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Nadir Matringe

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# Distinguished dihedral representations of $GL(2)$ over a p-adic field

November 3, 2006

## 1 Introduction

Let  $F$  be a finite extension of a p-adic field,  $K$  a quadratic extension of  $F$ . The principal series representations of  $GL_2(K)$  distinguished for  $GL_2(F)$  are well known, it's also known that the Steinberg representation is distinguished ( cf.[A-T] for a summary of these results due to Y.Flicker, J.Hakim and D.Prasad,).

Moreover there is a complete characterisation of distinguishedness in terms of the epsilon factor (due to J.Hakim) and in terms of base change from representations of the unitary group  $U(2, K/F)$  (due to Y.Flicker, cf.[A-T] for a local proof).

Using those, we give here a description of dihedral representations on the parameter's side (those which the Langlands correspondence associate with 2 dimension induced representations of the Weil group of  $K$ , cf.(2) p.122 in [G-L]) distinguished with respect to  $GL_2(F)$ .

Every distinguished representation of  $GL_2(K)$  is parametrised by a regular multiplicative character  $\omega$  of a quadratic extension  $L$  of  $K$ .

We show ( theorem 5.1) that such a representation is distinguished for  $GL_2(F)$  if and only if one can choose  $L$  to be biquadratic over  $F$  and  $\omega$  trivial on the invertible elements of the two other quadratic extensions of  $F$  in  $L$ .

The results we prove here are theorems 3.3, 5.1 and 6.1. The method is to isolate first representations distinguished for  $GL_2(F)_+$  using theorem 3.1, then to determinate those who are  $GL_2(F)$ -distinguished using theorems 4.1 and proposition 4.1.

Thus, in the case of odd residual characteristic, we obtain every supercuspidal distinguished representations.

We also observe ( see proposition 5.3) that if we consider the principal series as parametrised by a multiplicative character of a two dimensional semi-simple commutative algebra over  $K$ , the statement for distinguishedness is the same as for the supercuspidal dihedral representations.

We then give a generalisation of theorem 5.1 to dihedral representations ( non necessarily supercuspidal) in theorem 6.1.

## 2 Preliminaries

### 2.1 Generalities

We consider  $F$  a finite extension of  $\mathbb{Q}_p$ , and  $K$  a quadratic extension of  $F$  in an algebraic closure  $\bar{F}$  of  $\mathbb{Q}_p$ .

If  $L$  is a quadratic extension of  $K$  in  $\bar{F}$ , then to every character  $\omega$  of  $L^*$ , we associate a representation of  $GL_2(K)$  via the Weil representation ( cf.[J-L] p.144).

Such a representation is called dihedral.

We note  $\theta$  the conjugation of  $F/K$ .

For  $A$  is a ring, we note  $A^*$  the group of its invertible elements.

For  $E_2$  a finite extension of a local field  $E_1$ , we note respectively  $Tr_{E_2/E_1}$  and  $N_{E_2/E_1}$  the trace and norm of  $E_2$  over  $E_1$ .

We also note  $Gal(E_2/E_1)$  the Galois group of  $E_2$  over  $E_1$  when  $E_2/E_1$  is Galois, otherwise we note  $Aut_{E_1}(E_2/E_1)$  the group of automorphisms of the algebra  $E_2$  over  $E_1$ .

Moreover if  $E_2$  is quadratic over  $E_1$ , we note  $\eta_{E_2/E_1}$  the nontrivial character of  $E_1^*$  with kernel  $N_{E_2/E_1}(E_1^*)$ .

For  $n$  a positive integer, we note  $GL_n(K)_+$ , the subgroup with index two of  $GL_n(K)$ , of matrices whose determinant is a norm of  $K$  over  $F$ .

For  $\Pi$  a representation of a group  $G$ , we note  $\pi$  its class, and  $\Pi^\vee$  its smooth contragredient when  $\Pi$  is a smooth representation of a totally disconnected locally compact group.

If  $\phi$  is an automorphism of  $G$ , we note  $\Pi^\phi$  the representation of  $G$  given by  $\Pi \circ \phi$ .

If  $H$  is a subgroup of  $G$ , and  $\mu$  is a character of  $G$ , we say that a representation  $\Pi$  of  $G$  is  $\mu$ -distinguished for ( with respect to)  $H$  if there exists on the space of  $\Pi$  a linear fonctionnal  $L$  verifying for  $h$  in  $H$ ,  $L \circ \Pi(h) = \mu(h)L$ .

