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# Orders and dimensions for $sl(2)$ or $sl(3)$ module categories and Boundary Conformal Field Theories on a torus

R. Coquereaux\* † and G. Schieber‡ §

## Abstract

After giving a short description, in terms of action of categories, of some of the structures associated with  $sl(2)$  and  $sl(3)$  boundary conformal field theories on a torus, we provide tables of dimensions describing the semisimple and co-semisimple blocks of the corresponding weak bialgebras (quantum groupoids), tables of quantum dimensions and orders, and tables describing induction - restriction. For reasons of size, the  $sl(3)$  tables of induction are only given for theories with self-fusion (existence of a monoidal structure).

**Keywords:** quantum groupoids; weak Hopf algebras; quantum symmetries; fusion algebra; Coxeter-Dynkin diagrams; ADE; modular invariance; conformal field theories; categories.

## 1 Introduction

Our main and original purpose was to gather tables giving characteristic numbers for  $sl(3)$  boundary conformal field theories on a torus: dimensions of the blocks of the associated weak bialgebras, quantum dimensions, orders ("quantum mass"), induction tables, etc. a material that is hitherto scattered in a number of publications or unavailable. Our tables contain known results, that we have checked, but they also contain many new explicit formulae. The same is true for the expressions given in the last section, just before the tables. The corresponding data for  $sl(2)$  does not use much space, and we could easily summarize it, but the situation is different with  $sl(3)$ : to keep the size of this paper reasonable, we had sometimes to restrict ourselves to the case of Di Francesco - Zuber graphs with self-fusion (other cases will be described in [17]) and give only partial results for induction tables. Because we also needed a short introductory section discussing the underlying algebraic structures and giving our notations, we decided to describe boundary conformal field theories on a torus in terms of module categories (action of a monoidal category on a category) mostly adapting the relevant material from [38], while adding few things like the construction of weak bialgebras in terms of Hom spaces, or the description of the bimodule structure  $\mathcal{A} \times \mathcal{O} \times \mathcal{A} \mapsto \mathcal{O}$ , where  $\mathcal{A}$  is the fusion algebra and  $\mathcal{O}$  is the Ocneanu algebra of quantum symmetries. Independently of the interest of our tables of results, we hope that this presentation will provide a bridge between several mathematical or physical communities interested in those topics.

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## 2 The stage

In this paper  $\mathcal{A}_k$  is the fusion category of the affine algebra  $\widehat{sl}(2)$ , or  $\widehat{sl}(3)$ , at level  $k$ , or equivalently, the category of irreducible representations with non-zero  $q$ -dimension for the quantum groups  $SL(2)_q$  or  $SL(3)_q$  at roots of unity (set  $q = \exp(i\pi/\kappa)$ , with  $\kappa = k + 2$  for  $sl(2)$  and  $\kappa = k + 3$  for  $sl(3)$ ). This category is additive (existence of  $\oplus$ ), monoidal (existence of  $\otimes : \mathcal{A}_k \times \mathcal{A}_k \mapsto \mathcal{A}_k$ , with associativity constraints, unit object, etc.), tensorial ( $\otimes$  is a bifunctor), complex-linear, rigid (existence of duals), finite (finitely many irreducible objects), and semisimple, with irreducible unit object. It is also modular (braided, with invertible S-matrix) and ribbon (in particular balanced - or tortile). We refer to the literature [28], [26], [1] for a detailed description of these structures. The Grothendieck ring of this monoidal category comes with a special basis (corresponding to simple objects), it is usually called the fusion ring, or the Verlinde algebra. The corresponding structure constants, encoded by the so - called fusion matrices  $(N_n)_q^p$ , are therefore non - negative integers: NIM-reps in CFT terminology. The rigidity property of the category implies that  $(N_{\bar{n}})_{pq} = (N_n)_{qp}$ , where  $\bar{n}$  refers to the dual object i.e., , in our case, to the conjugate representation, so that the fusion ring is automatically a based  $\mathbb{Z}_+$  ring in the sense of [38] (maybe it would be better to call it “rigid”). In the case of  $sl(2)$ , this is a ring with one generator (corresponding to the fundamental representation), and fusion matrices are symmetric, because  $\bar{n} = n$ . In the case of  $sl(3)$ , it is convenient to use two generators that are conjugate to one another; they correspond to the two fundamental representations. Multiplication by the chosen generator is encoded by a particular fusion matrix  $N_1$ ; it is a finite size matrix of dimension  $r \times r$ , with  $r = k + 1$  for  $sl(2)$  and  $r = k(k + 1)/2$  for  $sl(3)$ . Since its elements are non negative integers, it can be interpreted as the adjacency matrix of a graph, which is the Cayley graph of multiplication by this generator, that we call the McKay graph of the category. Edges are non oriented in the case of  $sl(2)$  (rather, they carry both orientations) and are oriented in the case of  $sl(3)$ . Irreducible representations are denoted by  $\lambda_p$ , with  $p \geq 0$ , in the first case, and  $\lambda_{pq}$ , with  $p, q \geq 0$  in the next. Notice the shift of indices: the unit  $\mathbb{1}$  of the category corresponds to  $\lambda_0$  or to  $\lambda_{00}$ . One should certainly keep in mind the distinction between the monoidal category, with its objects and morphisms, and its Grothendieck ring, but they will often be denoted by the same symbol. Actually, the McKay graph itself is also denoted  $\mathcal{A}_k$ . In the  $sl(2)$  case, it can be identified with the Coxeter - Dynkin diagram  $A_r$ , with  $r = k + 1$ . In both  $sl(2)$  and  $sl(3)$  cases, it is a truncated Weyl chamber at level  $k$  (a Weyl alcove). It is often useful to think of  $\mathcal{A}_k$  as a category of representations of a would-be quantum object, that can be also denoted by the same symbol, although this may be quite misleading.

The next ingredient is a category  $\mathcal{E}$ , not necessarily monoidal, but we suppose that it is additive, semisimple and indecomposable, on which the previous one  $\mathcal{A}_k$  (which is monoidal) acts. Action of a monoidal category on a category has been described, under the name “module categories” by [19], and, in our context, by [38]. Using a slightly shorter description, we may say that we have such an action when we are given a (monoidal) functor from  $\mathcal{A}_k$  to the (monoidal) category of endofunctors of  $\mathcal{E}$ . The reader can think of this situation as being an analogue of the action of a group on a given space. Actually, it may be sometimes interesting to think that  $\mathcal{E}$  can be acted upon in more than one way, so that we can think of the action of  $\mathcal{A}_k$  as a particular “enrichment” of  $\mathcal{E}$ . The word “module” being used in so many different ways, we prefer to say that we have an action, or that  $\mathcal{E}$  is an actegory (another nice substantive coined by R. Street), and we shall freely use both terminologies. Irreducible objects of  $\mathcal{E}$  are boundary conditions for the corresponding Conformal Field Theory specified by  $\mathcal{A}_k$ . It is useful to assume, from now on, that the category  $\mathcal{E}$  is indecomposable (it is not equivalent to the direct sum of two non trivial categories with  $\mathcal{A}_k$  action). Since  $\mathcal{E}$  is additive, we have a Grothendieck group, also denoted by the same symbol. Because of the existence of an action, this (abelian) group has to be a module over the Grothendieck ring of  $\mathcal{A}_k$ , and it is automatically a  $\mathbb{Z}_+$  module: the structure constants of the module, usually called annulus coefficients in string theory articles [20], and described by (annular) matrices  $F_n = (F_n)_{ab}$ , are non negative integers. Let us consider the class of a particular simple object of  $\mathcal{A}_k$ , namely the generator  $n = 1$ , for  $sl(2)$  or  $n = (1, 0)$  for  $sl(3)$  (one of the two conjugated fundamental irreps). We interpret  $F_1$  (or  $F_{(1,0)}$ ) as the

adjacency matrix of a graph, called the McKay graph of the category  $\mathcal{E}$ . The rigidity property of  $\mathcal{A}_k$  implies that the module  $\mathcal{E}$  is rigid (or based [38]). In other words:  $(F_{\overline{n}})_{ab} = (F_n)_{ba}$ . In the case of  $sl(2)$ ,  $\mathbb{Z}_+$  modules for fusion rings at level  $k$  have been classified by [16] and [18]; McKay graphs are all the Coxeter - Dynkin diagram, plus some diagrams with loops (tadpoles), and  $F_1 = 2$  Cartan matrix  $- \mathbf{1}$ . The rigidity condition in the case of  $sl(2)$  implies that the matrix  $F_1$  is symmetric; this condition implies that non simply laced diagrams  $B_r$ ,  $C_r$ ,  $F_4$  and  $G_2$  should be rejected (this only means that they do not fit in the presented framework: see also [24] and references therein): we are left with the *ADE* diagrams and the tadpoles. A detailed analysis of the situation ([35], [38]) shows that the tadpole graphs do not give rise to any category endowed with an action of the monoidal categories of type  $sl(2)$ . As already mentioned, the category  $\mathcal{E}$  is not required to be monoidal, but there are cases where it is, so that it has a tensor product, compatible with the  $\mathcal{A}_k$  action. In another terminology, one says that the corresponding graphs have self - fusion (this is also related to the concept of flatness), or that they define “quantum subgroups” of  $sl(2)$ , whereas the others are only “quantum modules”. Like in the classical situation, we have a restriction functor  $\mathcal{A}_k \mapsto \mathcal{E}$  and an induction functor  $\mathcal{E} \mapsto \mathcal{A}_k$ . The cases where  $\mathcal{E}$  is monoidal correspond to the graphs  $A_r$  (with  $k = r - 1$ ),  $D_{even}$ , with  $k = 0 \bmod 4$ ,  $E_6$ , with  $k = 10$ , and  $E_8$ , with  $k = 28$ . This was already known, at the level of rings, in [39], [16] and was proved, at the categorical level, by [27]. The cases  $D_{odd}$ , with level  $k = 2 \bmod 4$ , and  $E_7$ , at level 16, are non monoidal actegories.  $\mathcal{A}_k$  is always modular, but the corresponding actegories are not, even when they happen to be monoidal; however, they always contain a subcategory which is modular (we shall come back to it). At the level of graphs, the  $D$  diagrams (even or odd) are  $\mathbb{Z}_2$  orbifolds of the  $A$  diagrams at the same level. In the case of  $sl(3)$ , the classification of  $\mathbb{Z}_+$  modules over the corresponding fusion rings at level  $k$  is not tractable (or not useful), however there is another route stemming from the classification of  $sl(3)$  modular invariants [21]. The graphs encoding all  $sl(3)$  module categories are called the di Francesco - Zuber diagrams [16]. Existence of the corresponding categories was shown by A. Ocneanu [34], actually one of the candidates had to be discarded, very much like the tadpole graphs of  $sl(2)$ . Several  $sl(3)$  actegories have monoidal structure (graphs with self - fusion), namely:  $\mathcal{A}_k$  itself, the  $\mathcal{D}_k$ , whose McKay diagrams are  $\mathbb{Z}_3$  orbifolds of those of  $\mathcal{A}_k$ , when  $k$  is divisible by 3, and three exceptional cases called  $\mathcal{E}_5$ ,  $\mathcal{E}_9$  and  $\mathcal{E}_{21}$ , at levels 5, 9 and 21. The other actegories (not monoidal) are: the series  $\mathcal{A}_k^*$ , for which the number of simple objects is equal to the number of self dual simple objects in  $\mathcal{A}_k$ , the  $\mathcal{D}_k$  series, when  $k = 1$  or  $2 \bmod 3$ , the series  $\mathcal{D}_k^*$ , for all  $k$ , and several modules of exceptionals called  $\mathcal{E}_5/3$ ,  $\mathcal{E}_9/3$ ,  $\mathcal{D}_9^t$  (a generalization of  $E_7$ ) and  $\mathcal{D}_9^{t*}$ . Some of the graphs of that system have double lines, like  $\mathcal{E}_9$ , so that it is not appropriate to say that Di Francesco - Zuber diagrams are the “simply laced” diagrams of type  $sl(3)$ : better to call them “higher ADE”. In all cases however, with self-fusion or not, the rigidity property implied by  $\mathcal{A}_k$  holds (the condition  $(F_{\overline{n}})_{ab} = (F_n)_{ba}$  does not forbid double lines). Taking quotients of the above diagrams by discrete groups gives higher analogues of the non *ADE* Dynkin diagrams which define modules over the Grothendieck ring of  $\mathcal{A}_k$ , but the rigidity condition is not satisfied and the corresponding category should not exist (see however the previous footnote). Classification of  $sl(4)$  module categories is also claimed to be completed [34].

