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On polynomial Torus Knots

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Abstract

We show that no torus knot of type $(2, n)$, $n > 3$ odd, can be obtained from a polynomial embedding $t \mapsto (f(t), g(t), h(t))$ where $(\deg(f), \deg(g)) \leq (3, n + 1)$. Eventually, we give explicit examples with minimal lexicographic degree.

keywords: Knot theory, polynomial curves, torus knots, parametrized space curve

Mathematics Subject Classification 2000: 14H50, 12D10, 26C10, 57M25

1 Introduction

The study of non compact knots began with Vassiliev [Va]. He proved that any non-compact knot type can be obtained from a polynomial embedding $t \mapsto (f(t), g(t), h(t)), t \in \mathbb{R}$. The proof uses Weierstrass approximation theorem on a compact interval, the degrees of the polynomials may be quite large, and the plane projections of the polynomial knots quite complicated.

Independently, Shastri [Sh] gave a detailed proof of this theorem, he also gave simple polynomial parametrizations of the trefoil and of the figure eight knot.

This is what motivated A. Ranjan and Rama Shukla [RS] to find small degree parametrizations of the simplest knots, the torus knots of type $(2, n)$, n odd, denoted by K_n . They proved that these knots can be attained from polynomials of degrees $(3, 2n - 2, 2n - 1)$. In particular, they obtain a parametrization of the trefoil K_3 analogous to Shastri's one. They also asked the natural question which is to find the minimal degrees of the polynomials representing a general torus knot of a given type (there is an analogous question in Vassiliev's paper [Va]).

The number of crossings of a plane projection of K_n is at least n (Bankwitz theorem, see [Re]). It is not difficult to see, using Bézout theorem, that this plane curve cannot be parametrized by polynomials of degrees smaller than $(3, n + 1)$.

Naturally, Rama Mishra ([Mi]) asked whether it was possible to parametrize the knot K_n by polynomials of degrees $(3, n + 1, m)$ when $n \equiv 1$, or $0 \pmod 3$.

In this paper, we shall prove the following result

Theorem. *If $n \neq 3$ is odd, the torus knot K_n cannot be represented by polynomials of degrees $(3, n + 1, m)$.*

Our method is based on the fact that all plane projections of K_n with the minimal number n of crossings have essentially the same diagram. This is a consequence of the now solved classical

Tait's conjectures [Mu, Ka, Pr, MT]. This allows us to transform our problem into a problem of real polynomial algebra.

As a conclusion, we give explicit parametrizations of K_3, K_5 and K_7 . By our result, they are of minimal degrees. We also give an explicit parametrization of K_9 with a plane projection possessing the minimal number of crossing points. This embedding is of smaller degree than those already known.

2 The principal result

If n is odd, the torus knot K_n of type $(2, n)$ is the boundary of a Moebius band twisted n times (see [Re, Ka, St]). The recently proved Tait's conjectures allow us to characterize plane projections

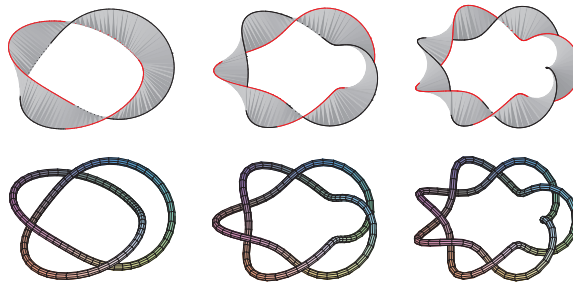


Figure 1: $K_n, n = 3, 5, 7$.

of K_n with the minimal number of crossings.

Lemma 1 *Let \mathcal{C} be a plane curve with n crossings parametrized by $\mathcal{C}(t) = (x(t), y(t))$. If \mathcal{C} is the projection of a knot K_n then there exist real numbers $s_1 < \dots < s_n < t_1 < \dots < t_n$, such that $\mathcal{C}(s_i) = \mathcal{C}(t_i)$.*

Proof. Let \mathcal{C} be a plane projection of a knot of type K_n with the minimal number n of crossings.

Using the Murasugi's theorem B ([Mu]) which says that a minimal projection of a prime alternating knot is alternating, we see that \mathcal{C} is alternating.

Then the Tait's flyping conjecture, proved by Menasco and Thistlethwaite ([MT, Pr]), asserts that \mathcal{C} is related to the standard diagram of K_n by a sequence of flypes. Let us recall that a flype is a transformation most clearly described by the following picture.