If  $\mu$  is trivial, we say that  $\Pi$  is distinguished for  $H$ .

### 2.2 Quadratic extensions of $K$

For  $L$  a quadratic extension of  $K$ , three cases arise:

1.  $L/F$  is biquadratic ( hence Galois), it contains  $K$  and two other quadratic extensions  $F$ ,  $K'$  and  $K''$ .

Its Galois group is isomorphic with  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , its non trivial elements are conjugations of  $L$  over  $K$ ,  $K'$  and  $K''$ .

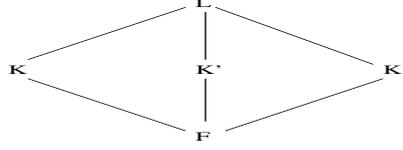


Figure 1:

The conjugation  $L$  over  $K$  extend those of  $K'$  and  $K''$  over  $F$ .

2.  $L/F$  is cyclic with Galois group isomorphic with  $\mathbb{Z}/4\mathbb{Z}$ .
3.  $L/F$  non Galois. Then its Galois Closure  $M$  is quadratic over  $L$  and the Galois group of  $M$  over  $F$  is dihedral with order 8.

To see this, we consider a morphism  $\tilde{\theta}$  from  $L$  to  $\bar{F}$  which extends  $\theta$ . Then if  $L' = \tilde{\theta}(L)$ ,  $L$  and  $L'$  are distinct, quadratic over  $K$  and generate  $M$  biquadratic over  $K$ .  $M$  is the Galois closure of  $L$  because any morphism from  $L$  into  $\bar{F}$ , either extends  $\theta$ , or the identity map of  $K$ , so that its image is either  $L$  or  $L'$ , so it is always included in  $M$ .

Finally the Galois group  $M$  over  $F$  cannot be abelian ( for  $L$  is not Galois), it is of order 8, and it's not the quaternion group which only has one element of order 2, whereas here the conjugations of  $M$  over  $L$  and  $L'$  are of order 2. Hence it is the dihedral group of order 8.

We deduce from this the following lattice:

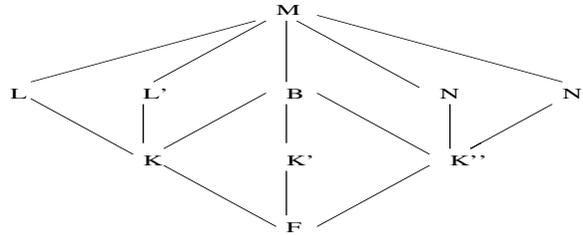


Figure 2:

Here  $M/K'$  is cyclic of degree 4,  $M/K$  and  $B/F$  are biquadratics.

In the case  $p$  odd,  $F$  has exactly three quadratic extensions which generate its unique biquadratic extension. If there exists  $L$  non Galois over  $F$ , then it implies that the cardinal  $q$  of the residual field  $F$  verifies  $q \equiv 3[4]$ , and  $M$  is generated over  $L$  by a primitive fourth root of unity in  $\bar{F}$ .

### 2.3 Quadratic characters

We wish to calculate how  $\eta_{L/K}$  restricts to  $F^*$  in the following two cases.

1. If  $L$  is biquadratic over  $F$ , then  $\eta_{L/K}$  has a trivial restriction to  $F^*$ .

Indeed, we have  $N_{L/K}(K'^*) = N_{K'/F}(K'^*)$  and  $N_{L/K}(K''^*) = N_{K''/F}(K''^*)$  because the conjugation of  $L$  over  $K$  extend those of  $K'$  and  $K''$  over  $F$ .

Both these groups are distinct from local class field theory and of index 2 in  $F^*$ , so that they generate this latter, but both are contained in  $N_{L/K}(L^*)$  which therefore contains  $F^*$ .

In other words  $\eta_{L/K}$  restricts trivially to  $F^*$ .

2. If  $L$  is cyclic over  $F$ , then  $\eta_{L/K}|_{F^*}$  is non trivial.

If it wasn't the case,  $F^*$  would be contained in  $N_{L/K}(L^*)$ , and composing with  $N_{K/F}$  on both sides,  $F^{*2}$  would be a subgroup of  $N_{L/F}(L^*)$ . But  $F^{*2}$  and  $N_{L/F}(L^*)$  have both index 4 in  $F^*$  and give different quotients ( $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  for the first and  $\mathbb{Z}/4\mathbb{Z}$  for the second), so that one cannot be contained in the other.

