On polynomial Torus Knots

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October 22, 2006

Abstract

We show that no torus knot of type $(2, 2n+1)$ can be obtained from a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ where $(\deg(f), \deg(g)) \leq (3, 2n+2)$. Eventually, we give explicit examples with minimal lexicographic degree.

keywords: Knot theory, polynomial curves, torus knots, parametrized space curve

1 Introduction

The study of non compact knots began with Vassiliev [Va]. He proved that any non-compact knot type can be obtained from by a polynomial embedding $t \mapsto (f(t), g(t), h(t)), t \in \mathbb{R}$. The proof uses Weierstrass approximation theorem on a compact interval, the degrees of the polynomials may be quite large, and the plane projections of the polynomial knots quite complicated.

Independently, Shastri [Sh] gave a detailed proof of this theorem, he also gave simple polynomial parametrizations of the trefoil and of the figure eight knot.

This is what motivated A. Ranjan and Rama Shukla [RS] to find small degree parametrizations of the simplest knots, the torus knots of type $(2, n), n$ odd, denoted by K_n. They proved that these knots can be attained from polynomials of degrees $(3, 2n-2, 2n-1)$. In particular, they obtain a parametrization of the trefoil K_3 analogous to Shastri's one.

As the number of crossings of a plane projection of K_n is at least n ([Re]), it is easy to see, using Bézout's theorem, that this plane curve cannot be parametrized by polynomials of degrees less than $(3, n+1)$.

Naturally, Rama Mishra ([Mi]) asked whether it was possible to parametrize the knot K_n by polynomials of degrees $(3, n+1, m)$ when $n \equiv 1$, or $0[\mod 3]$.

In this paper, we give a negative answer to this question for any $n \neq 3$.

As a conclusion, we give explicit parametrizations of optimal degrees of K_3, K_5 and K_7. We also give parametrizatios of K_9 with a plane projection possessing the minimal number of crossing points. These embeddings are of smaller degree than those already known.

Our method is based on the fact that all plane projections of K_n with the minimal number n of crossings have essentially the same diagram. This is a consequence of the now solved classical Tait's conjectures [Mu, Ka, Pr, MT]. This allows us to transform our problem into a problem of real polynomial algebra.

Unfortunately, our method cannot be generalized if there is no plane projection with n crossings (see the trefoil example, fig. 3)

2 The principal result

If n is odd, the torus knot of type $(2, n)$ noted K_n, is the boundary of a Moebius band twisted n times ([Re, Ka, St]).

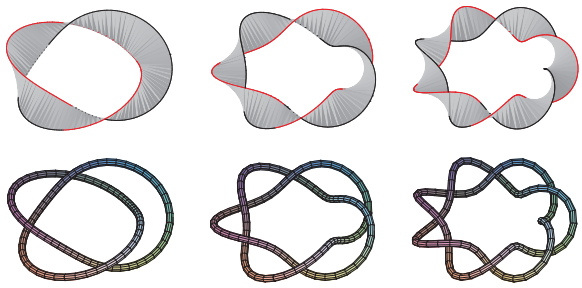


Figure 1: K_n, $n = 3, 5, 7$.

A theorem of Bankwitz (1930) says that the number of crossings of a plane projection of K_n is at least n (see [Re]). In this paper we shall consider polynomial knots, that is polynomial embeddings $\mathbb{R} \longrightarrow \mathbb{R}^3$, $t \mapsto (x(t), y(t), z(t))$. The closure of a polynomial knot in the one point compactification $\mathbf{S}^3$ of the space $\mathbb{R}^3$ is an ordinary knot (see [Va, Sh, RS] and figures at the end).

Lemma 1 *Let $\mathcal{C}$ be a plane polynomial curve with n crossings parametrized by $\mathcal{C}(t) = (x(t), y(t))$. Suppose $\mathcal{C}$ is the projection of a non trivial knot, and* $\deg x(t) \leq \deg y(t)$. *Then we have* $\deg x(t) \geq 3$. *If* $\deg x(t) = 3$, *then* $\deg y(t) \geq n + 1$.

