < s_1 < s_2 < 3\delta/2.$$

By induction, one sees that for any $k \in \mathbb{N}$ and any $\mathbf{y} \in \Omega$ with $\tau_-(\mathbf{y}) > \delta/2^k$ and $\tau_+(\mathbf{y}) > 2\delta - \delta/2^k$, the above formula holds for any $-\delta/2^k < s_1 < s_2 < 2\delta - \delta/2^k$. Similarly, one sees

that the formula holds true for any $\mathbf{y} \in \Omega$ with $\tau_+(\mathbf{y}) > \delta/2^k$ and $\tau_-(\mathbf{y}) > 2\delta - \delta/2^k$ and any $2\delta - \delta/2^k < s_1 < s_2 < -\delta/2^k$.

This shows finally that, for any $\mathbf{x} \in \Omega$ with $\tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) > 2\delta$ and any $-\tau_-(\mathbf{x}) < \underline{t} < \bar{t} < \tau_+(\mathbf{x})$ with $\bar{t} - \underline{t} < 2\delta$, the function $t \in (\underline{t}, \bar{t}) \mapsto \varrho_n \diamond f(\Phi(\mathbf{x}, t))$ is absolutely continuous with

$$[\varrho_n \diamond f](\Phi(\mathbf{x}, s_2)) - [\varrho_n \diamond f](\Phi(\mathbf{x}, s_1)) = \int_{s_1}^{s_2} g_n(\Phi(\mathbf{x}, s)) ds, \quad \text{for any } \underline{t} < s_1 < s_2 < \bar{t}.$$

Since $\delta > 0$ is arbitrary, the formula holds for any $\mathbf{x} \in \Omega$ and any $-\tau_-(\mathbf{x}) < t_1 < t_2 < \tau_+(\mathbf{x})$. By definition of $(\mathbf{T}, \mathcal{D}(\mathbf{T}))$, the proof is achieved. \square

According to the above Lemma, since \mathbf{T} is an extension of \mathcal{T}_{\max} , one has that $\varrho_n \diamond f \in \mathcal{D}(\mathcal{T}_{\max})$ for any $f \in L^1(\Omega, d\mu)$ and

$$\mathcal{T}_{\max}[\varrho_n \diamond f](\mathbf{x}) = - \int_0^{\tau_-(\mathbf{x})} \varrho_n'(s) f(\Phi(\mathbf{x}, -s)) ds.$$

Now, whenever $f \in \mathcal{D}(\mathcal{T}_{\max})$, one has the following more precise result:

Proposition A.4. If $f \in \mathcal{D}(\mathcal{T}_{\max})$, then $\varrho_n \diamond f \in \mathcal{D}(\mathcal{T}_{\max})$ with

$$(A.8) \quad [\mathcal{T}_{\max}(\varrho_n \diamond f)](\mathbf{x}) = [\varrho_n \diamond \mathcal{T}_{\max} f](\mathbf{x}), \quad (\mathbf{x} \in \Omega, n \in \mathbb{N}).$$

Proof. Since $\varrho_n \diamond \mathcal{T}_{\max} f \in L^1(\Omega, d\mu)$, it suffices to show that

$$\int_{\Omega} (\varrho_n \diamond f)(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega} \psi(\mathbf{x}) [\varrho_n \diamond \mathcal{T}_{\max} f](\mathbf{x}) d\mu(\mathbf{x}), \quad \forall \psi \in \mathfrak{Y}.$$

It appears useful to deal separately with the integrals over Ω_- , $\Omega_+ \cap \Omega_{-\infty}$ and $\Omega_{+\infty} \cap \Omega_{-\infty}$. Let $\psi \in \mathfrak{Y}$ be fixed. It is clear that, for any $\mathbf{y} \in \Gamma_-$ and any $0 < t < \tau_+(\mathbf{y})$, when $\mathbf{x} = \Phi(\mathbf{y}, t)$ then

$$(A.9) \quad \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} = \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)).$$

