Semigroups for general transport equations with abstract boundary conditions

Luisa Arlotti, Jacek Banasiak, Bertrand Lods

SEMIGROUPS FOR GENERAL TRANSPORT EQUATIONS WITH ABSTRACT BOUNDARY CONDITIONS

L. ARLOTTI, J. BANASIAK & B. LODS

ABSTRACT. We investigate C_0-semigroup generation properties of the Vlasov equation with general boundary conditions modeled by an abstract boundary operator H. For multiplicative boundary conditions we adapt techniques from [14] and in the case of conservative boundary conditions we show that there is an extension $\mathbf{A}$ of the free streaming operator $\mathbf{T}_H$ which generates a C_0-semigroup $(V_H(t))_{t\geqslant 0}$ in L^1. Furthermore, following the ideas of [4], we precisely describe its domain and provide necessary and sufficient conditions ensuring that $(V_H(t))_{t\geqslant 0}$ is stochastic.

1. INTRODUCTION

Let us consider the general transport equation

$$\frac{\partial}{\partial t}f(\mathbf{x},t)+\mathcal{F}(\mathbf{x})\cdot\nabla_{\mathbf{x}}f(\mathbf{x},t)=0 \qquad (\mathbf{x}\in\mathbf{\Omega},\ t>0), \tag{1.1a}$$

supplemented by the abstract boundary condition

$$f_{|\Gamma_-}(\mathbf{y},t)=H(f_{|\Gamma_+})(\mathbf{y},t), \qquad (\mathbf{y}\in\Gamma_-,t>0), \tag{1.1b}$$

and the initial condition

$$f(\mathbf{x},0)=f_0(\mathbf{x}), \qquad (\mathbf{x}\in\mathbf{\Omega}). \tag{1.1c}$$

Here $\mathbf{\Omega}$ is a smooth open subset of $\mathbb{R}^N$ endowed with a positive Radon measure $\mathrm{d}\mu(\cdot)$, $\Gamma_\pm$ are suitable boundaries of the phase space and the boundary operator H is a linear *bounded* operator between trace spaces $L^1_\pm$ corresponding to the boundaries $\Gamma_\pm$ (see Section 2 for details). The transport coefficient $\mathcal{F}$ is a *time independent* vector field $\mathcal{F}\ :\ \overline{\mathbf{\Omega}}\to\mathbb{R}^N$ satisfying the following general assumptions:

Assumption $\mathbf{H}_1$) $\mathcal{F}\ :\ \overline{\mathbf{\Omega}}\longrightarrow\mathbb{R}^N$ is *Lipschitz-continuous* with Lipschitz constant $\kappa>0$, i.e.

$$|\mathcal{F}(\mathbf{x}_1)-\mathcal{F}(\mathbf{x}_2)|\leqslant\kappa|\mathbf{x}_1-\mathbf{x}_2| \qquad \text{for any } \mathbf{x}_1,\mathbf{x}_2\in\overline{\mathbf{\Omega}}.$$

Assumption $\mathbf{H}_2$) The field $\mathcal{F}$ is *divergence-free* with respect to μ in the sense that

$$\int_{\mathbf{\Omega}}\mathcal{F}(\mathbf{x})\cdot\nabla_{\mathbf{x}}f(\mathbf{x})\mathrm{d}\mu(\mathbf{x})=0$$

for any Lipschitz continuous function f with compact support on $\mathbf{\Omega}$.

Keywords: Transport equation, Boundary conditions, C_0-semigroups, Characteristic curves.
AMS subject classifications (2000): 47D06, 47D05, 47N55, 35F05, 82C40.

A typical example of such a transport equation is the so-called Vlasov equation for which:

i) The phase space $\mathbf{\Omega}$ is given by the cylindrical domain $\mathbf{\Omega} = \mathcal{D} \times V \subset \mathbb{R}^6$ where $\mathcal{D}$ is a smooth open subset of $\mathbb{R}^3$, referred to as the *position space*, and V is a closed subset of $\mathbb{R}^3$, referred to as the *velocity space*. The measure $\mathrm{d}\mu$ is given by $\mathrm{d}\mu(\mathbf{x}) = \mathrm{d}x\mathrm{d}\beta(v)$ where $\mathrm{d}\beta(\cdot)$ is a suitable Radon measure on V.

ii) For any $\mathbf{x} = (x, v) \in \mathcal{D} \times V$,

$$\mathcal{F}(\mathbf{x}) = (v, \mathbf{F}(x, v)) \in \mathbb{R}^6 \tag{1.2}$$

where $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$ is a time independent force field over $\mathcal{D} \times V$ such that $\mathbf{H}_1)$ and $\mathbf{H}_2)$ are fulfilled.

The existence of solution to the transport equation (1.1a) is a classical matter when considering the whole space $\mathbf{\Omega} = \mathbb{R}^N$. In particular, the concept of renormalized solutions allows to consider irregular transport coefficient $\mathcal{F}(\cdot)$ (see [9] and the recent contributions [1, 13]) which is of particular relevance in fluid mechanics.

On the other hand, there are few results addressing the initial-boundary value problem (1.1), possibly due to difficulties created by the boundary conditions (1.1c). We mention here the seminal works by C. Bardos [6], and by R. Beals and V. Protopopescu [7] (see also [12]). Let us also mention more recent contributions [15] which also includes time-dependent transport coefficient, and [4, 14] dealing with the force-free ($\mathbf{F} \equiv 0$) Vlasov equation (1.2).

For $\mathbf{F} \neq 0$, the method of Beals and Protopopescu [7] provides the existence and a very precise description of a C_0-semigroup governing (1.1) for $\|H\| < 1$ while, for nonnegative boundary operator H with $\|H\| = 1$, it ensures the existence of a C_0-semigroup related to (1.1) without describing its generator. The method of [7] leaves totally open the multiplicative case $\|H\| > 1$. We also mention that the existing theories introduce restrictive assumptions on the characteristics of the equation. For instance, fields with 'too many' periodic trajectories create serious difficulties. They are however covered in a natural way by the theory presented here.

On the other hand, in the force-free case $\mathbf{F} = 0$, the case of conservative boundary conditions $\|H\| = 1$ has been solved in [4], while the multiplicative case has been addressed in [14]. The results of [4, 14] are based upon a relatively simple representation, inspired by the fundamental work of [16], of the resolvent of the free-streaming operator $\mathbf{T}_H$ (whose domain includes the boundary conditions (1.1c)) as a strongly convergent series.

The main objective of this work is to generalize the results of [4] and [14] to the general case $\mathbf{F} \neq 0$. Here again, the key ingredient is the derivation of a suitable representation of the resolvent of the free-streaming operator $\mathbf{T}_H$, see Theorem 3.5 and Proposition 3.9. We point out that the proof of Theorem 3.5 relies on a generalization of the result from [7, 12] which allows to compute integrals over $\mathbf{\Omega}$ via integration along the integral curves of $\mathcal{F}(\cdot)$ coming from the boundary $\partial\mathbf{\Omega}$, and which is free from some restrictive assumptions of *op. it.* In particular, we present a new proof of the Green formula which clarifies some points of the proofs in [7, 12].

The organization of the paper is as follows. In the following section (Section 2), we introduce the main tools used throughout the paper and present the aforementioned new results concerning *integration over the characteristic curves* of $\mathcal{F}$. Section 3 is dealing with various preliminary results. In particular, the question of the existence of trace results is addressed in Section 3.1. In

Section 3.2, we establish some basic existence results, mainly pertaining to *stationary versions* of (1.1) and, as a consequence, we derive a new proof of *Green formula*, generalizing that of [7, 12]. Section 3.3 is concerned with the *setting of the problem* and with the representation of the resolvent of the free-streaming operator $\mathbf{T}_H$ as a strongly convergent series. In Section 4, we are dealing with the transport equation (1.1) for *multiplicative boundary conditions*, generalizing the results of [14]. Finally, in Section 5, we consider the delicate question of *conservative boundary conditions*. We employ a strategy already used in [4], borrowing some tools to the so–called additive perturbation theory of substochastic semigroups [5].

Notations: We shall adopt the following notations throughout this paper: for any unbounded operator A, the domain of A shall be denoted by $\mathscr{D}(A)$ whereas $\sigma_p(A)$, $\sigma_r(A)$, $\sigma_c(A)$ and $\varrho(A)$ will denote respectively the point spectrum, the residual spectrum, the continuous spectrum and the resolvent set of A. For $\lambda \in \varrho(A)$, the resolvent of A will be denoted by either $\mathscr{R}(\lambda, A)$ or $(\lambda - A)^{-1}$. For any Banach spaces X and Y, $\mathscr{B}(X, Y)$ denotes the space of bounded linear operators form X to Y whereas $\mathscr{B}(X) = \mathscr{B}(X, X)$.

2. Integration along the characteristics

2.1. **Characteristic curves.** A crucial role in our study is played by the characteristic curves associated to the field

$$\mathcal{F} \,:\, \mathbf{x} \in \overline{\mathbf{\Omega}} \longmapsto \mathcal{F}(\mathbf{x}) \in \mathbb{R}^N.$$

Precisely, for any $\mathbf{x} \in \mathbf{\Omega}$ and $t \in \mathbb{R}$, consider the initial-value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\mathbf{X}(s) = \mathcal{F}(\mathbf{X}(s)), \qquad (s \in \mathbb{R}), \\ \mathbf{X}(t) = \mathbf{x}. \end{cases} \tag{2.1}$$

Since $\mathcal{F}$ is Lipschitz-continuous on $\overline{\mathbf{\Omega}}$, Eq. (2.1) has a unique solution and this allows to define the mapping $\mathbf{\Phi} : \mathbf{\Omega} \times \mathbb{R} \times I_{\mathbf{x},t} \to \mathbf{\Omega}$, $I_{\mathbf{x},t} \subset \mathbb{R}$, such that, for $(\mathbf{x}, t) \in \mathbf{\Omega} \times \mathbb{R}$, the mapping:

$$\mathbf{X}(\cdot) \,:\, s \in I_{\mathbf{x},t} \longmapsto \mathbf{\Phi}(\mathbf{x}, t, s)$$

is the only solution of Eq. (2.1). Note that, in general, $\mathbf{X}$ is only defined on a suitable neighborhood $I_{\mathbf{x},t}$ of the initial time t, which also depends on $\mathbf{x}$. This leads to the definition of existence times of the characteristic curves:

Definition 2.1. *For any $\mathbf{x} \in \mathbf{\Omega}$, define*

$$\tau_\pm(\mathbf{x}) = \inf\{s > 0\,; \mathbf{\Phi}(\mathbf{x}, 0, \pm s) \notin \mathbf{\Omega}\},$$

with the convention that $\inf \varnothing = \infty$, and set $\tau(\mathbf{x}) = \tau_+(\mathbf{x}) + \tau_-(\mathbf{x})$.

To shorten notation we put $I_{\mathbf{x}} = I_{\mathbf{x},0}$. In other words, $I_{\mathbf{x}} = (-\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$ is the maximal interval for which $\mathbf{\Phi}(\mathbf{x}, 0, s)$ lies in $\mathbf{\Omega}$ for any $s \in I_{\mathbf{x}}$ and $\tau(\mathbf{x})$ is the length of the interval $I_{\mathbf{x}}$. Notice that $0 \leqslant \tau_\pm(\mathbf{x}) \leqslant \infty$. Thus, for any $t \in \mathbb{R}$, the function $\mathbf{\Phi}$ is well-defined on the set

$$\{(\mathbf{x}, t, s)\,;\, \mathbf{x} \in \mathbf{\Omega},\, t \in \mathbb{R}\,,\, s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))\,\}\,.$$

Note that here we **do not** assume that the length of the interval $I_{\mathbf{x}} = (-\tau_-(\mathbf{x}),\, \tau_+(\mathbf{x}))$ is **finite**. In particular, $I_{\mathbf{x}} = \mathbb{R}$ for any stationary point $\mathbf{x}$ of $\mathcal{F}$, i.e. $\mathcal{F}(\mathbf{x}) = 0$. If $\tau(\mathbf{x})$ is finite, then the

function $\mathbf{X} : s \in I_{\mathbf{x}} \longmapsto \mathbf{\Phi}(\mathbf{x}, 0, s)$ is bounded since $\mathcal{F}$ is Lipschitz-continuous on $\overline{\mathbf{\Omega}}$. Moreover, still by virtue of the Lipschitz-continuity of $\mathcal{F}$ on $\overline{\mathbf{\Omega}}$, the only case when $\tau_\pm(\mathbf{x})$ is finite is when $\mathbf{\Phi}(\mathbf{x}, 0, \pm s)$ reaches the boundary $\partial\mathbf{\Omega}$. Then, we define $\mathbf{\Phi}(\mathbf{x}, 0, \tau_\pm(\mathbf{x})) \in \partial\mathbf{\Omega}$ as the following limit

$$\mathbf{\Phi}(\mathbf{x}, 0, \tau_\pm(\mathbf{x})) = \lim_{s \to \pm\tau_\pm(\mathbf{x})} \mathbf{\Phi}(\mathbf{x}, 0, s).$$

We note that, since $\mathcal{F}$ can be extended to a Lipschitz field (at least locally) around each point of $\partial\mathbf{\Omega}$, the points of the set $\{\mathbf{y} \in \partial\mathbf{\Omega}\,;\, \mathcal{F}(\mathbf{y}) = 0\}$ (introduced in [7, 12]) are equilibrium points of the extended field and cannot be reached in finite time.

Remark 2.2. *We emphasize that periodic trajectories which do not meet the boundaries have $\tau_\pm = \infty$ and thus are treated as infinite though geometrically they are bounded. Of course, in this case, the limit $\lim_{s \to \pm\tau_\pm(\mathbf{x})} \mathbf{\Phi}(\mathbf{x}, 0, s)$ does not exist for any $\mathbf{x}$.*

We finally mention that it is not difficult to prove that the mappings $\tau_\pm : \mathbf{\Omega} \to \mathbb{R}^+$ are lower semicontinuous and therefore measurable, see e.g., [5, p. 301]

The flow $\mathbf{\Phi}(\mathbf{x}, t, s)$ defines, at each instant t, a mapping of the phase space $\mathbf{\Omega}$ into itself. Through this mapping, to each point $\mathbf{x}$ there corresponds the point $\mathbf{x}_{s,t} = \mathbf{\Phi}(\mathbf{x}, t, s)$ reached at time s by a point which was in in $\mathbf{x}$ at the "initial" time t. This mapping is *one–to–one* and *measure-preserving* (Liouville's Theorem). More precisely, one can check that the flow $\mathbf{\Phi}$, defined on its maximal domain, has the following properties:

Proposition 2.3. *Let $\mathbf{x} \in \mathbf{\Omega}$ and $t \in \mathbb{R}$ be fixed. Then,*

i) $\mathbf{\Phi}(\mathbf{x}, t, t) = \mathbf{x}$.
ii) $\mathbf{\Phi}(\mathbf{\Phi}(\mathbf{x}, t, s_1), s_1, s_2) = \mathbf{\Phi}(\mathbf{x}, t, s_2), \quad \forall s_1, s_2 \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
iii) $\mathbf{\Phi}(\mathbf{x}, t, s) = \mathbf{\Phi}(\mathbf{x}, t - s, 0) = \mathbf{\Phi}(\mathbf{x}, 0, s - t), \qquad \forall s \in (t - \tau_-(\mathbf{x}), t + \tau_+(\mathbf{x}))$.
iv) $|\mathbf{\Phi}(\mathbf{x}_1, t, s) - \mathbf{\Phi}(\mathbf{x}_2, t, s)| \leqslant \exp(\kappa|t - s|)|\mathbf{x}_1 - \mathbf{x}_2|$ *for any* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{\Omega}$, $s - t \in I_{\mathbf{x}_1} \cap I_{\mathbf{x}_2}$.
v) For any $t, s \in \mathbb{R}$, the transformation $\mathbf{x} \in \mathbf{\Omega} \mapsto \mathbf{\Phi}(\mathbf{x}, t, s) \in \mathbf{\Omega}$ is measure-preserving with respect to μ (Liouville's Theorem).

Remark 2.4. *Note that Liouville's Theorem is equivalent to assumption $\mathbf{H}_2$) above. For instance, when $\mathrm{d}\mu$ is the Lebesgue measure over $\mathbb{R}^N$, then assumption $\mathbf{H}_2$) means precisely that $\mathrm{div}(\mathcal{F}(\mathbf{x})) = 0$ for any $\mathbf{x} \in \mathbf{\Omega}$ or, equivalently, that, for any $t, s \in \mathbb{R} \times I_{\mathbf{x},t}$, the Jacobian of the transformation $\mathbf{x} \mapsto \mathbf{\Phi}(\mathbf{x}, t, s)$ equals one .*

We define the incoming and outgoing part of the boundary $\partial\mathbf{\Omega}$ through the flow $\mathbf{\Phi}$:

Definition 2.5. *The incoming Γ_- and the outgoing Γ_+ parts of the boundary $\partial\mathbf{\Omega}$ are defined by:*

$$\Gamma_\pm := \{\mathbf{y} \in \partial\mathbf{\Omega}\,;\, \exists \mathbf{x} \in \mathbf{\Omega},\ \tau_\pm(\mathbf{x}) < \infty \text{ and } \mathbf{y} = \mathbf{\Phi}(\mathbf{x}, 0, \pm\tau_\pm(\mathbf{x}))\}\,. \tag{2.2}$$

Properties of $\mathbf{\Phi}$ and of $\tau_\pm$ imply that $\Gamma_\pm$ are Borel sets. It is possible to extend the definition of $\tau_\pm$ to $\Gamma_\pm$ as follows. If $\mathbf{x} \in \Gamma_-$ then we put $\tau_-(\mathbf{x}) = 0$ and denote $\tau_+(\mathbf{x})$ the length of the integral curve having $\mathbf{x}$ as its left end–point; similarly if $\mathbf{x} \in \Gamma_+$ then we put $\tau_+(\mathbf{x}) = 0$ and denote $\tau_-(\mathbf{x})$ the length of the integral curve having $\mathbf{x}$ as its right end–point. For technical reasons, we need to introduce the following assumption on the measure $\mathrm{d}\mu$:

Assumption $\mathbf{H}_3$) The set of integral curves hitting $\Gamma_- \cap \Gamma_+$ is of zero μ-measure, i.e.

$$\mathrm{d}\mu\left(\left\{\mathbf{x}\in\mathbf{\Omega}\,;\,\tau_\pm(\mathbf{x})<\infty\,;\,\mathbf{\Phi}(\mathbf{x},0,\tau_\pm(\mathbf{x}))\in\Gamma_-\cap\Gamma_+\right\}\right)=0.$$

Using Sard's theorem and arguing as in [6], one can prove that Assumption $\mathbf{H}_3$) is fulfilled if $\mathrm{d}\mu(\cdot)$ is absolutely continuous with respect to the Lebesgue measure over $\mathbb{R}^N$. Note however that, since the field $\mathcal{F}(\cdot)$ is only Lipschitz continuous, one has to invoke a generalized version of Sard's theorem for Hölder maps (see e.g. [8, Theorem 1.4]). In the same way, Assumption $\mathbf{H}_3$) is satisfied by suitable Hausdorff measures over manifolds of $\mathbb{R}^N$ (see [8]).