Proof. $x(t)$ must be non monotonic, so $\deg x(t) \geq 2$. Suppose that $x(t)$ is of degree 2. Then $x(t_i) = x(s_i)$ implies that $t_i + s_i$ is constant, and so the parameter values corresponding to the crossing points are ordered as

$$s_1 < \cdots < s_n < t_n < \cdots < t_1.$$

Using n Reidemeister's moves of type I ([Re, Ka]), we see that $\mathcal{C}$ can be reduced to a trivial diagram, a contradiction.

Suppose now that $\deg x(t) = 3$, then by Bézout's theorem ([Fi]), the curve $\mathcal{C}$ has at most $\deg y(t) - 1$ crossing points, and this implies that $\deg y(t) \geq n + 1$. □

The recently proved Tait's conjectures allow us to characterize plane projections of K_n with the minimal number of crossings.

Lemma 2 *Let $\mathcal{C}$ be a plane curve with n crossings parametrized by $\mathcal{C}(t) = (x(t), y(t))$. Then $\mathcal{C}$ is the projection of a knot K_n if and only if there exist real numbers $s_1 < \ldots < s_n < t_1 \ldots < t_n$, such that $\mathcal{C}(s_i) = \mathcal{C}(t_i)$.*

Proof. Let $\mathcal{C}$ be a plane projection of a knot of type K_n with the minimal number n of crossings. Using the Murasugi's theorem B ([Mu]) which says that a minimal projection of a prime alternating knot is alternating, we see that $\mathcal{C}$ is alternating.

Then the Tait's flyping conjecture, proved by Menasco and Thistlethwaite ([MT, Pr]) asserts that $\mathcal{C}$ is related to the standard diagram of K_n by a sequence of flypes. Let us recall that a flype is a transformation most clearly described by the following picture.
The standard diagram $\mathcal{S}$ of K_n has the property that there exist real numbers $s_1 < ... < s_n < t_1 < ... < t_n$ such that $\mathcal{S}(s_i) = \mathcal{S}(t_i)$. Now, let us we perform a flype on a knot diagram having this

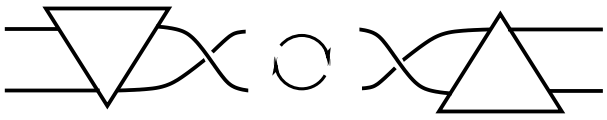


Figure 2: A flype

property. We see that, for any possible orientation of this diagram, the transformed diagram has the same property. □

In this paper, we shall prove the following result

Theorem. *If $n \neq 3$ is odd, the torus knot K_n cannot be represented by polynomials of degrees $(3, n+1, m)$.*

3 Proof of the main result

Our proof makes use of Chebyshev (unitary) polynomials:

3.1 Chebyshev Polynomials

Definition 1 *If $t = 2\cos\theta$, let $T_n(t) = 2\cos(n\theta)$ and $V_n(t) = \dfrac{\sin((n+1)\theta)}{\sin\theta}$.*

Remark 1 *T_n and V_n are both monic and have degree n. We have*

$$V_0(t) = 1, \quad V_1(t) = t, \quad V_{n+1}(t) = tV_n(t) - V_{n-1}(t).$$

We have also

$$T_0 = 2, \quad T_1 = t, \quad T_{n+1}(t) = tT_n(t) - T_{n-1}(t).$$

Furthermore $T_3(t) = t^3 - 3t$ and

$$V_n(t) = t^n - (n-1)t^{n-2} + \frac{1}{2}(n-2)(n-3)t^{n-4} - \cdots . \tag{1}$$

We shall also need the following lemmas which will be proved in the next paragraph.