Consequently, from Proposition 2.10,

$$\begin{aligned} \int_{\Omega_-} (\varrho_n \diamond f)(\mathbf{x}) \frac{d}{ds} \psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) &= \int_{\Gamma_-} d\mu(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} (\varrho_n \diamond f)(\Phi(\mathbf{y}, t)) \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \\ &= \int_{\Gamma_-} d\mu(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) dt \int_0^t \varrho_n(t-r) f(\Phi(\mathbf{y}, r)) dr \\ &= \int_{\Gamma_-} d\mu(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r)) dr \int_r^{\tau_+(\mathbf{y})} \frac{d}{dt} \psi(\Phi(\mathbf{y}, t)) \varrho_n(t-r) dt. \end{aligned}$$

Set

$$\chi_n(\mathbf{x}) = \int_0^{\tau_+(\mathbf{x})} \varrho_n(s) \psi(\Phi(\mathbf{x}, s)) ds.$$

Clearly, for almost every $\mathbf{y} \in \Gamma_-$, and any $0 < s < \tau_+(\mathbf{y})$,

$$(A.10) \quad \chi_n(\Phi(\mathbf{y}, s)) = \int_0^{\tau_+(\mathbf{y})-s} \varrho_n(r) \psi(\Phi(\mathbf{y}, r+s)) ds = \int_s^{\tau_+(\mathbf{y})} \varrho_n(t-s) \psi(\Phi(\mathbf{y}, t)) dt,$$

and one obtains easily that

$$\frac{d}{ds}\chi_n(\Phi(\mathbf{y}, s)) = \int_s^{\tau_+(\mathbf{y})} \varrho_n(t-s) \frac{d}{dt}[\psi(\Phi(\mathbf{y}, t))] dt.$$

Therefore,

$$\begin{aligned} \int_{\Omega_-} (\varrho_n \diamond f)(\mathbf{x}) \frac{d}{ds}\psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) &= \int_{\Gamma_-} d\mu(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\Phi(\mathbf{y}, r)) \frac{d}{dr}\chi_n(\Phi(\mathbf{y}, r)) dr \\ &= \int_{\Omega_-} f(\mathbf{x}) \frac{d}{ds}\chi_n(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) = \int_{\Omega_-} \chi_n(\mathbf{x}) [\mathcal{T}_{\max} f](\mathbf{x}) d\mu(\mathbf{x}) \end{aligned}$$

where, for the two last identities, we used (A.9) and the fact that $\chi_n \in \mathfrak{Y}$. Now,

$$\int_{\Omega_-} \chi_n(\mathbf{x}) [\mathcal{T}_{\max} f](\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega_-} \mathcal{T}_{\max} f(\mathbf{x}) d\mu(\mathbf{x}) \int_0^{\tau_+(\mathbf{x})} \varrho_n(r) \psi(\Phi(\mathbf{x}, r)) dr.$$

The use of representation formulae stated in Proposition 2.10 leads to

$$\begin{aligned} \text{(A.11)} \quad \int_{\Omega_-} \chi_n(\mathbf{x}) [\mathcal{T}_{\max} f](\mathbf{x}) d\mu(\mathbf{x}) &= \int_{\Omega_-} \psi(\mathbf{z}) d\mu(\mathbf{z}) \int_0^{\tau_-(\mathbf{z})} \varrho_n(r) [\mathcal{T}_{\max} f](\Phi(\mathbf{z}, -r)) dr \\ &= \int_{\Omega_-} \psi(\mathbf{z}) [\varrho_n \diamond \mathcal{T}_{\max} f](\mathbf{z}) d\mu(\mathbf{z}). \end{aligned}$$

Consider now the integral over $\Omega_{-\infty} \cap \Omega_+$. From Proposition 2.10,

$$\begin{aligned} \int_{\Omega_{-\infty} \cap \Omega_+} (\varrho_n \diamond f)(\mathbf{x}) \frac{d}{ds}\psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) &= \\ &= - \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^\infty (\varrho_n \diamond f)(\Phi(\mathbf{y}, -t)) \frac{d}{dt}\psi(\Phi(\mathbf{y}, -t)) dt \\ &= - \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^\infty \frac{d}{dt}\psi(\Phi(\mathbf{y}, -t)) dt \int_0^\infty \varrho_n(s) f(\Phi(\mathbf{y}, -t-s)) ds \\ &= - \int_{\Gamma_{+\infty}} d\mu_+(\mathbf{y}) \int_0^\infty \frac{d}{dt}\psi(\Phi(\mathbf{y}, -t)) dt \int_t^\infty \varrho_n(r-t) f(\Phi(\mathbf{y}, -r)) dr. \end{aligned}$$