Let us pause to develop a tentative pedagogical analogy that should make natural the next result. Consider a finite group, a subgroup, and the corresponding homogenous space (the space of right cosets, for example). The group can be fibered as a principal bundle over the coset space, the structure group being the chosen subgroup. This subgroup has representations, in particular irreducible ones. For any such, let us say  $a$ , one can build an associated vector bundle, and consider the space  $\Gamma_a$  of its sections. It carries a representation of the big group, although not irreducible (theory of induced representations). For the particular choice of the trivial representation of the small group, call it 0, the space of sections  $\Gamma_0$  is an algebra, namely the space of functions  $\mathcal{F}$  on the coset. Moreover every space of sections, say  $\Gamma_a$ , is a module over the algebra  $\mathcal{F}$ . In our case we have a non commutative geometry which is still, in a sense, finite, but the situation is similar. Simple objects, labelled by  $a$ , of the module category  $\mathcal{E}$  can be thought as points of a graph, as irreducible representations of a would-be quantum subgroup of  $su(2)$  or  $su(3)$  at some root of unity, or as spaces of sections  $\Gamma_a$  above a quantum

space determined by the pair  $(\mathcal{A}_k, \mathcal{E})$ , i.e., as modules over some particular algebra  $\mathcal{F}$  which is an algebra in a monoidal category ( $\mathcal{A}_k$  in our case), and right modules over  $\mathcal{F}$  form an additive category  $Mod_{\mathcal{A}_k}(\mathcal{F})$ . Reciprocally, we have the following theorem proved in [38] under actually weaker assumptions than those listed previously for the action of  $\mathcal{A}_k$  on  $\mathcal{E}$ : there exists a semisimple indecomposable algebra  $\mathcal{F}$ , belonging to the set of objects of  $\mathcal{A}_k$  such that the module categories  $\mathcal{E}$  and  $Mod_{\mathcal{A}_k}(\mathcal{F})$  are equivalent. It is shown in [27] that  $\mathcal{E}$  is monoidal (self-fusion) if and only if  $\mathcal{F}$  is commutative (terminological warning: in [27] this property was required, and such algebras were called rigid, this requirement and terminology was abandoned in [38]). Later, we shall describe  $\mathcal{F}$  as an object in  $\mathcal{A}$ , actually a monoid in  $\mathcal{A}$ , for all  $sl(2)$  and  $sl(3)$  actegories, in particular for those that are monoidal, but we shall not use the previous equivalence. Simple summands of  $\mathcal{F}$  play the role of quantum Klein invariants. The same algebra  $\mathcal{F}$ , called Frobenius algebra, or Frobenius monoid, plays a prominent role in the approach of [20]. Notice that module categories associated with *ADE* Dynkin diagrams for  $sl(2)$ , or with Di Francesco - Zuber diagrams for  $sl(3)$  are never modular (unless  $\mathcal{E} = \mathcal{A}$  itself) but they contained a subcategory  $J$ , also denoted  $Mod_{\mathcal{A}_k}^0(\mathcal{F})$  which is modular. When  $\mathcal{E}$  is monoidal, the subring of its Grothendieck ring associated with this modular subcategory is called the modular subring (or subalgebra) and is also denoted  $J$ .

The third and final needed ingredient is the centralizer category of  $\mathcal{E}$  with respect to the action of  $\mathcal{A}_k$ . It is sometimes called the “dual category” (not a very good name) and is defined as the category of module functors from  $\mathcal{E}$  to itself: these endofunctors should be functors  $F$  “commuting” with the action of  $\mathcal{A}_k$ , i.e., such that  $F(\lambda_n \otimes \lambda_a)$  is isomorphic with  $\lambda_n \otimes F(\lambda_a)$ , for  $\lambda_n \in Ob(\mathcal{A}_k)$  and  $\lambda_a \in Ob(\mathcal{E})$ , via a family of morphisms  $c_{\lambda_n, \lambda_m}$  obeying triangular and pentagonal constraints. We simply call  $\mathcal{O} = Fun_{\mathcal{A}_k}(\mathcal{E}, \mathcal{E})$  this centralizer category<sup>1</sup>, but one should remember that its definition involves both  $\mathcal{A}_k$  and  $\mathcal{E}$ . Because of the previous compatibility property, if  $\mathcal{E}$  is a left actegory over  $\mathcal{A}_k$ , it is probably better to consider it as a right actegory over  $\mathcal{O}$  (actually over its opposite, since we are permuting the two factors). The category  $\mathcal{O}$  is additive, semi-simple in our case, and monoidal (use composition of functors as tensor product).  $\mathcal{E}$  is therefore both a module category over  $\mathcal{A}_k$  and over  $\mathcal{O}$ . The Grothendieck group of  $\mathcal{E}$  is therefore not only a  $\mathbb{Z}_+$  module over the fusion ring, but also a  $\mathbb{Z}_+$  module over the Grothendieck ring of  $\mathcal{O}$ , called the Ocneanu ring (or algebra) of quantum symmetries and denoted by the same symbol. Structure constants of the ring of quantum symmetries are encoded by matrices  $O_x$ , called “matrices of quantum symmetries”; structure constants of the module, with respect to the action of quantum symmetries, are encoded by the so called “dual annular matrices”  $S_x$ . The next problem is to find a way to describe explicitly this centralizer category. The solution lies in the construction of a finite dimensional weak bialgebra  $\mathcal{B}$ , which is going to be such that the monoidal category  $\mathcal{A}_k$  can be realized as  $Rep(\mathcal{B})$ , and also such that the monoidal category  $\mathcal{O}$  can be realized as  $Rep(\widehat{\mathcal{B}})$  where  $\widehat{\mathcal{B}}$  is the dual of  $\mathcal{B}$ . These two algebras are finite dimensional (actually semisimple in our case) and one algebra structure (say  $\widehat{\mathcal{B}}$ ) can be traded against a coalgebra structure on its dual.  $\mathcal{B}$  is a weak bialgebra, not a bialgebra, because  $\Delta \mathbb{1} \neq \mathbb{1} \otimes \mathbb{1}$ , where  $\Delta$  is the coproduct in  $\mathcal{B}$ , and  $\mathbb{1}$  is its unit. Actually, in our cases, it is not only a weak bialgebra but a weak Hopf algebra (we can define an antipode, with the expected properties [2], [31], [32], [33]). One categorical construction of  $\mathcal{B}$  is given in [38]. We propose another one that should lead to the same bialgebra, and may be simpler. Label irreducible objects  $\lambda_-$  of categories  $\mathcal{A}_k$  by  $m, n, \dots$ , of  $\mathcal{E}$  by  $a, b, \dots$ , and of  $\mathcal{O}$  by  $x, y, \dots$ . Call  $H_{ab}^m = Hom(\lambda_n \otimes \lambda_a, \lambda_b)$ , the “horizontal space of type  $n$  from  $a$  to  $b$ ” (also called space of essential paths of type  $n$  from  $a$  to  $b$ , space of admissible triples, or triangles. . .) Call  $V_{ab}^x = Hom(\lambda_a \otimes \lambda_x, \lambda_b)$  the “vertical space of type  $x$  from  $a$  to  $b$ ”. We just take these horizontal and vertical spaces as vector spaces and consider the graded sums  $H^m = \sum_{ab} H_{ab}^m$  and  $V^x = \sum_{ab} V_{ab}^x$ . To construct the weak bialgebra, we take the (graded) endomorphism algebras  $\mathcal{B} = \sum_m End(H^m)$  and  $\widehat{\mathcal{B}} = \sum_x End(V^x)$ . For obvious reasons,  $\mathcal{B}$  and  $\widehat{\mathcal{B}}$  are sometimes called “algebra of double triangles”. Existence of the bialgebra structure (compatibility) rests on the properties of the pairing, or, equivalently, on the

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<sup>1</sup>For  $sl(2)$ , the structure of  $Fun_{\mathcal{A}_k}(\mathcal{E}_1, \mathcal{E}_2)$ , where  $\mathcal{E}_{1,2}$  can be distinct module categories was obtained by [35].

properties of the coefficients<sup>2</sup> (Ocneanu cells) obtained by pairing two bases of matrix units<sup>3</sup> for the two products. Being obtained by pairing double triangles, Ocneanu cells (generalized  $6J$  symbols) are naturally associated with tetrahedra with two types (black “b”, or white “w”) of vertices, so that edges  $bb$ ,  $bw$  or  $ww$  refer to labels  $n$ ,  $a$ ,  $x$  of  $\mathcal{A}$ ,  $\mathcal{E}$  and  $\mathcal{O}$ . Using the two product and coproduct structures<sup>4</sup> one can obtain representation categories  $Rep(\mathcal{B})$  and  $Rep(\widehat{\mathcal{B}})$  respectively equivalent to  $\mathcal{A}_k$  and  $\mathcal{O}$ . As already discussed, the character ring of the first (the fusion algebra) is always described by a (generalized) Dynkin diagram of type  $\mathcal{A}$ . The character ring of the next (the algebra of quantum symmetries) is not always commutative, it has two generators, called chiral, in the case of  $sl(2)$ , together with their complex conjugated in the case of  $sl(3)$ . The Cayley graph of multiplication by the two chiral generators (two types of lines), called the Ocneanu graph of  $\mathcal{E}$ , encodes the structure. As already mentioned,  $\mathcal{B}$  is weak. This should not be a surprise since not any monoidal category arises as representation category of a bialgebra, however (at least when the category has finitely many simple objects, and this is our case), it can be realized as the category of representations of a weak bialgebra. In the next paragraph, we shall see that the knowledge of a modular invariant is sufficient to reconstruct the character rings of  $\mathcal{A}_k$  (that we already know), of  $\mathcal{O}$ , and the semisimple and co semisimple structure of  $\mathcal{B}$ . The  $\mathcal{A}_k \times \mathcal{O}$  module corresponding to  $\mathcal{E}$  itself can be recovered from the study of the source and target subalgebras of  $\mathcal{B}$ . Results obtained in operator algebra by [35] and [4, 5, 6] have been translated to a categorical language by [38]: choice of a braiding or of the opposite braiding in the category  $\mathcal{A}_k$  can be used to construct two tensor functors from  $(\mathcal{A}_k)$  to  $\mathcal{O}$ , called  $\alpha^{L,R}$  (“alpha induction” in the language of [5]). Here our presentation differs from [38], because we find easier to think that there exists a functor  $\mathcal{A}_k \times \mathcal{O} \times \mathcal{A}_k \mapsto \mathcal{O}$ , so that the previous  $\alpha^{L,R}$  are obtained as particular cases ( $\mathbb{1}_A \times \mathbb{1}_O \times \mathcal{A}_k$  or to  $\mathcal{A}_k \times \mathbb{1}_O \times \mathbb{1}_A$ ). At the level of Grothendieck rings, we have a bimodule property, that reads (we only use labels to denote the corresponding irreducible objects):  $m x n = \sum_y (W_{xy})_{mn} y$ , where  $m, n$  refer to irreducible objects of  $\mathcal{A}_k$ ,  $x, y$  to irreducible objects of  $\mathcal{O}$ , and where  $W_{xy}$  constitute a family of so - called toric matrices, with matrix elements  $(W_{xy})_{mn}$ , again non negative integers. When both  $x$  and  $y$  refer to the unit object (that we label 0), one recovers the modular invariant partition function  $Z = W_{00}$  of conformal field theory. As explained in [40], when one or two indices  $x$  and  $y$  are non trivial, toric matrices are interpreted as partition functions on a torus, in a conformal theory of type  $\mathcal{A}_k$ , with boundary conditions specified by  $\mathcal{E}$ , but with defects (one or two) specified by  $x$  and  $y$ . Only  $Z$  is modular invariant (it commutes with the generator  $S$  and  $T$  of  $SL(2, \mathbb{Z})$  in the Hurwitz - Verlinde representation). Toric matrices were first introduced and calculated by Ocneanu (unpublished). Various methods to compute or define them can be found in [8], [40]. Ref. [11] gives explicit expressions for all  $W_{x0}$ , for all members of the  $sl(2)$  family. The modular invariant partition functions themselves have been known for many years: we have the  $ADE$  classification of [7] for  $sl(2)$ , and the classification [21] for  $sl(3)$ , encoded by Di Francesco - Zuber diagrams. Left and right associativity constraints  $(m(nxp)q) = (mn)x(pq)$  for the  $\mathcal{A} \times \mathcal{A}$  bimodule structure of  $\mathcal{O}$  can be written in terms of fusion and toric matrices; a particular case of this equation reads  $\sum_x (W_{0x})_{\lambda\mu} W_{x0} = N_\lambda Z N_\mu^{tr}$ , called “equation of modular splitting”, was presented by A.Ocneanu in Bariloche (2000). Given fusion matrices  $N_p$  (known in general) and a modular invariant matrix  $Z = W_{00}$ , solving this equation, i.e., finding the  $W_{x0}$ , allows one to reconstruct the character ring of  $\mathcal{O}$ . A practical method to solve this equation is given in [24], with several  $sl(2)$  and  $sl(3)$  examples. Left and right chiral categories  $\mathcal{O}_{L,R}$  are defined, using alpha-induction functors, as additive and monoidal subcategories of  $\mathcal{O}$  whose objects are direct summands of  $\alpha_{LR}(\lambda)$ , for all  $\lambda$  in  $\mathcal{A}_k$ . They are not braided but their intersection, the ambichiral subcategory  $\mathcal{J}$  is. When  $\mathcal{E}$  is monoidal, the Grothendieck ring of  $\mathcal{J}$ , called ambichiral, is isomorphic with the modular subalgebra  $J$  already defined.

<sup>2</sup>Constructions of  $\mathcal{B}$ , inspired from [36], and using these properties, were given in [40] and [13].

<sup>3</sup>Definition of cells involve normalization choices: the spaces  $H_{ab}^m$  are not always one-dimensional, moreover one may decide to use bases made of vector proportional to matrix units rather than matrix units themselves.

<sup>4</sup>In the operator algebra community, one would usually define a star operation and a scalar product on  $\mathcal{B}$ , so that both products could be defined on the same underlying vector space [36].