Lemma A. *Let $s \neq t$ be real numbers such that $T_3(s) = T_3(t)$. For any integer k, we have*

$$\frac{T_k(t) - T_k(s)}{t - s} = \frac{2}{\sqrt{3}} \sin\frac{k\pi}{3}\, V_{k-1}(s+t).$$

Lemma B. *Let $n \geq 3$ be an integer.*
Let $s_1 < s_2 < \cdots < s_n$ and $t_1 < \cdots < t_n$ be real numbers such that $T_3(s_i) = T_3(t_i)$. Let $u_i = t_i + s_i$. We have

$$\sum_{i=1}^{n} u_i^2 < n + 4, \ \sum_{i=1}^{n} u_i^4 < n + 22. \tag{2}$$

3.2 Proof of the theorem

Proof. We shall prove this result by reducing it to a contradiction. Suppose the plane curve $\mathcal{C}$ parametrized by $x = P(t),\ y = Q(t)$ where $\deg P = 3,\ \deg Q = n+1$ is a plane projection of K_n. If the polynomial P was monotonic, $\mathcal{C}$ would have no crossings, which is absurd. We can also suppose that $Q(t)$ is monic. Therefore, we can suppose that

$$P(t) = T_3(t), \quad Q(t) = T_{n+1}(t) + a_n T_n(t) + \cdots + a_1 T_1(t) + a_0.$$

By Bézout's theorem, the curve $\mathcal{C}$ has at most $(3-1)(n+1-1)/2 = n$ double points. As it has at least n crossings, we see that it has exactly n crossings and therefore is a minimal crossing diagram of K_n. According to the lemma **2**, there exist real numbers $s_1 < \cdots < s_n,\ t_1 < \cdots < t_n,\ s_i < t_i$ such that $P(s_i) = P(t_i),\ Q(s_i) = Q(t_i)$. Let $u_i = t_i + s_i,\ 1 \leq i \leq n$. We have

$$\frac{y(t_i) - y(s_i)}{t_i - s_i} = \frac{T_{n+1}(t_i) - T_{n+1}(s_i)}{t_i - s_i} + \sum_{k=0}^{n} a_k \frac{T_k(t_i) - T_k(s_i)}{t_i - s_i}.$$

so by lemma 2, $u_1, \ldots, u_n$ are the roots of the polynomial

$$R(u) = \frac{2}{\sqrt{3}} \sin\frac{(n+1)\pi}{3}\, V_n(u) + \sum_{k=0}^{n-1} a_k \frac{2}{\sqrt{3}} \sin\frac{k\pi}{3}\, V_k(u).$$

We have to consider several cases.

Case: $\mathbf{n} \equiv 2$[mod 3]. $R(u)$ has degree at most $n-1$. This is a contradiction.

Case: $\mathbf{n} \equiv 1$[mod 3]. In this case the polynomial $R(u)$ can be written

$$\begin{aligned} R(u) &= V_n(u) + a_{n-1}V_{n-1}(u) - a_{n-3}V_{n-3}(u) - \cdots + a_0 \\ &= u^n + a_{n-1}u^{n-1} - (n-1)u^{n-2} + \cdots . \end{aligned}$$

Therefore we get

$$\sum_{1 \leq i \leq n} u_i = -a_{n-1}, \quad \sum_{1 \leq i < j \leq j} u_i u_j = -(n-1),$$

and then

$$\sum_{i=1}^{n} u_i^2 = \left(\sum_{i=1}^{n} u_i\right)^2 - 2\sum_{i<j} u_i u_j = a_{n-1}^2 + 2(n-1) \geq 2(n-1).$$

According to lemma **B** we also have $\sum_i u_i^2 \leq n+4$, we get a contradiction for $n > 6$.