## 2.4 Weil's representation

Let  $L$  be a quadratic extension of  $K$ , then for any character  $\omega$  of  $L^*$ , we associate an irreducible representation  $\Pi(\omega)$  of  $GL_2(K)$  (cf. [J-L]), with central character  $\omega|_{K^*}\eta_{L/K}$ .

If  $\omega$  is regular for  $N_{L/K}$ , then  $\Pi(\omega)$  is supercuspidal, otherwise there exists a character  $\mu$  of  $K^*$  so that  $\omega = \mu \circ N_{L/K}$ , and then  $\pi(\omega)$  is the principal series  $\pi(\mu, \mu\eta_{L/K})$ .

The conjugation of  $K$  over  $F$  naturally extends to an involutive automorphism of  $GL_2(K)$  which we also note  $\theta$ .

Here we want to determinate  $\Pi(\omega)^\theta$ .

Suppose there exists  $\tilde{\theta}$  an element of  $Aut_F(L/F)$  which extends  $\theta$ , we then have:

**Proposition 2.1** *If  $\theta$  extends to an element  $\tilde{\theta}$  of  $Aut_F(L/F)$ , then  $\Pi(\omega)^\theta$  is isomorphic with  $\Pi(\omega^{\tilde{\theta}})$ .*

**Proof:**

Following [J-L],  $\Pi(\omega)$  is the induced of  $r(\omega, \psi_F)$  from  $Gl_2(K)_+$  the group  $GL_2(K)$ .

The space of this representation  $\mathbf{S}(L, \omega)$  is constituted by the continuous functions  $f$  with compact support from  $L^*$  to complex numbers verifying, for  $x$  in  $L$  and  $y$  in  $Ker(N_{L/K})$ ,  $f(xy) = \omega^{-1}(y)f(x)$ .

Then the mapping which associates to  $f$  in  $\mathbf{S}(L, \omega)$  the function  $f^\theta = f \circ \tilde{\theta}$  is an equivariant morphism between  $r(\omega, \psi_F)^\theta$  and  $r(\omega^{\tilde{\theta}}, \psi_F)$ .

We then see  $\pi(\omega)^\theta = [Ind_{GL_2(K)_+}^{GL_2(K)}(r(\omega, \psi_F))]^\theta \approx [Ind_{GL_2(K)_+}^{GL_2(K)}(r(\omega, \psi_F)^\theta)] \approx Ind_{GL_2(K)_+}^{GL_2(K)}(r(\omega^{\tilde{\theta}}, \psi_F)) = \pi(\omega^{\tilde{\theta}})$  where  $Ind_{GL_2(K)_+}^{GL_2(K)}$  designs the induced representation from  $GL_2(K)_+$  to  $GL_2(K)$ .

**Remark :**

It is not always true that  $\theta$  extends to an element  $\tilde{\theta}$  of  $Aut_F(L/F)$ .

For instance, take  $F$  local with residual characteristic  $q \equiv 3[4]$ , and let  $\pi_F$  be a prime element (generating the maximal ideal of the integers ring).

We choose  $K = F(\pi_F^{1/2})$  and  $L = F(\pi_F^{1/4})$ .

Let  $\tilde{\theta}$  be a  $F$  linear morphism extending  $\theta$  to  $L$ , with values in  $\bar{F}$ . Then  $\tilde{\theta}(\pi_F^{1/4}) = i\pi_F^{1/4}$ , where  $i$  is a primitive fourth root of unity.

Indeed  $i^2 = -1$ , because  $\tilde{\theta}(\pi_F^{1/2}) = \theta(\pi_F^{1/2}) = -\pi_F^{1/2}$ .

Moreover,  $i$  cannot be in  $L$ : indeed, this element is a root of unity with order prime to  $q$ , thus it would imply that the residual field of  $L$ , which is the one of  $F$  as  $L/F$  is totally ramified, contains a primitive fourth root of unity.

This cannot happen because 4 does not divide  $q - 1$ .

We conclude that any  $F$  linear morphism extending  $\theta$  to  $L$ , sends  $L$  onto  $F(i\pi_F^{1/4})$  which is distinct from  $L$ , and hence cannot be in  $Aut_F(L/F)$ .

### 3 Representations distinguished by a character

#### 3.1 Definitions and preliminary results

The following theorem due to Y.Flicker [A-T] ( th. 1.3) will be of constant use.