### 3 Notations and miscellaneous results

#### 3.1 Notations (summary)

From now on we shall work at the level of Grothendieck groups, or rings, but use for them the same notation as for the categories themselves. So, we have a commutative and associative algebra  $\mathcal{A}$  with<sup>5</sup> a base  $\lambda_n$ , structure constants  $(N_n)_{pq}$ , an associative algebra  $\mathcal{O}$ , with a base  $o_x$ , structure constants  $(O_x)_{yz}$ , a vector space  $\mathcal{E}$  with a base  $\sigma_a$  which is a module over  $\mathcal{A}$  and  $\mathcal{O}$ , with structure constants  $(F_n)_{ab}$  and  $(S_x)_{ab}$ . When  $\mathcal{E}$  has self fusion, its structure constants are  $(E_a)_{bc}$ . The ring  $\mathcal{O}$  is a  $\mathcal{A}$  bimodule with structure coefficients  $(W_{xy})_{m,n}$ . The modular invariant partition function is  $Z = W_{00}$ . Like before, the notation  $\mathcal{E}$  refers to a generic example, unless it denotes an exceptional case (the context should be clear).  $\mathcal{E}$  being chosen, the numbers of irreducible objects in categories  $\mathcal{A}$ ,  $\mathcal{E}$  and  $\mathcal{O}$  are respectively denoted  $r_{\widehat{\mathcal{A}}}$ ,  $r = r_E$  and  $r_{\mathcal{O}}$ . They are the number of vertices of the associated graphs. In the case of  $\widehat{sl}(2)_k$ ,  $r_{\widehat{\mathcal{A}}} = k + 1$ . In the case of  $\widehat{sl}(3)_k$ ,  $r_{\widehat{\mathcal{A}}} = (k + 1)(k + 2)/2$ . The script notation using the level  $k$  as an index can be generalized to all  $sl(N)$  theories but it is incompatible with the traditional notation for  $sl(2)$  (the Dynkin diagrams), where the index refers to the rank. We have the identifications:  $E_6 = \mathcal{E}_{10}$ ,  $E_7 = \mathcal{D}_{16}^t$  (and sometimes  $\mathcal{E}_{16}$ ),  $E_8 = \mathcal{E}_{28}$ ,  $A_r = \mathcal{A}_{k=r-1}$ ,  $D_{s+2} = \mathcal{D}_{k=2s}$ . There are no  $\mathcal{D}$  cases with odd level in the  $sl(2)$  family. Notations for higher ADE diagrams of type  $sl(3)$  were given in the previous section. Generalized Coxeter numbers (altitudes) are  $\kappa = k + N$  for members of  $sl(N)$  family at level  $k$ . More generally, if  $\mathfrak{g}$  is a Lie algebra with dual Coxeter number  $h$ , we would write  $\kappa = k + h$  for all members of the higher ADE system associated with the affine Lie algebra  $\widehat{\mathfrak{g}}$  at level  $k$ .

#### 3.2 Modular blocks

Call  $Z = W_{00}$  the modular invariant. It is a matrix  $Z_{mn}$  indexed by (classes of) irreducible objects  $m, n \in Irr(\mathcal{A})$ . It can also be written as a sesquilinear quadratic form  $\sum_{mn} \chi_{\overline{m}} Z_{\overline{mn}} \chi_n$  (the partition function). The following results are attributed to [35] and [4, 5, 6]:

$$r_E = Tr(Z) \quad r_{\mathcal{O}} = Tr(ZZ^t)$$

The Grothendieck ring of  $\mathcal{A}$  is commutative but the one of  $\mathcal{O}$ , which is not necessarily commutative, is isomorphic to the direct sum of matrix algebras of sizes  $Z_{mn}$ . For example, take  $sl(2)$  at level 8; if we are given  $Z = |\chi_0 + \chi_8|^2 + |\chi_2 + \chi_6|^2 + 2|\chi_4|^2$ , we know a priori that  $r_{\mathcal{A}} = 9$ ,  $r_E = 6$ ,  $r_{\mathcal{O}} = 12$ , and that the algebra of quantum symmetries is isomorphic with  $\bigoplus_{x=1}^{x=8} \mathbb{C}_x \oplus M(2, \mathbb{C})$ . Actually this is the  $D_6$  module of  $A_9$ . The collection  $K$  of those irreducible objects  $\lambda_n$  of  $\mathcal{A}$  that appear on the diagonal of  $Z$ , with multiplicity  $Z_{nn}$ , is called the multiset<sup>6</sup> of exponents of  $Z$  (or exponents of  $\mathcal{E}$ ). Forgetting multiplicities, the matrix  $Z$  defines a partition on this set: two exponents  $m, n$  are in the same modular block iff  $Z_{mn} \neq 0$ . For instance, at level 10, in the case of  $E_6$ , we have  $Z = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2$  and therefore a partition of  $K = \{0, 6, 3, 7, 4, 10\}$  into three subsets  $\{0, 6\}$ ,  $\{3, 7\}$ ,  $\{4, 10\}$ . At level 17, in the case of  $E_7$ , we have  $Z = |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 + (\overline{\chi}_8)(\chi_2 + \chi_{14}) + (\overline{\chi}_2 + \overline{\chi}_{14})\chi_8$ , so we have a partition of  $K = \{0, 16, 4, 12, 6, 10, 8\}$  into four subsets  $\{0, 16\}$ ,  $\{4, 12\}$ ,  $\{6, 10\}$ ,  $\{8\}$ . The modular block of the origin (containing 0) is denoted  $K_0$ . We observe that when  $\mathcal{E}$  is monoidal (self-fusion),  $Z$  is a sum of squares and there is a one to one correspondence between modular blocks  $K_a$  and the modules  $\Gamma_a$  relative to the irreducible objects  $a$  belonging to the modular subalgebra  $J$  of the Grothendieck ring of  $\mathcal{E}$ . For instance the first modular block  $K_0 = \{0, 6\}$  of  $E_6$  corresponds to the algebra  $\mathcal{F} = \Gamma_0 = \lambda_0 \oplus \lambda_6$ . This is not so when there is no self-fusion: the first modular block  $K_0$  of  $E_7$  is  $\{0, 16\}$  although  $\mathcal{F} = \Gamma_0 = \lambda_0 \oplus \lambda_8 \oplus \lambda_{16}$ .

<sup>5</sup>In the previous section, we used the notation  $\lambda_n, \lambda_a, \lambda_x$  for  $\lambda_n, \sigma_a, o_x$ .

<sup>6</sup>Exponents are conventionally defined as indices  $n$  shifted by +1 for  $sl(2)$ , or by +(1, 1) for  $sl(3)$ .

### 3.3 Dimensions of horizontal and vertical spaces $H$ and $V$ , dimension of the weak Hopf algebra $\mathcal{B}$

The horizontal space  $H = \bigoplus_n H_n$ , coming in the construction of the first algebra structure on the weak Hopf algebra  $\mathcal{B}$  was defined before, in terms of categorical data. In the  $sl(2)$  case  $H$  can be realized as the (Ocneanu) vector space of essential paths on ADE graphs but also as the vector space underlying the Gelfand-Ponomarev preprojective algebra associated with the corresponding unoriented quiver. In the first realization  $H_n$  is defined as a particular subspace of the space *Paths* of all paths on the graph  $\mathcal{E}$ , whereas in the second construction it is defined as a quotient. Identification stems, for instance, from the fact that dimensions of these finite dimensional vector spaces, calculated according to the two definitions, are equal. In the case of  $sl(3)$ , the grading label of the horizontal space  $H_n$  refers to a pair of integers  $(n_1, n_2)$ , specifying an irreducible representation (it can also be seen as a Young tableau), this suggests a generalization of the notion of preprojective algebras associated with quivers.

In the case of  $sl(2)$ , dimensions  $d_n = \dim H_n$ ,  $d_H = \sum_n d_n$  and  $d_{\mathcal{B}} = \sum_n d_n^2$ , where  $n$  runs in the set of irreducible objects of  $\mathcal{A}_k$ , and  $d_x = \dim V_x$ ,  $d_V = \sum_x d_x$  and  $d_{\widehat{\mathcal{B}}} = \sum_x d_x^2$ , where  $x$  runs in the set of irreducible objects of  $\mathcal{O}$  have been calculated first by [35], then by [8], [40], [11]. In the case of  $sl(3)$ , they have been calculated by [35], [12], [41] and [22]. One check, of course, that  $d_{\mathcal{B}} = d_{\widehat{\mathcal{B}}}$  in all cases, since the underlying vector space is the same. Surprisingly, one also observes<sup>7</sup> that  $d_H = d_V$  in most cases. The collection of known results giving  $d_H$  for all  $sl(2)$  cases can be condensed into a closed formula by using a recent result obtained by [30] for the dimensions of preprojective algebras associated with ADE quivers<sup>8</sup>. To compute  $d_n$ , the pedestrian approach, that works in all cases, is to calculate the annular matrices  $F_n$ , describing the module action of  $\mathcal{A}_k$  on  $\mathcal{E}$ , using recursion formulae giving irreps of  $sl(2)$  or  $sl(3)$ , then to sum over all matrix elements, since  $H_{ab}^m = \text{Hom}(\lambda_n \otimes \lambda_a, \lambda_b)$ . To compute  $d_x$ , one has first to determine  $\mathcal{O}$ , for instance by solving the modular splitting equation, then the dual annular matrices  $S_x$  describing the module action of  $\mathcal{O}$  on  $\mathcal{E}$ , and finally to sum over all matrix elements, since  $V_{ab}^x = \text{Hom}(\lambda_a \otimes \lambda_x, \lambda_b)$ .

#### 3.3.1 $sl(2)$

We shall give the values of  $d_n, d_x, d_H, d_V$  and  $d_{\mathcal{B}}$  in tables, but thanks to the above identification with the vector space underlying the preprojective algebra of quiver theories, we have a closed formula for  $d_H$  that work for all  $sl(2)$  cases, namely

$$d_H = \frac{\kappa(\kappa + 1)r}{6}$$

Recall that  $k$  is the level,  $r = k + 1$  is the rank (the number of vertices), and  $\kappa = k + 2$  is the Coxeter number. In terms of the dimension  $\dim(E)$  of the Lie group corresponding to the chosen ADE diagram, and using the Kostant formula  $\dim(E) = (\kappa + 1)r$ , we can also write  $d_H = \frac{\kappa \dim(E)}{6}$ . For instance,  $E_6 \rightarrow 156 = 78 \frac{12}{6}$   $E_7 \rightarrow 399 = 133 \frac{18}{6}$   $E_8 \rightarrow 1240 = 248 \frac{30}{6}$ . For  $A_r$  graphs, the rank  $r = k + 1 = \kappa - 1$  so that  $d_H(A_r) = (\kappa - 1)\kappa(\kappa + 1)/6$ , this can be obtained directly from the fact that  $d_n = (n + 1)(k + 1 - n)$ , the trivial representation being labelled  $n = 0$ . Notice that  $2\kappa$  is the period (in  $n$ ) of matrices  $F_n$ . It is interesting to summarize how the general formula for  $d_H$  is obtained in quiver theories [30]: one constructs a generating function for the  $d_n$  (matrix Hilbert series) and obtains  $d_H$  as twice the sum of matrix elements of the inverse of the Cartan matrix of the chosen graph ( $2\mathbf{1} - F_1$ ); this, in turn, is given by the Freudenthal - de Vries strange formula, which, using Kostant relation, can be written  $\kappa(\kappa + 1)r/12$ , hence the result.

<sup>7</sup>When it is not so, in particular when the graph  $\mathcal{E}$  is a  $\mathbb{Z}_2$  or  $\mathbb{Z}_3$  orbifold, one knows how to “correct” this curious linear sum rule, which was first observed in the case of  $sl(2)$  by [40].

<sup>8</sup>Warning: the preprojective algebra is a multiplicative structure on  $H$ , at least for  $sl(2)$  cases, it cannot be identified with either  $\mathcal{B}$  or  $\widehat{\mathcal{B}}$ , see also [10].



We also found explicit expressions giving  $d_{\mathcal{B}}$  for the different series (see tables): in the case of  $A_{k+1}$ , the given formula comes immediately from the fact that we know that  $d_p = (p+1)(k+1-p)$ ; for the  $D$  cases, the results come from the fact that  $2d_p(D) - d_p(A)$  is a superposition of two arithmetic sequences (one has to distinguish even and odd cases).

### 3.3.2 $sl(3)$

Dimensions are later given in tables. In the case of  $d_H$ , there is no known formula that works in all cases. We however obtained the following closed result for  $\mathcal{A}_k$  diagrams:  $d_H = (k+1)(k+2)(k+3)(k+4)(k+5)(k^2+6k+14)/1680$  or, using  $\kappa = k+3$ ,

$$d_H = (\kappa-2)(\kappa-1)\kappa(\kappa+1)(\kappa+2)(\kappa^2+5)/1680$$

There is no Lie group theory associated with the Di Francesco Zuber diagrams, nevertheless it is natural to call rank the quantity  $r = (k+1)(k+2)/2$ , since it is the number of vertices of the  $\mathcal{A}_k$  Weyl alcove. Our previous result giving  $d_H$  for  $\mathcal{A}_k$  uses the fact that the sum of matrix elements of the inverse of  $(3\mathbb{1} - (F_1 + F_1^t)/2)$  can be shown to be equal to  $r(\kappa+1)(\kappa+2)/60$ .

Still in the  $\mathcal{A}_k$  case, one can show that the recurrence formula  $d_{(p,q)} = d_{(p+1,q-1)} - d_{(p-q,q-1)} + d_{(p-q,q)}$  holds when  $p \geq q+1 \geq 2$ . We did not find any simple generic formula giving the horizontal dimension  $d_n$  of arbitrary  $(p,q)$  blocks of  $\mathcal{B}$ . However, there are simple enough (distinct) expressions for blocks of type  $(p,0)$ ,  $(p,1)$ ,  $(p,2)$ , etc, for instance  $d_{(p,0)} = d_{(0,p)} = (k+2-p)(k+1-p)(1+p)(2+p)/4$ . This was enough to show that  $d_{\mathcal{B}}$ , the sum of the  $d_{(p,q)}^2$ , should be a polynomial in  $k$  or  $\kappa$  with rational coefficients, and to guess several divisors (the same as for  $d_H$ ); the coefficients of the conjectured 12th degree polynomial that appears in the  $sl(3)$  table were determined from a study of the first cases, but we checked it against direct calculation of  $d_{\mathcal{B}}$  (i.e., determining all the  $d_n$ ) for high values of  $\kappa$ .