Case: $\mathbf{n} \equiv 0$[mod 3]. In this last case we have

$$R(u) = V_n(u) - a_{n-2}V_{n-2}(u) - a_{n-3}V_{n-3}(u) + \cdots$$

Les us denote $S_k = \sum_{i=1}^{n} u_i^k$ the Newton sums of the roots of the polynomial R. Using the classical Newton formulas ([FS]), we obtain

$$S_1 = \sigma_1 = 0,\ S_2 = \sigma_1^2 - 2\sigma_2 = -2\sigma_2,\ S_4 = 2\sigma_2^2 - 4\sigma_4,$$

where σ_i are the coefficients of R. On the other hand, it is not difficult (eq. 1) to see that

$$\sigma_2 = -(a_{n-2} + n - 1),\ \sigma_4 = (n-3)\, a_{n-2} + \frac{(n-2)(n-3)}{2}$$

and then

$$S_4 = 2(a_{n-2} + 2)^2 + 6n - 18 \geq 6n - 18.$$

By the lemma **B**, we deduce that $22 + n > 6n - 18$, i.e. $n = 3$. □

3.3 Proof of lemmas A and B

We shall use the following lemma that says that parameters of crossing points lie on an ellipse.

Lemma 3 (Lissajous ellipse) *Let $s \neq t$ be complex numbers such that*

$$T_3(t) = T_3(s).$$

There exists a complex number α such that

$$s = 2\cos(\alpha + \pi/3),\ \ t = 2\cos(\alpha - \pi/3).$$

Furthermore, α is real if and only if s and t are both real, and then $t > s$ if and only if $\sin\alpha > 0$.

Proof. We have

$$\frac{T_3(t) - T_3(s)}{t - s} = t^2 + s^2 + st - 3. \tag{3}$$

Then, if $T_3(t) = T_3(s)$, $t \neq s$, we get

$$\frac{3}{2}(t+s)^2 + \frac{1}{2}(t-s)^2 = 2(t^2 + s^2 + st) = 6.$$

That means

$$\left[\frac{t+s}{2}\right]^2 + \left[\frac{t-s}{2\sqrt{3}}\right]^2 = 1$$

Then there exists a complex number α such that

$$\cos\,\alpha = \frac{t+s}{2},\ \sin\,\alpha = \frac{t-s}{2\sqrt{3}},$$

that is

$$t = 2\cos(\alpha - \pi/3),\ \ s = 2\cos(\alpha + \pi/3).$$

α is real if and only if $\cos\,\alpha$ and $\sin\,\alpha$ are both real that is to say, iff s and t are real. In this case: $t > s \Leftrightarrow \sin\alpha > 0$. □

Proof of lemma A.
Let $s < t$ be real numbers such that $T_3(s) = T_3(t)$. According to lemma 3 **(ellipse)**, there exists a real number α such that

$$t = 2\cos(\alpha - \pi/3),\ \ s = 2\cos(\alpha + \pi/3).$$

We have

$$T_k(t) - T_k(s) = 2\big(\cos(k(\alpha - \pi/3)) - \cos(k(\alpha + \pi/3))\big) = 4\sin\frac{k\pi}{3}\,\sin k\alpha,$$

and then

$$\frac{T_k(t) - T_k(s)}{t - s} = \frac{T_k(t) - T_k(s)}{T_1(t) - T_1(s)} = \frac{\sin k\alpha}{\sin\alpha} \cdot \frac{\sin\dfrac{k\pi}{3}}{\sin\dfrac{\pi}{3}} = \frac{2}{\sqrt{3}}\sin\frac{k\pi}{3}\,V_{k-1}(u)$$

where $u = 2\cos\alpha$. And eventually we have

$$t + s = 2\big(\cos(\alpha - \pi/3) + \cos(\alpha + \pi/3)\big) = 4\cos\alpha\,\cos\frac{\pi}{3} = 2\cos\alpha = u.$$

□

In order to prove lemma B, we shall use the following lemma which describes the geometrical configuration. Let us denote $s(\alpha) = 2\cos(\alpha + \frac{\pi}{3}),\ t(\alpha) = 2\cos(\alpha - \frac{\pi}{3})$.