**Theorem 3.1** *Let  $\Pi$  be an irreducible admissible representation of  $GL_2(K)$ , such that  $c_\pi$  is trivial on  $F^*$ . Then  $\pi^\theta = \pi^\vee$  if and only if  $\pi$  is distinguished or  $\eta_{K/F}$ -distinguished for  $GL_2(F)$ .*

Let  $GL_2(F)_+$  be the subgroup of index two in  $GL_2(F)$ , it is clear that if a representation of  $GL_2(K)$  is distinguished or  $\eta_{K/F}$ -distinguished for  $GL_2(F)$ , it is distinguished for  $GL_2(F)_+$ , the reverse is true.

**Proposition 3.1** *(cf.[P], p.71)*

*A representation of  $GL_2(K)$  is distinguished or  $\eta_{K/F}$ -distinguished for  $GL_2(F)$  if and only if it is distinguished for  $GL_2(F)_+$ .*

**Proof:**

We show the non trivial implication.

Let  $s$  be an element of  $GL_2(F)$  whose determinant is not a norm and let  $\Pi$  be a  $GL_2(F)_+$ -distinguished representation.

Let  $L_+$  be the  $GL_2(F)_+$ -invariant linear form on the space of  $\Pi$ , two cases arise:

1. If  $\Pi^\vee(s)L_+ = -L_+$ , then for  $h$  in  $GL_2(F) \setminus GL_2(F)_+$ , we have  $\Pi^\vee(h)L_+ = \Pi^\vee(h)\Pi^\vee(s^2)L_+ = \Pi^\vee(hs)\Pi^\vee(s)L_+ = -L_+$  because  $hs$  is in  $GL_2(F)_+$  ( here we also note  $\Pi^\vee$  the non smooth contragredient).  $\Pi$  is therefore  $\eta_{K/F}$ -distinguished for  $GL_2(F)$ .
2. Otherwise  $\Pi^\vee(s)L_+ \neq -L_+$ , and  $L_+ + \Pi^\vee(s)L_+$  is fixed under the action of  $GL_2(F)$ .

Theorem 3.1 takes the following form:

**Theorem 3.2** *Let  $\Pi$  be an irreducible admissible representation of  $GL_2(K)$ . Then  $\Pi$  is  $GL_2(F)_+$ -distinguished if and only if  $\pi^\theta = \pi^\vee$  and  $c_\Pi$  restricts trivially to  $F^*$ .*

#### 3.2 Description of the $GL_2(F)_+$ -distinguished representations

**Theorem 3.3** *A supercuspidal dihedral representation  $\Pi$  of  $GL_2(K)$  is  $GL_2(F)_+$ -distinguished if and only if there exists a quadratic extension  $L$  of  $K$  bi-quadratic on  $F$ , and a multiplicative character  $\omega$  of  $L$  trivial on  $N_{L/K'}(K'^*)$  or on  $N_{L/K''}(K''^*)$ , such that  $\pi = \pi(\omega)$ .*

**Proof:**

Let  $L$  be a quadratic extension of  $K$  and  $\omega$  a regular multiplicative of  $L$  such that  $\pi = \pi(\omega)$ , we note  $\sigma$  the conjugation of  $L$  over  $K$ , three cases show up:

1.  $L/F$  si biquadratic.

we note  $\sigma'$  the conjugation of  $L$  over  $K'$  and  $\sigma''$  the conjugation of  $L$  over  $K''$ ,  $\sigma'$  and  $\sigma''$  both extend  $\theta$ , and thus can play  $\tilde{\theta}$ 's role in proposition 1.1.

The condition  $\pi^\vee = \pi^\theta$  which one can also read  $\pi(\omega^{-1}) = \pi(\omega^{\tilde{\theta}})$ , is then equivalent to  $\omega^{\sigma'} = \omega^{-1}$  or  $\omega^{\sigma''} = \omega^{-1}$ .

This is equivalent to  $\omega$  trivial on  $N_{L/K'}(K'^*)$  and on  $N_{L/K''}(K''^*)$ .

As  $\eta_{L/K}$ ,  $\eta_{L/K'}$ , and  $\eta_{L/K''}$  are trivial  $F^*$ , we have  $c_{\pi|F^*} = \omega|_{F^*} \eta_{L/K}|_{F^*} = 1$  for such a representation .

2.  $L/F$  is cyclic, the regularity of  $\omega$  makes the condition  $\pi(\omega^{-1}) = \pi(\omega^{\tilde{\theta}})$  impossible.