The main aim of the present discussion is to represent $\mathbf{\Omega}$ as a collection of characteristics running between points of Γ_- and Γ_+ so that the integral over $\mathbf{\Omega}$ can be split into integrals over Γ_- (or Γ_+) and along the characteristics. However, we cannot do this in a precise way now since, in general, the sets Γ_+ and Γ_- do not provide a partition of $\partial\mathbf{\Omega}$. In spite of assumption $\mathbf{H}_3$), there still may be too many characteristics which extend to infinity on either side. Since we have not assumed $\mathbf{\Omega}$ to be bounded, Γ_- or Γ_+ may be empty and also we may have characteristics running from $-\infty$ to $+\infty$ such as periodic ones. Thus, in general characteristics starting from Γ_- or ending at Γ_+ would not fill the whole $\mathbf{\Omega}$ and, to proceed, we have to construct an auxiliary set by extending $\mathbf{\Omega}$ into the time domain and use the approach of [7] which is explained below.

2.2. **Integration along characteristics.** For any $0<T<\infty$, we define the domain

$$\mathbf{\Omega}_T=\mathbf{\Omega}\times(0,T)$$

and the measure $\mathrm{d}\mu_T=\mathrm{d}\mu\otimes\mathrm{d}t$ on $\mathbf{\Omega}_T$. Consider the vector field over $\mathbf{\Omega}_T$:

$$Y=\frac{\partial}{\partial t}+\mathcal{F}(\mathbf{x})\cdot\nabla_{\mathbf{x}}=\mathscr{A}(\xi)\cdot\nabla_\xi$$

where $\mathscr{A}(\xi)=(1,\mathcal{F}(\mathbf{x}))$ for any $\xi=(t,\mathbf{x})$. We can define the characteristic curves of $\mathscr{A}$ as the solution $\xi(s)=(t(s),\mathbf{X}(s))$ to the system $\dfrac{\mathrm{d}}{\mathrm{d}s}\xi(s)=\mathscr{A}(\xi(s))$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}s}t(s)=1,\quad\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{X}(s)=\mathcal{F}(\mathbf{X}(s)),\qquad(s\in\mathbb{R}),$$

with

$$t(0)=t,\quad\mathbf{X}(0)=\mathbf{x}.$$

It is clear that the solution $\xi(s)$ to such a system is given by

$$t(s)=s+t,\qquad\mathbf{X}(s)=\mathbf{\Phi}(\mathbf{x},0,s),$$

and we can define the flow of solution $\mathbf{\Psi}(\xi,0,s)=(s+t,\mathbf{\Phi}(\mathbf{x},0,s))$ associated to $\mathscr{A}$ and the existence times of the characteristic curves of Y are defined, for any $\xi=(t,\mathbf{x})\in\mathbf{\Omega}_T$, as

$$\ell_\pm(\xi)=\inf\{s>0,(\pm s+t,\mathbf{\Phi}(\mathbf{x},0,\pm s))\notin\mathbf{\Omega}_T\}.$$

The flow $\mathbf{\Psi}(\cdot,\cdot,\cdot)$ enjoys, *mutatis mutandis*, the properties listed in Proposition 2.3. Moreover, since $\mathscr{A}$ is clearly Lipschitz continuous on $\overline{\mathbf{\Omega}_T}$, no characteristic of Y can escape to infinity in finite time. In other words, all characteristic curves of Y now have finite lengths. Indeed, if $\mathbf{\Phi}(\mathbf{x},0,\pm s)$ does not reach $\partial\mathbf{\Omega}$, then the characteristic curve $\mathbf{\Psi}(\xi,0,\pm s)$ enters or leaves $\mathbf{\Omega}_T$

through the bottom $\mathbf{\Omega}\times\{0\}$, or through the top $\mathbf{\Omega}\times\{T\}$ of it. Precisely, it is easy to verify that for $\xi=(t,\mathbf{x})\in\mathbf{\Omega}_T$ we have

$$\ell_+(\xi)=\tau_+(\mathbf{x})\wedge(T-t)\quad\text{ and }\quad \ell_-(\xi)=\tau_-(\mathbf{x})\wedge t,$$

where $\wedge$ denotes minimum. This clearly implies $\sup\{\ell_\pm(\xi)\,;\,\xi\in\mathbf{\Omega}_T\}\leqslant T$. Define now

$$\Sigma_{\pm,T}=\{\zeta\in\partial\mathbf{\Omega}_T\,;\,\exists\xi\in\mathbf{\Omega}_T\ \text{such that}\ \zeta=\mathbf{\Psi}(\xi,0,\pm\ell_\pm(\xi))\}.$$

The definition of $\Sigma_{\pm,T}$ is analogous to $\Gamma_\pm$ with the understanding that the charateristic curves now correspond to the vector field $\mathscr{A}$. In other words, $\Sigma_{-,T}$ (resp. $\Sigma_{+,T}$) is the subset of $\partial\mathbf{\Omega}_T$ consisting of all left (resp. right) limits of characteristic curves of $\mathscr{A}$ in $\mathbf{\Omega}_T$ whereas Γ_- (resp. Γ_+) is the subset of $\partial\mathbf{\Omega}$ consisting of all left (resp. right) limits of characteristic curves of $\mathcal{F}$ in $\mathbf{\Omega}$. The main difference (and the interest of such a lifting to $\mathbf{\Omega}_T$) is the fact that *each characteristic curve of $\mathscr{A}$ does reach the boundaries $\Sigma_{\pm,T}$ in finite time*. The above formulae allow us to extend functions $\ell_\pm$ to $\Sigma_{\pm,T}$ in the same way as we extended the functions $\tau_\pm$ to $\Gamma_\pm$. With these considerations, we can represent, up to a set of zero measure, the phase space $\mathbf{\Omega}_T$ as

$$\begin{aligned}\mathbf{\Omega}_T&=\{\mathbf{\Psi}(\xi,0,s)\,;\,\xi\in\Sigma_{-,T}\,,\,0<s<\ell_+(\xi)\}\\&=\{\mathbf{\Psi}(\xi,s,0)\,;\,\xi\in\Sigma_{+,T}\,,\,0<s<\ell_-(\xi)\}.\end{aligned}\tag{2.3}$$

With this realization one can prove the following:

Proposition 2.6. *Let $T>0$ be fixed. There are unique positive Borel measures $\mathrm{d}\nu_\pm$ on $\Sigma_{\pm,T}$ such that $\mathrm{d}\mu_T=\mathrm{d}\nu_+\otimes\mathrm{d}s=\mathrm{d}\nu_-\otimes\mathrm{d}s$.*

PROOF: For any $\delta>0$, define $\mathscr{F}_\delta$ as the set of all bounded Borel subsets E of $\Sigma_{-,T}$ such that $\ell_+(\xi)>\delta$ for any $\xi\in E$. Let us now *fix* $E\in\mathscr{F}_\delta$. For all $0<\sigma\leqslant\delta$ put

$$E_\sigma=\{\mathbf{\Psi}(\xi,0,s)\,;\,\xi\in E,0<s\leqslant\sigma\}.$$

Clearly E_σ is a measurable subset of $\mathbf{\Omega}_T$. Define the mapping $h:\ \sigma\in(0,\delta]\mapsto h(\sigma)=\mu_T(E_\sigma)$ with $h(0)=0$. If σ_1 and σ_2 are two positive numbers such that $\sigma_1+\sigma_2\leqslant\delta$, then

$$E_{\sigma_1+\sigma_2}\setminus E_{\sigma_1}=\{\mathbf{\Psi}(\xi,0,s)\,;\,\xi\in E,\sigma_1<s\leqslant\sigma_1+\sigma_2\}=\{\mathbf{\Psi}(\eta,0,\sigma_1)\,;\,\eta\in E_{\sigma_2}\}.$$

The properties of the flow $\mathbf{\Psi}$ (see Proposition 2.3) ensure that the mapping $\eta\mapsto\mathbf{\Psi}(\eta,0,\sigma_1)$ is one-to-one and measure preserving, so that

$$\mu_T(E_{\sigma_1+\sigma_2}\setminus E_{\sigma_1})=\mu_T(E_{\sigma_2})=h(\sigma_2).$$

Since $E_{\sigma_1+\sigma_2}=E_{\sigma_1}\cup(E_{\sigma_1+\sigma_2}\setminus E_{\sigma_1})$, we immediately obtain

$$h(\sigma_1+\sigma_2)=h(\sigma_1)+h(\sigma_2)\qquad\text{for any}\qquad\sigma_1,\,\sigma_2>0\ \text{with}\ \sigma_1+\sigma_2\leqslant\delta.\tag{2.4}$$

This is the well-known Cauchy equation, though defined only on an interval of the real line. It can be solved in a standard way using non negativity instead of continuity, yielding:

$$h(\sigma)=c_E\sigma\qquad\text{for any}\quad 0<\sigma\leqslant\delta$$

where $c_E=h(\delta)/\delta$. We define $\nu_-(E)=c_E$. It is not difficult to see that, with the above procedure, the mapping $\nu_-(\cdot)$ defines a positive measure on the ring $\mathscr{F}=\bigcup_{\delta>0}\mathscr{F}_\delta$ of all the Borel subsets of $\Sigma_{-,T}$ on which the function $\ell_+(\xi)$ is bounded away from 0. Such a measure ν_- can be uniquely extended to the σ-algebra of the Borel subsets of $\Sigma_{-,T}$ (see e.g. [10, Theorem

A, p. 54]). Consider now a Borel subset E of $\Sigma_{-,T}$ and a Borel subset I of $\mathbb{R}^+$, such that for all $\xi \in E$ and $s \in I$ we have $0 < s < \ell_+(\xi)$. Then

$$E \times I = \{\mathbf{\Psi}(\xi, 0, s)\,;\, \xi \in E, s \in I\} \subset \mathbf{\Omega}_T.$$

Thanks to the definition of $\nu_-(\cdot)$, we can state that $\mu_T(E \times I) = \nu_-(E)\mathrm{meas}(I)$ where $\mathrm{meas}(I)$ denotes the Borel measure of $I \subset \mathbb{R}$. This shows that $\mathrm{d}\mu_T = \mathrm{d}\nu_- \otimes \mathrm{d}s$. Similarly we can define a measure ν_+ on $\Sigma_{+,T}$ and prove that $\mathrm{d}\mu_T = \mathrm{d}\nu_+ \otimes \mathrm{d}s$. The uniqueness of the measures $\mathrm{d}\nu_\pm$ is then obvious. □

Remark 2.7. *Note that the above construction of the Borel measures* $\mathrm{d}\nu_\pm$ *differs from that of* [12, Lemmas XI.3.1 & 3.2], [7, Propositions 7 & 8].

Next, by the cylindrical structure of $\mathbf{\Omega}_T$, the measures $\mathrm{d}\nu_\pm$ can be written as $\mathrm{d}\nu_\pm = \mathrm{d}\mu_\pm \otimes \mathrm{d}t$ where $\mathrm{d}\mu_\pm$ are Borel measures on $\Gamma_\pm$ [12, p. 408]. This leads to the following

Lemma 2.8. *There are unique positive Borel measures* $\mathrm{d}\mu_\pm$ *on* $\Gamma_\pm$ *such that, for any* $f \in L^1(\mathbf{\Omega}_T, \mathrm{d}\mu_T)$

$$\begin{aligned}\int_{\mathbf{\Omega}_T} f(\mathbf{x},t)\mathrm{d}\mu_T(\mathbf{x},t) = \int_0^T \mathrm{d}t \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})\wedge t} f(\mathbf{\Phi}(\mathbf{y},s,0), t-s)\mathrm{d}s \\ + \int_{\mathbf{\Omega}} \mathrm{d}\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x})\wedge T} f(\mathbf{\Phi}(\mathbf{x},s,0), T-s)\mathrm{d}s,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\int_{\mathbf{\Omega}_T} f(\mathbf{x},t)\mathrm{d}\mu_T(\mathbf{x},t) = \int_0^T \mathrm{d}t \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})\wedge(T-t)} f(\mathbf{\Phi}(\mathbf{y},0,s), t+s)\mathrm{d}s \\ + \int_{\mathbf{\Omega}} \mathrm{d}\mu(\mathbf{x}) \int_0^{\tau_+(\mathbf{x})\wedge T} f(\mathbf{\Phi}(\mathbf{x},0,s), s)\mathrm{d}s.\end{aligned} \tag{2.6}$$

The above fundamental result allows to compute integral over the cylindrical phase-space $\mathbf{\Omega}_T$ through integration along the characteristic curves. Let us now generalize it to the phase space $\mathbf{\Omega}$. Here the main difficulty stems from the fact that the characteristic curves of the vector field $\mathcal{F}$ are no longer assumed to be of finite length. In order to extend Lemma 2.8 to possibly infinite existence times, first we prove the following:

Lemma 2.9. *Let* $T > 0$ *be fixed. Then,* $\tau_+(\mathbf{x}) < T$ *for any* $\mathbf{x} \in \mathbf{\Omega}$ *if and only if* $\tau_-(\mathbf{x}) < T$ *for any* $\mathbf{x} \in \mathbf{\Omega}$.

PROOF: Assume that $T > \tau_+(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{\Omega}$ and that there is $\mathbf{z} \in \mathbf{\Omega}$ such that $\tau_-(\mathbf{z}) \geqslant T$. One can assume without loss of generality that $\tau_-(\mathbf{z}) > T$. Indeed, if $\tau_-(\mathbf{z}) = T$, since $\mathbf{\Omega}$ is open, the orbit passing through $\mathbf{z}$ can be continued beyond ensuring the existence of $\mathbf{z}' \in \mathbf{\Omega}$ with $\tau_-(\mathbf{z}') > T$. Now, if $\tau_-(\mathbf{z}) > T$, for any $T < t < \tau_-(\mathbf{z})$, $\mathbf{y} = \mathbf{\Phi}(\mathbf{z}, t, 0) = \mathbf{\Phi}(\mathbf{z}, 0, -t) \in \mathbf{\Omega}$ and $\mathbf{\Phi}(\mathbf{y}, 0, s) = \mathbf{\Phi}(\mathbf{z}, t, s) = \mathbf{\Phi}(\mathbf{z}, t-s, 0) \in \mathbf{\Omega}$ for all $0 < s < t$. This leads to the contradiction that $\tau_+(\mathbf{z}) \geqslant t > T$. We proceed in the same way for the converse implication. □

The above lemma allows to prove a representation formula for integral of the type $\int_{\mathbf{\Omega}} f\,\mathrm{d}\mu$ in terms of integrals over $\Gamma_\pm$. Hereafter, the support of a measurable function f defined on $\mathbf{\Omega}$ is defined as $\mathrm{Supp} f = \mathbf{\Omega} \setminus \omega$ where ω is the maximal open subset of $\mathbf{\Omega}$ on which f vanishes $\mathrm{d}\mu$–almost everywhere.

Proposition 2.10. *Let $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. Assume that there exists $\tau_0 > 0$ such that $\tau_\pm(\mathbf{x}) < \tau_0$ for any $\mathbf{x} \in \mathrm{Supp}(f)$. Then,*

$$\begin{aligned}\int_{\mathbf{\Omega}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f\left(\mathbf{\Phi}(\mathbf{y}, s, 0)\right) \mathrm{d}s \\ &= \int_{\Gamma_-} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y}, 0, s))\mathrm{d}s.\end{aligned} \tag{2.7}$$

PROOF: For any $T > \tau_0$, define the domain $\mathbf{\Omega}_T = \mathbf{\Omega} \times (0, T)$. Since $T < \infty$, it is clear that $f \in L^1(\mathbf{\Omega}_T, \mathrm{d}\mu\mathrm{d}t)$ and, by (2.5), we get

$$T \int_{\mathbf{\Omega}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_0^T \mathrm{d}t \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{t\wedge\tau_-(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y}, s, 0))\mathrm{d}s + \int_{\mathbf{\Omega}} \mathrm{d}\mu(\mathbf{x}) \int_0^{\tau_-(\mathbf{x})} f(\mathbf{\Phi}(\mathbf{x}, s, 0))\mathrm{d}s.$$

Since the formula is valid for any $T > \tau_0$, differentiating with respect to T leads to the first assertion. The second assertion is proved in the same way by using formula (2.6). □

To drop the finiteness assumption on $\tau_\pm(\mathbf{x})$, first we introduce the sets

$$\mathbf{\Omega}_\pm = \{\mathbf{x} \in \mathbf{\Omega}\,;\, \tau_\pm(\mathbf{x}) < \infty\}, \qquad \mathbf{\Omega}_{\pm\infty} = \{\mathbf{x} \in \mathbf{\Omega}\,;\, \tau_\pm(\mathbf{x}) = \infty\},$$

and

$$\Gamma_{\pm\infty} = \{\mathbf{y} \in \Gamma_\pm\,;\, \tau_\mp(\mathbf{y}) = \infty\}.$$

One gets

Proposition 2.11. *Let $f \in L^1(\mathbf{\Omega}, \mathrm{d}\mu)$. Then*

$$\int_{\mathbf{\Omega}_\pm} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_\pm} \mathrm{d}\mu_\pm(\mathbf{y}) \int_0^{\tau_\mp(\mathbf{y})} f\left(\mathbf{\Phi}(\mathbf{y}, 0, \mp s)\right) \mathrm{d}s, \tag{2.8}$$

and

$$\int_{\mathbf{\Omega}_\pm \cap \mathbf{\Omega}_{\mp\infty}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_{\pm\infty}} \mathrm{d}\mu_\pm(\mathbf{y}) \int_0^{\infty} f\left(\mathbf{\Phi}(\mathbf{y}, 0, \mp s)\right) \mathrm{d}s. \tag{2.9}$$

PROOF: Assume first $f \geqslant 0$. Let us fix $T > 0$. It is clear that $\mathbf{x} \in \mathbf{\Omega}$ satisfies $\tau_+(\mathbf{x}) < T$ if and only if $\mathbf{x} = \mathbf{\Phi}(\mathbf{y}, s, 0)$, with $\mathbf{y} \in \Gamma_+$ and $0 < s < T \wedge \tau_-(\mathbf{y})$. Then, by Proposition 2.10,

$$\int_{\{\tau_+(\mathbf{x}) < T\}} f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{T\wedge\tau_-(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y}, s, 0))\mathrm{d}s.$$