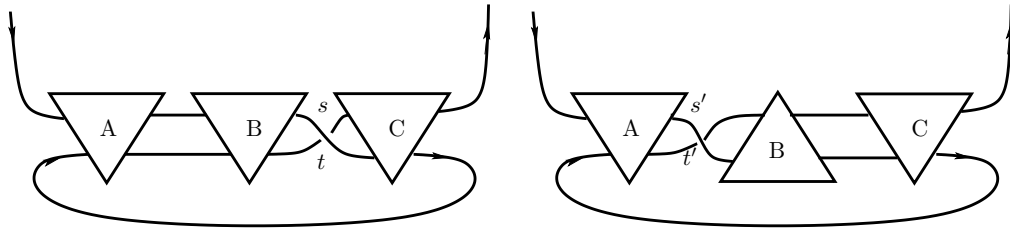
Figure 2: A flype

The standard diagram \mathcal{S}_0 of K_n has the property (cf [Re]) that there exist real numbers $s_1 < \dots < s_n < t_1 < \dots < t_n$ such that $\mathcal{S}_0(s_i) = \mathcal{S}_0(t_i)$. It is alternating.

Let \mathcal{S} be a diagram with real parameters $s_1 < \dots < s_n < t_1 < \dots < t_n$ such that $\mathcal{S}(s_i) = \mathcal{S}(t_i)$, and let us perform a flype of a part B of \mathcal{S}

For any $(a, b, c) \in A \times B \times C$ we have

$$s_a < s_b < s < s_c < t_a < t_b < t < t_c.$$

Figure 3: Flype on the part B

After the flype on B , we have new parameters corresponding to the crossing points satisfying

$$s'_a < s' < s'_b < s'_c < t'_a < t' < t'_b < t'_c.$$

The transformed diagram \mathcal{S}' has the same property: there exist real parameters $s_1 < \dots < s_n < t_1 < \dots < t_n$, such that $\mathcal{S}'(s_i) = \mathcal{S}'(t_i)$.

So then, after any sequence of flypes, the transformed diagram will have the same property. \square

In this paper we shall consider polynomial knots, that is to say, polynomial embeddings $\mathbb{R} \rightarrow \mathbb{R}^3$, $t \mapsto (x(t), y(t), z(t))$. Polynomial knots are non-compact subsets of \mathbb{R}^3 . The closure of a polynomial knot in the one point compactification \mathbf{S}^3 of the space \mathbb{R}^3 is an ordinary knot (see [Va, Sh, RS] and figures at the end).

Lemma 2 *Let \mathcal{C} be a plane polynomial curve with n crossings parametrized by $\mathcal{C}(t) = (x(t), y(t))$. Suppose that \mathcal{C} is the projection of K_n and $\deg x(t) \leq \deg y(t)$. Then we have $\deg x(t) \geq 3$. If $\deg x(t) = 3$, then $\deg y(t) \geq n + 1$.*

Proof. $x(t)$ must be non-monotonic, so $\deg x(t) \geq 2$. Suppose that $x(t)$ is of degree 2. Then $x(t_i) = x(s_i)$ implies that $t_i + s_i$ is constant, and so the parameter values corresponding to the crossing points are ordered as

$$s_1 < \dots < s_n < t_n < \dots < t_1.$$

We have a contradiction according to lemma 1

Suppose now that $\deg x(t) = 3$. The crossing points of the curve \mathcal{C} correspond to parameters (s, t) , $s \neq t$, that are common points of the curves of degrees 2 and $\deg y(t) - 1$:

$$\frac{x(t) - x(s)}{t - s} = 0, \quad \frac{y(t) - y(s)}{t - s} = 0.$$

By Bézout theorem ([Fi]), the number of such points are at most $2 \times (\deg y(t) - 1)$. (s, t) and (t, s) are distinct points and correspond to the same crossing point. So, the curve \mathcal{C} has at most $\deg y(t) - 1$ crossing points, and this implies that $\deg y(t) \geq n + 1$. \square

3 Proof of the main result

Our proof makes use of Chebyshev (monic) polynomials.

3.1 Chebyshev Polynomials

Definition 1 If $t = 2 \cos \theta$, let $T_n(t) = 2 \cos(n\theta)$ and $V_n(t) = \frac{\sin((n+1)\theta)}{\sin \theta}$.