Indeed one would have  $\omega^{\tilde{\theta}} = \omega^{-1}$ , which would implie  $\omega^\sigma = \omega$  for  $\sigma^2 = \theta$ , and so  $\omega$  would be trivial on the kernel of  $N_{L/K}$  from Hilbert's theorem 90.

$\Pi$  can therefore not be  $GL_2(F)_+$ -distinguised.

3.  $L/K$  is not Galois ( which implies  $q \equiv 3[4]$  in the case p odd), we note again  $\theta$  the conjugation of  $B$  over  $K'$  which extends the one of  $K$  over  $F$ .

Let  $\pi_{B/K}$  be the representation of  $GL_2(B)$  which is the base change lifting of  $\pi$  to  $B$ . As  $\pi_{B/K} = \pi(\omega \circ N_{M/L})$ , if  $\omega \circ N_{M/L} = \mu \circ N_{M/B}$  for a character  $\mu$  of  $B^*$ , then  $\pi(\omega) = \pi(\mu)$  ( cf.[G-L], (3) p.123) and we are brought back to case 1.

Otherwise  $\omega \circ N_{M/L}$  is regular for  $N_{M/B}$ .

If  $\pi$  was  $GL_2(F)_+$ -distinguised, taht is  $\pi^\theta = \pi^\vee$  and  $c_{\pi|F^*}$ , we would have  $\pi_{B/K}^\theta = \pi_{B/K}^\vee$  and  $c_{\pi_{B/K}} = c_\pi \circ N_{B/K}$  from theorem 1 of [G-L].

As  $N_{B/K}(K'^*) = N_{K'/F}(K'^*)$  for the conjugation of  $B$  over  $K$  extends that of  $K'$  over  $F$ , one would deduce that  $c_{\pi_{B/K}}$  would be trivial on  $K'^*$  and theorem 2.2 would implie that  $\pi_{B/K}$  would be  $GL_2(K')_+$ -distinguised.

That would contradict case 2 because  $M/K'$  is cyclic.

## 4 Distinguished representations

We described in the previous section the supercuspidal dihedral representations of  $GL_2(K)$  which are  $GL_2(F)_+$  distinguished.

We want to characterize those who are  $GL_2(F)$ -distinguished among them.

### 4.1 Definitions and useful results

We refer to [J-L] for definitions and basic properties of  $\epsilon$  factors attached to an irreducible admissible representation of  $GL_2(K)$ , and to [T] for those of  $\epsilon$  factors attached to a multiplicative character of a local field.

The  $\epsilon$  used here for representations of  $GL_2(K)$  is the one described in [J-L] evaluated at  $s = 1/2$  and the  $\epsilon$  attached to a multiplicative character of a local field is Langlands'  $\epsilon_L$  described in [T].

We will use the three following results.

The first, due to J.Hakim can be found in [H], page 8.

Here we replaced  $\gamma$  with  $\epsilon$  because both are equal for supercuspidal representations:

**Theorem 4.1** *Let  $\Pi$  be a supercuspidal irreducible representation of  $GL_2(K)$ , and  $\psi$  a nontrivial character of  $K$  trivial on  $F$ . Then  $\Pi$  is distinguished if and only if  $\epsilon(\Pi \otimes \chi, \psi) = 1$  for every character  $\chi$  of  $K^*$  trivial on  $F^*$ .*

The second, due to Fröhlich and Queyrut, is in [F-Q], page 130 :

**Theorem 4.2** *Let  $L_2$  be a quadratic extension of  $L_1$  which is a quadratic extension of  $\mathbb{Q}_p$ , then if  $\psi_{L_2}$  is the standard character of  $L_2$  and if  $\Delta$  is an element of  $L_2^*$  with  $Tr_{L_2/L_1}(\Delta) = 0$ , we then have  $\epsilon(\chi, \psi_{L_2}) = \chi(\Delta)$  for every character  $\chi$  of  $L_2^*$  trivial on  $L_1^*$ .*

The third is a corollary of proposition 3.1 of [A-T]:

**Proposition 4.1** *There exists no supercuspidal representation of  $GL_2(K)$  which is distinguished and  $\eta_{K/F}$ -distinguished at the same time.*

## 5 Description of distinguished representations

**Theorem 5.1** *A dihedral supercuspidal representation  $\Pi$  of  $GL_2(K)$  is  $GL_2(F)$ -distinguished if and only if there exists a quadratic extension  $L$  of  $K$  bi-quadratic over  $F$ , and a regular multiplicative character  $\omega$  of  $L$  trivial on  $K'^*$  or on  $K''^*$ , such that  $\pi = \pi(\omega)$ .*

**Proof:**

From the second section, we can suppose that  $\pi = \pi(\omega)$ , for  $\omega$  a regular multiplicative character of a quadratic extension  $L$  of  $K$  biquadratic over  $F$ , with  $\omega$  trivial on  $N_{L/K'}(K'^*)$  or on  $N_{L/K''}(K''^*)$ .