Since $f \geqslant 0$, the inner integral is increasing with T and, using the monotone convergence theorem, we let $T \to \infty$ to get

$$\int_{\mathbf{\Omega}_+} f(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f\left(\mathbf{\Phi}(\mathbf{y}, s, 0)\right) \mathrm{d}s$$

which coincides with (2.8) since $\mathbf{\Phi}(\mathbf{y}, s, 0) = \mathbf{\Phi}(\mathbf{y}, 0, -s)$. We proceed in the same way integration on Γ_- and get the second part of (2.8). Next we consider the set

$$\Delta = \{\mathbf{x} \in \mathbf{\Omega}\,;\, \mathbf{x} = \mathbf{\Phi}(\mathbf{y}, s, 0),\ \mathbf{y} \in \mathbf{\Omega}_{+\infty},\ 0 < s < T\}.$$

Proposition 2.10 asserts that

$$\int_{\Delta} f(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_{+\infty}} \mathrm{d}\mu_+(\mathbf{y}) \int_0^T f(\mathbf{\Phi}(\mathbf{y}, s, 0)) \mathrm{d}s.$$

Letting again $T \to \infty$, we get (2.9). We extend the results to arbitrary f by linearity. □

Finally, with the following, we show that it is possible to transfer integrals over Γ_- to Γ_+:

Proposition 2.12. *For any $\psi \in L^1(\Gamma_-, \mathrm{d}\mu_-)$,*

$$\int_{\Gamma_- \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) \mathrm{d}\mu_-(\mathbf{y}) = \int_{\Gamma_+ \setminus \Gamma_{+\infty}} \psi(\mathbf{\Phi}(\mathbf{z}, \tau_-(\mathbf{z}), 0)) \mathrm{d}\mu_+(\mathbf{z}). \tag{2.10}$$

PROOF: For any $\epsilon > 0$, let f_ϵ be the function defined on $\mathbf{\Omega}_+ \cap \mathbf{\Omega}_-$ by

$$\psi_\epsilon(\mathbf{x}) = \begin{cases} \dfrac{\psi(\mathbf{\Phi}(\mathbf{x}, \tau_-(\mathbf{x}), 0))}{\tau_+(\mathbf{x}) + \tau_-(\mathbf{x})} & \text{if} \quad \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) > \epsilon, \\ 0 & \text{else.} \end{cases}$$

Since $\psi_\epsilon \in L^1(\mathbf{\Omega}_+ \cap \mathbf{\Omega}_-, \mathrm{d}\mu)$, Eqs. (2.8) and (2.9) give

$$\begin{aligned} \int_{\mathbf{\Omega}_+ \cap \mathbf{\Omega}_-} \psi_\epsilon(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}) &= \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\tau_+(\mathbf{y})} \psi(\mathbf{y}) \frac{\mathrm{d}s}{\tau_+(\mathbf{y})} \\ &= \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) \mathrm{d}\mu_-(\mathbf{y}). \end{aligned}$$

In the same way,

$$\begin{aligned} \int_{\mathbf{\Omega}_+ \cap \mathbf{\Omega}_-} \psi_\epsilon(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}) &= \int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \psi(\mathbf{\Phi}(\mathbf{y}, \tau_-(\mathbf{y}), 0)) \frac{\mathrm{d}s}{\tau_-(\mathbf{y})} \\ &= \int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} \psi(\mathbf{\Phi}(\mathbf{y}, \tau_-(\mathbf{y}), 0)) \mathrm{d}\mu_-(\mathbf{y}), \end{aligned}$$

which leads to

$$\int_{\{\tau_-(\mathbf{y}) > \epsilon\} \setminus \Gamma_{+\infty}} \psi(\mathbf{\Phi}(\mathbf{y}, \tau_-(\mathbf{y}), 0)) \mathrm{d}\mu_+(\mathbf{y}) = \int_{\{\tau_+(\mathbf{y}) > \epsilon\} \setminus \Gamma_{-\infty}} \psi(\mathbf{y}) \mathrm{d}\mu_-(\mathbf{y})$$

for any $\epsilon > 0$. Passing to the limit as $\epsilon \to 0$ we get the conclusion. □

3. Trace operators and basic existence results

3.1. **The maximal transport operator and trace results.** Now we define the transport operator on $X = L^1(\mathbf{\Omega}, \mathrm{d}\mu)$ by

$$\begin{cases} \mathbf{T}_{\max}: & \mathscr{D}(\mathbf{T}_{\max}) \subset X \longrightarrow X \\ & f \longmapsto \mathbf{T}_{\max} f(\mathbf{x}) = -\mathcal{F}(\mathbf{x}) \cdot \nabla f(\mathbf{x}) \end{cases}$$

with domain $\mathscr{D}(\mathbf{T}_{\max}) = \{f \in X\,;\, \mathbf{T}_{\max} f \in X\}$ where $\mathbf{T}_{\max}$ is to be understood in the sense of distributions. It is a classical matter (see e.g. [5, Proposition 10.13]) that $\mathscr{C}^1(\mathbf{\Omega}) \cap \mathscr{D}(\mathbf{T}_{\max})$ is dense in $(\mathscr{D}(\mathbf{T}_{\max}), \|\cdot\|_{\mathscr{D}})$ where $\|\cdot\|_{\mathscr{D}}$ is the graph norm. Let us state the following trace result:

Proposition 3.1. *Let $f \in \mathscr{D}(\mathbf{T}_{\max})$. Then the limit*

$$\lim_{s \to 0+} f(\mathbf{\Phi}(\mathbf{y}, s, 0))$$

exists for almost every $\mathbf{y} \in \Gamma_+$. Similarly, $\lim_{s\to 0+} f(\mathbf{\Phi}(\mathbf{y}, 0, s))$ exists for almost every $\mathbf{y} \in \Gamma_-$.

Proof: Let $(f_n)_n \subset \mathscr{C}^1(\mathbf{\Omega}) \cap \mathscr{D}(\mathbf{T}_{\max})$ be such that $\|f_n - f\|_{\mathscr{D}} \to 0$. Then Eq. (2.8) yields

$$\int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |f_n(\mathbf{\Phi}(\mathbf{y}, s, 0)) - f(\mathbf{\Phi}(\mathbf{y}, s, 0))|\, \mathrm{d}s$$
$$+ \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |\mathbf{T}_{\max} f_n(\mathbf{\Phi}(\mathbf{y}, s, 0)) - \mathbf{T}_{\max} f(\mathbf{\Phi}(\mathbf{y}, s, 0))|\, \mathrm{d}s \underset{n\to\infty}{\longrightarrow} 0$$

since $\mathbf{T}_{\max} f$ and $\mathbf{T}_{\max} f_n$ both belong to X. Consequently, for almost every $\mathbf{y} \in \Gamma_+$ (up to a subsequence, still denoted by f_n) we get

$$\begin{cases} f_n(\mathbf{\Phi}(\mathbf{y}, \cdot, 0)) \longrightarrow f(\mathbf{\Phi}(\mathbf{y}, \cdot, 0)) \\ \mathbf{T}_{\max} f_n(\mathbf{\Phi}(\mathbf{y}, \cdot, 0)) \longrightarrow \mathbf{T}_{\max} f(\mathbf{\Phi}(\mathbf{y}, \cdot, 0)) \quad \text{in} \quad L^1((0, \tau_-(\mathbf{y})), \mathrm{d}s) \end{cases}$$

as $n \to \infty$. Let us fix $\mathbf{y} \in \Gamma_+$ for which this holds. Passing again to a subsequence, we may assume that $f_n(\mathbf{\Phi}(\mathbf{y}, s, 0))$ converges (pointwise) to $f(\mathbf{\Phi}(\mathbf{y}, s, 0))$ for almost every $s \in (0, \tau_-(\mathbf{y}))$. Let us fix such a s_0. Then,

$$f_n(\mathbf{\Phi}(\mathbf{y}, s, 0)) - f_n(\mathbf{\Phi}(\mathbf{y}, s_0, 0)) = \int_{s_0}^{s} [\mathbf{T}_{\max} f_n](\mathbf{\Phi}(\mathbf{y}, r, 0)) \mathrm{d}r \qquad \forall s \in (0, \tau_-(\mathbf{y})).$$

Now, the right-hand-side has a limit as $n \to \infty$ so that the first term on the left-hand side also must converge as $n \to \infty$. Thus, for any $s \in (0, \tau_-(\mathbf{y}))$, we have

$$\lim_{n\to\infty} f_n(\mathbf{\Phi}(\mathbf{y}, s, 0)) = f(\mathbf{\Phi}(\mathbf{y}, s, 0))$$

and

$$f(\mathbf{\Phi}(\mathbf{y}, s, 0)) = f(\mathbf{\Phi}(\mathbf{y}, s_0, 0)) + \int_{s_0}^{s} [\mathbf{T}_{\max} f](\mathbf{\Phi}(\mathbf{y}, r, 0)) \mathrm{d}r. \tag{3.1}$$

As a direct direct consequence, the limit $\lim_{s\to 0+} f(\mathbf{\Phi}(\mathbf{y}, s, 0))$ exists and equals

$$f(\mathbf{\Phi}(\mathbf{y}, s_0, 0)) - \int_0^{s_0} [\mathbf{T}_{\max} f](\mathbf{\Phi}(\mathbf{y}, r, 0)) \mathrm{d}r.$$

It is easy to check that this limit does not depend on s_0. The existence of $\lim_{s\to 0+} f(\mathbf{\Phi}(\mathbf{y},0,s))$ for almost every $\mathbf{y}\in\Gamma_-$ follows by the same argument. □

The above proposition allows to define the trace operators.

Definition 3.2. *For any $f\in\mathscr{D}(\mathbf{T}_{\max})$, define the traces $\mathsf{B}^{\pm}f$ by*

$$\mathsf{B}^{\pm}f(\mathbf{y}):=\lim_{s\to 0+} f(\mathbf{\Phi}(\mathbf{y},\pm s,0))=\lim_{s\to 0+} f(\mathbf{\Phi}(\mathbf{y},0,\mp s))$$

for any $\mathbf{y}\in\Gamma_\pm$ for which the limits exist.

Note that, as we saw in the proof of Proposition 3.1, for any $f\in\mathscr{D}(\mathbf{T}_{\max})$ and a.e. $\mathbf{z}\in\Gamma_+$,

$$\mathsf{B}^{+}f(\mathbf{z})=f(\mathbf{\Phi}(\mathbf{z},t,0))-\int_0^t[\mathbf{T}_{\max}f](\mathbf{\Phi}(\mathbf{z},s,0))\mathrm{d}s,\qquad \forall t\in(0,\tau_-(\mathbf{z})).$$

In the same way, for a.e. $\mathbf{y}\in\Gamma_-$

$$\mathsf{B}^{-}f(\mathbf{y})=f(\mathbf{\Phi}(\mathbf{y},0,t))+\int_0^t[\mathbf{T}_{\max}f](\mathbf{\Phi}(\mathbf{y},0,s))\mathrm{d}s,\qquad \forall t\in(0,\tau_+(\mathbf{y})).$$

Note that the above representation allows us to represent $\mathbf{T}_{\max}$ as the derivation along the characteristics.

Proposition 3.3. *Let $f\in\mathscr{D}(\mathbf{T}_{\max})$. Then, for any $\mathbf{x}\in\mathbf{\Omega}_\pm$, one has*

$$\mathbf{T}_{\max}f(\mathbf{x})=\lim_{s\to 0} s^{-1}\left(f(\mathbf{\Phi}(\mathbf{x},\pm s,0))-f(\mathbf{x})\right), \tag{3.2}$$

where the limit exists in X.

PROOF: Let $\mathbf{x}\in\mathbf{\Omega}_+$ and let $\mathbf{y}=\mathbf{\Phi}(\mathbf{x},0,\tau_+(\mathbf{x}))$. Using Formula (3.1), with $s_0=\tau_+(\mathbf{x})$ and $s=t+\tau_+(\mathbf{x})$, one has

$$f(\mathbf{\Phi}(\mathbf{x},t,0))=f(\mathbf{\Phi}(\mathbf{x}))+\int_0^t[\mathbf{T}_{\max}f](\mathbf{\Phi}(\mathbf{x},r,0))\mathrm{d}r\qquad(-\tau_-(\mathbf{x})<t<\tau_-(\mathbf{x})).$$

Therefore, the limit $\lim_{t\to 0^+} t^{-1}\left(f(\mathbf{\Phi}(\mathbf{x},t,0))-f(\mathbf{x})\right)=\lim_{t\to 0^+} t^{-1}\int_0^t[\mathbf{T}_{\max}f](\mathbf{\Phi}(\mathbf{x},r,0))\mathrm{d}r$ exists in X and is equal to $\mathbf{T}_{\max}f(\mathbf{x})$. One proceeds in the same way to prove the result when $\mathbf{x}\in\mathbf{\Omega}_-$, for which

$$f(\mathbf{\Phi}(\mathbf{x},0,t))=f(\mathbf{\Phi}(\mathbf{x}))-\int_0^t[\mathbf{T}_{\max}f](\mathbf{\Phi}(\mathbf{x},0,r))\mathrm{d}r\qquad(-\tau_-(\mathbf{x})<t<\tau_-(\mathbf{x})),$$

so that $\lim_{t\to 0^+} t^{-1}\left(f(\mathbf{\Phi}(\mathbf{x},0,t))-f(\mathbf{x})\right)=\mathbf{T}_{\max}f(\mathbf{x})$. □

Lemma 2.8 provides the existence of Borel measures $\mathrm{d}\mu_\pm$ on $\Gamma_\pm$ which allow us to define the natural trace spaces associated to Problem (1.1), namely,

$$L^1_\pm:=L^1(\Gamma_\pm,\mathrm{d}\mu_\pm).$$

We note, however, that for $f\in X$, the traces $\mathbf{y}\in\Gamma_\pm\longmapsto\mathsf{B}^{\pm}f(\mathbf{y})$ not necessarily belong to $L^1_\pm$.

3.2. **Basic existence results.** Let $\mathbf{T}_0$ be the free streaming operator with *no re–entry boundary conditions*:

$$\mathbf{T}_0\varphi = \mathbf{T}_{\max}\varphi \qquad \text{for any } \varphi \in \mathscr{D}(\mathbf{T}_0),$$

where the domain $\mathscr{D}(\mathbf{T}_0)$ is defined by

$$\mathscr{D}(\mathbf{T}_0) = \{\psi \in \mathscr{D}(\mathbf{T}_{\max})\,;\, \mathsf{B}^-\psi = 0\}.$$

We state the following generation result, whose proof is postponed to the Appendix of this paper:

Theorem 3.4. *The operator $(\mathbf{T}_0, \mathscr{D}(\mathbf{T}_0))$ is the generator of a nonnegative C_0-semigroup of contractions $(U_0(t))_{t\geqslant 0}$ in X given by*

$$U_0(t)f(\mathbf{x}) = f(\mathbf{\Phi}(\mathbf{x}, t, 0))\chi_{\{t<\tau_-(\mathbf{x})\}}(\mathbf{x}), \qquad (\mathbf{x} \in \mathbf{\Omega},\ f \in X),$$

where χ_A denotes the characteristic function of a set A.

One can now state the following result.

Theorem 3.5. *Let $u \in L^1_-$ and $g \in X$ be given. Then the function*

$$f(\mathbf{x}) \,=\, \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t)\, g(\mathbf{\Phi}(\mathbf{x}, t, 0))\mathrm{d}t \,+\, \chi_{\{\tau_-(\mathbf{x})<\infty\}} \exp(-\lambda\tau_-(\mathbf{x}))u(\mathbf{\Phi}(\mathbf{x}, \tau_-(\mathbf{x}), 0))$$

*is the **unique** solution $f \in \mathscr{D}(\mathbf{T}_{\max})$ of the boundary value problem:*

$$\begin{cases} (\lambda - \mathbf{T}_{\max})f = g \\ \mathsf{B}^- f = u \end{cases} \tag{3.3}$$

where $\lambda > 0$. Moreover, $\mathsf{B}^+ f \in L^1_+$ and

$$\|\mathsf{B}^+ f\|_{L^1_+} + \lambda\|f\|_X \leqslant \|u\|_{L^1_-} + \|g\|_X. \tag{3.4}$$

Furthermore, if $g \geqslant 0$ and $u \geqslant 0$, then (3.4) *turns into equality.*

PROOF: Let us write $f = f_1 + f_2$ with $f_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t)\, g(\mathbf{\Phi}(\mathbf{x}, t, 0))\mathrm{d}t$, and

$$f_2(\mathbf{x}) = \chi_{\{\tau_-(\mathbf{x})<\infty\}} \exp(-\lambda\tau_-(\mathbf{x}))u(\mathbf{\Phi}(\mathbf{x}, \tau_-(\mathbf{x}), 0)), \qquad (\mathbf{x} \in \mathbf{\Omega}).$$

According to Theorem 3.4, $f_1 = (\lambda - \mathbf{T}_0)^{-1}g$, i.e. $f_1 \in \mathscr{D}(\mathbf{T}_{\max})$ with $(\lambda - \mathbf{T}_{\max})f_1 = g$ and $\mathsf{B}^- f_1 = 0$. Therefore, to prove that f is a solution of (3.3) it suffices to check that $f_2 \in \mathscr{D}(\mathbf{T}_{\max})$, $(\lambda - \mathbf{T}_{\max})f_2 = 0$ and $\mathsf{B}^- f_2 = u$. Arguing as in the proof of Theorem 3.4 (see Appendix), we easily see that

$$f_2(\mathbf{\Phi}(\mathbf{y}, 0, t)) = \exp(-\lambda t)u(\mathbf{y}), \qquad (\mathbf{y} \in \Gamma_-,\ \ 0 < t < \tau_+(\mathbf{y})). \tag{3.5}$$

Then Proposition 2.3 yields

$$\begin{aligned}\int_{\mathbf{\Omega}} \psi\, \mathbf{T}_{\max} f_2 \,\mathrm{d}\mu &= \int_{\mathbf{\Omega}} f_2(\mathbf{x})\mathcal{F}(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_-} f_2(\mathbf{x})\mathcal{F}(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x})\\ &= \int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})} f_2(\mathbf{\Phi}(\mathbf{y},0,t))\mathcal{F}(\mathbf{\Phi}(\mathbf{y},0,t))\cdot\nabla\psi(\mathbf{\Phi}(\mathbf{y},0,t))\mathrm{d}t\\ &= \int_{\Gamma_-} u(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})} e^{-\lambda t}\mathcal{F}(\mathbf{\Phi}(\mathbf{y},0,t))\cdot\nabla\psi(\mathbf{\Phi}(\mathbf{y},0,t))\mathrm{d}t.\end{aligned}$$