## 3.4 Quantum dimensions

### 3.4.1 General relations

As usual,  $q$ -integers are denoted  $[n] = (q^n - q^{-n})/(q - q^{-1}) = \sin(n\pi/k)/\sin(\pi/k)$  with  $q = \exp[i\pi/\kappa]$ . One can define quantum dimensions  $\mu$  of the vertices of  $\mathcal{E}$  from the Perron Frobenius eigenvector of its adjacency matrix. If the graph  $\mathcal{E}$  has self-fusion, they can be calculated from the quantum dimensions of the unit (which is 1) and of the generator, which is

$$\beta = [2] = 2 \cos(\pi/\kappa) \quad \text{for } sl(2), \quad \beta = [3] = 1 + 2 \cos(2\pi/\kappa) \quad \text{for } sl(3),$$

by using the character property of  $\mu$ . The number  $\beta^2$  is called the Jones index. Let  $\mu_a = \dim_{\mathcal{E}}(\sigma_a)$  the quantum dimension of the vertex  $\sigma_a$ . Call also order of  $\mathcal{E}$  the quantity

$$|\mathcal{E}| = \sum_a \mu_a^2$$

Graphs  $\mathcal{A}$  and  $\mathcal{O}$  describing fusion and quantum symmetries have self-multiplication, and quantum dimensions  $\mu_n = \dim_{\mathcal{A}}(\lambda_n)$  and  $\mu_x = \dim_{\mathcal{O}}(o_x)$  for their vertices can be obtained in a similar way. In the particular case  $\mathcal{E} = \mathcal{A}$ , the  $\mu_n$  can be obtained from the modular generator  $S$  (a unitarizing matrix for the adjacency matrix of the graph):  $\mu_n = S_{0n}/S_{00}$ . For  $sl(2)$ ,  $\mu_n = [n+1]$ . For  $sl(3)$ ,  $\mu_{p,q} = [p+1][q+1][p+q+2]/[2]$ . These numbers are called statistical dimensions in the context of subfactor theory. Unitarity of  $S$  implies

$$|\mathcal{A}| = \sum_n \mu_n^2 = 1/S_{00}^2$$

and several explicit expressions given later in this section.

Call  $\epsilon$  the intertwiner describing induction-restriction between  $\mathcal{A}$  and  $\mathcal{E}$  (also called essential matrix relative to the unit vertex). It is a rectangular matrix with  $r_A$  lines and  $r_E$  columns.

Essential matrices  $\mathbf{e}_a$  are obtained from annular matrices  $F^n$  as follows:  $(\mathbf{e}_a)_{nb} = (F^n)_{ab}$ . The intertwiner is  $\mathbf{e} = \mathbf{e}_0$ . From induction, one obtains  $\dim(\Gamma_a) = \sum_n (\mathbf{e})_{na} \mu_n$ . It is convenient to write  $\lambda_n \uparrow \Gamma_a$  when the space of sections  $\Gamma_a$  contains  $\lambda_n$  in its reduction, i.e., when the integer  $(\mathbf{e})_{na} \geq 1$ . Possible multiplicities being understood, we shall write

$$\dim(\Gamma_a) = \sum_{\lambda_n \uparrow \Gamma_a} \mu_n$$

In particular, for the space of functions  $\mathcal{F}$  over the “discrete non commutative space”:  $\mathcal{A}/\mathcal{E}$ , we have  $\dim(\Gamma_0) = \sum_n (\mathbf{e})_{n0} \mu_n$  and we shall write (see also [9]):

$$|\mathcal{A}/\mathcal{E}| = \dim(\Gamma_0) = \dim(\mathcal{F}) = \sum_{\lambda_n \uparrow \Gamma_0} \mu_n$$

Then, we have the following result :

$$\dim_{\mathcal{E}}(a) = \frac{\dim \Gamma_a}{\dim \Gamma_0} = \frac{\dim \Gamma_a}{|\mathcal{A}/\mathcal{E}|}$$

Moreover,  $|\mathcal{A}/\mathcal{E}| = |\mathcal{A}|/|\mathcal{E}|$ . To obtain the quantum dimensions  $\mu_a$  of vertices of  $\mathcal{E}$ , and therefore the order  $|\mathcal{E}|$ , it is therefore enough to know the  $\mu_n$  for  $\mathcal{A}$  diagrams and the induction rules.

Example: take  $\mathcal{E} = E_6$ , and consider the central vertex  $\sigma_2$  (the triple point). Its quantum dimension can be calculated using Perron Frobenius, or directly, using  $\sigma_2 = \sigma_1^2 - \sigma_0$ :

$$\dim_E(\sigma_2) = \dim_E(\sigma_1)^2 - \dim_E(\sigma_0) = (2\cos(\pi/12))^2 - 1 = 1 + \sqrt{3}$$

but one can also obtain from  $A_{11}$ : induction from  $\sigma_0$  gives  $\Gamma_0 = \lambda_0 \oplus \lambda_6$ , so that  $|A_{11}/E_6| = [1] + [7] = 3 + \sqrt{3}$ , and induction from  $\sigma_2$  gives  $\Gamma_2 = \lambda_2 \oplus \lambda_4 \oplus \lambda_6 \oplus \lambda_8$ , so

$$\dim_E(\sigma_2) = \frac{[3] + [5] + [7] + [9]}{[1] + [7]} = \frac{6 + 4\sqrt{3}}{3 + \sqrt{3}} = 1 + \sqrt{3}.$$

Call  $J$  the modular subalgebra of  $\mathcal{E}$ . For graphs with self - fusion, we have both  $|\mathcal{O}| = |\mathcal{E}| \times |\mathcal{E}|/|J|$  and  $|\mathcal{O}| = |\mathcal{A}|$ , so that we have also

$$|\mathcal{A}|/|\mathcal{E}| = |\mathcal{E}|/|J|$$

Using quantum dimensions of  $\mathcal{E}$ , one can compute  $|J| = \sum_{\sigma_c \in J \subset \mathcal{E}} \mu_c^2 = \sum_{\sigma_c \in J} |\Gamma_c|^2 / |\Gamma_0|^2$ , but the relation for  $|\mathcal{O}|$  can be written  $|J| = |\mathcal{A}|/|\Gamma_0|^2$ , so that comparing the two expressions implies

$$|\mathcal{A}| = \sum_{c \in J} |\Gamma_c|^2$$

### 3.4.2 Trigonometric identities

The following identity, which seems to belong to the folklore of CFT, implies the previous result. Let  $Z$  be a modular invariant matrix of an  $sl(N)$  system at level  $k$ , with or without self-fusion. Its matrix elements  $(Z)_{m,n}$  are indexed by a pair of irreps  $\lambda_m$  and  $\lambda_n$ . For  $sl(N)$ , these are of course multi-indices. As before, call  $\mu_m$  the quantum dimension of  $\lambda_m$ . Then we have the following :

$$\sum_{m,n} \mu_m (Z)_{m,n} \mu_n = \sum_m \mu_m^2$$

The right hand side of this identity is  $|\mathcal{A}_k|$  and can be calculated from the modular matrix  $S$ .

$$|\mathcal{A}| = \frac{\kappa}{2} \frac{1}{\sin^2\left(\frac{\pi}{\kappa}\right)} \quad \text{for } sl(2), \text{ with } \kappa = k + 2$$

$$|\mathcal{A}| = \frac{3}{256} \kappa^2 \frac{1}{\sin^6\left(\frac{\pi}{\kappa}\right)} \frac{1}{\cos^2\left(\frac{\pi}{\kappa}\right)} \quad \text{for } sl(3), \text{ with } \kappa = k + 3$$

Using known expressions for the quantum dimensions  $\mu_n$  and the above values for the order  $|\mathcal{A}|$ , the previous identity involving the partition function  $Z$  leads, after simplification by a non trivial common denominator, to a set of *ADE* or generalized *ADE* trigonometric identities.

*ADE* trigonometric identities, i.e., identities of *sl(2)* type, read<sup>9</sup>

$$\sum_{m,n=1}^r (Z)_{m,n} \sin(m\pi/\kappa) \sin(n\pi/\kappa) = \kappa/2$$

For instance, at level 28, i.e., for  $\kappa = 30$ , we have three identities, for  $A_{29}$ ,  $D_{16}$  and  $E_8$ , reading respectively:

$$\begin{aligned} 15 &= \sum_{n=1}^{n=29} \left\{ \sin\left(n\frac{\pi}{30}\right) \right\}^2 \\ 15 &= \sum_{\substack{n=1,3,5,\dots \\ n \text{ odd}}}^{n=13} \left\{ \sin\left(n\frac{\pi}{30}\right) + \sin\left((30-n)\frac{\pi}{30}\right) \right\}^2 + 2\left\{ \sin\left(15\frac{\pi}{30}\right) \right\}^2 \\ 15 &= \left\{ \sin\left(7\frac{\pi}{30}\right) + \sin\left(13\frac{\pi}{30}\right) + \sin\left(17\frac{\pi}{30}\right) + \sin\left(23\frac{\pi}{30}\right) \right\}^2 + \left\{ \sin\left(1\frac{\pi}{30}\right) + \sin\left(11\frac{\pi}{30}\right) + \sin\left(19\frac{\pi}{30}\right) + \sin\left(29\frac{\pi}{30}\right) \right\}^2 \end{aligned}$$

Higher *ADE* trigonometric identities of *sl(3)* type read<sup>9</sup>:

$$\sum_{\substack{n=(n_1, n_2) \\ m=(m_1, m_2)}} (Z)_{m,n} \sin\left(m_1\frac{\pi}{\kappa}\right) \sin\left(m_2\frac{\pi}{\kappa}\right) \sin\left((m_1+m_2)\frac{\pi}{\kappa}\right) \sin\left(n_1\frac{\pi}{\kappa}\right) \sin\left(n_2\frac{\pi}{\kappa}\right) \sin\left((n_1+n_2)\frac{\pi}{\kappa}\right) = 3\kappa^2/64$$

For instance, at level  $k = 21$ , i.e., for  $\kappa = 24$ , we have three identities: for  $\mathcal{A}_{21}$ ,  $\mathcal{D}_{21}$  and  $\mathcal{E}_{21}$ . Setting  $s_n = \sin(n\pi/24)$ , they read respectively

$$\begin{aligned} 27 &= \sum_{p=1, q=1}^{p=22, q=24-p-1} (s_p s_q s_{p+q})^2 \\ 27 &= \sum'_{p,q,p \neq 8} (s_p s_q s_{p+q} + s_{24-p-q} s_p s_{24-q} + s_q s_{24-p-q} s_{24-p})^2 + 3(s_8 s_8 s_{16})^2 \\ 27 &= (2s_1 s_7 s_8 + 2s_5 s_8 s_{13} + 2s_5 s_{11} s_{16} + 2s_1 s_{16} s_{17} + 2s_8 s_{11} s_{19} + 2s_7 s_{16} s_{23})^2 + \\ &\quad + (s_1^2 s_2 + s_5^2 s_{10} + 2s_2 s_{11} s_{13} + s_7^2 s_{14} + 2s_7 s_{10} s_{17} + 2s_5 s_{14} s_{19} + s_{11}^2 s_{22} + 2s_1 s_{22} s_{23})^2 \end{aligned}$$

In the  $\mathcal{D}_{21}$  case,  $\sum'$  runs over  $\mathbb{Z}_3$  orbits<sup>10</sup> of vertices  $[p, q]$  with triality zero:  $p + 2q = 0 \pmod{3}$ .

### 3.4.3 Products of quantum dimensions

Product of quantum dimensions are also of interest. In the *sl(2)* case at level  $k$ , the general expression is easy to obtain. In the *sl(3)* cases, or for other Lie algebras, they can be obtained as a by - product of a discriminant formula recently discovered by [25]. Define:

$$D = \left( \sum_{\lambda_n \in \mathcal{A}} \mu_n^2 \right)^r / \prod_{\lambda_n \in \mathcal{A}} \mu_n^2$$

where  $r$  is the total number of irreps of the fusion algebra,  $r = k+1$  for *sl(2)*,  $r = (k+1)(k+2)/2!$  for *sl(3)*. In the case of *sl(2)*, using previous results for  $\sum_{\lambda_n \in \mathcal{A}} \mu_n^2$  and a classical trigonometric identity giving

$$\prod_{\lambda_n \in \mathcal{A}} \mu_n^2 = [2^{-(\kappa-1)} \kappa (\text{Sin}(\pi/\kappa))^{-(\kappa-1)}]^2$$

<sup>9</sup>Warning: In these expressions, we shifted by  $+(1)$ , or  $+(1, 1)$ , the indices labelling irreducible representations.

<sup>10</sup> $\mathbb{Z}_3$  invariance implies  $(s_p s_q s_{p+q})^2 = (s_{k+3-p-q} s_p s_{k+3-q})^2 = (s_q s_{k+3-p-q} s_{k+3-p})^2$ . One can choose a fundamental domain such that  $27 = \sum_{p=1}^7 c(p, p) + \sum_{p=1}^6 c(p+3, p) + \sum_{p=1}^5 c(p, p+3) + \sum_{p=1}^4 c(p+6, p) + \sum_{p=1}^3 c(p, p+6) + \sum_{p=1}^2 c(p+9, p) + \sum_{p=1}^1 c(p+9, p) + \frac{1}{3}c(8, 8)$ ; with  $c(p, q) = 9 (s_p s_q s_{p+q})^2$ .

where results are written in terms of the Coxeter number  $\kappa$ , one finds

$$D = 2^{\kappa-1} \kappa^{\kappa-3} \quad \text{for } sl(2)$$

More generally, it was shown recently in [25], that  $D$  is the square of the determinant of the matrix  $S_{mn}/S_{m0}$  (this is the matrix of “quantum conjugacy classes”) which, in the case of  $sl(2)$  could also be defined as the discriminant of the characteristic polynomial of the adjacency matrix of the graph  $A_r$ . Example: Take  $A_{11}$  of the  $sl(2)$  system, the characteristic polynomial of the adjacency matrix is  $-s^{11} + 10s^9 - 36s^7 + 56s^5 - 35s^3 + 6s$  and its discriminant is 10567230160896, indeed equal to  $2^{11} 12^9$ . General expressions for  $D$ , which is always an integer, are obtained in the quoted reference. For  $sl(2)$ , one recovers the previous expression, whereas [25] gives

$$D = 3^{(\kappa-2)(\kappa-1)/2} \kappa^{(\kappa-4)(\kappa-2)} \quad \text{for } sl(3)$$

where  $\kappa = k + 3$ . Since  $|\mathcal{A}_k|$  is already known, one can use this Gepner formula for  $D$  to provide an efficient way for obtaining the square product of quantum dimensions. For  $sl(3)$  we find:

$$\prod_{\lambda_n \in \mathcal{A}} \mu_n^2 = 16^{-(\kappa-2)(\kappa-1)} \kappa^{3(\kappa-2)} \left( \cos\left(\frac{\pi}{\kappa}\right) \sin^3\left(\frac{\pi}{\kappa}\right) \right)^{-(\kappa-2)(\kappa-1)}$$

### 3.5 Methodological remarks

When the McKay graph of the category  $\mathcal{E}$ , i.e., the matrix  $F_1$ , is known, all annular matrices are obtained from the  $sl(2)$  or  $sl(3)$  recurrence formulae (see for instance [14]). This determines the horizontal dimensions  $d_n$  and therefore the block structure of the associated quantum groupoid  $\mathcal{B}$  for the multiplication  $\circ$ . This also gives, by induction, the quantum dimensions  $\mu_a$  of irreducible objects and the order  $|\mathcal{E}|$ .