Let  $\psi_K$  be the standard character of  $K$ ,  $\psi_L$  the one of  $L$ , and  $a$  a non null element of  $K$  such that  $Tr_{K/F}(a) = 0$ , which implies  $Tr_{L/K'}(a) = Tr_{L/K''}(a) = 0$ .

we note  $(\psi_K)_a$  the character trivial on  $F$  given by  $(\psi_K)_a(x) = (\psi_K)(ax)$ .

To see if  $\pi(\omega)$  is distinguished, we use Hakim's criterion ( th.3.1).

So let  $\chi$  be a character of  $K^*$  trivial on  $F^*$ , we have  $\pi(\omega) \otimes \chi = \pi(\omega \times \chi \circ N_{L/K})$  and we note  $\mu = \omega \times \chi \circ N_{L/K}$ .

**i)** if  $\omega|_{K'^*} = 1$ : on a  $\epsilon(\pi(\omega) \otimes \chi, (\psi_K)_a) = \epsilon(\pi(\mu), (\psi_K)_a) = \epsilon(\pi(\mu), \psi_K) \mu(a) \eta_{L/K}(a)$ . Now  $\epsilon(\pi(\mu), \psi_K) = \lambda(L/K, \psi_K) \epsilon(\mu, \psi_L)$  ( cf.[J-L] p.150), where the Langlands-Deligne factor  $\lambda(L/K, \psi_K)$  equals  $\epsilon(\eta_{L/K}, \psi_K)$  divided by its module.

As  $\eta_{L/K}|_{F^*} = 1$  et  $\mu|_{K'^*} = 1$ , from theorem 4.2, we have that  $\epsilon(\mu, \psi_L) = \mu(a)$  and  $\epsilon(\eta_{L/K}, \psi_K) = \eta_{L/K}(a)$ .

We deduce that  $\epsilon(\pi(\omega) \otimes \chi, (\psi_K)_a) = \mu(a)^2 \eta_{L/K}(a)^2 = 1$  for  $a^2$  is in  $F$ .  $\pi(\omega)$  is therefore distinguished.

**ii)** If  $\omega|_{K'^*} = \eta_{L/K'}$  : Let  $\chi'$  be a character of  $K^*$  which extends  $\eta_{K/F}$ , then  $\chi' \circ N_{L/K}$  equals  $\eta_{K/F} \circ N_{K'/F}$  on  $K'$  because the conjugation of  $L$  over  $K$  extends the one of  $K'$  over  $F$ .

But  $\eta_{K/F} \circ N_{K'/F}$  is trivial on the image of  $N_{L/K'}$  from the identity  $N_{K'/F} \circ N_{L/K'} = N_{K/F} \circ N_{L/K}$ , but not trivial for  $N_{K'/F}$  is not the kernel  $N_{K/F}(K^*)$  of  $\eta_{K/F}$  from local class field theory.

Thus  $\omega \times \chi' \circ N_{L/K}$  is trivial on  $K'$ , and we deduce that  $\pi(\omega) \otimes \chi' = \pi(\omega \times \chi' \circ N_{L/K})$  is distinguished from i).

This implies that  $\pi(\omega)$  is  $\eta_{K/F}$ -distinguished and thus not distinguished from proposition 3.1.

The cases  $\omega|_{K''^*} = 1$  and  $\omega|_{K''^*} = \eta_{L/K''}$  are handled as well.

## 6 The principal series

Representations of the principal series of  $GL_2(K)$  distinguished for  $GL_2(F)$  are well known, and described for example in proposition 4.2 of [A-T].