Again, as in the proof of Theorem 3.4, we get that

$$\int_{\mathbf{\Omega}} \psi\, \mathbf{T}_{\max} f_2 \mathrm{d}\mu = \lambda\int_{\Gamma_-} u(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})} e^{-\lambda t}\psi(\mathbf{\Phi}(\mathbf{y},0,t))\mathrm{d}t = \lambda\int_{\mathbf{\Omega}} f_2\psi\mathrm{d}\mu.$$

This proves that $f_2\in\mathscr{D}(\mathbf{T}_{\max})$ and $(\lambda-\mathbf{T}_{\max})f_2=0$. Consequently, f is a solution to (3.3). To prove that the solution is unique, it is sufficient to prove that the only solution $\psi\in\mathscr{D}(\mathbf{T}_{\max})$ to

$$(\lambda-\mathbf{T}_{\max})\psi=0,\quad \mathsf{B}^-\psi=0$$

is $\psi=0$. This follows from the fact that such a solution ψ actually belongs to $\mathscr{D}(\mathbf{T}_0)$ while $\lambda\in\varrho(\mathbf{T}_0)$. Finally, it remains to prove (3.4). Using (3.5) and the fact that f_2 vanishes on $\mathbf{\Omega}_{-\infty}$, we infer from (2.8) that

$$\begin{aligned}\lambda\int_{\mathbf{\Omega}}|f_2|\mathrm{d}\mu = \lambda\int_{\mathbf{\Omega}_-}|f_2|\mathrm{d}\mu &= \lambda\int_{\Gamma_-}\mathrm{d}\mu_-(\mathbf{y})\int_0^{\tau_+(\mathbf{y})} e^{-\lambda t}|u(\mathbf{y})|\mathrm{d}t\\ &= \int_{\Gamma_-}|u(\mathbf{y})|\left(1-e^{-\lambda\tau_+(\mathbf{y})}\right)\mathrm{d}\mu_-(\mathbf{y}).\end{aligned}\tag{3.6}$$

Define $h:\mathbf{y}\in\Gamma_-\longmapsto h(\mathbf{y})=|u(\mathbf{y})|e^{-\lambda\tau_+(\mathbf{y})}$. It is clear that h vanishes on $\Gamma_{-\infty}$ and $h(\mathbf{y})\leqslant|u(\mathbf{y})|$ for a.e. $\mathbf{y}\in\Gamma_-$. In particular, $h\in L^1_-$ and, according to (2.10),

$$\begin{aligned}\int_{\Gamma_-} h(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y}) &= \int_{\Gamma_-\setminus\Gamma_{-\infty}} h(\mathbf{y})\mathrm{d}\mu_-(\mathbf{y}) = \int_{\Gamma_+\setminus\Gamma_{+\infty}} h(\mathbf{\Phi}(\mathbf{z},\tau_-(\mathbf{z}),0))\mathrm{d}\mu_+(\mathbf{z})\\ &= \int_{\Gamma_+\setminus\Gamma_{+\infty}} e^{-\lambda\tau_-(\mathbf{z})}|u(\mathbf{\Phi}(\mathbf{z},\tau_-(\mathbf{z}),0))|\mathrm{d}\mu_+(\mathbf{z})\\ &= \int_{\Gamma_+}|\mathsf{B}^+f_2(\mathbf{z})|\mathrm{d}\mu_+(\mathbf{z}) = \|\mathsf{B}^+f_2\|_{L^1_+}.\end{aligned}$$

Combining this with (3.6) leads to

$$\lambda\,|f_2\|_X + \|\mathsf{B}^+f_2\|_{L^1_+} = \|u\|_{L^1_-}.\tag{3.7}$$

Now, let us show that $\mathsf{B}^+ f_1 \in L^1_+$ and $\|\mathsf{B}^+ f_1\|_{L^1_+} + \lambda\|f\|_X \leqslant \|g\|_X$. For any $\mathbf{y} \in \Gamma_+$ and $0 < t < \tau_-(\mathbf{y})$, we see, as above, that

$$\begin{aligned} f_1(\mathbf{\Phi}(\mathbf{y},t,0)) &= \int_0^{\tau_-(\mathbf{y})-t} \exp(-\lambda s) g(\mathbf{\Phi}(\mathbf{y},s+t,0))\mathrm{d}s \\ &= \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t)) g(\mathbf{\Phi}(\mathbf{y},s,0))\mathrm{d}s. \end{aligned}$$

This shows that $\mathsf{B}^+ f_1(\mathbf{y}) = \lim_{t\to 0^+} f_1(\mathbf{\Phi}(\mathbf{y},t,0)) = \int_0^{\tau_-(\mathbf{y})} \exp(-\lambda s)) g(\mathbf{\Phi}(\mathbf{y},s,0))\mathrm{d}s$. According to Proposition 2.11,

$$\int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |g(\mathbf{\Phi}(\mathbf{y},s,0))|\,\mathrm{d}s = \int_{\mathbf{\Omega}_+} |g|\,\mathrm{d}\mu$$

which, since $\exp(-\lambda(s-t))|g(\mathbf{\Phi}(\mathbf{y},s,0))| \leqslant |g(\mathbf{\Phi}(\mathbf{y},s,0))|$, implies $\mathsf{B}^+ f_1 \in L^1_+$. Let us now assume $g \geqslant 0$. Then $f_1 \geqslant 0$,

$$\lambda\,|f_1\| = \lambda \int_{\mathbf{\Omega}} f_1\,\mathrm{d}\mu = \lambda \int_{\mathbf{\Omega}_+} f_1\,\mathrm{d}\mu + \lambda \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu + \lambda \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu.$$

Using similar arguments to those used in the study of f_2, we have

$$\lambda \int_{\mathbf{\Omega}_+} f_1\,\mathrm{d}\mu = \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\mathbf{\Phi}(\mathbf{y},t,0))\,(1-\exp(-\lambda t))\,\mathrm{d}t,$$

which, by Proposition 2.11, implies

$$\lambda \int_{\mathbf{\Omega}_+} f_1\,\mathrm{d}\mu = \int_{\mathbf{\Omega}_+} g\,\mathrm{d}\mu - \int_{\Gamma_+} \mathsf{B}^+ f_1\,\mathrm{d}\mu_+.$$

Similar argument shows that

$$\lambda \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu = \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} g\,\mathrm{d}\mu,$$

while the equality

$$\lambda \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} f_1\,\mathrm{d}\mu = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} g\,\mathrm{d}\mu,$$

follows since this case behaves as the whole space case. This shows that $\lambda\,|f\|_X = \|g\|_X - \|\mathsf{B}^+ f\|_{L^1_+}$ for $g \geqslant 0$. In general, defining

$$F_1(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda s)\;|g(\mathbf{\Phi}(\mathbf{x},s,0))|\,\mathrm{d}s, \qquad (\mathbf{x}\in\mathbf{\Omega}),$$

we obtain $\|\mathsf{B}^+ f_1\|_{L^1_+} + \lambda\|f_1\|_X \leqslant \|B^+ F_1\|_{L^1_+} + \lambda\,|F_1\|_X = \|g\|_X$, which combined with (3.7), gives (3.4). □

Let us note that, with the notation of Theorem 3.5, we have

$$\int_{\Gamma_+} \mathsf{B}^+ f \mathrm{d}\mu_+ + \lambda \int_{\mathbf{\Omega}} f \,\mathrm{d}\mu = \int_{\Gamma_-} u \,\mathrm{d}\mu_- + \int_{\mathbf{\Omega}} g \,\mathrm{d}\mu. \tag{3.8}$$

Indeed, for nonnegative u and g, (3.4) turns out to be an identity which is precisely (3.8). Then, for arbitrary $u \in L^1_-$ and $g \in X$, we get (3.8) by splitting functions into positive and negative parts. This leads to the following generalization of Green's formula:

Proposition 3.6 (Green's formula). *Let $f \in \mathscr{D}(\mathbf{T}_{\max})$ be such that $\mathsf{B}^- f \in L^1_-$. Then $\mathsf{B}^+ f \in L^1_+$ and*

$$\int_{\mathbf{\Omega}} \mathbf{T}_{\max} f \mathrm{d}\mu = \int_{\Gamma_-} \mathsf{B}^- f \mathrm{d}\mu_- - \int_{\Gamma_+} \mathsf{B}^+ f \,\mathrm{d}\mu_+$$

PROOF: For given $f \in \mathscr{D}(\mathbf{T}_{\max})$, we obtain the result by setting $u = \mathsf{B}^- f \in L^1_-$ and $g = (\lambda - \mathbf{T}_{\max}) f \in X$ in Eq. (3.8). □

Remark 3.7. *If $\mathrm{d}\mu$ is the Lebesgue measure on $\mathbb{R}^N$, the above formula leads to a better understanding of the measures $\mathrm{d}\mu_\pm$. Indeed, comparing it to the classical Green's formula (see e.g.* [6]*), one sees that the restriction of $\mathrm{d}\mu_\pm$ on the set $\Sigma_\pm = \{\mathbf{y} \in \partial\mathbf{\Omega}\,;\, \pm\mathcal{F}(\mathbf{y}) \cdot n(\mathbf{y}) > 0\}$ is equal to*

$$\mathrm{d}\mu_{\pm|\Sigma_\pm} = (\pm\mathcal{F}(\mathbf{y}) \cdot n(\mathbf{y}))\, \mathrm{d}\gamma(\mathbf{y})$$

where $\mathrm{d}\gamma(\cdot)$ is the surface Lebesgue measure on $\partial\mathbf{\Omega}$.

We conclude this section with a result similar to Theorem 3.5. Precisely, for the boundary value problem with data given on Γ_+, we have the following generalization of [4, Lemma 2.2]

Proposition 3.8. *Given $h \in L^1_+$, let*

$$f(\mathbf{x}) = \begin{cases} h(\mathbf{\Phi}(\mathbf{x}, 0, \tau_+(\mathbf{x}))\dfrac{\tau_-(\mathbf{x}) e^{-\tau_+(\mathbf{x})}}{\tau_-(\mathbf{x}) + \tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) + \tau_+(\mathbf{x}) < \infty, \\ h(\mathbf{\Phi}(\mathbf{x}, 0, \tau_+(\mathbf{x}))e^{-\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x}) = \infty \text{ and } \tau_+(\mathbf{x}) < \infty, \\ 0 & \text{if } \tau_+(\mathbf{x}) = \infty. \end{cases}$$

Then, $f \in \mathscr{D}(\mathbf{T}_{\max})$, $\mathsf{B}^- f = 0$, and $\mathsf{B}^+ f = h$, with $\|f\|_X \leqslant \|h\|_{L^1_+}$ and $\|\mathbf{T}_0 f\|_X \leqslant \|h\|_{L^1_+}$.

PROOF: Let $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$. Since

$$\tau_-(\mathbf{\Phi}(\mathbf{y}, 0, t)) = t, \qquad \text{while} \qquad \tau_+(\mathbf{\Phi}(\mathbf{y}, 0, t)) = \tau_+(\mathbf{y}) - t$$

we have

$$f(\mathbf{\Phi}(\mathbf{y}, 0, t)) = h\left(\mathbf{\Phi}(\mathbf{y}, 0, \tau_+(\mathbf{y}))\right) \frac{t e^{t - \tau_+(\mathbf{y})}}{\tau_+(\mathbf{y})} \chi_{\{\tau_+(\mathbf{y}) < \infty\}} \tag{3.9}$$

and $\mathsf{B}^- f(\mathbf{y}) = \lim_{t \to 0} f(\mathbf{\Phi}(\mathbf{y}, 0, t)) = 0$ for a.e. $\mathbf{y} \in \Gamma_-$. In the same way, given $\mathbf{z} \in \Gamma_+$ and $0 < t < \tau_+(\mathbf{z})$, we have

$$\tau_+(\mathbf{\Phi}(\mathbf{z}, t, 0)) = t, \qquad \text{while} \qquad \tau_-(\mathbf{\Phi}(\mathbf{z}, t, 0)) = \tau_-(\mathbf{z}) - t.$$

Consequently,

$$f(\mathbf{\Phi}(\mathbf{z},t,0)) = h(\mathbf{z})\left[\frac{(\tau_-(\mathbf{z})-t)e^{-t}}{\tau_-(\mathbf{z})}\chi_{\{\tau_-(\mathbf{z})<\infty\}} + e^{-t}\chi_{\{\tau_-(\mathbf{z})=\infty\}}\right],$$

so that $\mathsf{B}^+ f(\mathbf{z}) = \lim_{t\to 0} f(\mathbf{\Phi}(\mathbf{z},t,0)) = h(\mathbf{z})$ for a.e. $\mathbf{z}\in\Gamma_+$. Now, Eq. (3.2) yields $f\in\mathscr{D}(\mathbf{T}_{\max})$ and

$$\mathbf{T}_{\max} f(\mathbf{x}) = \begin{cases} -h(\mathbf{\Phi}(\mathbf{x},0,\tau_+(\mathbf{x})))\, e^{-\tau_+(\mathbf{x})}\dfrac{1+\tau_-(\mathbf{x})}{\tau_-(\mathbf{x})+\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x})+\tau_+(\mathbf{x})<\infty,\\ -h(\mathbf{\Phi}(\mathbf{x},0,\tau_+(\mathbf{x})))\, e^{-\tau_+(\mathbf{x})} & \text{if } \tau_-(\mathbf{x})=\infty \text{ and } \tau_+(\mathbf{x})<\infty,\\ 0 & \text{if } \tau_+(\mathbf{x})=\infty. \end{cases}$$

Next we show $\|f\|_X \leqslant \|h\|_{L^1_+}$ and $\|\mathbf{T}_{\max} f\|_X \leqslant \|h\|_{L^1_+}$. First we notice that

$$\int_{\mathbf{\Omega}} |f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+} |f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) + \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} |f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}),$$

since $f(\mathbf{x})=0$ whenever $\tau_+(\mathbf{x})=\infty$. Now, according to the integration formula (2.8),

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_-} |h(\mathbf{\Phi}(\mathbf{x},0,\tau_+(\mathbf{x})))|\frac{\tau_-(\mathbf{x})e^{-\tau_+(\mathbf{x})}}{\tau_-(\mathbf{x})+\tau_+(\mathbf{x})}$$
$$= \int_{\Gamma_+\setminus\Gamma_{+\infty}} \mathrm{d}\mu(\mathbf{z}) \int_0^{\tau_-(\mathbf{z})} \frac{|h(\mathbf{z})|}{\tau_-(\mathbf{z})}(\tau_-(\mathbf{z})-s)e^{-s}\mathrm{d}s = \int_{\Gamma_+\setminus\Gamma_{+\infty}} |h(\mathbf{z})|\frac{e^{-\tau_-(\mathbf{z})}+\tau_-(\mathbf{z})-1}{\tau_-(\mathbf{z})}\mathrm{d}\mu(\mathbf{z})$$
$$\leqslant \int_{\Gamma_+\setminus\Gamma_{+\infty}} |h(\mathbf{z})|\mathrm{d}\mu(\mathbf{z}).$$

In the same way, according to Eq. (2.9),

$$\int_{\mathbf{\Omega}_+\cap\mathbf{\Omega}_{-\infty}} |f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) = \int_{\Gamma_{+\infty}} \mathrm{d}\mu_+(\mathbf{z}) \int_0^{\infty} |h(\mathbf{z})|e^{-s}\mathrm{d}s = \int_{\Gamma_{+\infty}} |h(\mathbf{z})|\mathrm{d}\mu_+(\mathbf{z}).$$

One obtains thus that

$$\|f\|_X = \int_{\mathbf{\Omega}} |f(\mathbf{x})|\mathrm{d}\mu(\mathbf{x}) \leqslant \int_{\Gamma_+\setminus\Gamma_{+\infty}} |h(\mathbf{z})|\mathrm{d}\mu(\mathbf{z}) + \int_{\Gamma_{+\infty}} |h(\mathbf{z})|\mathrm{d}\mu(\mathbf{z}) = \|h\|_{L^1_+}.$$

One proceeds in the same way to show that $\|\mathbf{T}_{\max} f\|_X \leqslant \|h\|_{L^1_+}$. □

3.3. **Transport equations with abstract boundary conditions.** For any *(linear) bounded boundary operator* $H\in\mathscr{B}(L^1_+,L^1_-)$, define $\mathbf{T}_H$ as

$$\mathbf{T}_H\varphi = \mathbf{T}_{\max}\varphi \qquad \text{for any } \varphi\in\mathscr{D}(\mathbf{T}_H),$$

where

$$\mathscr{D}(\mathbf{T}_H) = \left\{\psi\in\mathscr{D}(\mathbf{T}_{\max})\,;\, \mathsf{B}^+\psi\in L^1_+, \mathsf{B}^-\psi = H\mathsf{B}^+\psi\right\}.$$

For any $\lambda > 0$, we define the operators:

$$\begin{cases} M_\lambda \ : & L^1_- \longrightarrow L^1_+ \\ & u \longmapsto [M_\lambda u](\mathbf{y}) = u(\mathbf{\Phi}(\mathbf{y}, \tau_-(\mathbf{y}), 0)) \exp(-\lambda \tau_-(\mathbf{y}))\, \chi_{\{\tau_-(\mathbf{y})<\infty\}}, \quad (\mathbf{y} \in \Gamma_+)\ ; \end{cases}$$

$$\begin{cases} \Xi_\lambda \ : & L^1_- \longrightarrow X \\ & u \longmapsto [\Xi_\lambda u](\mathbf{x}) = u(\mathbf{\Phi}(\mathbf{x}, \tau_-(\mathbf{x}), 0)) \exp(-\lambda \tau_-(\mathbf{x}))\, \chi_{\{\tau_-(\mathbf{x})<\infty\}}, \quad (\mathbf{x} \in \mathbf{\Omega})\ ; \end{cases}$$

$$\begin{cases} G_\lambda \ : & X \longrightarrow L^1_+ \\ & f \longmapsto [G_\lambda f](\mathbf{z}) = \displaystyle\int_0^{\tau_-(\mathbf{z})} f(\mathbf{\Phi}(\mathbf{z}, s, 0)) \exp(-\lambda s) \mathrm{d}s, \quad (\mathbf{z} \in \Gamma_+)\ ; \end{cases}$$

and

$$\begin{cases} C_\lambda \ : & X \longrightarrow X \\ & f \longmapsto [C_\lambda f](\mathbf{x}) = \displaystyle\int_0^{\tau_-(\mathbf{x})} f(\mathbf{\Phi}(\mathbf{x}, s, 0)) \exp(-\lambda s) \mathrm{d}s, \quad (\mathbf{x} \in \mathbf{\Omega}). \end{cases}$$