Integration by parts shows again that

$$\begin{aligned} \text{(A.12)} \quad \int_{\Omega_{-\infty} \cap \Omega_+} (\varrho_n \diamond f)(\mathbf{x}) \frac{d}{ds}\psi(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) &= \int_{\Omega_{-\infty} \cap \Omega_+} f(\mathbf{x}) \frac{d}{ds}\chi_n(\Phi(\mathbf{x}, s)) \Big|_{s=0} d\mu(\mathbf{x}) \\ &= \int_{\Omega_{-\infty} \cap \Omega_+} \chi_n(\mathbf{x}) [\mathcal{T}_{\max} f](\mathbf{x}) d\mu(\mathbf{x}) = \int_{\Omega_{-\infty} \cap \Omega_+} \psi(\mathbf{z}) [\varrho_n \diamond \mathcal{T}_{\max} f](\mathbf{z}) d\mu(\mathbf{z}). \end{aligned}$$

Finally, to deal with the integral over $\Omega_{-\infty} \cap \Omega_{+\infty}$, set $g(\mathbf{x}) = \frac{d}{ds}\psi(\Phi(\mathbf{x}, s))\Big|_{s=0}$ for any $\mathbf{x} \in \Omega_{-\infty} \cap \Omega_{+\infty}$. One proves easily that

$$\begin{aligned} \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} (\varrho_n \diamond f)(\mathbf{x})g(\mathbf{x})d\mu(\mathbf{x}) &= \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g(\mathbf{x})d\mu(\mathbf{x}) \int_0^\infty f(\Phi(\mathbf{x}, -s))\varrho_n(s)ds \\ &= \int_0^\infty \varrho_n(s)ds \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} g(\mathbf{x})f(\Phi(\mathbf{x}, -s))d\mu(\mathbf{x}) \end{aligned}$$

which, thanks to Liouville's property, leads to

$$\begin{aligned} \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} (\varrho_n \diamond f)(\mathbf{x})g(\mathbf{x})d\mu(\mathbf{x}) &= \int_0^\infty \varrho_n(s)ds \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} f(\mathbf{x})g(\Phi(\mathbf{x}, s))d\mu(\mathbf{x}) \\ &= \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} f(\mathbf{x}) \int_0^\infty \varrho_n(s)g(\Phi(\mathbf{x}, s))ds. \end{aligned}$$

Setting now

$$\vartheta_n(\mathbf{x}) = \int_0^\infty \varrho_n(t)\psi(\Phi(\mathbf{x}, t))dt$$

one sees that $\vartheta_n \in \mathfrak{V}$ and

$$\frac{d}{ds}\vartheta_n(\Phi(\mathbf{x}, s))\Big|_{s=0} = \int_0^\infty \varrho_n(t)g(\Phi(\mathbf{x}, t))dt.$$

Therefore, the above identity reads:

$$\begin{aligned} \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} (\varrho_n \diamond f)(\mathbf{x})\frac{d}{ds}\psi(\Phi(\mathbf{x}, s))\Big|_{s=0} d\mu(\mathbf{x}) &= \\ \text{(A.13)} \quad &= \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} f(\mathbf{x})\frac{d}{ds}\vartheta_n(\Phi(\mathbf{x}, s))\Big|_{s=0} d\mu(\mathbf{x}) \\ &= \int_{\Omega_{-\infty} \cap \Omega_{+\infty}} \psi(\mathbf{x})[\varrho_n \diamond \mathcal{T}_{\max}f](\mathbf{x})d\mu(\mathbf{x}). \end{aligned}$$

Combining (A.11), (A.12) and (A.13) leads to (A.8). \square

The above results allow now to prove our main result Proposition A.1 (see also Prop. 3.2). Recall that we only have to prove that any $f \in \mathcal{D}(\mathcal{T}_{\max})$ belongs also to $\mathcal{D}(\mathbf{T})$ and $\mathbf{T}f = \mathcal{T}_{\max}f$. Let us fix $f \in \mathcal{D}(\mathcal{T}_{\max})$ and define, for any $n \geq 1$,

$$f_n = \varrho_n \diamond f$$

so that, from Propositions A.3 and A.4

$$\text{(A.14)} \quad \lim_{n \rightarrow \infty} \|f_n - f\| + \|\mathcal{T}_{\max}f_n - \mathcal{T}_{\max}f\| = 0.$$