The result is the following:

**Proposition 6.1** *Let  $\lambda$  and  $\mu$  be two characters of  $K^*$ , whose quotient is not the module of  $K$  or its inverse. The principal series representation  $\Pi(\lambda, \mu)$  of  $GL_2(K)$  is  $GL_2(F)$ -distinguished either when  $\lambda = \mu^{-\theta}$  or when  $\lambda$  and  $\mu$  have a trivial restriction to  $F^*$ .*

Now one can construct the principal series  $\Pi(\lambda, \mu)$  via the Weil representation ( cf.[B] p.523 à 557), in this case  $(\lambda, \mu)$  identifies with a character of  $K^* \times K^*$ .

This way of parametrising irreducible representations of  $GL_2(K)$  with multiplicitive characters of two-dimensional semi-simple commutative algebras over  $K$ , includes the principal series (for the algebra  $K \times K$ ) and the dihedral representations (for quadratic extensions of  $K$ ).

Let  $L$  be a quadratic extension of  $K$  biquadratic over  $F$ , as we are here interested with  $GL_2(F)$ -distinguishedness, we consider the following  $F$ -algebras.

### 1. the algebra $K \times K$

One note  $Aut_F(K \times K)$  its automorphisms group. The elements of this group are  $(x, y) \mapsto (x, y)$ ,  $(x, y) \mapsto (y, x)$ ,  $(x, y) \mapsto (x^\theta, y^\theta)$ ,  $(x, y) \mapsto (y^\theta, x^\theta)$ , and  $Aut_F(K \times K)$  is isomorphic with  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

The three sub-algebras fixed by non trivial elements of  $Aut_F(K \times K)$  are  $K$  via the natural diagonal inclusion, the twisted form  $\tilde{K}$  of  $K$  given by  $x \mapsto (x, x^\theta)$ , and  $F \times F$ .

### 2. l'algèbre $L$

The group  $Gal(B_F/K)$  of its automorphisms is isomorphic with  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

The three sub-algebras fixed by non trivial elements of  $Gal(B_F/K)$  are  $K$ ,  $K'$  et  $K''$ .

We then observe that proposition 5.2 for the principal series has the same statement that the one for theorem 4.1:

**Proposition 6.2** *A principal series representation  $\Pi(\lambda, \mu)$  of  $GL_2(K)$  is  $GL_2(F)$ -distinguished if and only if the multiplicative character  $(\lambda, \mu)$  is trivial one the invertible elements of one of the two intermediate sub-algebras of  $K \times K$  distinct from  $K$ .*

We now study dihedral non supercuspidal representations.

Let  $\Pi$  be such a representation, there exists a quadratic extension  $L$  over  $K$  and a non regular multiplicative character  $\omega$  of  $L$  such that  $\pi = \pi(\omega)$ .

If  $\mu$  is a character of  $K^*$  such that  $\omega = \mu \circ N_{L/K}$ , then  $\pi = \pi(\mu, \mu\eta_{L/K})$ .

Three cases arise:

1. For  $L$  biquadratic over  $F$ , we show that  $\omega$  restricts trivially to  $K'^*$  or  $K''^*$  if and only if  $(\mu, \mu\eta_{L/K})$  restricts trivially to  $\tilde{K}^*$  or to  $F^* \times F^*$ .

We have the following equivalences:

- $\omega(K'^*) = 1 \Leftrightarrow \mu(N_{L/K}(K'^*)) = 1 \Leftrightarrow \mu(N_{K'/F}(K'^*)) = 1$  because the conjugation of  $L/K$  extends the one of  $K'/F$ , and so  $\omega(K'^*) = 1 \Leftrightarrow \mu|_{F^*} = 1$  or  $\eta_{K'/F}$ .
- $\omega(K''^*) = 1 \Leftrightarrow \mu(N_{L/K}(K''^*)) = 1 \Leftrightarrow \mu(N_{K''/F}(K''^*)) = 1$  because the conjugation of  $L/K$  extends the one of  $K''/F$ , and so  $\omega(K''^*) = 1 \Leftrightarrow \mu|_{F^*} = 1$  or  $\eta_{K''/F}$ .

- $(\mu, \mu\eta_{L/K})$  trivial on  $\tilde{K}^* \iff \mu^\theta \mu\eta_{L/K} = 1 \iff \mu \circ N_{K/F} = \eta_{L/K}$ . We deduce that  $\mu \circ N_{L/F} = 1$ , which implies that  $\mu|_{F^*} = 1$ ,  $\eta_{K/F}$ ,  $\eta_{K'/F}$  or  $\eta_{K''/F}$ , but the first two possibilities are excluded by the identity  $\mu \circ N_{K/F} = \eta_{L/K}$ .