Lemma 4 *Let $\alpha, \alpha' \in [0, \pi]$ be such that $s(\alpha) < s(\alpha')$, and $t(\alpha) < t(\alpha')$. Then $\alpha > \alpha'$, and* $\frac{2\pi}{3} > \frac{\alpha + \alpha'}{2} > \frac{\pi}{3}$.

Proof. We have

$$\begin{aligned} t(\alpha') - t(\alpha) &= 4\sin(\frac{\alpha + \alpha'}{2} - \frac{\pi}{3}) \cdot \sin(\frac{\alpha - \alpha'}{2}) > 0, \\ s(\alpha') - s(\alpha) &= 4\sin(\frac{\alpha + \alpha'}{2} + \frac{\pi}{3}) \cdot \sin(\frac{\alpha - \alpha'}{2}) > 0. \end{aligned}$$

Therefore, if $\alpha < \alpha'$ we get $\frac{\alpha + \alpha'}{2} < \frac{\pi}{3}$ and $\frac{\alpha + \alpha'}{2} > \frac{2\pi}{3}$. This is a contradiction, and consequently $\alpha > \alpha'$.

We thus deduce that $\frac{\alpha + \alpha'}{2} > \frac{\pi}{3}$. By the second inequality we get $\frac{\alpha + \alpha'}{2} < \frac{2\pi}{3}$. □

Proof of lemma B.

Let $s_1 < \ldots < s_n$ and $t_1 < \ldots t_n$ be such that $T_3(s_i) = T_3(t_i)$. Using lemma **4** there are $\alpha_1 > \cdots > \alpha_n \in]0, \pi[$ such that $s_i = s(\alpha_i),\ t_i = t(\alpha_i)$ and we have

$$\frac{2\pi}{3} > \frac{\alpha_1 + \alpha_2}{2} > \alpha_2 > \cdots > \alpha_{n-1} > \frac{\alpha_{n-1} + \alpha_n}{2} > \frac{\pi}{3}.$$

We thus have only two possibilities: $\frac{\pi}{2} \geq \alpha_{n-1} > \alpha_n > 0$ or $\pi > \alpha_1 > \alpha_2 \geq \frac{\pi}{2}$.

On the other hand, we get

$$\cos(x)^2 + \cos(y)^2 = 1 - \cos(x + y)^2 + 2\cos(x)\cos(y)\cos(x + y).$$

Case 1. $\frac{\pi}{2} \geq \alpha_{n-1} > \alpha_n > 0$**.**

We get $\cos(\alpha_n) \geq 0$, $\cos(\alpha_{n-1}) \geq 0$ and $\cos(\alpha_{n-1} + \alpha_n) < -\frac{1}{2}$ so eq. 3.3 becomes

$$\cos(\alpha_{n-1})^2 + \cos(\alpha_n)^2 \leq 1 - \cos(\alpha_{n-1} + \alpha_n)^2 \leq \frac{3}{4}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \cos(\alpha_i)^2 &= \cos(\alpha_1)^2 + \sum_{i=2}^{n-2} \cos(\alpha_i)^2 + (\cos(\alpha_{n-1})^2 + \cos(\alpha_n)^2) \\ &\leq 1 + (n-3) \cdot \frac{1}{4} + \frac{3}{4} = \frac{1}{4}(n+4). \end{aligned}$$