Thanks to Hölder's inequality, all these operators are bounded on their respective spaces. Note that Ξ_λ is a lifting operator which, to a given $u \in L^1_-$ associates a function $f = \Xi_\lambda u \in \mathscr{D}(\mathbf{T}_{\max})$ whose trace on Γ_- is exactly u (Theorem 3.5). The operator M_λ transfers functions defined on Γ_- to functions defined on Γ_+ and, using Theorem 3.4, it is easy to see that C_λ coincides with the resolvent of $\mathbf{T}_0$, i.e. $C_\lambda f = (\lambda - \mathbf{T}_0)^{-1} f$ for any $f \in X$, $\lambda > 0$. In particular, $\mathrm{Rank}(C_\lambda) \subset \mathscr{D}(\mathbf{T}_{\max})$. Moreover, still using Theorem 3.5, we see that $G_\lambda f = \mathsf{B}^+ C_\lambda f$ for any $f \in X$ and $M_\lambda u = \mathsf{B}^+ \Xi_\lambda u$ for any $u \in L^1_+$. Finally, we see that G_λ *is surjective* for any $\lambda > 0$. Indeed, according to Proposition 3.8, we have that for any $g \in L^1_+$, there is an $f \in \mathscr{D}(\mathbf{T}_{\max})$, such that $\mathsf{B}^+ f = g$ and $\mathsf{B}^- f = 0$. The latter property means that $f \in \mathscr{D}(\mathbf{T}_0)$ so that, for any $\lambda > 0$, there is $\psi \in X$ such that $f = \mathscr{R}(\lambda, \mathbf{T}_0)\psi$. In this case, $g = \mathsf{B}^+ f = G_\lambda \psi$ and $\|\psi\|_X \leqslant \lambda \|f\|_X + \|\mathbf{T}_0 f\|_X \leqslant (1+\lambda)\|g\|_X$. The above operators allow to solve the boundary-value problem

$$\begin{cases} (\lambda - \mathbf{T}_{\max}) f = g \\ \mathsf{B}^- f = H \mathsf{B}^+ f \end{cases} \tag{3.10}$$

where $g \in X$ and $\lambda > 0$. Precisely, we have

Proposition 3.9. *Let $g \in X$ be given. Assume that for some $\lambda_0 > 0$ the series $\sum_{n=0}^\infty (M_{\lambda_0} H)^n G_{\lambda_0} g$ converges in L^1_+. Then the function*

$$f = C_{\lambda_0} g + \Xi_{\lambda_0} H \left(\sum_{n=0}^\infty (M_{\lambda_0} H)^n G_{\lambda_0} g \right) \tag{3.11}$$

is a solution of $(\lambda_0 - \mathbf{T}_H) f = g$. If, moreover, $H \geqslant 0$, then the thesis is valid for all $\lambda \geqslant \lambda_0$.

PROOF: Define $\mathcal{S}_{\lambda_0} g = \sum_{n=0}^\infty (M_{\lambda_0} H)^n G_{\lambda_0} g$. By assumption, $\mathcal{S}_{\lambda_0} g \in L^1_+$ so that $H \mathcal{S}_{\lambda_0} g \in L^1_-$. Then, as we have already seen, both $C_{\lambda_0} g$ and $\Xi_{\lambda_0} H \mathcal{S}_{\lambda_0} g$ belong to $\mathscr{D}(\mathbf{T}_{\max})$. This shows that $f = C_{\lambda_0} g + \Xi_{\lambda_0} H \mathcal{S}_{\lambda_0} g \in \mathscr{D}(\mathbf{T}_{\max})$. Furthermore,

$$\mathsf{B}^- f = \mathsf{B}^- C_{\lambda_0} g + \mathsf{B}^- \Xi_{\lambda_0} H \mathcal{S}_{\lambda_0} g = H \mathcal{S}_{\lambda_0} g$$

since $C_{\lambda_0} g \in \mathscr{D}(\mathbf{T}_0)$ and $\mathsf{B}^- \Xi_{\lambda_0} u = u$ for all $u \in L^1_-$. In a similar way,

$$\mathsf{B}^+ f = \mathsf{B}^+ C_{\lambda_0} g + \mathsf{B}^+ \Xi_{\lambda_0} H \mathcal{S}_{\lambda_0} g = G_{\lambda_0} g + M_{\lambda_0} H \mathcal{S}_{\lambda_0} g = \sum_{n=0}^{\infty} (M_{\lambda_0} H)^n G_{\lambda_0} g = \mathcal{S}_{\lambda_0} g,$$

so that $\mathsf{B}^- f = H \mathsf{B}^+ f$, i.e. $f \in \mathscr{D}(\mathbf{T}_H)$. Finally, Theorems 3.4 and 3.5 assert respectively that $(\lambda_0 - \mathbf{T}_{\max}) C_{\lambda_0} g = g$ and $(\lambda_0 - \mathbf{T}_{\max}) \Xi_{\lambda_0} u = 0$ for any $u \in L^1_+$ so that f solves (3.10) with $\lambda = \lambda_0$. The statement for $H \geqslant 0$ follows from the fact that M_λ and G_λ decrease with λ and thus the series in (3.11) converges for any $\lambda \, geq \lambda_0$. □

As a consequence, one gets the following generation result for contractive boundary operators already stated in [7, 12]

Theorem 3.10. *Assume H to be strictly contractive, i.e. $\|H\|_{\mathscr{B}(L^1_+, L^1_-)} < 1$. Then $\mathbf{T}_H$ generates a C_0-semigroup of contractions $(U_H(t))_{t \geqslant 0}$ and the resolvent $(\lambda - \mathbf{T}_H)^{-1}$ is given by*

$$(\lambda - \mathbf{T}_H)^{-1} = C_\lambda + \Xi_\lambda H \left(\sum_{n=0}^{\infty} (M_\lambda H)^n G_\lambda \right) \qquad \textit{for any} \quad \lambda > 0 \tag{3.12}$$

where the series is convergent in $\mathscr{B}(X)$.

PROOF: It is easy to see that $\|M_\lambda\| \leqslant 1$ for any $\lambda > 0$. In particular, $\|M_\lambda H\| < 1$ for any $\lambda > 0$ and the series $\sum_{n=0}^{\infty} (M_\lambda H)^n G_\lambda$ converges in $\mathscr{B}(X, L^1_+)$. Fix now $g \in X$ and let $f \in X$ be given by (3.11). Proposition 3.9 ensures that f is a solution of (3.10) while (3.4) implies that

$$\lambda \, |f\|_X \leqslant \|g\|_X + \|\mathsf{B}^- f\|_{L^1_-} - \|\mathsf{B}^+ f\|_{L^1_+}.$$

Since $\|\mathsf{B}^- f\|_{L^1_-} = \|H \mathsf{B}^+ f\|_{L^1_-} < \|B^+ f\|_{L^1_+}$, we get $\lambda \|f\|_X < \|g\|_X$ or, equivalently,

$$\|(\lambda - \mathbf{T}_H) f\|_X \geqslant \lambda \|f\|_X.$$

Therefore $\mathbf{T}_H$ is a densely defined dissipative operator (recall that $\mathscr{D}(\mathbf{T}_H)$ contains the set of compactly supported continuous functions) of X. Moreover, the range of $(\lambda - \mathbf{T}_H)$ is exactly X according to Proposition 3.9 so that the Lumer-Phillips Theorem leads to the generation result. Now, the representation (3.12) is nothing else but (3.11). □

Remark 3.11. *Hadamard's criterion ensures that the series in* (3.12) *converges in $\mathscr{B}(X)$ for any $\lambda > 0$ and any boundary operator H such that $r_\sigma(M_\lambda H) < 1$.*

4. MULTIPLICATIVE BOUNDARY CONDITIONS

In this section, we consider the general case $\|H\|_{\mathscr{B}(L^1_+, L^1_-)} \geqslant 1$, and we provide, in the spirit of [14], a sufficient condition on H ensuring that $\mathbf{T}_H$ generates a C_0-semigroup in X. Let χ_ϵ denote the following multiplication operator in L^1_+:

$$[\chi_\epsilon u](\mathbf{y}) = \begin{cases} u(\mathbf{y}) & \text{if } \tau_-(\mathbf{y}) \leqslant \epsilon, \\ 0 & \text{else,} \end{cases}$$

for any $u \in L^1_+$ and any $\epsilon > 0$. Our main result is the following.

Theorem 4.1. *Let $H \in \mathscr{B}(L^1_+, L^1_-)$ with $\|H\| \geqslant 1$. If*

$$\limsup_{\epsilon \to 0} \|H\chi_\epsilon\|_{\mathscr{B}(L^1_+, L^1_-)} < 1, \tag{4.1}$$

then $\mathbf{T}_H$ generates a C_0-semigroup $(V_H(t))_{t \geqslant 0}$ in X such that

$$\|V_H(t)\| \leqslant \frac{\|H\|}{1 - \|H\chi_\epsilon\|} \exp\left[-\frac{t}{\varepsilon} \log\left(\frac{1 - \|H\chi_\epsilon\|}{\|H\|}\right)\right] \qquad (t \geqslant 0), \tag{4.2}$$

for any $\epsilon > 0$ such that $\|H\chi_\epsilon\| < 1$.

The strategy to prove this result is adapted from [14] and consists in performing a suitable change of the unknown function in (1.1) (similar to that used in [12, Chapter XIII]) so that the new unknown satisfies an equivalent evolution problem (4.4) but with a boundary operator which is contractive, provided the assumption (4.1) holds. More precisely, for any $0 < q < 1$, define the multiplication operator in L^1_+:

$$\mathcal{M}_q \,:\, L^1_+ \ni u \mapsto [\mathcal{M}_q u](\mathbf{y}) = \exp\{(\tau_-(\mathbf{y}) \wedge k) \log q\} u(\mathbf{y}) \in L^1_+,$$

where k is a positive real number to be fixed later. Let $\mathcal{Z}_q$ be defined by

$$\mathcal{Z}_q \,:\, X \ni \varphi \mapsto [\mathcal{Z}_q \varphi](\mathbf{x}) = \exp\{(\tau_-(\mathbf{x}) \wedge k) \log q\} \varphi(\mathbf{x}) \in X.$$

Since $\mathcal{M}_q \in \mathscr{B}(L^1_+)$, it is possible to define the free streaming operator $\mathbf{T}_{H\mathcal{M}_q}$ associated to the boundary operator $H\mathcal{M}_q \in \mathscr{B}(L^1_-, L^1_+)$ and the *absorption operator*

$$\mathbf{A}_{H,q}\psi(\mathbf{x}) = \mathbf{T}_{H\mathcal{M}_q}\psi(\mathbf{x}) - \log q \, \chi_{\{\tau_-(\mathbf{x}) \leqslant k\}} \psi(\mathbf{x}), \qquad \psi \in \mathscr{D}(\mathbf{A}_{H,q}),$$

where

$$\mathscr{D}(\mathbf{A}_{H,q}) = \mathscr{D}(\mathbf{T}_{H\mathcal{M}_q}) = \left\{\psi \in \mathscr{D}(\mathbf{T}_{\max})\,;\, \mathsf{B}^{\pm}\psi \in L^1_{\pm}, \mathsf{B}^{-}\psi = H\mathcal{M}_q \mathsf{B}^{+}\psi\right\}.$$

The unbounded operators $\mathbf{T}_H$ and $\mathbf{A}_{H,q}$ are related by the following lemma.

Lemma 4.2. *For any $0 < q < 1$, $\mathcal{Z}_q^{-1}\mathscr{D}(\mathbf{T}_H) = \mathscr{D}(\mathbf{A}_{H,q})$ and $\mathbf{T}_H = \mathcal{Z}_q \mathbf{A}_{H,q} \mathcal{Z}_q^{-1}$.*

PROOF: Let $0 < q < 1$ be fixed. One sees easily that $\mathcal{Z}_q$ is a continuous bijection from X onto itself. Its inverse is given by

$$\mathcal{Z}_q^{-1} \,:\, \varphi \in X \mapsto \mathcal{Z}_q^{-1}\varphi(\mathbf{x}) = \exp\{-(\tau_-(\mathbf{x}) \wedge k) \, \log q\} \varphi(\mathbf{x}) \in X.$$

Note that $\mathcal{Z}_q^{-1} \in \mathscr{B}(X)$ because $\sup\{\tau_-(\mathbf{x}) \wedge k \,;\, \mathbf{x} \in \mathbf{\Omega}\} \leqslant k$. Now, let $\varphi \in \mathscr{D}(\mathbf{T}_H)$ and $\psi = \mathcal{Z}_q^{-1}\varphi$. First we show that $\psi \in \mathscr{D}(\mathbf{T}_{\max})$. From (3.2), on the set $\{\mathbf{x} \in \mathbf{\Omega}\,;\, \tau_-(\mathbf{x}) \leqslant k\} \subset \mathbf{\Omega}_-$,

$$\begin{aligned}
\mathbf{T}_{\max}\psi(\mathbf{x}) &= \lim_{s \to 0} s^{-1} \left(\psi(\mathbf{\Phi}(\mathbf{x}, s, 0)) - \psi(\mathbf{x})\right) \\
&= \lim_{s \to 0} s^{-1} \left(e^{-(\tau_-(\mathbf{\Phi}(\mathbf{x},s,0)) \wedge k) \log q} \varphi(\mathbf{\Phi}(\mathbf{x}, s, 0)) - e^{-\tau_-(\mathbf{x}) \log q} \varphi(\mathbf{x})\right).
\end{aligned}$$

Since $\tau_-(\mathbf{\Phi}(\mathbf{x}, s, 0)) = \tau_-(\mathbf{x}) - s$ for any $0 \leqslant s < \tau_-(\mathbf{x})$, we get

$$\begin{aligned}
\mathbf{T}_{\max}\psi(\mathbf{x}) &= e^{-\tau_-(\mathbf{x}) \log q} \lim_{s \to 0} s^{-1} \left(e^{s \log q} \varphi(\mathbf{\Phi}(\mathbf{x}, s, 0)) - \varphi(\mathbf{x})\right) \\
&= e^{-\tau_-(\mathbf{x}) \log q} \left(\log q \varphi(\mathbf{x}) + \mathbf{T}_{\max}\varphi(\mathbf{x})\right)
\end{aligned}$$

for any $\mathbf{x} \in \mathbf{\Omega}$ with $\tau_-(\mathbf{x}) \leqslant k$. Since

$$\psi(\mathbf{x}) = e^{-\tau_-(\mathbf{x})\log q}\varphi(\mathbf{x})\chi_{\{\tau_-(x)\leqslant k\}} + e^{-k\log q}\varphi(\mathbf{x})\chi_{\{\tau_-(\mathbf{x})>k\}}$$

we obtain $\mathbf{T}_{\max}\psi(\mathbf{x}) = e^{-(\tau_-(\mathbf{x})\wedge k)\log q}\,(\log q\ \varphi(\mathbf{x})\chi_{\{\tau_-(\mathbf{x})\leqslant k\}} + \mathbf{T}_{\max}\varphi(\mathbf{x}))$, i.e.

$$\mathbf{T}_{\max}\mathcal{Z}_q^{-1}\varphi = \mathcal{Z}_q^{-1}\left(\log q\,\varphi\,\chi_{\{\tau_-\leqslant k\}} + \mathbf{T}_{\max}\varphi\right) \tag{4.3}$$

so that $\psi \in \mathscr{D}(\mathbf{T}_{\max})$. Moreover, since $\tau_-(\mathbf{y}) \wedge k = 0$ for any $\mathbf{y} \in \Gamma_-$, it is clear that

$$\mathsf{B}^-\psi(\mathbf{y}) = \lim_{s\to 0^+}\psi(\mathbf{\Phi}(\mathbf{y},0,s)) = \lim_{s\to 0^+}\exp(-s\log q)\varphi(\mathbf{\Phi}(\mathbf{y},0,s)) = \mathsf{B}^-\varphi(\mathbf{y}).$$

Now, for any $\mathbf{z} \in \Gamma_+$, using that $\tau_-(\mathbf{\Phi}(\mathbf{z},s,0)) = \tau_-(\mathbf{z}) - s$ for any $0 < s < \tau_-(\mathbf{z})$, we have

$$\begin{aligned}\mathsf{B}^+\psi(\mathbf{z}) &= \lim_{s\to 0^+}\psi(\mathbf{\Phi}(\mathbf{z},s,0) = \lim_{s\to 0^+}\exp\left(-\left((\tau_-(\mathbf{z})\wedge k) - s\right)\log q\right)\varphi(\mathbf{\Phi}(\mathbf{z},0,s))\\ &= \exp\left(-\left(\tau_-(\mathbf{z})\wedge k\right)\log q\right)\mathsf{B}^+\varphi(\mathbf{z}),\end{aligned}$$

i.e. $\mathsf{B}^+\psi = \mathcal{M}_q^{-1}\mathsf{B}^+\varphi$. Consequently $\mathsf{B}^\pm\psi \in L^1_\pm$ and $\mathsf{B}^-\psi = H\mathcal{M}_q\mathsf{B}^+\psi$. This proves that $\psi \in \mathscr{D}(\mathbf{A}_{H,q})$ i.e. $\mathcal{Z}_q^{-1}\mathscr{D}(\mathbf{T}_H) \subset \mathscr{D}(\mathbf{A}_{H,q})$. The converse inclusion is proved similarly. Finally, for any $\varphi \in \mathscr{D}(\mathbf{T}_H)$, (4.3) reads $\mathbf{T}_{H\mathcal{M}_q}\mathcal{Z}_q^{-1}\varphi = \mathcal{Z}_q^{-1}\left(\log q\,\chi_{\{\tau_-\leqslant k\}}\varphi + \mathbf{T}_H\varphi\right)$, i.e.

$$\mathcal{Z}_q\mathbf{A}_{H,q}\mathcal{Z}_q^{-1}\varphi = \mathbf{T}_H\varphi \qquad\qquad \forall\varphi \in \mathscr{D}(\mathbf{T}_H)$$

which completes the proof. □

Remark 4.3. *Note that the characteristic function $\chi_{\{\tau_-\leqslant k\}}$ in the definition of $\mathbf{A}_{H,q}$ is missing in the force-free case studied in* [14] *but has to be considered if one wishes to take into account characteristic curves with infinite length.*

The above lemma shows that the evolution problem

$$\begin{cases}\dfrac{\partial}{\partial t}\varphi(\mathbf{x},t) + \mathcal{F}(\mathbf{x})\cdot\nabla_{\mathbf{x}}\varphi(\mathbf{x},t) + \log q\,\chi_{\{\tau_-(\mathbf{x})\leqslant k\}}\varphi(\mathbf{x},t) = 0\\ \mathsf{B}^-\varphi = H\mathcal{M}_q\mathsf{B}^+\varphi\\ \varphi(\mathbf{x},0) = \exp\{-(\tau_-(\mathbf{x})\wedge k)\log q\}\psi_0(\mathbf{x}), \qquad (\mathbf{x}\in\mathbf{\Omega})\end{cases} \tag{4.4}$$

is equivalent, by the change of variables, to problem (1.1). Consequently, to prove that $\mathbf{T}_H$ is a generator of a C_0-semigroup $(V_H(t))_{t\geqslant 0}$ in X, it suffices to show that $\mathbf{A}_{H,q}$ generates a C_0-semigroup $(V_{H,q}(t))_{t\geqslant 0}$ in X (for some $0 < q < 1$). Moreover, by Theorem 3.10, it is enough to find $0 < q < 1$ such that $\|H\mathcal{M}_q\| < 1$. We are now in position to prove Theorem 4.1.