In most cases the graph of  $\mathcal{E}$  is not known *a priori*: one has to start from a given modular invariant, i.e., the partition function  $Z$ , and solve the modular splitting equation (see for instance [24]). This determines the Ocneanu quantum symmetries, i.e., the ring corresponding to the centralizer category  $\mathcal{O}$ , and therefore the vertical dimensions  $d_x$  describing the block structure of  $\widehat{\mathcal{B}}$ , for the multiplication  $\widehat{\circ}$ . Usually the McKay graph of  $\mathcal{E}$  is itself obtained as a by product of this calculation, from the requirement that the vector space spanned by its vertices should be a module on both the fusion ring and the ring of quantum symmetries.

There is a huge number of structure constants (cells) of different kinds for  $\mathcal{B}$ , respectively associated with tetrahedra with black and white vertices. The above method bypasses their calculation, and this is fortunate because they are only known explicitly when  $\mathcal{E} = \mathcal{A}_k$ , in the  $sl(2)$  case. Their determination in the other cases would anyway involve non canonical choices (bases in horizontal or vertical spaces).

Given a graph, candidate to be the McKay graph of an  $sl(3)$  actegory  $\mathcal{E}$ , there is a method, due to A. Ocneanu [34], that allows one to see whether or not it is so. The condition is that the graph should carry an  $sl(3)$  self-connection, i.e., a family of complex numbers (one for every oriented triangle) obeying two sets of conditions known as the “small pocket” and the “large pocket” equations. If these conditions fail, either the corresponding category does not exist, or if it exists, it is not an  $sl(3)$  actegory. Self - connections (that can also be used to define  $sl(3)$  - or higher - analogues of preprojective algebras) have been determined for all  $sl(3)$  cases by [37]. These calculations, that we have later reproduced in some cases, are not given in the tables.

As mentioned in the first section, another possibility to obtain  $\mathcal{E}$  is to define an algebra  $\mathcal{F}$  in the category  $\mathcal{A}_k$ . Our methods tell us what  $\mathcal{F}$  is, as an object of  $\mathcal{A}_k$ , and this information appears in our  $sl(3)$  tables (and of course for  $sl(2)$  as well), but we neither give nor study the multiplication map  $\mathcal{F} \otimes \mathcal{F} \mapsto \mathcal{F}$ . In many  $sl(3)$  cases, and like for  $sl(2)$ , it could be obtained from the study of conformal embeddings of affine algebras. In particular the three exceptional  $sl(3)$  examples with self-fusion (monoidal structure), namely  $\mathcal{E}_5$ ,  $\mathcal{E}_9$  and  $\mathcal{E}_{21}$  correspond to the embeddings of  $A_2$ , at levels  $k = 5, 9, 21$  into  $A_5$ ,  $E_6$  and  $E_7$  respectively, at level 1, and the  $\mathcal{D}_3$  example to an embedding into  $spin(8)$ . The above four embeddings have been studied in the

context of subfactor theory by [43]. See also [42]. We remind the reader that for a given Lie algebra  $\mathfrak{g}$  and a given level  $k$ , the conformal charge is  $c = \frac{\dim(\mathfrak{g})k}{k+h}$  where  $h$  is the dual Coxeter number of  $\mathfrak{g}$ , not to be confused with the generalized Coxeter number or altitude  $\kappa = k + h$  of a graph (or an actegory) associated with  $\mathfrak{g}$  at level  $k$ . In particular,  $c = 3k/(k + 2)$  for  $sl(2)$  and  $c = 8k/(k + 3)$  for  $sl(3)$ .

## 3.6 Tables

### 3.6.1 The $sl(2)$ family

There are two tables. Notations should be clear. Most results are hardly new. In particular, the dimensions  $d_n$  and  $d_v$  given in table 1 were obtained already in [35], [8], [40] and [11]. The compact expression for  $d_H$  can be seen in [30], but it is obtained there as the solution of a problem given in the context of quivers theory. Several quantum dimensions and orders given in table 2 can also be found in [11] and [41]. Nevertheless, both tables contain new entries and explicit generic formulae, for instance the closed expressions giving  $d_B$ . We also list the well known partition functions  $Z$ , but to save space, we use short notations like  $|\dots + [n] + \dots|^2$  where  $[n] = (n-1)$ , stands for the  $\widehat{sl}(2)$  characters  $\chi_n$ . The same remarks hold for  $\widehat{sl}(3)$  partition functions (also well known) recalled in the next tables.

### 3.6.2 The $sl(3)$ family

Many results concerning individual  $sl(3)$  graphs can be found in the original article [16] and in the book [3], chap.17. Other aspects of members of this system, in particular their quantum symmetries, are described in [34], [23], [14], [24] and [22]. Here we are mostly interested in giving tables describing the block structure of the quantum groupoid  $\mathcal{B}$ , i.e., dimensions  $d_n$ ,  $d_x$ , quantum dimensions  $\mu_n$  and orders. Many such results were certainly obtained by [35], and several examples are studied in [12] and [41]. Results presented here are complete only for the cases with self - fusion, for which we also give tables describing induction rules; the other cases should be made available in [17] and [22]. Some of these results already appeared in [12]. In both  $sl(2)$  and  $sl(3)$  tables, cases with self-fusion (monoidal structure) are underlined.

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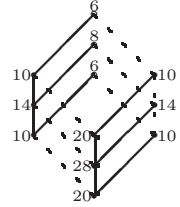
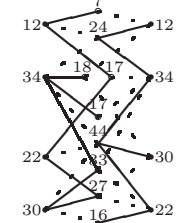
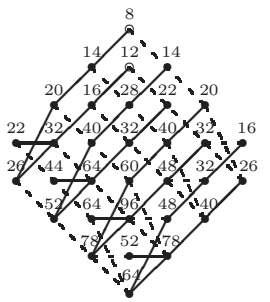
Graph	$d_n$	$d_x$	$d_H = \kappa(\kappa + 1)r/6$	$d_V - d_H$	$d_B = d_{\mathcal{B}}$
$\underline{A_1} = \underline{A_2}$ $\underline{A_2} = \underline{A_3}$ $\underline{A_3} = \underline{A_4}$ $\underline{A_4} = \underline{A_5}$ $\underline{A_5} = \underline{A_6}$ $\underline{A_{10}} = \underline{A_{11}}$ $\underline{A_{16}} = \underline{A_{17}}$ $\underline{A_{28}} = \underline{A_{29}}$ $\underline{A_k} = \underline{A_{r=k+1}}$	$(2, 2)$ $(3, 4, 3)$ $(4, 6, 6, 4)$ $(5, 8, 9, 8, 5)$ $(6, 10, 12, 12, 10, 6)$ $(11, 20, 27, 32, 35, 36, 35, 32, 27, 20, 11)$ $(17, 32, 45, 56, 65, 72, 77, 80, 81, 80, 77, 72, 65, 56, 45, 32, 17)$ $(29, 56, 81, 104, 125, 144, 161, 176, 189, 200, 209, 216, 221, 224; \text{sym.})$ $d_n = (n+1)(k+1-n), \quad n = 0, \dots, k$	$(2, 2)$ $(3, 4, 3)$ $(4, 6, 6, 4)$ $(5, 8, 9, 8, 5)$ $(6, 10, 12, 12, 10, 6)$ $\dots$ $\dots$ $\dots$ $d_x = d_n$	$4$ $10$ $20$ $35$ $56$ $286 = 2^1 11^1 13^1$ $3^1 17^1 19^1$ $5^1 29^1 31^1$ $(\kappa - 1)\kappa(\kappa + 1)/6$	$0$ $0$ $0$ $0$ $0$ $0$ $0$ $0$	$8 = 2^3$ $34 = 2^1 17^1$ $104 = 2^3 13^1$ $259 = 7^1 37^1$ $560 = 2^4 5^1 7^1$ $8294 = 2^1 11^1 13^1 29^1$ $3^1 5^1 13^1 17^1 19^1$ $17^1 29^1 31^1 53^1$ $\kappa(\kappa^4 - 1)/30$
$\underline{D_4} = \underline{D_4}$ $\underline{D_8} = \underline{D_6}$ $\underline{D_{16}} = \underline{D_{10}}$ $\underline{D_{28}} = \underline{D_{16}}$ $\underline{D_k} = \underline{D_{r_e = \frac{k}{2} + 2}}$	$(4, 6; 8; 6, 4)$ $(6, 10, 14, 16; 18; 16, 14, 10, 6)$ $(10, 18, 26, 32, 38, 42, 46, 48; \text{sym.})$ $(16, 30, 44, 56, 68, 78, 88, 96, 104, 110, 116, 120, 124, 126; \text{sym.})$ $(r, \kappa, \dots; \frac{1}{2}(1 + \frac{\kappa}{2})^2; \dots, \kappa, r)$	$(4, 6; 4, 4) (4, 6; 4, 4)$ $(6, 10, 14, 16; 9, 9) (6, 10, 14, 16; 9, 9)$ $(10, 18, 26, 32, 38, 42, 46, 48; 25, 25)_2$ $\dots \dots$ $(r, \kappa, \dots; \frac{1}{4}(1 + \frac{\kappa}{2})^2, \frac{1}{4}(1 + \frac{\kappa}{2})^2)_2$	$28 = 4^1 7^1$ $110 = 2^1 5^1 11^1$ $570 = 2^1 3^1 5^1 19^1$ $2480 = 2^4 5^1 31^1$ $\kappa(\kappa + 1)(\kappa + 2)/12$	$(4 + 4)$ $(9 + 9)$ $(25 + 25)$ $(64 + 64)$ $\frac{1}{2}(1 + \frac{\kappa}{2})^2$	$168 = 2^3 3^1 7^1$ $1500 = 2^2 3^1 5^3$ $2^2 5501^1$ $2^5 7757^1$ $\frac{(2+\kappa)(120+\kappa(28+\kappa(26+\kappa(17+4\kappa))))}{480}$
$\underline{D_6} = \underline{D_5}$ $\underline{D_{10}} = \underline{D_7}$ $\underline{D_k} = \underline{D_{r_o = \frac{k}{2} + 2}}$	$(5, 8, 11, 12, 11, 8, 5)$ $(7, 12, 17, 20, 23, 24, 23, 20, 17, 12, 7)$ $(r, \kappa, \dots, \kappa, r)$	$(5, 8, 11, 12, 11, 8, 5)$ $(7, 12, 17, 20, 23, 24, 23, 20, 17, 12, 7)$ $(r, k + 2, \dots, k + 2, r)$	$60 = 2^2 3^1 5^1$ $182 = 2^1 7^1 13^1$ $\kappa(\kappa + 1)(\kappa + 2)/12$	$0$ $0$ $0$	$564 = 2^2 3^1 47^1$ $3398 = 2^1 1699^1$ $\frac{\kappa(176+\kappa(80+\kappa(60+\kappa(25+4\kappa))))}{480}$
$\underline{E_{10}} = \underline{E_6}$	$(6, 10, 14, 18, 20, 20, 20, 18, 14, 10, 6)$		$156 = 2^2 3^1 13^1$	$0$	$2512 = 2^4 157^1$
$\underline{E_{16}} = \underline{E_7}$	$(7, 12, 17, 22, 27, 30, 33, 34, 35, 34, 33, 30, 27, 22, 17, 12, 7)$		$399 = 3^1 7^1 19^1$	$0$	$10905 = 3^1 5^1 727^1$
$\underline{E_{28}} = \underline{E_8}$	$(8, 14, 20, 26, 32, 38, 44, 48, 52, 56, 60, 62, 64, 64, 64; \text{sym.})$		$1240 = 2^3 5^1 31$	$0$	$63136 = 2^5 1973^1$

Table 1: Horizontal dimensions, vertical dimensions, and bialgebras dimensions for  $sl(2)$  cases.