Case 2. $\pi > \alpha_1 > \alpha_2 \geq \frac{\pi}{2}$**.**

We get $\cos(\alpha_1) \leq 0$, $\cos(\alpha_2) \leq 0$ and $\cos(\alpha_1 + \alpha_2) < -\frac{1}{2}$ so eq. 3.3 becomes

$$\cos(\alpha_1)^2 + \cos(\alpha_2)^2 \leq 1 - \cos(\alpha_1 + \alpha_2)^2 \leq \frac{3}{4}$$

that is,

$$S_2 = \sum_{i=1}^{n} (2\cos(\alpha_i))^2 \leq n + 4.$$

Analogously, we get $\cos(x)^4 + \cos(y)^4 \leq (\cos(x)^2 + \cos(y)^2)^2$, and we deduce:

$$S_4 = \sum_{i=1}^{n} (2\cos(\alpha_i))^4 \leq n + 22.$$

□

4 Parametrized models of K_3, K_5 and K_7, K_9 ...

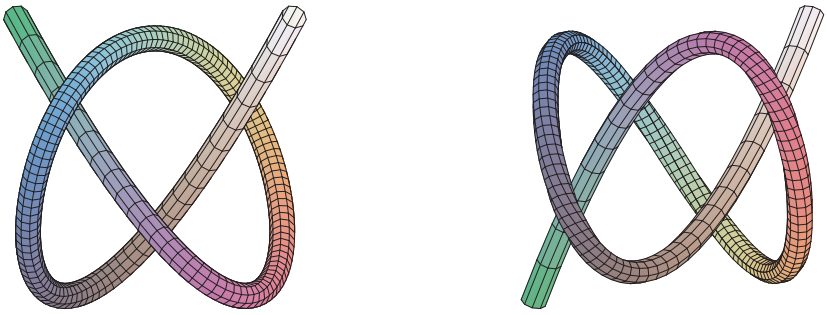
We get parametrizations of K_n: $\mathcal{C} = (X(t), Y(t), Z(t))$, with n crossings obtained for parameter values satisfying the hypothesis of lemma 2. All of our curves are symmetric with respect to the y-axis.

4.1 Parametrization of K_3

We can parametrize K_3 by $x = T_3(t), y = T_4(t), z = T_5(t)$. It is a Lissajous space curve (compare [Sh]).

We can in fact notice that there exists a projection of our trefoil K_3 with four crossings and a complicated ordering of the parameter values corresponding to the crossings.

The plane curve $(T_3(t), T_5(t))$ has 4 crossing points but notice that there do not exist real numbers $s_1 < s_2 < s_3, t_1 < t_2 < t_3$, such that $x(s_i) = x(t_i),\ z(s_i) = z(t_i)$. This example shows that



bottom view of K_3 face view of K_3

Figure 3: Knot K_3

our method cannot be generalized when the projections of K_n have at least $n + 1$ crossing points.

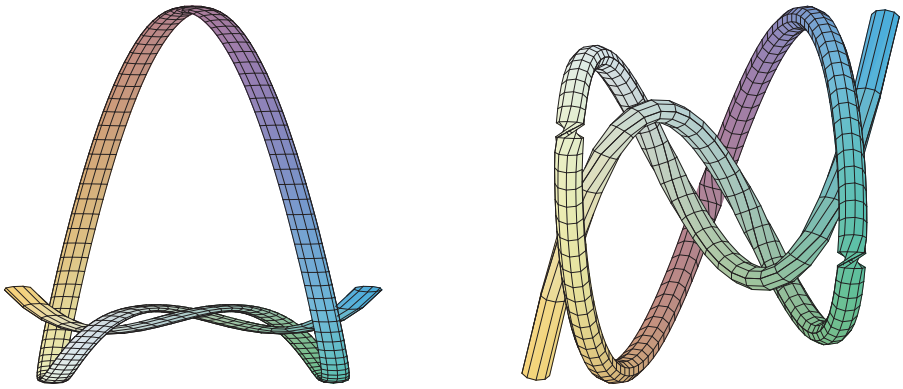
4.2 Parametrization of K_5

Let

$$\begin{aligned} X(t) &= T_3(t) \\ Y(t) &= T_8(t) - 2\,T_6(t) + 2.189\,T_4(t) - 2.170\,T_2(t), \\ Z(t) &= T_7(t) - 0.56\,T_5(t) - 0.01348\,T_1(t). \end{aligned}$$

The curve $(X(t), Y(t))$ has exactly 5 double points.