PROOF OF THEOREM 4.1: Define $\mathcal{Q} = \{\,0 < q < 1\ ;\ \|H\mathcal{M}_q\| < 1\}$. As explained above, Lemma 4.2 and Theorem 3.10 imply that if $\mathcal{Q} \neq \varnothing$, then $\mathbf{T}_H$ generates a C_0-semigroup $(V_H(t))_{t\geqslant 0}$ such that

$$V_H(t) = \mathcal{Z}_q V_{H,q}(t)\mathcal{Z}_q^{-1} \qquad (t \geqslant 0,\ q \in \mathcal{Q}), \tag{4.5}$$

where $(V_{H,q}(t))_{t\geqslant 0}$ is a C_0-semigroup in X with generator $\mathbf{A}_{H,q}$ $(q \in \mathcal{Q})$. Using assumption (4.1), let us fix $\epsilon > 0$ so that $\|H\chi_\epsilon\| < 1$ and choose k to be larger than ϵ. Then, for any $0 < q < 1$,

$$\|H\mathcal{M}_q\| \leqslant \|H\chi_\epsilon\,\mathcal{M}_q\| + \|H\,(I-\chi_\epsilon)\,\mathcal{M}_q\| \leqslant \|H\chi_\epsilon\| + \|H\|\,\|(I-\chi_\epsilon)\,\mathcal{M}_q\|.$$

Moreover

$$\begin{aligned}\|(I-\chi_\epsilon)\mathcal{M}_q\| &= \sup\{\exp[(\tau_-(\mathbf{y})\wedge k)\log q]\,;\,\mathbf{y}\in\Gamma_+ \text{ and } (\tau_-(\mathbf{y})\wedge k)\geqslant\epsilon\}\\ &\leqslant \exp(\epsilon\log q) \qquad\qquad (0<q<1).\end{aligned}$$

Consequently,

$$\|H\mathcal{M}_q\| \leqslant \|H\chi_\epsilon\| + \|H\|\exp(\epsilon\log q)$$

and $q\in\mathcal{Q}$ provided

$$(\epsilon\log q) < \log\left(\frac{1-\|H\chi_\epsilon\|}{\|H\|}\right). \tag{4.6}$$

Therefore, $\mathcal{Q}\neq\varnothing$ and $\mathbf{T}_H$ is a generator of a C_0-semigroup $(V_H(t))_{t\geqslant 0}$ in X. On the other hand, since $\mathbf{A}_{H,q}\psi = \mathbf{T}_{H\mathcal{M}_q}\psi - \log q\,\chi_{\{\tau_-\leqslant k\}}\psi$ for any $\psi\in\mathscr{D}(\mathbf{A}_{H,q})$, and since $\mathbf{T}_{H\mathcal{M}_q}$ generates a C_0-semigroup of contractions, we see that

$$\|V_{H,q}(t)\| \leqslant \exp(-t\,\log q) \qquad\qquad (t\geqslant 0,\ q\in\mathcal{Q}).$$

Next, we see that

$$\|\mathcal{Z}_q\|\leqslant 1 \qquad \text{and} \qquad \|\mathcal{Z}_q^{-1}\|\leqslant\exp(-k\log q),\quad (q\in\mathcal{Q}),$$

hence (4.5) implies $\|V_H(t)\|\leqslant\exp(-(k+t)\log q)$ for any $t\geqslant 0$ and any $q\in\mathcal{Q}$. Noting that the set $\mathcal{Q}$ is independent of k (actually it depends only on ϵ and H through (4.6)), we may let k go to ϵ so that

$$\|V_H(t)\|\leqslant\exp(-(\varepsilon+t)\log q) \qquad\qquad (t\geqslant 0,\ q\in\mathcal{Q}).$$

Now, for any fixed $\varepsilon>0$, optimizing the free parameter $\log q$ in (4.6) we obtain (4.2). □

The estimate (4.2) on $\|V_H(t)\|$ certainly is not optimal and can be improved for some geometries of the phase space. One such case is described in the corollary below.

Corollary 4.4. *Assume that* $\inf\{\tau_-(\mathbf{y})\,;\,\mathbf{y}\in\Gamma_+\}=\ell_0>0$. *Then, for any boundary operator* $H\in\mathscr{B}(L^1_+,L^1_-)$, $\mathbf{T}_H$ *generates a* C_0*-semigroup* $(V_H(t))_{t\geqslant 0}$ *in* X *such that*

$$\|V_H(t)\|\leqslant\max\{1,\|H\|\}\exp\left(\max\{0,\log\|H\|\}t/\ell_0\right) \qquad (t\geqslant 0).$$

PROOF: According to Theorem 3.10, it suffices to prove the result for $\|H\|\geqslant 1$. Noting that

$$\|H\chi_\varepsilon\| = \begin{cases}0 & \text{if } 0<\varepsilon<\ell_0\\ \|H\| & \text{if } \varepsilon\geqslant\ell_0,\end{cases}$$

we immediately see that

$$\inf\left\{\frac{1}{\epsilon}\log\left(\frac{1-\|H\chi_\varepsilon\|}{\|H\|}\right)\,;\,\|H\chi_\epsilon\|<1\right\} = -\frac{\log\|H\|}{\ell_0}.$$

The proof becomes now a straightforward application of Theorem 4.1. □

This corollary shows that if $\mathbf{\Omega}$ is a phase space in which the lengthes of characteristic curves are bounded away from 0, then the general transport equation (1.1) is **well-posed for any bounded boundary operator** $H\in\mathscr{B}(L^1_+,L^1_-)$. Let us illustrate the above result with a few examples.

Example 4.5. In the first example we consider the force–free Vlasov equation in a slab of thickness $2a$, $(a > 0)$. In such a case, $\mathbf{\Omega} = \{\mathbf{x} = (x, v) \in \mathbb{R}^2\,;\, -a < x < a\,,\, -1 < v < 1\}$ and $\mathcal{F}(\mathbf{x}) = (v, 0)$. It is not difficult to see (see for instance [14, Section 4.1]) that the above Corollary applies in this case since $\inf\{\tau_-(\mathbf{y})\,;\, \mathbf{y} \in \Gamma_+\} = 2a > 0$.

Consider now an example of the Vlasov equation with a non trivial force term for which Corollary 4.4 still applies.

Example 4.6. Let us consider the following two-dimensional phase space:

$$\mathbf{\Omega} = \{\mathbf{x} = (x, y) \in \mathbb{R}^2\,;\, x^2 + y^2 < 2 \text{ and } -1 < y < 1\}$$

with the field $\mathcal{F}(\mathbf{x}) = (-y, x)$ for any $\mathbf{x} = (x, y) \in \mathbf{\Omega}$. In such a case, the characteristic curves are *circular*, namely

$$\mathbf{\Phi}(\mathbf{x}, 0, s) = (x\cos s - y\sin s, x\sin s + y\cos s), \qquad \mathbf{x} = (x, y), \qquad s \in \mathbb{R}.$$

In particular, for any $\mathbf{x} = (x, y) \in \mathbf{\Omega}$ such that $x^2 + y^2 < 1$, one has $\tau_\pm(\mathbf{x}) = \infty$. Moreover,

$$\Gamma_\pm = \{(x, -1)\,;\, -1 < \pm\, x < 0\} \cup \{(x, 1)\,;\, 0 < \pm\, x < 1\}.$$

In this case, one can easily check that $\inf\{\tau_-(\mathbf{y})\,;\, \mathbf{y} \in \Gamma_+\} = \pi/2$.

5. Conservative boundary conditions

In this section we consider the case of conservative boundary conditions. Note that such boundary conditions arise naturally in the study of gas dynamics [12] and are typically associated to a boundary operator H such that

$$\|H\psi\| = \|\psi\|, \qquad \text{for any} \quad \psi \in L^1_+,\ \psi \geqslant 0.$$

Theorem 4.1 does not apply to such boundary operators since

$$\|H\chi_\epsilon\| = \|\chi_\epsilon\| = 1 \quad \text{for any } \epsilon > 0.$$

Therefore to deal with the generation properties of the operator $\mathbf{T}_H$, we shall proceed in a different way adapting techniques used in [4] in the force-free case. From now on, we adopt the following assumptions, which are more specific than the condition above.

Assumption 5.1.

*(a) The boundary operator $H \in \mathscr{B}(L^1_+, L^1_-)$ is **positive**.*
(b) $\|H\| = 1$.
(c) If $f \in L^1_+$ is non-negative and $Hf = 0$, then $f = 0$.

Under these hypotheses we can prove the existence result given in Theorem 5.2. This result, with different proof, can be found in [7]. A less general version of it has been obtained also in [4, Theorem 2.8]. The proof in the case including the force field $\mathbf{F}$, which we present below for self-consistency of the paper, is the same as in [4] since it only uses the series representation of the resolvent of $\mathbf{T}_H$ and the generation result for contractive boundary operators (Theorem 3.10).

For any $0 < r < 1$, let $(V_r(t))_{t\geqslant 0}$ be the C_0-semigroup of X generated by $\mathbf{T}_{rH}$ (whose existence is given by Theorem 3.10).

Theorem 5.2. *Let H satisfy Assumption* (5.1)*. Then, for any $t \geqslant 0$ and any $f \in X$ the limit $V_H(t)f = \lim_{r\nearrow 1} V_r(t)f$ exists in X and defines a substochastic semigroup $(V_H(t))_{t\geqslant 0}$. If $(\mathbf{A}, \mathscr{D}(\mathbf{A}))$ is the generator of $(V_H(t))_{t\geqslant 0}$, then its resolvent is given by*

$$(\lambda - \mathbf{A})^{-1} f = C_\lambda f + \sum_{n=0}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda f \qquad \text{for any } f \in X,\ \lambda > 0, \tag{5.1}$$

where the series converges in X.

PROOF: According to Theorem 3.10, for any $0 < r < 1$ and any fixed $\lambda > 0$, the resolvent of $\mathbf{T}_{rH}$ is given by

$$(\lambda - \mathbf{T}_{rH})^{-1} = C_\lambda + \sum_{n=0}^{\infty} r^{n+1} B_\lambda H (M_\lambda H)^n G_\lambda$$

with $\sup_{0<r<1} \|(\lambda - \mathbf{T}_{rH})^{-1}\| \leqslant \lambda^{-1}$. Then, for any $f \geqslant 0$, the function $r \in (0,1) \mapsto (\lambda - \mathbf{T}_{rH})^{-1} f$ is non-negative and non-decreasing so that the following limit exists

$$\mathscr{R}(\lambda) f = \lim_{r\nearrow 1} (\lambda - \mathbf{T}_{rH})^{-1} f = C_\lambda f + \sum_{n=0}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda f$$

where the series converges absolutely because of the monotone convergence theorem. It is easy to check that $\|\mathscr{R}(\lambda) f\| \leqslant \lambda^{-1} \|f\|$. Now, for $f = f^+ - f^-$, we define $\mathscr{R}(\lambda) f = \mathscr{R}(\lambda) f^+ - \mathscr{R}(\lambda) f^-$ so that $\mathscr{R}(\lambda)$ is a linear and bounded operator in X with $\|\mathscr{R}(\lambda)\| \leqslant \lambda^{-1}$. Furthermore, the range of $\mathscr{R}(\lambda)$ is dense in X since it contains the $\mathscr{C}_0^\infty(\mathbf{\Omega})$. Indeed, if $f \in \mathscr{C}_0^\infty(\mathbf{\Omega})$, then $(\lambda - \mathbf{T}_{rH}) f = (\lambda - \mathbf{T}_H) f = g$ is independent of $0 < r < 1$, so that $(\lambda - \mathbf{T}_{rH})^{-1} g = f \to \mathscr{R}(\lambda) g$ as $r \nearrow 1$. Now, thanks to Trotter-Kato Theorem, there exists an operator $(\mathbf{A}, \mathscr{D}(\mathbf{A}))$ which generates a C_0-semigroup $(V_H(t))_{t\geqslant 0}$ in X and such that $\mathscr{R}(\lambda) = (\lambda - \mathbf{A})^{-1}$ for any $\lambda > 0$ and $V_H(t) f = \lim_{r\nearrow 1} V_r(t) f$, for any $t \geqslant 0$. □

Remark 5.3. *We note that the expression* (5.1) *implies that $(V_H(t))_{t\geqslant 0}$ does not depend on the choice of the approximating sequence of semigroups $(V_r(t))_{t\geqslant 0}$. Indeed, for any sequence of non-negative boundary operators $(H_n)_n \subset \mathscr{B}(L^1_+, L^1_-)$ with $H_n f \nearrow H f$ as $n \to \infty$ for any nonnegative $f \in L^1_+$, one can check that $V_{H_n}(t)$ converges strongly to $V_H(t)$.*

Remark 5.4. *Note that, in contrast to what happens in the force-free case* [4, Theorem 2.8] *we cannot say at this moment that $(\mathbf{A}, \mathscr{D}(\mathbf{A}))$ is an extension of $(\mathbf{T}_H, \mathscr{D}(\mathbf{T}_H))$. This, however, will become clear by Theorem 5.8.*

Remark 5.5. *Note that, arguing as in* [4, Corollary 2.10]*, we can show that, for any $\lambda > 0$, the series $\sum_{n=0}^{\infty} \Xi_\lambda H (M_\lambda H)^n$ defines a bounded linear operator from L^1_+ to X whose norm is less than (or equal to) $(\lambda + 1)/\lambda$.*

5.1. **Characterization of $\mathscr{D}(\mathbf{A})$.** In this section we characterize the domain of $\mathbf{A}$ by adapting the extensions techniques used in [4, Section 3] in the force-free case. Such extension techniques are similar to those introduced in [3] in a different context (see also [5]). Precisely, let us denote by E_- the set of all measurable functions defined on $(\Gamma_-, \mathrm{d}\mu_-)$ taking values in the extended set of reals $\mathbb{R} \cup \{\pm\infty\}$. It is clear that $L^1_- \subset \mathsf{E}_-$. In the sequel we shall denote $\Xi := \Xi_1 \in \mathscr{B}(L^1_-, X)$. Through Ξ, we define the set $\mathsf{F}_- \subset \mathsf{E}_-$ as follows: $f \in \mathsf{F}_-$ if and only if for any non-negative and non-decreasing sequence $(f_n)_n \subset L^1_-$, satisfying $\sup_n f_n = |f|$ we have $\sup_n \Xi f_n \in X$. Such a sequence will be called a Ξ-*approximating sequence of* f.

Definition 5.6. *For any $f \in \mathsf{F}_-$, $f \geqslant 0$, we define $\Pi f := \sup_n \Xi f_n \in X$, for any Ξ–approximating sequence $(f_n)_n$ of f. If $f = f_+ - f_-$, we define Πf as $\Pi f = \Pi f_+ - \Pi f_-$.*

Note that, from [4, Lemma 3.1], the operator Π is well-defined from F_- to X in the sense that the value of Π does not depend on the choice of the Ξ-approximating sequence of f.

In the same way, we define the set $\mathsf{E}_+ \supset L^1_+$ to be the set of all extended real-valued measurable functions defined on $(\Gamma_+, \mathrm{d}\mu_+)$. Now, through the boundary operator H, we construct a subset F_+ of E_+ as the set of all functions $\psi \in \mathsf{E}_+$ such that $\sup_n H\psi_n \in \mathsf{F}_-$ for any non-negative and nondecreasing sequence $(\psi_n)_n$ of L^1_+ such that $\sup_n \psi_n = |\psi|$. Such a sequence will be called an *H-approximating sequence of* ψ. We have the following definition

Definition 5.7. *For any $\psi \in \mathsf{F}_+$, $\psi \geqslant 0$, define $\mathsf{H}\psi := \sup_n H\psi_n \in \mathsf{F}_-$, for any H-approximating sequence (ψ_n) of ψ. If $\psi = \psi_+ - \psi_-$, we define $\mathsf{H}\psi$ as $\mathsf{H}\psi = \mathsf{H}\psi_+ - \mathsf{H}\psi_-$.*

Here again, the above operator is well-defined by virtue of [4, Lemma 3.4]. We are now in position to precisely describe the domain of $\mathbf{A}$.

Theorem 5.8. *Let Assumption 5.1 be satisfied. Then $\varphi \in \mathscr{D}(\mathbf{A})$ if and only if*

(1) $\varphi \in \mathscr{D}(\mathbf{T}_{\max})$, $\mathsf{B}^{\pm}\varphi \in \mathsf{F}_{\pm}$,
(2) $\mathsf{B}^-\varphi = \mathsf{H}\mathsf{B}^+\varphi$
(3) $\lim_{n\to\infty} \|\Pi\mathsf{H}(\mathsf{M}\mathsf{H})^n\mathsf{B}^+\varphi\|_X = 0$.

Moreover, for any $\varphi \in \mathscr{D}(\mathbf{A})$, $\mathbf{A}\varphi = \mathbf{T}_{\max}\varphi$.

PROOF: We refer the reader to [4, Theorem 3.6] for the proof of the above Theorem. Actually, the main ingredient of the proof is the representation formula (3.12) whereas the explicit expressions of the operators M_λ, Ξ_λ, C_λ and G_λ do not play any role in the proof. Note that, though the range of M is $\mathsf{E}_+ \subsetneq \mathsf{F}_+$, it can be check that, for any φ satisfying 1) and 2), the sequence $(\Pi\mathsf{H}(\mathsf{M}\mathsf{H})^n\mathsf{B}^+\varphi)_n$ is well-defined. □

An important consequence of the above characterization is that it explains the link between the domains of $\mathbf{T}_H$ and that of $\mathbf{A}$.