Graph	induction	$q - \dim$	$ \mathcal{E} $	$ \mathcal{A}/\mathcal{E} $	$ \mathcal{J} $
$\underline{A_2} = \underline{A_3}$ $\underline{A_3} = \underline{A_4}$ $\underline{A_4} = \underline{A_5}$ $\underline{A_5} = \underline{A_6}$ $\underline{A_k} = \underline{A_{r=k+1}}$ $\underline{A_{10}} = \underline{A_{11}}$ $\underline{A_{16}} = \underline{A_{17}}$ $\underline{A_{28}} = \underline{A_{29}}$	<p>.</p> <p>.</p> <p>.</p> <p>.</p> <p>.</p> <p>.</p> <p>.</p> <p>.</p> <p>.</p>	<p><math>1, \sqrt{2}, 1</math></p> <p><math>1, \frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(1 + \sqrt{5}), 1</math></p> <p><math>1, \sqrt{3}, 2, \sqrt{3}, 1</math></p> <p><math>1, 2 \cos(\frac{\pi}{7}), 1 + 2 \cos(\frac{\pi}{7}), 1 + 2 \cos(\frac{\pi}{7}), 1</math></p> <p><math>[1], [2], \dots, [2], [1]</math></p> <p><math>1, \sqrt{2 + \sqrt{3}}, 1 + \sqrt{3}, \sqrt{3(2 + \sqrt{3})}, 2 + \sqrt{3}; 2\sqrt{2 + \sqrt{3}}; + \text{sym.}</math></p> <p>...</p> <p>...</p>	<p>4</p> <p><math>5 + \sqrt{5}</math></p> <p>12</p> <p>18.59</p> <p><math>(\kappa/2) \csc^2(\pi/\kappa)</math></p> <p><math>24(2 + \sqrt{3})</math></p> <p><math>9 \csc^2(\pi/18) = 298.47</math></p> <p><math>30(12 + 5\sqrt{5} + \sqrt{3(85 + 38\sqrt{5})})</math></p>	<p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p> <p>1</p>	<p><math>=  \mathcal{A} </math></p> <p><math>=  \mathcal{A} </math></p> <p><math>=  \mathcal{A} </math></p> <p><math>=  \mathcal{A} </math></p> <p><math>=  \mathcal{A} </math></p> <p><math>=  \mathcal{A} </math></p> <p><math>=  \mathcal{A} </math></p> <p><math>=  \mathcal{A} </math></p>
$\underline{D_4} = \underline{D_4}$ $\underline{D_8} = \underline{D_6}$ $\underline{D_{16}} = \underline{D_{10}}$ $\underline{D_{28}} = \underline{D_{16}}$	<p>.</p> <p>...</p> <p>...</p>	<p><math>\frac{1}{2}(1 + \sqrt{5})</math></p> <p><math>\sqrt{\frac{1}{2}(5 + \sqrt{5})}, \frac{1}{2}(3 + \sqrt{5}), \sqrt{5 + 2\sqrt{5}}; \frac{1}{2}(1 + \sqrt{5})</math></p> <p><math>\frac{1}{2}(1 + \sqrt{5})</math></p> <p><math>[1], [2], [3], [4], [5], [6], [7], [8]; [9]/2, [9]/2</math></p> <p>.....</p>	<p><math>\frac{ \underline{A_8} }{2} = 6</math></p> <p><math>\frac{ \underline{A_9} }{2} = 5(3 + \sqrt{5})</math></p> <p><math>\frac{ \underline{A_{17}} }{2} = 149.23</math></p> <p><math>\frac{ \underline{A_{29}} }{2}</math></p>	<p>2</p> <p>2</p> <p>2</p> <p>2</p>	<p><math>\frac{ \underline{A_8} }{4}</math></p> <p><math>\frac{ \underline{A_9} }{4}</math></p> <p><math>\frac{ \underline{A_{17}} }{4}</math></p> <p><math>\frac{ \underline{A_{29}} }{4}</math></p>
$D_6 = D_5$ $D_{10} = D_7$	<p>.</p> <p>.</p>	<p><math>1, \sqrt{2 + \sqrt{2}}, 1 + \sqrt{2}; \sqrt{\frac{1 + \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}}</math></p> <p><math>1, \sqrt{2 + \sqrt{3}}, 1 + \sqrt{3}, \sqrt{3(2 + \sqrt{3})}, 2 + \sqrt{3}; \frac{\sqrt{2 + \sqrt{3}}}{\sqrt{2 + \sqrt{3}}}</math></p>	<p><math> \underline{A_7} /2 = 2 \csc^2(\pi/8) = 13.65</math></p> <p><math> \underline{A_{11}} /2 = 3 \csc^2(\pi/12)</math></p>	<p>.</p> <p>.</p>	<p>.</p> <p>.</p>
$\underline{E_{10}} = \underline{E_6}$ $\underline{E_{16}} = \underline{E_7}$ $\underline{E_{28}} = \underline{E_8}$	<p>.</p> <p>.</p> <p>.</p>	<p><math>q = \exp(\frac{4\pi}{18})</math></p> <p><math>q = \exp(\frac{4\pi}{30})</math></p>	<p><math>4(3 + \sqrt{3})</math></p> <p><math> \mathcal{E}  = 2([2]^2 + [4]^2 + [4]^2/[3]^2) = 38.46</math></p> <p><math> \underline{A_{17}}  =  \underline{D_{10}}   \underline{D_{10}}  /  \mathcal{J} </math></p> <p><math> \underline{D_{10}}  =  \underline{A_{17}} /2</math></p> <p><math>\frac{1}{2}(15(3 + \sqrt{5}) + \sqrt{30(65 + 29\sqrt{5})})</math></p>	<p><math>3 + \sqrt{3}</math></p> <p><math>[1] + [9] + [17] = 7.75</math></p> <p><math>\frac{1}{2}(3(5 + \sqrt{5}) + \sqrt{150 + 66\sqrt{5}})</math></p>	<p>4</p> <p><math> \mathcal{J}  =  \underline{D_{10}} /2 =  \underline{A_{17}} /4</math></p> <p><math> \mathcal{J}  = [1]^2 + [3]^2 + [5]^2 + [7]^2 + [9]^2/4 + [9]^2/4</math></p> <p><math>= [1]^2 + [3]^2 + [5]^2 + [7]^2 + [9]^2/4 + [9]^2/4</math></p> <p><math>\frac{1}{2}(5 + \sqrt{5})</math></p>

Table 2: Quantum dimensions for  $sl(2)$  cases.

<i>Graph</i>	$\kappa = k + 2$	$r_E, r_A, r_O$	$\mathcal{Z}$	$\mathcal{F}$
$\underline{A_k} = A_{r=k+1}$	$k + 2$	$k + 1, k + 1, k + 1$	$\sum_{i=1}^{k+1}  i ^2$	$\lambda_0$
$\underline{D_k} = D_{\substack{r_{\text{even}}=k/2+2}}$	$2r - 2$	$\frac{k}{2} + 2, k + 1, k + 4$	$\sum_{i \text{ odd}=1}^{\frac{k}{2}-1}  i + (k + i) ^2 + 2 \frac{k}{2} + 1 ^2$	$\lambda_0 \oplus \lambda_k$
$\underline{D_k} = D_{\substack{r_{\text{odd}}=k/2+2}}$	$2r - 2$	$\frac{k}{2} + 2, k + 1, k + 1$	$\sum_{i \text{ odd}=1}^{k+1}  i ^2 +  \frac{k}{2} + 1 ^2 + \sum_{i \text{ even}=2}^{\frac{k}{2}-1} i(\overline{k+i}) + (k+i)\overline{i}$	$\lambda_0 \oplus \lambda_k$
$\underline{E_{10}} = E_6$	<b>12</b>	6, 11, 12	$ 1 + 7 ^2 +  4 + 8 ^2 +  5 + 11 ^2$	$\lambda_0 \oplus \lambda_6$
$E_{16} = D_{16}^\dagger = E_7$	18	7, 17, 17	$ 1 + 17 ^2 +  5 + 13 ^2 +  7 + 11 ^2 +  9 ^2 + ((3 + 15) \overline{9} + h.c.)$	$\lambda_0 \oplus \lambda_8 \oplus \lambda_{16}$
$\underline{E_{28}} = E_8$	<b>30</b>	8, 29, 32	$ 1 + 11 + 19 + 29 ^2 +  7 + 13 + 17 + 23 ^2$	$\lambda_0 \oplus \lambda_{10} \oplus \lambda_{18} \oplus \lambda_{28}$

Table 3: Complements for  $sl(2)$  cases.



Graph	$\kappa$	$r_E, r_A, r_O$	$d_H$	$d_V - d_H$	$d_B = d_{\mathcal{B}}$	$ \mathcal{E} $	$ \mathcal{A}/\mathcal{E} $	$ J $
$\underline{\mathcal{A}}_1$	4	3, 3, 3	9	0	27	3	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_2$	5	6, 6, 6	45	0	351	$\frac{3}{2}(5 + \sqrt{5})$	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_3$	6	10, 10, 10	164	0	2920	36	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_4$	7	15, 15, 15	486	0	17 766	106.027	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_5$	8	21, 21, 21	1 242	0	85 644	$48(3 + 2\sqrt{2})$	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_6$	9	28, 28, 28	2 838	0	344 826	671.56	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_9$	12	55, 55, 55	21 307	0	10 517 299	$432(7 + 4\sqrt{3})$	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_{21}$	24	253, 253, 253	2 729 870	0	41 644 127 980	$288 \left( 18 + 10\sqrt{3} + \sqrt{6(97 + 56\sqrt{3})^2} \right)$	1	$=  \mathcal{E} $
$\underline{\mathcal{A}}_k$	$k + 3$	$r_E = r_A = r_O = \frac{(k+1)(k+2)}{2}$	$\frac{(\kappa-2)(\kappa-1)\kappa(\kappa+1)(\kappa+2)(\kappa^2+5)}{1680}$	0	$d_B(\mathcal{A}_k)$	$3 \frac{\kappa^2 c s c^6(\pi/\kappa) s e c^2(\pi/\kappa)}{256}$	1	$=  \mathcal{E} $
$\mathcal{A}_1^*$	4	1, 3, 3	3	0	3	1	3	
$\mathcal{A}_2^*$	5	2, 6, 6	15	0	39	$\frac{1}{2}(5 + \sqrt{5})$	3	
$\mathcal{A}_3^*$	6	2, 10, 10	36	0	144	2	18	
$\mathcal{A}_4^*$	7	3, 15, 15	102	0	798	9.295	11.405	
$\mathcal{A}_5^*$	8	3, 21, 21	204	0	2376	4	$12(3 + 2\sqrt{2})$	
$\mathcal{A}_6^*$	9	4, 28, 28	442	0	8578	$19.234$	34.914	
$\mathcal{A}_7^*$	10	4, 36, 36	780	0	21360	$5 + \sqrt{5}$	$15(7 + 3\sqrt{5})$	
$\mathcal{A}_{k \geq 1}^*$	$k + 3$	$\lceil \frac{k+1}{2} \rceil, r_A, r_A$	$\begin{cases} \frac{(\kappa-2)\kappa^2(\kappa+2)(\kappa^2+4)}{1280} & k \text{ impair} \\ \frac{(\kappa-1)(\kappa+1)((\kappa-1)^2+4)((\kappa+1)^2+4)}{1280} & k \text{ pair} \end{cases}$	0	.	.	.	.
$\mathcal{D}_1$	4	1, 3, 3	$\frac{1}{3}d_H(\mathcal{A}_1) = 3$	0	$\frac{1}{9}d_B(\mathcal{A}_1) = 3$	$\frac{1}{3} \mathcal{A}_1  = 1$	3	
$\mathcal{D}_2$	5	2, 6, 6	$\frac{1}{3}d_H(\mathcal{A}_2) = 15$	0	$\frac{1}{9}d_B(\mathcal{A}_2) = 39$	$\frac{1}{3} \mathcal{A}_2  = \frac{1}{2}(5 + \sqrt{5})$	3	
$\mathcal{D}_4$	7	5, 15, 15	$\frac{1}{3}d_H(\mathcal{A}_4) = 162$	0	$\frac{1}{9}d_B(\mathcal{A}_4) = 1974$	$\frac{1}{3} \mathcal{A}_4  = 35.342$	3	
$\mathcal{D}_5$	8	7, 21, 21	$\frac{1}{3}d_H(\mathcal{A}_5) = 414$	0	$\frac{1}{9}d_B(\mathcal{A}_5) = 9516$	$\frac{1}{3} \mathcal{A}_5  = 16(3 + 2\sqrt{2})$	3	
$\mathcal{D}_{k=1,2 \bmod 3}$	$k + 3$	$\frac{1}{3}r_A, r_A, r_A$	$\frac{1}{3}d_H(\mathcal{A}_k)$	0	$\frac{1}{9}d_B(\mathcal{A}_k)$	$\frac{1}{3} \mathcal{A}_k $	3	
$\mathcal{D}_1^*$	4	3, 3, 3	$3 d_H(\mathcal{A}_1^*) = 9$	0	$9 d_B(\mathcal{A}_1^*) = 27$	3	1	
$\mathcal{D}_2^*$	5	6, 6, 6	$3 d_H(\mathcal{A}_2^*) = 45$	0	$9 d_B(\mathcal{A}_2^*) = 351$	$\frac{3}{2}(5 + \sqrt{5})$	1	
$\mathcal{D}_4^*$	7	9, 15, 15	$3 d_H(\mathcal{A}_4^*) = 306$	0	$9 d_B(\mathcal{A}_4^*) = 7 182$	8.588	12.344	
$\mathcal{D}_5^*$	8	9, 21, 21	$3 d_H(\mathcal{A}_5^*) = 612$	0	$9 d_B(\mathcal{A}_5^*) = 21 384$	12	$4(3 + 2\sqrt{2})$	
$\mathcal{D}_{k=1,2 \bmod 3}^*$	$k + 3$	$3r_{A^*}, r_A, r_A$	$3 d_H(\mathcal{A}_k^*)$	0	$9 d_B(\mathcal{A}_k^*)$	.	.	.
$\underline{\mathcal{D}}_3$	6	6, 10, 18	96	36	1 032	$\frac{1}{3} \mathcal{A}_3  = 12$	3	4
$\underline{\mathcal{D}}_6$	9	12, 28, 36	1 218	174	64 698	$\frac{1}{3} \mathcal{A}_6  = 223.853$	3	74.617
$\underline{\mathcal{D}}_9$	12	21, 55, 63	8 193	622	1 573 275	$\frac{1}{3} \mathcal{A}_9  = 144(7 + 4\sqrt{3})$	3	$48(7 + 4\sqrt{3})$
$\underline{\mathcal{D}}_{k=0 \bmod 3}$	$k + 3$	$\frac{r_{A^*}-1}{3} + 3, r_A, 3r_E$	.	.	.	$= \frac{1}{3} \mathcal{A}_k $	3	$\frac{1}{3} \mathcal{E} $
$\mathcal{D}_3^*$	6	6, 10, 18	$3 d_H(\mathcal{A}_3^*) = 108$	36	$9 d_B(\mathcal{A}_3^*) = 1 296$	6	6	
$\mathcal{D}_6^*$	9	12, 18, 36	$3 d_H(\mathcal{A}_6^*) = 1326$	186	$9 d_B(\mathcal{A}_6^*) = 77 202$	16.336	41.107	
$\mathcal{D}_9^*$	12		$3 d_H(\mathcal{A}_9^*)$	.	$9 d_B(\mathcal{A}_9^*)$	.	.	.
$\mathcal{D}_{k=0 \bmod 3}^*$	$k + 3$	$3r_{A^*}, r_A, r_O(\mathcal{D}_k)$	$3 d_H(\mathcal{A}_k^*)$	.	$9 d_B(\mathcal{A}_k^*)$	.	.	.