In conclusion we have found a curve of degree $(3, 7, 8)$. Using our theorem, this curve has minimal degree. R. Mishra showed the existence of such an example ([RS, Mi]).



bottom view face view

Figure 4: Knot K_5

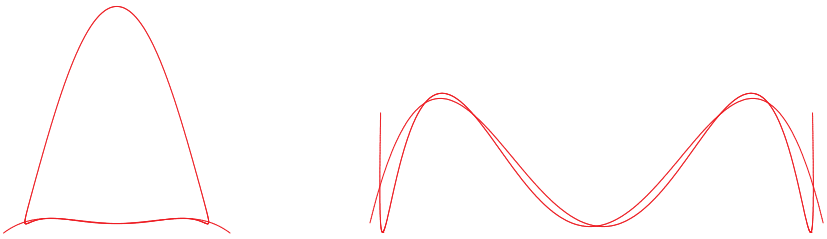
4.3 Parametrization of K_7

We choose

$$\begin{aligned} X &= T_3(t), \\ Y &= T_{10}(t) - 2.360\,T_8(t) + 4.108\,T_6(t) - 6.037\,T_4(t) + 7.397\,T_2(t), \\ Z &= -6.263\,T_{11}(t) - 22.42\,T_7(t) + 23.42\,T_5(t) + 6.263\,T_1(t). \end{aligned}$$

The curve $(X(t), Y(t))$ has exactly 7 double points for values (t_i, s_i) and we have the following

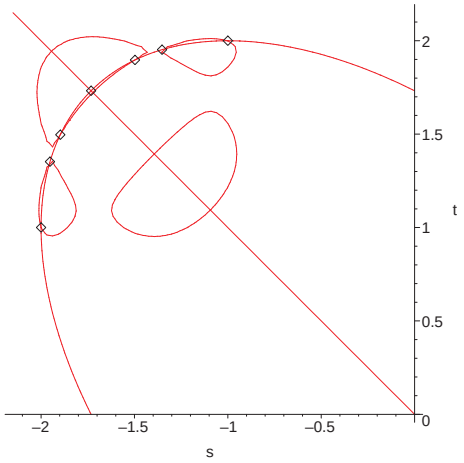
n	1	2	3	4	5	6	7
$\cos(\alpha_i)$	- 1/2	- 3/10	- 2/10	0	2/10	3/10	1/2
s_i	- 2.0	- 1.952	- 1.897	- 1.732	- 1.497	- 1.352	- 1.0
t_i	1.0	1.352	1.497	1.732	1.897	1.952	2.0
$Z(t_i) - Z(s_i)$	- 3.0	3.305	- 3.394	3.464	- 3.394	3.305	- 3.0



Bottom view Zoom on the bottom view

Figure 5: Knot K_7

In conclusion we found a curve of degree $(3, 10, 11)$. Using our theorem, this curve has minimal degree.



The values of the parameters corresponding to the double points are obtained as intersection points between the ellipse

$$t^2 + s^2 + st - 3 = 0$$

and the curve of degree 9:

$$(Y(t) - Y(s))/(t - s) = 0$$

Figure 6: Knot K_7

4.4 Parametrization of K_9

We choose polynomials of degree $3, 13, 14$.

$$\begin{aligned} X &= T_3(t), \\ Y &= T_{14}(t) - 4.516\,T_{12}(t) + 12.16\,T_{10}(t) - 24.46\,T_8(t) + 39.92\,T_6(t) - 55.30\,T_4(t) + 66.60\,T_2(t), \\ Z &= 5.161\,T_5(t) + 1.397\,T_1(t) - 5.161\,T_7(t) - 2.389\,T_{11}(t) + T_{13}(t) \end{aligned}$$