Proposition 5.9. *Let $\varphi \in \mathscr{D}(\mathbf{A})$ be such that $\varphi_{|\Gamma_\pm} \in L^1_\pm$. Then $\varphi \in \mathscr{D}(\mathbf{T}_H)$, i.e. $\varphi_{|\Gamma_-} = H(\varphi_{|\Gamma_+})$. More precisely, $\varphi \in \mathscr{D}(\mathbf{T}_H)$ if and only if $\varphi \in \mathscr{D}(\mathbf{A})$ and the series $\sum_{n=0}^\infty (M_1 H)^n G_1 f$ is convergent in L^1_+ where $f = (1-\mathbf{A})\varphi$. In particular, $\mathbf{A} = \mathbf{T}_H$ if and only if $\sum_{n=0}^\infty (M_1 H)^n G_1 f$ converges in L^1_+ for any $f \in X$.*

PROOF: Let $\varphi \in \mathscr{D}(\mathbf{A})$. According to Theorem 5.8, $\mathsf{B}^-\varphi = \mathsf{H}\mathsf{B}^+\varphi$ which, since $\mathsf{B}^-\varphi \in L^1_-$, reads $\varphi_{|\Gamma_-} = H(\varphi_{|\Gamma_+})$. Since $\varphi \in \mathscr{D}(\mathbf{T}_{\max})$, it is then clear that $\varphi \in \mathscr{D}(\mathbf{T}_H)$.

Assume now that $\varphi \in \mathscr{D}(\mathbf{T}_H)$. As above, $\varphi \in \mathscr{D}(\mathbf{A})$ and $\varphi_{|\Gamma_\pm} \in L^1_\pm$. Let $f = (1-\mathbf{A})\varphi$ and let $\psi_n = \sum_{k=0}^{n+1}(M_1H)^kG_1f$ for $n \geqslant 0$. Assume for a while that $f \geqslant 0$. We can show that

$$\sup_n \psi_n = \mathsf{B}^+\varphi \in L^1_+$$

which implies the convergence of the series $\sum_{k=0}^{\infty}(M_1H)^kG_1f$ in L^1_+, which extends for arbitrary f by linearity. Conversely, let $\varphi \in \mathscr{D}(\mathbf{A})$ and $f = (1-\mathbf{A})\varphi$. If $\sum_{k=0}^{\infty}(M_1H)^kG_1f$ converges in L^1_+, then we get $\mathsf{B}^+\varphi \in L^1_+$ in the same way and, from the first part, $\varphi \in \mathscr{D}(\mathbf{T}_H)$. □

The above result shows that $(\mathbf{A}, \mathscr{D}(\mathbf{A}))$ is an extension of $(\mathbf{T}_H, \mathscr{D}(\mathbf{T}_H))$. Moreover, if $\mathbf{T}_H$ does not generate a C_0-semigroup in X, then the set

$$\mathscr{D}(\mathbf{A}) \setminus \mathscr{D}(\mathbf{T}_H) = \{f \in \mathscr{D}(\mathbf{A})\,;\, f_{|\Gamma_\pm} \notin L^1_\pm\} \neq \varnothing,$$

and if $\mathbf{T}_H$ is not closed, then there exists $\varphi \in \mathscr{D}(\overline{\mathbf{T}_H})$ such that $\varphi_{|\Gamma_\pm} \notin L^1_\pm$. The main scope of the following section is to determine the necessary and sufficient condition on H ensuring the stochasticity of $(V_H(t))_{t\geqslant 0}$.

5.2. **Stochasticity of $(V_H(t))_{t\geqslant 0}$.** In this section, we assume that, besides Assumption 5.1, H satisfies conservativeness assumption mentioned at the beginning of this section, i.e.

$$\|H\psi\|_{L^1_-} = \|\psi\|_{L^1_+} \qquad \text{for any } \psi \in L^1_+,\ \psi \geqslant 0. \tag{5.2}$$

In such a case, one expects the semigroup $(V_H(t))_{t\geqslant 0}$ to be *stochastic*, that is,

$$\int_{\mathbf{\Omega}} V_H(t)f\,\mathrm{d}\mu = \int_{\mathbf{\Omega}} f\,\mathrm{d}\mu \qquad (f \in X). \tag{5.3}$$

Indeed, a consequence of Green's formula (Proposition 3.6) is that

$$\int_{\mathbf{\Omega}} \mathbf{T}_H f\,\mathrm{d}\mu = 0 \qquad \text{for any } f \in \mathscr{D}(\mathbf{T}_H).$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}V_H(t)f = \mathbf{A}V_H(t)f$ for any $t \geqslant 0$ and any $f \in \mathscr{D}(\mathbf{A})$, (5.3) should be true at least when $\mathbf{A} = \overline{\mathbf{T}_H}$ (see [16]). In this section we give necessary and sufficient conditions ensuring (5.3) to hold. For any $f \in X$, $f \geqslant 0$, we define

$$\beta(f) := \lim_{n\to\infty} \int_{\Gamma_+} (M_1H)^nG_1f(\mathbf{y})\mathrm{d}\mu_+(\mathbf{y}) \geqslant 0. \tag{5.4}$$

This limit exists since $\|M_1H\| \leqslant 1$ so that the the right-hand-side of (5.4) is a decreasing numerical sequence. For arbitrary $f \in X$, $\beta(f)$ is defined by linearity. We have

Theorem 5.10. *The C_0-semigroup $(V_H(t))_{t\geqslant 0}$ is **stochastic** in X if and only if $\beta(f) = 0$ for any $f \in X$.*

PROOF: Let us fix $f \in X$, $f \geqslant 0$ and let $\varphi = (1-\mathbf{A})^{-1} f$. For any $n \geqslant 1$, define

$$\varphi_n = \mathscr{R}(1, \mathbf{T}_0) f + \sum_{k=0}^{n} \Xi H (M_1 H)^k G_1 f = \mathscr{R}(1, \mathbf{T}_0) f + \Xi H \sum_{k=0}^{n} (M_1 H)^k G_1 f.$$

According to (5.1), we have $\varphi_n \to \varphi$ in X and $\varphi_n \in \mathscr{D}(\mathbf{T}_{\max})$ with $\mathbf{T}_{\max}\varphi_n + f = \varphi_n$ for any $n \geqslant 1$. Now, set

$$u_n := \sum_{k=0}^{n} H (M_1 H)^k G_1 f \in L^1_-, \quad \text{and} \quad \psi_n = \sum_{k=0}^{n+1} (M_1 H)^k G_1 f \in L^1_+.$$

Then it is clear that $\psi_n = \mathsf{B}^+ \varphi_n$ and $\varphi_n = \mathscr{R}(1, \mathbf{T}_0) f + \Xi u_n$. Consequently, Green's formula (Proposition 3.6) yields

$$\int_{\mathbf{\Omega}} \varphi_n \mathrm{d}\mu = \int_{\mathbf{\Omega}} f \mathrm{d}\mu + \int_{\Gamma_-} u_n \mathrm{d}\mu_- - \int_{\Gamma_+} \psi_n \mathrm{d}\mu_+. \tag{5.5}$$

Since $u_n = H\psi_{n-1}$ and $\psi_n \geqslant 0$, (5.2) yields

$$\int_{\mathbf{\Omega}} \varphi_n \mathrm{d}\mu = \int_{\mathbf{\Omega}} f \mathrm{d}\mu + \int_{\Gamma_+} (\psi_{n-1} - \psi_n) \mathrm{d}\mu_+.$$

Now, using that $\psi_n - \psi_{n-1} = (M_1 H)^{n+1} G_1 f$ and passing to the limit as $n \to \infty$, we obtain

$$\int_{\mathbf{\Omega}} \varphi \, \mathrm{d}\mu = \int_{\mathbf{\Omega}} f \mathrm{d}\mu - \beta(f). \tag{5.6}$$

Consequently, $\beta(f) = 0$ if and only if $\|(1-\mathbf{A})^{-1} f\|_X = \|f\|_X$. Now, it is easy to see that the stochasticity of $(V_H(t))_{t \geqslant 0}$ is equivalent to the property that $\|(1-\mathbf{A})^{-1} f\|_X = \|f\|_X$ for any nonnegative $f \in X$. □

Remark 5.11. *The above proof makes more precise the meaning of the functional β. As we saw above, for any $f \in X$ and $\varphi = (1-\mathbf{A})^{-1} f$:*

$$\beta(f) = \int_{\mathbf{\Omega}} \mathbf{A}\varphi(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}).$$

Remark 5.12. *Since G_1 is surjective according to Proposition 3.8, we have $\beta(f) = 0$ for any $f \in X$ if and only if $\|(M_1 H)^n g\|_{L^1_+} \longrightarrow 0$ for any $g \in L^1_+$.*

Proposition 5.13. *Assume that H is conservative. Then, the following are equivalent:*

1) $(V_H(t))_{t \geqslant 0}$ *is stochastic;*
2) $\mathbf{A} = \overline{\mathbf{T}_H}$*;*
3) $\displaystyle\int_{\mathbf{\Omega}} \mathbf{A}\varphi = 0$ *for any* $\varphi \in \mathscr{D}(\mathbf{A})$.

PROOF: The equivalence between 1) and 3) is nothing but (5.6). Let us prove the implication 1) $\Rightarrow$ 2). Take $\varphi \in \mathscr{D}(\mathbf{A})$, the implication is proven if we are able to construct a sequence $(\varphi_n)_n \subset \mathscr{D}(\mathbf{T}_H)$ such that

$$\begin{cases} \varphi_n \longrightarrow \varphi \qquad (n \to \infty) \\ (1 - \mathbf{T}_H)\varphi_n \longrightarrow f = (1 - \mathbf{A})\varphi \qquad \text{in } X. \end{cases} \tag{5.7}$$

For any $n \in \mathbb{N}$, define $g_n = (M_1 H)^n G_1 f \in L^1_+$. Then, using Proposition 3.8, for any $n \geqslant 1$, there exists $\psi_n \in \mathscr{D}(\mathbf{T}_{\max})$ such that $\mathsf{B}^- \psi_n = 0$ and $\mathsf{B}^+ \psi_n = g_n$ with $\|\psi_n\|_X \leqslant \|g_n\|_{L^1_+}$ and $\|\mathbf{T}_0 \psi_n\|_X \leqslant \|g_n\|_{L^1_+}$. As in [4, Proposition 4.4], we can define

$$\varphi_n = \mathscr{R}(1, \mathbf{T}_0) f + \sum_{k=0}^{n-1} \Xi H (M_1 H)^k G_1 f - \psi_n$$

and show that $\varphi_n \in \mathscr{D}(\mathbf{T}_H)$. Since $(V_H(t))_{t \geqslant 0}$ is assumed to be stochastic, from Theorem 5.10 we infer that $\|g_n\|_{L^1_+} \to 0$ as $n \to \infty$ so that $\psi_n \to 0$ and $\mathbf{T}_0 \psi_n \to 0$. Then it is easy to see that $(\varphi_n)_n$ satisfies (5.7). This proves that 1) $\Rightarrow$ 2). Finally we explained the idea underlying the converse implication 2) $\Rightarrow$ 1) at the beginning of this subsection (see the considerations after formula (5.3)). We refer to [4, Proposition 2.11] for a detailed proof using both Green's formula and a density argument. □

Now we discuss spectral properties $M_\lambda H$ which ensure stochasticity of $(V_H(t))_{t \geqslant 0}$. The proof of the following can be seen as a simple adaptation of that of [4, Theorem 4.5], where the explicit expressions of the various operators Ξ_λ, M_λ, G_λ do not play any role but the main idea goes back to [11] (see also [5, Theorem 4.3]). We provide it here for the sake of completeness.

Theorem 5.14. *1) For any $\lambda > 0$, $1 \notin \sigma_p(M_\lambda H)$;*
2) $1 \in \varrho(M_\lambda H)$ for some/all $\lambda > 0$ if and only if $\mathbf{A} = \mathbf{T}_H$;
3) $1 \in \sigma_c(M_\lambda H)$ for some/all $\lambda > 0$ if and only if $\mathbf{A} = \overline{\mathbf{T}_H} \neq \mathbf{T}_H$.
4) $1 \in \sigma_r(M_\lambda H)$ for some/all $\lambda > 0$ if and only if $\mathbf{A} \supsetneq \overline{\mathbf{T}_H}$.

PROOF: 1) The fact that 1 cannot belong to the point spectrum of $M_\lambda H$ ($\lambda > 0$) is a simple consequence of Assumption 5.1 (c) and of the inclusion $\{\lambda > 0\} \subset \varrho(A)$.

2) If there exists $\lambda > 0$ such that $1 \in \varrho(M_\lambda H)$, then, since the series (3.12) converges in the norm topology to $(\lambda - \mathbf{T}_H)^{-1}$, we have $\mathbf{A} = \mathbf{T}_H$. Conversely, assume that $\mathbf{A} = \mathbf{T}_H$. Then, for any $f \in X$, the series $\sum_{n=0}^\infty (M_\lambda H)^n G_\lambda f$ converges in L^1_+ according to Proposition 5.9. Now, since G_λ is surjective, the series $\sum_{n=0}^\infty (M_\lambda H)^n g$ converges in L^1_+ for any $g \in L^1_+$. Denoting by $\mathcal{R}(\lambda) g$ the limit, we see from the Banach-Steinhaus Theorem that $\mathcal{R}(\lambda) \in \mathfrak{B}(L^1_+)$ and that $\mathcal{R}(\lambda)(1 - M_\lambda H) = (1 - M_\lambda H)\mathcal{R}(\lambda)$, which proves that $1 \in \varrho(M_\lambda H)$.

3) Let $\lambda > 0$ be such that $1 \in \sigma_c(M_\lambda H)$. Then

$$\overline{(1 - M_\lambda H) L^1_+} = L^1_+. \tag{5.8}$$

Let $\varphi \in \mathscr{D}(\mathbf{A})$ be given and let $f \in X$ be such that $\varphi = (\lambda - \mathbf{A})^{-1} f$. Since $G_\lambda f \in L^1_+$, there is a sequence $(\phi_n)_n \subset L^1_+$ such that $\|\phi_n - M_\lambda H \phi_n - G_\lambda f\|_{L^1_+} \to 0$. Now, define $g_n =$

$\phi_n - M_\lambda H\phi_n - G_\lambda f$. According to Proposition 3.8, there exists $f_n \in \mathscr{D}(\mathbf{T}_{\max})$ such that $\mathsf{B}^+ f_n = g_n$ and $\mathsf{B}^- f_n = 0$. Moreover, $f_n \to 0$ and $\mathbf{T}_0 f_n \to 0$. Now, setting

$$\varphi_n = f_n + \mathscr{R}(\lambda, \mathbf{T}_0) f + \Xi_\lambda H\phi_n$$

we see that $(\varphi_n)_n \subset \mathscr{D}(\mathbf{T}_H)$ and $(\lambda - \mathbf{T}_H)\varphi_n \to f$. Furthermore

$$\Xi_\lambda H\phi_n = \sum_{k=0}^{\infty} \Xi_\lambda H(M_\lambda H)^k(\phi_n - M_\lambda H\phi_n) = \sum_{k=0}^{\infty} \Xi_\lambda H(M_\lambda H)^k(g_n + G_\lambda f),$$

where both above series are convergent by Remark 5.5. Using again Remark 5.5, we see that $\sum_{k=0}^{\infty} \Xi_\lambda H(M_\lambda H)^k g_n \to 0$ so that $\varphi_n \to \varphi$ and this proves that $\mathbf{A} = \overline{\mathbf{T}_H}$.

Conversely, assume $\mathbf{A} = \overline{\mathbf{T}_H} \neq \mathbf{T}_H$ and let $g \in L^1_+$. Define $g_n = \sum_{k=0}^{n-1} (M_1 H)^k g$. Then, $g_n \in L^1_+$ and, clearly $(1 - M_1 H)g_n = g - (M_1 H)^n g$. Since $\beta(f) = 0$, according to Remark 5.12, one has $\|(1 - M_1 H)g_n - g\|_{L^1_+} \to 0$ so that (5.8) holds. Since $\mathbf{A} \neq \mathbf{T}_H$, one has $1 \in \sigma(M_1 H) \setminus \sigma_p(M_1 H)$ which proves that $1 \in \sigma_c(M_1 H)$.

4) The last assertions is now clear since all the possibilities have been exhausted. □

As in [4, Corollary 4.6], we provide here a useful criterion (see [4, Section 5] for several application in the force–free case).

Corollary 5.15. *$(V_H(t))_{t\geqslant 0}$ is stochastic if and only if $1 \notin \sigma_p((M_\lambda H)^\star)$ for any $\lambda > 0$. Moreover, if $(V_H(t))_{t\geqslant 0}$ is not stochastic, then there exists a **non-negative** $\gamma \in (L^1_+)^\star$, $\gamma \neq 0$, such that $\gamma = (M_\lambda H)^\star \gamma$.*

APPENDIX A: PROOF OF THEOREM 3.4

We prove here the Theorem 3.4 announced in Section 3.2. The proof is divided into three steps:

• *Step 1.* Let us first check that the family of operators $(U_0(t))_{t\geqslant 0}$ is a nonnegative contractive C_0-semigroup in X. Thanks to Proposition 2.3, we can prove that, for any $f \in X$ and any $t \geqslant 0$, the mapping $U_0(t)f : \mathbf{\Omega} \to \mathbb{R}$ is measurable and the semigroup properties $U_0(0)f = f$ and $U_0(t)U_0(s)f = U_0(t+s)f$ $(t, s \geqslant 0)$ hold. Let us now show that $\|U_0(t)f\|_X \leqslant \|f\|_X$. We have

$$\|U_0(t)f\|_X = \int_{\mathbf{\Omega}_+} |U_0(t)f| \mathrm{d}\mu + \int_{\mathbf{\Omega}_- \cap \mathbf{\Omega}_{+\infty}} |U_0(t)f| \mathrm{d}\mu + \int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} |U_0(t)f| \mathrm{d}\mu.$$

Propositions 2.11 and 2.3 yield

$$\begin{aligned}
\int_{\mathbf{\Omega}_+} |U_0(t)f| \mathrm{d}\mu &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} |U_0(t)f(\mathbf{\Phi}(\mathbf{y}, s, 0))| \mathrm{d}s \\
&= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\max(0, \tau_-(\mathbf{y}) - t)} |f(\mathbf{\Phi}(\mathbf{y}, s+t, 0))| \mathrm{d}s \\
&\leqslant \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_t^{\max(t, \tau_-(\mathbf{y}))} |f(\mathbf{\Phi}(\mathbf{y}, r, 0))| \mathrm{d}r \leqslant \int_{\mathbf{\Omega}_+} |f| \mathrm{d}\mu.
\end{aligned}$$

In the same way we obtain

$$\int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} |U_0(t)f|\mathrm{d}\mu = \int_{\Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^\infty |U_0(t)f(\mathbf{\Phi}(\mathbf{y},0,s))|\mathrm{d}s = \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} |f|\mathrm{d}\mu,$$

and

$$\int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} |U_0(t)f|\mathrm{d}\mu = \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} |f|\mathrm{d}\mu.$$

This proves contractivity of $U_0(t)$. Let us now show that $U_0(t)f$ is continuous, i.e.

$$\lim_{t\to 0} \|U_0(t)f - f\|_X = 0.$$

It is enough to show that this property holds for any $f \in \mathscr{C}_0^\infty(\mathbf{\Omega})$. In this case, $\lim_{t\to 0} U_0(t)f(\mathbf{x}) = f(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{\Omega}$. Moreover, $\sup_{\mathbf{x}\in\mathbf{\Omega}} |U_0(t)f(\mathbf{x})| \leqslant \sup_{\mathbf{x}\in\mathbf{\Omega}} |f(\mathbf{x})|$ and the support of $U_0(t)f$ is bounded, so that the Lebesgue dominated convergence theorem leads to the result. This proves that $(U_0(t))_{t\geqslant 0}$ is a C_0-semigroup of contractions in X. Let $\mathbf{A}_0$ denote its generator.