Table 4: Dimensions and quantum masses for  $sl(3)$  cases :  $\mathcal{A}$ ,  $\mathcal{D}$  series and conjugated.

$$d_B(\mathcal{A}_k) = \frac{(\kappa-2)(\kappa-1)\kappa^2(\kappa+1)(\kappa+2)(1052+325\kappa^2+58\kappa^4+5\kappa^6)}{4435200}$$

<i>Graph</i>	$\kappa$	$r_E, r_A, r_O$	$d_H$	$d_V - d_H$	$d_B = d_{\hat{B}}$	$ \mathcal{E} $	$ \mathcal{A}/\mathcal{E} $	$ J $
$\mathcal{D}_9^t$	12	17, 55, 63	7 001	.	1 167 355	$72(2 + \sqrt{3})$	$6(2 + \sqrt{3})$	.
$\mathcal{D}_9^{t*}$	12	11, 55, 63	.	.	.	.	.	.
$\underline{\mathcal{E}_5}$	8	12, 21, 24	720	0	29 376	$12(2 + \sqrt{2})$	$2(2 + \sqrt{2})$	6
$\mathcal{E}_5/3$	8	4, 21, 24	$\frac{1}{3} d_H(\mathcal{E}_5) = 240$	0	$\frac{1}{9} d_B(\mathcal{E}_5) = 3\,264$	$\frac{1}{3}  \mathcal{E}_5  = 4(2 + \sqrt{2})$	$6(2 + \sqrt{2})$	.
$\underline{\mathcal{E}_9}$	12	12, 55, 72	4 656	792	518 976	$36(2 + \sqrt{3})$	$12(2 + \sqrt{3})$	3
$\mathcal{E}_9/3$	12	12, 55, 72	5 616	936	754 272	$\frac{1}{3}  \mathcal{E}_9  = 12(2 + \sqrt{3})$	$36(2 + \sqrt{3})$	.
$\underline{\mathcal{E}_{21}}$	24	24, 253, 288	288 576	0	480 701 952	$24 \left( 18 + 10\sqrt{3} + \sqrt{6(97 + 56\sqrt{3})} \right)$	$12 \left( 18 + 10\sqrt{3} + \sqrt{6(97 + 56\sqrt{3})} \right)$	2

Table 5: Dimensions and quantum masses for exceptional  $sl(3)$  cases.

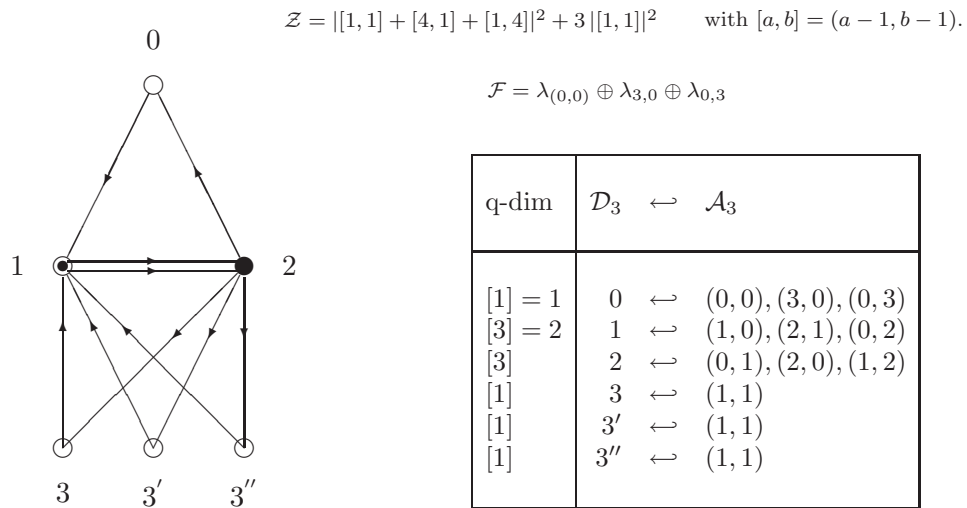


Figure 1: The  $\mathcal{D}_3$  graph, quantum dimensions and  $\mathcal{D}_3 \leftrightarrow \mathcal{A}_3$  induction rules.

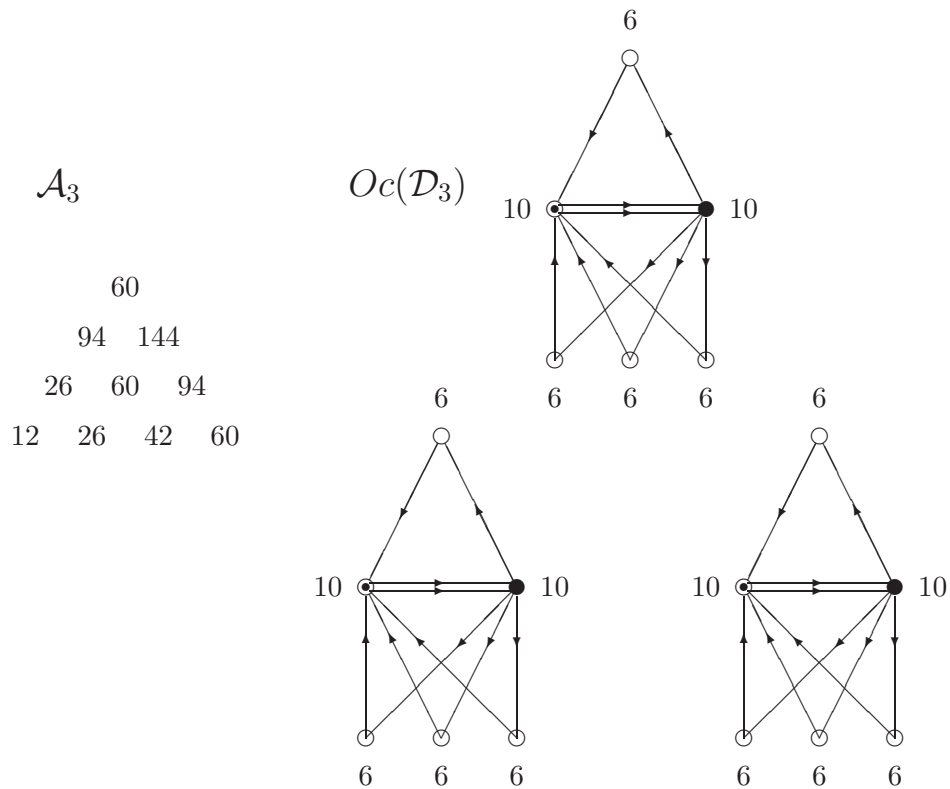
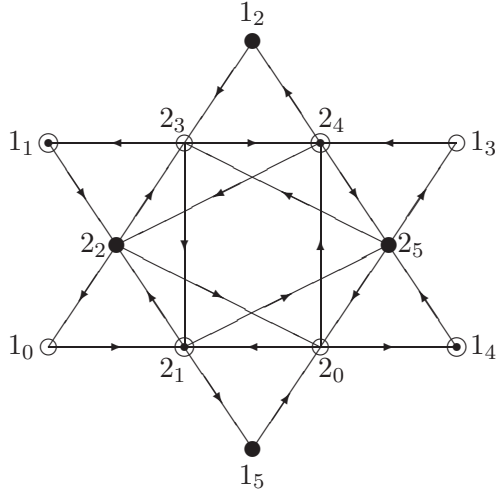


Figure 2: Dimension  $d_n$  and  $d_x$  of the blocks for  $\mathcal{D}_3$ .

$$\begin{aligned} \mathcal{Z} &= |[1, 1] + [3, 3]|^2 + |[1, 3] + [4, 3]|^2 + |[2, 3] + [6, 1]|^2 \\ &+ |[4, 1] + [1, 4]|^2 + |[3, 2] + [1, 6]|^2 + |[3, 1] + [3, 4]|^2 \\ &\text{with } [a, b] = (a - 1, b - 1). \end{aligned}$$

$$\mathcal{F} = \lambda_{(0,0)} \oplus \lambda_{2,2}$$



q-dim	$\mathcal{E}_5 \leftrightarrow \mathcal{A}_5$
[1] = 1	$1_0 \leftrightarrow (0, 0), (2, 2)$
[1]	$1_1 \leftrightarrow (0, 2), (3, 2)$
[1]	$1_2 \leftrightarrow (1, 2), (5, 0)$
[1]	$1_3 \leftrightarrow (3, 0), (0, 3)$
[1]	$1_4 \leftrightarrow (2, 1), (0, 5)$
[1]	$1_5 \leftrightarrow (2, 0), (2, 3)$
[3] = $1 + \sqrt{2}$	$2_0 \leftrightarrow (1, 1), (3, 0), (2, 2), (1, 4)$
[3]	$2_1 \leftrightarrow (1, 0), (2, 1), (1, 3), (3, 2)$
[3]	$2_2 \leftrightarrow (0, 1), (1, 2), (3, 1), (2, 3)$
[3]	$2_3 \leftrightarrow (1, 1), (0, 3), (2, 2), (4, 1)$
[3]	$2_4 \leftrightarrow (0, 2), (2, 1), (4, 0), (1, 3)$
[3]	$2_5 \leftrightarrow (2, 0), (1, 2), (3, 1), (0, 4)$

Figure 3: The  $\mathcal{E}_5$  graph, quantum dimensions and  $\mathcal{E}_5 \leftrightarrow \mathcal{A}_5$  induction rules.

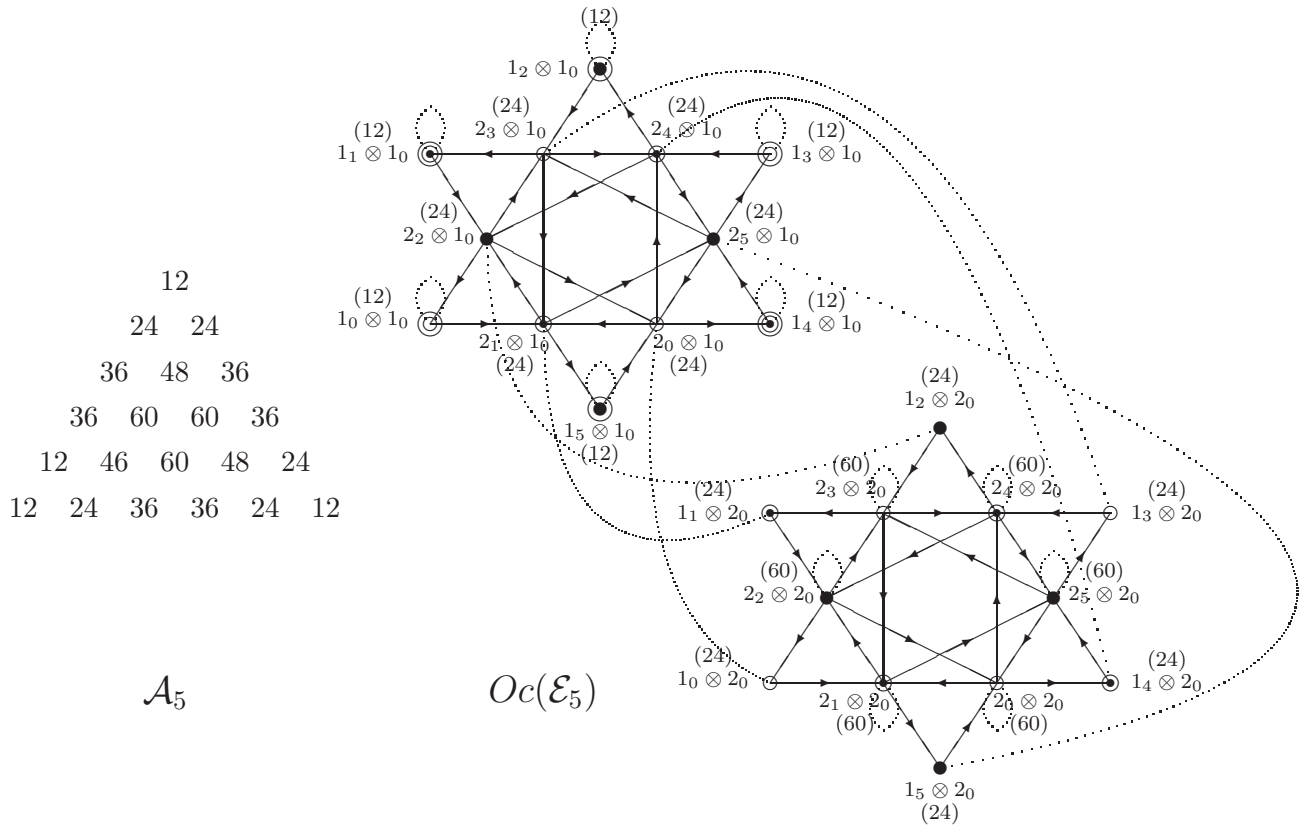
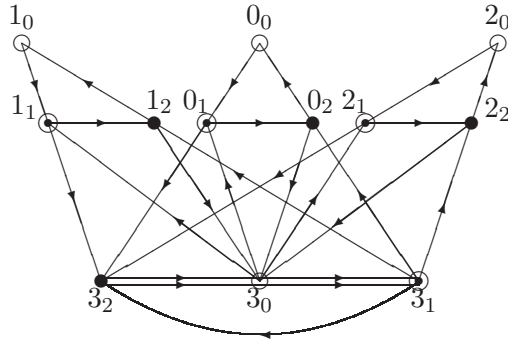


Figure 4: Dimension  $d_n$  and  $d_x$  of the blocks for  $\mathcal{E}_5$ .