The curve $(X(t), Y(t))$ has exactly 9 double points for values (t_i, s_i) and we have:

n	1	2	3	4	5	6	7	8	9
$\cos(\alpha_i)$	-1/2	-3/10	-2/10	-1/10	0	1/10	2/10	3/10	1/2
s_i	-2.	-1.952	-1.897	-1.823	-1.732	-1.623	-1.497	-1.352	-1.000
t_i	1.000	1.352	1.497	1.623	1.732	1.823	1.897	1.952	2.
$Z(t_i) - Z(s_i)$	-3.000	3.306	-3.394	3.447	-3.464	3.447	-3.394	3.306	-3.000

Conclusion

We have found minimal degree polynomial curves for toric knots K_n, $n = 3, 5, 7$. For degree 9, one can prove that it is minimal under the assumption that the projection $(X(t), Y(t))$ has exactly 9 double points. We have similar constructions for higher degrees.

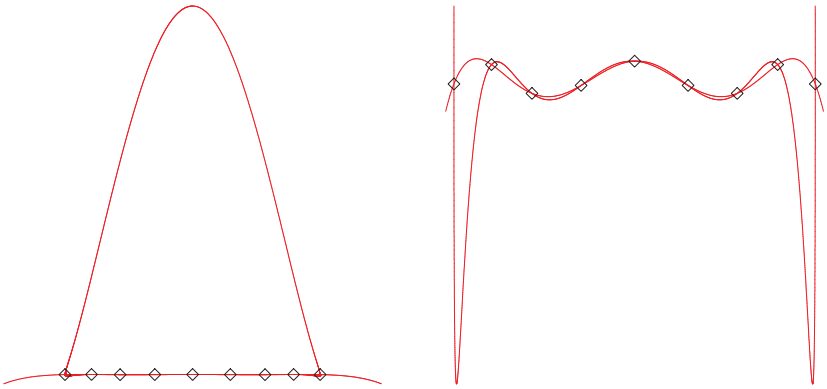
Acknowledgments

We would like to thank Julien Marché for fruitfull discussions on knot theory.

References

[FS] Faddeev, D., Sominskij, I. *Problems in higher algebra*, W. H. Freeman, 1965.

[Fi] G. Fischer, *Plane Algebraic Curves*, A.M.S. Student Mathematical Library Vol 15, 2001.



Bottom view Zoom on the bottom view

Figure 7: Knot K_9

[Ka] L. Kaufmann, *On knots,* Princeton University Press, 1987.

[MT] W.M. Menasco, M.B. Thistlethwaite, *The Tait Flyping conjecture,* Bull. Amer. Math Soc. **25 (2)** (1991), 403-412.

[Mi] R. Mishra, *Minimal Degree Sequence for Torus knots,* Journal of Knot Theory and its Ramifications, Vol. 9, No 6 (2000), 759-769.

[Mu] K. Murasugi, *Jones polynomials and classical conjectures in knot theory,* Topology, 26 (1987), 187-194.

[Pr] J.H. Przytycki, *History of the knot theory from Vandermonde to Jones,* Aportaciones Matemáticas Comunicaciones, 11 (1992), 173-195.

[RS] A. Ranjan and R. Shukla, *On polynomial Representation of Torus Knots,* Journal of knot theory and its ramifications, Vol. 5 (2) (1996) 279-294.

[Re] K. Reidemeister, *Knot Theory*, Springer-Verlag, Berlin, Heidelberg New-York, 1974.

[Sh] A.R. Shastri, *Polynomial Representation of knots,* Tôhoku Math. J. **44** (1992), 11-17.

[St] H. Steinhaus, *Mathematical Snapshots*, 3rd edition, New York: Dover 1999.

[Va] V. A. Vassiliev, *Cohomology of Knot Spaces,* Theory of singularities and its Applications, Advances Soviet Maths Vol 1, (1990)