• *Step 2.* To show that $\mathscr{D}(\mathbf{A}_0) \subset \mathscr{D}(\mathbf{T}_0)$ let $f \in \mathscr{D}(\mathbf{A}_0)$, $\lambda > 0$ and $g = (\lambda - \mathbf{A}_0)f$. Then,

$$f(\mathbf{x}) = \int_0^{\tau_-(\mathbf{x})} \exp(-\lambda t)\, g(\mathbf{\Phi}(\mathbf{x},t,0))\mathrm{d}t, \qquad (\mathbf{x} \in \mathbf{\Omega}).$$

Let $\mathbf{y} \in \Gamma_-$ and $0 < t < \tau_+(\mathbf{y})$. Noting that $t = \tau_-\left(\mathbf{\Phi}(\mathbf{y},0,t)\right)$, by Proposition 2.3 we obtain

$$\begin{aligned} f(\mathbf{\Phi}(\mathbf{y},0,t)) &= \int_0^t \exp(-\lambda s)\, g(\mathbf{\Phi}(\mathbf{\Phi}(\mathbf{y},0,t),s,0)\mathrm{d}s = \int_0^t \exp(-\lambda s)\, g(\mathbf{\Phi}(\mathbf{y},0,t-s))\mathrm{d}s \\ &= \int_0^t \exp(-\lambda(t-s))\, g(\mathbf{\Phi}(\mathbf{y},0,s))\mathrm{d}s. \end{aligned}$$

Consequently, $\lim_{t\to 0^+} f(\mathbf{\Phi}(\mathbf{y},0,t)) = 0$ a.e. $\mathbf{y} \in \Gamma_-$, i.e. $\mathsf{B}^- f = 0$. Next we show that $\langle \mathbf{T}_{\max}, \psi\rangle = \langle \lambda f - g, \psi\rangle$ for any $\psi \in \mathscr{C}_0^\infty(\mathbf{\Omega})$, where $\langle\cdot,\cdot\rangle$ denotes the usual duality product of X. Indeed

$$\begin{aligned} \int_{\mathbf{\Omega}} \psi\, \mathbf{T}_{\max} f \,\mathrm{d}\mu &= \int_{\mathbf{\Omega}} f(\mathbf{x})\mathcal{F}(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \\ &= \int_{\mathbf{\Omega}_+} f(\mathbf{x})\mathcal{F}(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) + \int_{\mathbf{\Omega}_-\cap\mathbf{\Omega}_{+\infty}} f(\mathbf{x})\mathcal{F}(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \\ &\qquad + \int_{\mathbf{\Omega}_{-\infty}\cap\mathbf{\Omega}_{+\infty}} f(\mathbf{x})\mathcal{F}(\mathbf{x})\cdot\nabla\psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Arguing as in Step 1, we observe that, for any $\mathbf{y} \in \Gamma_+$ and $0 < t < \tau_-(\mathbf{y})$,

$$f(\mathbf{\Phi}(\mathbf{y},t,0)) = \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t)) g(\mathbf{\Phi}(\mathbf{y},s,0))\mathrm{d}s,$$

and, by Proposition 2.11,

$$\begin{aligned} I_1 &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} f(\mathbf{\Phi}(\mathbf{y},t,0))\mathcal{F}(\mathbf{\Phi}(\mathbf{y},t,0)) \cdot \nabla\psi(\mathbf{\Phi}(\mathbf{y},t,0))\mathrm{d}t \\ &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} \mathcal{F}(\mathbf{\Phi}(\mathbf{y},t,0)) \cdot \nabla\psi(\mathbf{\Phi}(\mathbf{y},t,0))\mathrm{d}t \times \\ &\qquad \times \int_t^{\tau_-(\mathbf{y})} \exp(-\lambda(s-t)) g(\mathbf{\Phi}(\mathbf{y},s,0))\mathrm{d}s. \end{aligned}$$

Recall that, according to the definition of $\mathbf{\Phi}(\cdot,\cdot,\cdot)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi\left(\mathbf{\Phi}(\mathbf{y},t,0)\right) = -\mathcal{F}(\mathbf{\Phi}(\mathbf{y},t,0)) \cdot \nabla\psi(\mathbf{\Phi}(\mathbf{y},t,0)),$$

for a.e. $\mathbf{y} \in \Gamma_+$ and all $t > 0$, so that

$$\begin{aligned} I_1 &= -\int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\mathbf{\Phi}(\mathbf{y},s,0))\mathrm{d}s \int_0^s \exp(-\lambda(s-t)) \frac{\mathrm{d}}{\mathrm{d}t}\left(\psi(\mathbf{\Phi}(\mathbf{y},t,0))\right) \mathrm{d}t \\ &= \int_{\Gamma_+} \mathrm{d}\mu_+(\mathbf{y}) \int_0^{\tau_-(\mathbf{y})} g(\mathbf{\Phi}(\mathbf{y},s,0)) \times \\ &\qquad \times \left\{ \lambda \int_0^s \exp(-\lambda(s-t))\psi(\mathbf{\Phi}(\mathbf{y},t,0))\mathrm{d}t - \psi(\mathbf{\Phi}(\mathbf{y},s,0)) \right\} \mathrm{d}s. \end{aligned}$$

Using again Proposition 2.11, we obtain

$$I_1 = -\int_{\mathbf{\Omega}_+} \left(g(\mathbf{x}) - \lambda f(\mathbf{x})\right) \psi(\mathbf{x})\mathrm{d}\mu(\mathbf{x}). \tag{A.1}$$

Let us now compute I_2. Proposition 2.11 yields

$$\begin{aligned} I_2 &= \int_{\Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\infty} f(\mathbf{\Phi}(\mathbf{y},0,t))\mathcal{F}(\mathbf{\Phi}(\mathbf{y},0,t)) \cdot \nabla\psi(\mathbf{\Phi}(\mathbf{y},0,t))\mathrm{d}t \\ &= \int_{\Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\infty} \mathcal{F}(\mathbf{\Phi}(\mathbf{y},t,0)) \cdot \nabla\psi(\mathbf{\Phi}(\mathbf{y},t,0))\mathrm{d}t \times \\ &\qquad \times \int_0^t \exp(-\lambda(t-s)) g(\mathbf{\Phi}(\mathbf{y},0,s))\mathrm{d}s, \end{aligned}$$

and, as above,

$$\begin{aligned} I_2 &= -\int_{\Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\infty} g(\mathbf{\Phi}(\mathbf{y},0,s))\mathrm{d}s \int_s^{\infty} \exp(-\lambda(t-s)) \frac{\mathrm{d}}{\mathrm{d}t}\left(\psi(\mathbf{\Phi}(\mathbf{y},0,t))\right) \mathrm{d}t \\ &= \int_{\Gamma_{-\infty}} \mathrm{d}\mu_-(\mathbf{y}) \int_0^{\infty} g(\mathbf{\Phi}(\mathbf{y},0,s)) \times \\ &\qquad \times \left\{ \lambda \int_s^{\infty} \exp(-\lambda(t-s))\psi(\mathbf{\Phi}(\mathbf{y},0,t))\mathrm{d}t - \psi(\mathbf{\Phi}(\mathbf{y},0,s)) \right\} \mathrm{d}s, \end{aligned}$$

which results in

$$I_2 = -\int_{\mathbf{\Omega}_- \cap \mathbf{\Omega}_{+\infty}} (g(\mathbf{x}) - \lambda f(\mathbf{x}))\, \psi(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}). \tag{A.2}$$

Finally, since

$$f(\mathbf{x}) = \int_0^\infty \exp(-\lambda t) g\left(\mathbf{\Phi}(\mathbf{x}, t, 0)\right) \mathrm{d}t \qquad \text{for any} \quad \mathbf{x} \in \mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty},$$

we argue as above to get

$$\begin{aligned} I_3 &= \int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} g(\mathbf{z}) \mathrm{d}\mu(\mathbf{z}) \int_0^\infty \exp(-\lambda t) \frac{\mathrm{d}}{\mathrm{d}t} \left(\psi(\mathbf{\Phi}(\mathbf{z}, 0, t))\right) \mathrm{d}t \\ &= -\int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} g(\mathbf{z}) \mathrm{d}\mu(\mathbf{z}) + \lambda \int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} g(\mathbf{z}) \mathrm{d}\mu(\mathbf{z}) \int_0^\infty \exp(-\lambda t) g(\mathbf{\Phi}(\mathbf{z}, t, 0)) \mathrm{d}t, \end{aligned}$$

which gives

$$I_3 = -\int_{\mathbf{\Omega}_{-\infty} \cap \mathbf{\Omega}_{+\infty}} (g(\mathbf{x}) - \lambda f(\mathbf{x}))\, \psi(\mathbf{x}) \mathrm{d}\mu(\mathbf{x}). \tag{A.3}$$

Combining (A.1)–(A.3) leads to

$$\int_{\mathbf{\Omega}} \psi\, \mathbf{T}_{\max} f \, \mathrm{d}\mu = -\int_{\mathbf{\Omega}} (g(\mathbf{x}) - \lambda f(\mathbf{x}))\, \mathrm{d}\mu(\mathbf{x})$$

which proves that $f \in \mathscr{D}(\mathbf{T}_{\max})$ and $(\lambda - \mathbf{T}_{\max}) f = g$. Since $\mathsf{B}^- f = 0$, we see that $f \in \mathscr{D}(\mathbf{T}_0)$ and $\mathbf{A}_0 f = \mathbf{T}_0 f = \lambda f - g$.

• *Step 3.* Let us show now the converse inclusion $\mathscr{D}(\mathbf{T}_0) \subset \mathscr{D}(\mathbf{A}_0)$. Let $f \in \mathscr{D}(\mathbf{T}_0)$. For almost every $\mathbf{y} \in \Gamma_-$ and any $0 < s < s_0 < \tau_+(\mathbf{y})$ we have

$$f(\mathbf{\Phi}(\mathbf{y}, 0, s)) - f(\mathbf{\Phi}(\mathbf{y}, 0, s_0)) = -\int_{s_0}^s f(\mathbf{\Phi}(\mathbf{y}, 0, \sigma)) \mathrm{d}\sigma. \tag{A.4}$$

Letting $\mathbf{x} = \mathbf{\Phi}(\mathbf{y}, 0, s_0)$ and $t = s - s_0 \in (0, \tau_-(\mathbf{y}))$, this yields

$$f(\mathbf{\Phi}(\mathbf{x}, t, 0)) - f(\mathbf{x}) = \int_0^t (\mathbf{T}_{\max} f)(\mathbf{\Phi}(\mathbf{x}, r, 0)) \mathrm{d}r.$$

According to the explicit expression of $U_0(t)$, this means that

$$U_0(t) f(\mathbf{x}) - f(\mathbf{x}) = \int_0^t U_0(r) \mathbf{T}_{\max} f(\mathbf{x}) \mathrm{d}r \tag{A.5}$$

holds for any $\mathbf{x} \in \mathbf{\Omega}_-$, and $t < \tau_-(\mathbf{x})$. In the same way, taking $s = 0$ in (A.4) and setting again $\mathbf{x} = \mathbf{\Phi}(\mathbf{y}, 0, s_0)$ and $t = s_0 = \tau_-(\mathbf{x})$, we get

$$-f(\mathbf{x}) = -\int_{\tau_-(\mathbf{x})}^0 (\mathbf{T}_{\max} f)(\mathbf{\Phi}(\mathbf{x}, r, 0) \mathrm{d}r.$$

This shows that Eq. (A.5) holds true for any $\mathbf{x} \in \mathbf{\Omega}_-$ and any $t \geqslant \tau_-(\mathbf{x})$. Now, let us show that (A.5) is still valid for $\mathbf{x} \in \mathbf{\Omega}_{-\infty}$. Let us choose a sequence $(f_k)_k \subset \mathscr{D}(\mathbf{T}_{\max}) \cap \mathscr{C}^1(\mathbf{\Omega})$ converging to f in the graph norm of $\mathbf{T}_{\max}$. It is clear that f_k fulfills (A.5) for any $\mathbf{x} \in \mathbf{\Omega}_{-\infty}$, i.e

$$U_0(t)f_k(\mathbf{x}) - f_k(\mathbf{x}) = \int_0^t U_0(r)\mathbf{T}_{\max}f_k(\mathbf{x})\mathrm{d}r, \qquad \text{for any } \mathbf{x} \in \mathbf{\Omega}_{-\infty}, t > 0,\ k \in \mathbb{N}.$$

Set

$$G(t)f(\mathbf{x}) = \left| U_0(t)f(\mathbf{x}) - f(\mathbf{x}) - \int_0^t U_0(r)\mathbf{T}_{\max}f(\mathbf{x})\mathrm{d}r \right|,$$

for any $\mathbf{x} \in \mathbf{\Omega}_{-\infty}$, $t > 0$. We have

$$\begin{aligned}\int_{\mathbf{\Omega}_{-\infty}} G(t)f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) &\leqslant \int_{\mathbf{\Omega}_{-\infty}} G(t)(f - f_k)(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) + \int_{\mathbf{\Omega}_{-\infty}} G(t)f_k(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) \\ &\leqslant \|U_0(t)(f - f_k)\|_X + \|f - f_k\|_X + \int_0^t \|U_0(r)\mathbf{T}_{\max}(f - f_k)\|_X \mathrm{d}r.\end{aligned}$$

Since the right-hand side term goes to zero as $k \to \infty$, $\int_{\mathbf{\Omega}_{-\infty}} G(t)f(\mathbf{x})\mathrm{d}\mu(\mathbf{x}) = 0$ and therefore $G(t)f(\mathbf{x}) = 0$ for almost every $\mathbf{x} \in \mathbf{\Omega}_{-\infty}$. This shows that (A.5) holds true for almost every $\mathbf{x} \in \mathbf{\Omega}_{-\infty}$ and any $t \geqslant 0$. Consequently, $f \in \mathscr{D}(\mathbf{A}_0)$ with $\mathbf{A}_0 f = \mathbf{T}_{\max} f$.

Acknowledgments. The research of B. Lods was supported by a *Marie Curie Intra-European Fellowship* within the 6th E. C. Framework Program, Contract Number MEIF-CT-2003-500923.

REFERENCES

[1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, *Invent. Math.* **158** 227–260, 2004.

[2] L. Arlotti, The Cauchy problem for the linear Maxwell–Bolztmann equation, *J. Differential Equations* **69** 166–184, 1987.

[3] ———, A perturbation theorem for positive contraction semigroups on L^1-spaces with applications to transport equation and Kolmogorov's differential equations, *Acta Appl. Math.* **23** 129–144, 1991.

[4] L. Arlotti, B. Lods, Substochastic semigroups for transport equations with conservative boundary conditions, *J. Evol. Equations* **5** 485–508, 2005.

[5] J. Banasiak, L. Arlotti, **Perturbations of positive semigroups with applications**, Springer-Verlag, 2006.

[6] C. Bardos, Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; théorèmes d'approximation; application à l'équation de transport. *Ann. Sci. École Norm. Sup.* **3** 185–233, 1970.

[7] R. Beals, V. Protopopescu, Abstract time-dependent transport equations, *J. Math. Anal. Appl.* **121** 370–405, 1987.

[8] B. Bojarski, P. Hajlasz and P. Strzelecki, Sard's theorem for mappings in Hölder and Sobolev spaces. *Manuscripta Math.* **118**, 383–397, 2005.

[9] R. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** 511–547, 1989.

[10] P. R. Halmos, **Measure Theory**, Van Nostrand, Toronto, 3rd ed., 1954.

[11] G. Frosali, C. van der Mee, F. Mugelli, A characterization theorem for the evolution semigroup generated by the sum of two unbounded operators, *Math. Methods Appl. Sci.* **27** 669–685, 2004.

[12] W. Greenberg, C. Van der Mee and V. Protopopescu, **Boundary Value Problems in Abstract Kinetic theory**, Birkhäuser Verlag, Basel, 1987.

[13] C. Le Bris, P.-L. Lions, Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications, *Ann. Mat. Pura Appl.* **183** 97–130, 2004.
[14] B. Lods, Semigroup Generation Properties of Streaming Operators with Noncontractive Boundary Conditions, *Math. Comput. Modelling* **42** 1441–1462, 2005.
[15] C. Van der Mee, Time dependent kinetic equations with collision terms relatively bounded with respect to collision frequency, *Transp. Theory Stat. Phys.* **30** 63–90, 2001.
[16] J. Voigt, **Functional analytic treatment of the initial boundary value problem for collisionless gases**, München, Habilitationsschrift, 1981.

LUISA ARLOTTI
DIPARTIMENTO DI INGEGNERIA CIVILE, UNIVERSITÀ DI UDINE, VIA DELLE SCIENZE 208,
33100 UDINE, ITALY.
luisa.arlotti@uniud.it

JACEK BANASIAK
SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF KWAZULU–NATAL,
DURBAN 4041, SOUTH AFRICA.
banasiak@ukzn.ac.za

BERTRAND LODS
DIPARTIMENTO DI MATEMATICA, POLITECNICO DI TORINO, CORSO DUCA DEGLI ABRUZZI, 24,
10129 TORINO, ITALY.
lods@calvino.polito.it