$$\mathcal{Z} = [1, 1] + [1, 10] + [10, 1] + [2, 5] + [5, 2] + [5, 5] \|^2 + 2 \|[3, 3] + [3, 6] + [6, 3]\|^2$$

$$\text{with } [a, b] = (a - 1, b - 1)$$

$$\mathcal{F} = \lambda_{(0,0)} \oplus \lambda_{(1,4)} \oplus \lambda_{(4,1)} \oplus \lambda_{(4,4)} \oplus \lambda_{(0,9)} \oplus \lambda_{(9,0)}$$



q-dim	$\mathcal{E}_9 \leftrightarrow \mathcal{A}_9$
[1] = 1	$0_0 \leftrightarrow (0, 0), (4, 1), (1, 4), (4, 4), (9, 0), (0, 9)$
[1]	$1_0 \leftrightarrow (2, 2), (5, 2), (2, 5)$
[1]	$2_0 \leftrightarrow (2, 2), (5, 2), (2, 5)$
$3 + 2\sqrt{3}$	$3_0 \leftrightarrow (1, 1), (3, 0), (0, 3), (6, 0), (0, 6), (7, 1), (1, 7), (6, 3), (3, 6), (2, 2)_2, (4, 1)_2, (1, 4)_2, (5, 2)_2, (2, 5)_2, (4, 4)_2, (3, 3)_3$
$[3] = 1 + \sqrt{3}$	$0_1 \leftrightarrow (1, 0), (4, 0), (1, 3), (3, 2), (0, 5), (5, 1), (2, 4), (4, 3), (3, 5), (0, 8), (8, 1), (5, 4)$
[3]	$1_1 \leftrightarrow (2, 1), (1, 3), (3, 2), (5, 1), (2, 4), (4, 3), (1, 6), (6, 2), (3, 5)$
[3]	$2_1 \leftrightarrow (2, 1), (1, 3), (3, 2), (5, 1), (2, 4), (4, 3), (1, 6), (6, 2), (3, 5)$
$3 + \sqrt{3}$	$3_1 \leftrightarrow (0, 2), (2, 1), (4, 0), (1, 3), (0, 5), (5, 1), (7, 0), (1, 6), (6, 2), (3, 5), (5, 4), (2, 7), (3, 2)_2, (2, 4)_2, (4, 3)_2$
[3]	$0_2 \leftrightarrow (0, 1), (3, 1), (0, 4), (5, 0), (2, 3), (4, 2), (1, 5), (3, 4), (8, 0), (5, 3), (4, 5), (1, 8)$
[3]	$1_2 \leftrightarrow (1, 2), (3, 1), (2, 3), (4, 2), (1, 5), (6, 1), (3, 4), (5, 3), (2, 6)$
[3]	$2_2 \leftrightarrow (1, 2), (3, 1), (2, 3), (4, 2), (1, 5), (6, 1), (3, 4), (5, 3), (2, 6)$
$3 + \sqrt{3}$	$3_2 \leftrightarrow (2, 0), (1, 2), (3, 1), (0, 4), (5, 0), (1, 5), (6, 1), (0, 7), (5, 3), (2, 6), (7, 2), (4, 5), (2, 3)_2, (4, 2)_2, (3, 4)_2$

Figure 5: The  $\mathcal{E}_9$  graph, quantum dimensions and  $\mathcal{E}_9 \leftrightarrow \mathcal{A}_9$  induction rules.

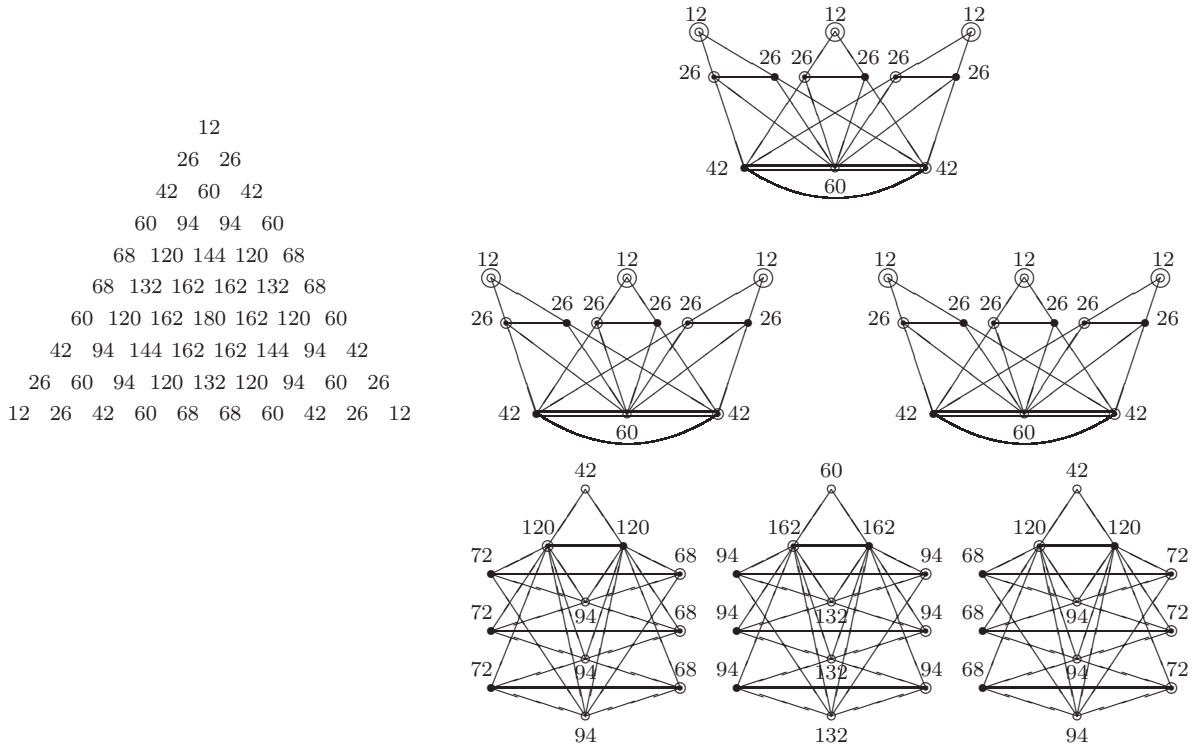
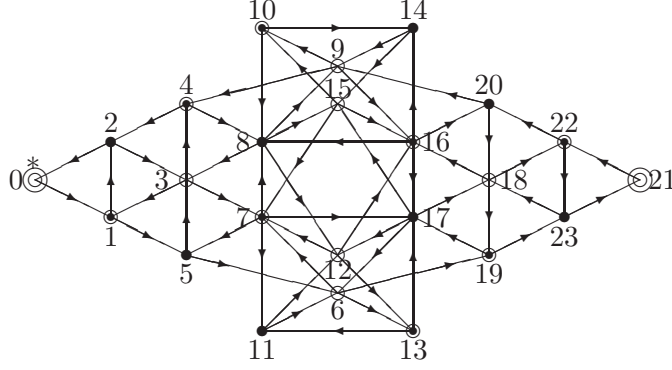


Figure 6: Dimensions  $d_n$  and  $d_x$  for the  $\mathcal{E}_9$  graph.

$$\begin{aligned} \mathcal{Z} = & \quad |[1, 1] + [5, 5] + [7, 7] + [10, 10] + [1, 22] + [22, 1] + [2, 11] + [11, 2] + [5, 14] + [14, 5] + [7, 10] + [10, 7]|^2 \\ & + \quad |[1, 7] + [7, 1] + [1, 16] + [16, 1] + [5, 8] + [8, 5] + [5, 11] + [11, 5] + [7, 16] + [16, 7] + [8, 11] + [11, 8]|^2 \end{aligned} \quad \text{with } [a,b] = (a-1,b-1)$$



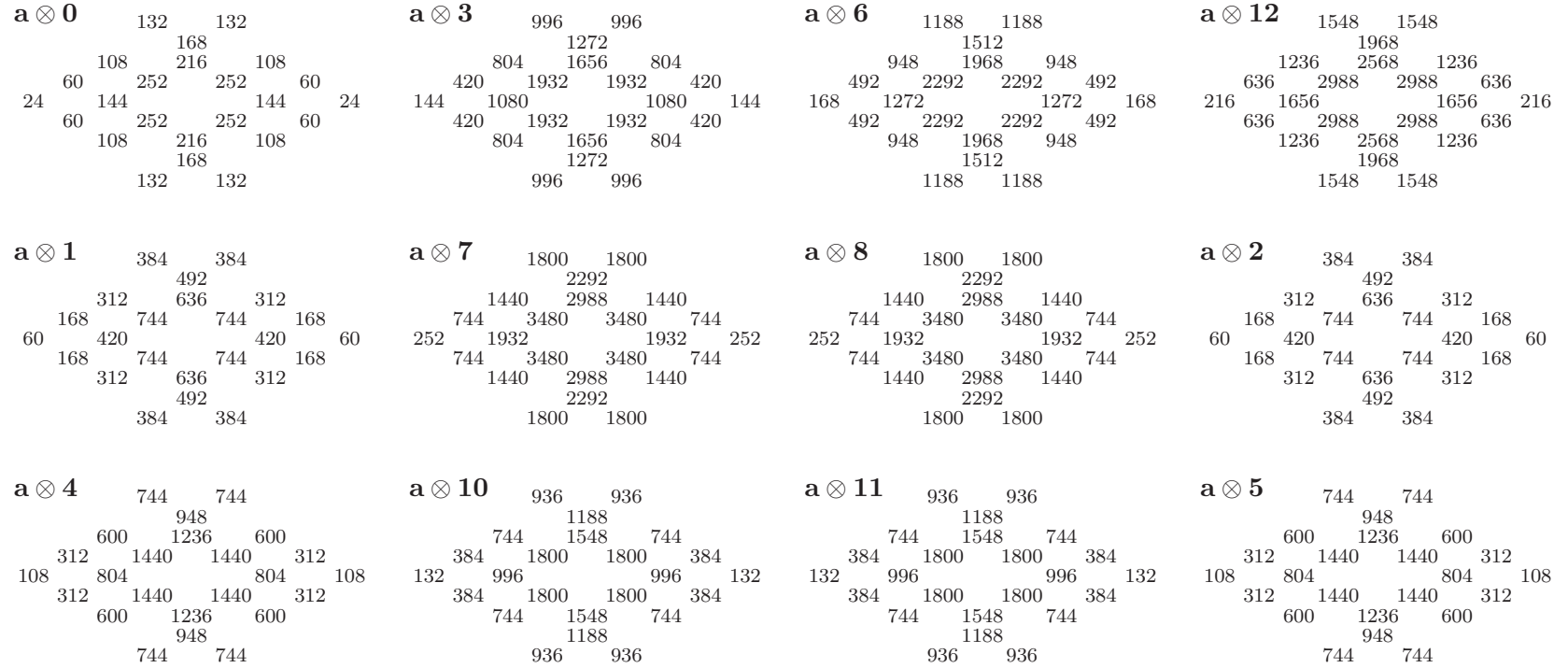
$$\begin{aligned} \mathcal{F} = & \quad \lambda_{(0,0)} \oplus \lambda_{(4,4)} \oplus \lambda_{(6,6)} \oplus \lambda_{(9,9)} \oplus \lambda_{(0,21)} \oplus \lambda_{(21,0)} \\ & \oplus \quad \lambda_{(1,10)} \oplus \lambda_{(10,1)} \oplus \lambda_{(4,13)} \oplus \lambda_{(13,4)} \oplus \lambda_{(6,9)} \oplus \lambda_{(9,6)} \end{aligned}$$

$\mathcal{E}_{21} \leftarrow \mathcal{A}_{21}$
$0 \leftarrow (0, 0), (4, 4), (10, 1), (1, 10), (6, 6), (9, 6), (6, 9), (13, 4), (4, 13), (10, 10), (21, 0), (0, 21)$
$21 \leftarrow (6, 0), (0, 6), (7, 4), (4, 7), (10, 4), (4, 10), (15, 0), (0, 15), (10, 7), (7, 10), (15, 6), (6, 15)$

Figure 7: The  $\mathcal{E}_{21}$  graph and  $\mathcal{E}_{21} \leftarrow \mathcal{A}_{21}$  induction rules (for vertices  $\in J$ ).

24
60 60
108 144 108
168 252 252 168
240 384 432 384 240
312 528 636 636 528 312
384 672 852 912 852 672 384
444 804 1056 1188 1188 1056 804 444
492 912 1236 1440 1512 1440 1236 912 492
528 996 1380 1656 1800 1800 1656 1380 996 528
552 1056 1488 1824 2040 2112 2040 1824 1488 1056 552
552 1080 1548 1932 2208 2352 2352 2208 1932 1548 1080 552
528 1056 1548 1968 2292 2496 2568 2496 2292 1968 1548 1056 528
492 996 1488 1932 2292 2544 2676 2676 2544 2292 1932 1488 996 492
444 912 1380 1824 2208 2496 2676 2736 2676 2496 2208 1824 1380 912 444
384 804 1236 1656 2040 2352 2568 2676 2676 2568 2352 2040 1656 1236 804 384
312 672 1056 1440 1800 2112 2352 2496 2544 2496 2352 2112 1800 1440 1056 672 312
240 528 852 1188 1512 1800 2040 2208 2292 2292 2208 2040 1800 1512 1188 852 528 240
168 384 636 912 1188 1440 1656 1824 1932 1968 1932 1824 1656 1440 1188 912 636 384 168
108 252 432 636 852 1056 1236 1380 1488 1548 1548 1488 1380 1236 1056 852 636 432 252 108
60 144 252 384 528 672 804 912 996 1056 1080 1056 996 912 804 672 528 384 252 144 60
24 60 108 168 240 312 384 444 492 528 552 552 528 492 444 384 312 240 168 108 60 24

Figure 8: Dimension of the blocks labelled by vertices of the  $\mathcal{A}_{21}$  graph for  $\mathcal{E}_{21}$ .

Figure 9: Dimensions  $d_x$  for blocks labelled by  $Oc(\mathcal{E}_{21})$ .

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