Conversely if  $\mu|_{F^*} = \eta_{K'/F}$  or  $\eta_{K''/F}$ , then  $\mu \circ N_{K/F}$  is a character of order two of  $K^*$  which cannot be trivial from local class field theory. As the equalities  $N_{L/F} = N_{L/K'} \circ N_{K'/K} = N_{L/K''} \circ N_{K''/K}$  imply that  $\mu \circ N_{K/F}$  is trivial on  $N_{L/K}(L^*)$ , it is therefore  $\eta_{L/K}$ .

Eventually  $(\mu, \mu\eta_{L/K})$  trivial on  $\tilde{K}^* \iff \mu|_{F^*} = \eta_{K'/F}$  or  $\eta_{K''/F}$ .

- Also  $(\mu, \mu\eta_{L/K})$  trivial on  $F^* \times F^* \iff \mu|_{F^*} = 1$  because we have already seen that  $\eta_{L/K}$  is trivial on  $F^*$ .
- Finally these equivalences show that  $\omega(K'^*) = 1$  or  $\omega(K''^*) = 1 \iff (\mu, \mu\eta_{L/K})$  trivial on  $\tilde{K}^*$  or on  $F^* \times F^*$ .

2. If  $L$  is cyclic over  $F$ . One shows that  $\pi(\omega) = \pi(\mu, \mu\eta_{L/K})$  is distinguished if and only if  $\mu|_{F^*}$  generates the cyclic group of the characters of  $F^*/N_{L/F}(L^*)$ .

- It is not possible for  $(\mu, \mu\eta_{L/K})$  to be trivial on  $F^* \times F^*$  because we saw in the preliminaries that  $\eta_{L/K}$  is not trivial on  $F^*$ .

- $(\mu, \mu\eta_{L/K})$  trivial on  $\tilde{K}^* \iff \mu \circ N_{K/F} = \eta_{L/K}$ .

We deduce as before that  $\mu$  is a character of  $F^*/N_{L/F}(L^*)$ .

As  $F^*/N_{L/F}(L^*)$  is cyclic of order four, the same is true for its characters group.

As  $\mu \circ N_{K/F} = \eta_{L/K}$ , we deduce that  $\mu|_{F^*}$  is non trivial, moreover if  $\mu|_{F^*}$  was of order 2, it would be equal to  $\eta_{K/F}$  which is the unique element with order 2 of the characters group of  $F^*/N_{L/F}(L^*)$ , which contradicts  $\mu \circ N_{K/F} = \eta_{L/K}$ . We deduce that  $\mu \circ N_{K/F} = \eta_{L/K} \implies \mu|_{F^*}$  generates the dual group of  $F^*/N_{L/F}(L^*)$ .

- Conversely if  $\mu|_{F^*}$  is of order four in the dual group of  $F^*/N_{L/F}(L^*)$ , we deduce that  $\mu \circ N_{K/F}$  is a character of  $K^*$  trivial on  $N_{L/K}(L^*)$ , but not trivial because it would imply  $\mu|_{F^*} = 1$  or  $\eta_{K/F}$ , namely  $\mu|_{F^*}$  with order less than 2. On conclude that  $\mu \circ N_{K/F} = \eta_{L/K}$ .

3. If  $L$  is not Galois over  $F$ . As  $\omega \circ N_{M/K} = \mu \circ N_{M/F} = \mu' \circ N_{M/B}$ , where  $\mu' = \mu \circ N_{B/F}$ , we conclude as in the case 3. of the proof of theorem 2.3 that  $\pi(\omega) = \pi(\mu')$  and we are brought back to case 1. because  $B$  is biquadratic over  $F$ .

Thus we have the following general theorem:

**Theorem 6.1** *A dihedral representation  $\Pi$  of  $GL_2(K)$  is  $GL_2(F)$ -distinguished if and only if  $\pi = \pi(\omega)$  for some multiplicative character  $\omega$  of a quadratic extension  $L$  over  $K$  verifying i) or ii):*

- i)  $L/F$  is biquadratic, and  $\omega|_{K'^*} = 1$  or  $\omega|_{K''^*} = 1$ ,
- ii)  $L/F$  is cyclic and  $\omega = \mu \circ N_{L/K}$  for  $\mu$  a character of  $K^*$  whose restriction to  $F^*$  generates the dual group of  $F^*/N_{L/F}(L^*)$ .

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