Recall that we set

$$\Omega_\delta = \{\mathbf{x} \in \Omega; \min(\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) > \delta\}$$

for any fixed $\delta > 0$. Define now, for any $t \in (-\delta, \delta)$ and any h such that $|t + h| < \delta$

$$\mathcal{S}(f, h) := \int_{\Omega_\delta} \left| f(\Phi(\mathbf{x}, t + h)) - f(\Phi(\mathbf{x}, t)) + \int_t^{t+h} \mathcal{T}_{\max} f(\Phi(\mathbf{x}, s)) ds \right| d\mu(\mathbf{x}).$$

Then, it is easy to see that, with obvious notations,

$$\mathcal{S}(f, h) \leq \mathcal{S}(f - f_n, h) + \mathcal{S}(f_n, h), \quad \forall n \geq 1.$$

Note that

$$\mathcal{S}(f_n, h) = 0, \quad \forall n \in \mathbb{N}, |t + h| < \delta.$$

Indeed, according to Prop. A.4, $f_n \in \mathcal{D}(\mathbf{T})$ so that

$$f_n(\Phi(\mathbf{x}, t + h)) - f_n(\Phi(\mathbf{x}, t)) = - \int_t^{t+h} [\mathbf{T}f_n](\Phi(\mathbf{x}, s)) ds, \quad \text{a.e. } \mathbf{x} \in \Omega, \quad |t + h| < \delta.$$

Since $\mathbf{T}f_n = \mathcal{T}_{\max} f_n$, one gets that $\mathcal{S}(f_n, h) = 0$. Moreover, according to (A.14), and since $\mathcal{T}_{\max} f_n = \varrho_n \diamond \mathcal{T}_{\max} f$, one has

$$\lim_{n \rightarrow \infty} \mathcal{S}(f - f_n, h) = 0.$$

Therefore,

$$(A.15) \quad \mathcal{S}(f, h) = 0 \quad \forall t + h \in (-\delta, \delta).$$

In particular, letting h go to zero, one deduces that

$$\lim_{h \rightarrow 0} \int_{\Omega_\delta} \left| \frac{f(\Phi(\mathbf{x}, t + h)) - f(\Phi(\mathbf{x}, t))}{h} + \mathcal{T}_{\max} f(\Phi(\mathbf{x}, t)) \right| d\mu(\mathbf{x}) = 0.$$

Here again, since δ is arbitrary, this proves that

$$\lim_{h \rightarrow 0} \frac{f(\Phi(\cdot, t + h)) - f(\Phi(\cdot, t))}{h} = -\mathcal{T}_{\max} f(\Phi(\cdot, t)),$$

where the convergence occurs in $L^1(\Omega, d\mu)$. Therefore, $\mathcal{T}_{\max} f(\Phi(\cdot, t))$ is the strong derivative in the L^1 -sense of $f(\Phi(\cdot, t))$. Applying again [8, Theorem 2.40], one can conclude as we did in the proof of Proposition A.3. that actually the limit occurs also pointwise almost everywhere for $t = 0$, i.e.

$$\lim_{h \rightarrow 0} \frac{f(\Phi(\mathbf{x}, h)) - f(\Phi(\mathbf{x}, t))}{h} = -\mathcal{T}_{\max} f(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Omega_\delta.$$

Moreover, by virtue of Eq. (A.15), one has

$$f(\Phi(\mathbf{x}, t + h)) - f(\Phi(\mathbf{x}, t)) = \int_t^{t+h} [\mathcal{T}_{\max} f](\Phi(\mathbf{x}, s)) ds$$

for almost every $\mathbf{x} \in \Omega_\delta$ and any $t \in (-\delta, \delta)$ and $|t + h| < \delta$. Again, since δ is arbitrary, the mild formulation of Proposition 3.2 (1) holds true with $g = -\mathcal{T}_{\max} f$. Therefore, $f \in \mathcal{D}(\mathbf{T})$ with $\mathbf{T}f = \mathcal{T}_{\max} f$.

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