Remark 1 T_n and V_n are both monic and have degree n . We have

$$V_0 = 1, \quad V_1 = t, \quad V_{n+1} = tV_n - V_{n-1}. \quad (1)$$

We have also

$$T_0 = 2, \quad T_1 = t, \quad T_{n+1} = tT_n - T_{n-1}.$$

For $n \geq 2$, let $V_n = t^n + a_n t^{n-2} + b_n t^{n-4} + \dots$. Using recurrence formula 1, we get

$$a_{n+1} = a_n - 1, \quad b_{n+1} = b_n - a_{n-1}$$

so by induction,

$$V_n = t^n - (n-1)t^{n-2} + \frac{1}{2}(n-2)(n-3)t^{n-4} + \dots. \quad (2)$$

We shall also need the following lemmas which will be proved in the next paragraph.

Lemma A. Let $s \neq t$ be real numbers such that $T_3(s) = T_3(t)$. For any integer k , we have

$$\frac{T_k(t) - T_k(s)}{t - s} = \frac{2}{\sqrt{3}} \sin \frac{k\pi}{3} V_{k-1}(s+t).$$

Lemma B. Let $n \geq 3$ be an integer.

Let $s_1 < s_2 < \dots < s_n$ and $t_1 < \dots < t_n$ be real numbers such that $T_3(s_i) = T_3(t_i)$. Let $u_i = t_i + s_i$.

We have

$$\sum_{i=1}^n u_i^2 \leq n + 4, \quad \sum_{i=1}^n u_i^4 \leq n + 22.$$

3.2 Proof of the theorem

Proof. We shall prove this result by reducing it to a contradiction. Suppose the plane curve \mathcal{C} parametrized by $x = P(t)$, $y = Q(t)$ where $\deg P = 3$, $\deg Q = n + 1$ is a plane projection of K_n .

By translation on t , one can suppose that $P(t) = t^3 - \alpha t + \beta$. If the polynomial P was monotonic, \mathcal{C} would have no crossings, which is absurd. Therefore $\alpha > 0$. Dividing t by $\rho = \sqrt{3}/\sqrt{\alpha}$, one has $P(t) = \rho^3(t^3 - 3t) + \mu$. By translating the origin and scaling x , one can now suppose that $P(t) = t^3 - 3t = T_3(t)$.

By translating the origin and scaling y , we can also suppose that $Q(t)$ is monic and write

$$P(t) = T_3(t), \quad Q(t) = T_{n+1}(t) + a_n T_n(t) + \dots + a_1 T_1(t).$$

By Bézout theorem, the curve \mathcal{C} has at most $(3-1)(n+1-1)/2 = n$ double points. As it has at least n crossings, we see that it has exactly n crossings and therefore is a minimal crossing diagram of K_n . According to the lemma 1, there exist real numbers $s_1 < \dots < s_n$, $t_1 < \dots < t_n$, $s_i < t_i$ such that $P(s_i) = P(t_i)$, $Q(s_i) = Q(t_i)$. Let $u_i = t_i + s_i$, $1 \leq i \leq n$. We have

$$\frac{Q(t_i) - Q(s_i)}{t_i - s_i} = \frac{T_{n+1}(t_i) - T_{n+1}(s_i)}{t_i - s_i} + \sum_{k=1}^n a_k \frac{T_k(t_i) - T_k(s_i)}{t_i - s_i}.$$

so by lemma **A**, u_1, \dots, u_n are the distinct roots of the polynomial

$$R(u) = \varepsilon_{n+1} V_n(u) + \sum_{k=1}^n a_k \varepsilon_k V_{k-1}(u), \quad (3)$$

where

$$\varepsilon_k = \frac{2}{\sqrt{3}} \sin \frac{k\pi}{3}.$$

Remark 2 Note that $\varepsilon_k = V_{k-1}(1)$ is the 6-period sequence $\varepsilon_0 = 0, \varepsilon_1 = 1, \varepsilon_2 = 1, 0, -1, -1, \dots$

We have to consider several cases.

▷ **Case $n \equiv 2 \pmod{3}$.** $\varepsilon_{n+1} = 0$ and $R(u)$ has degree at most $n - 1$. This is a contradiction.

▷ **Case $n \equiv 1 \pmod{3}$.** In this case, $n \equiv 1 \pmod{6}$ and $\varepsilon_{n+1} = \varepsilon_n = 1, \varepsilon_{n-1} = 0$. Thus $R(u)$ can be written as

$$\begin{aligned} R(u) &= V_n(u) + a_n V_{n-1}(u) - a_{n-2} V_{n-3}(u) - \dots + a_2 V_1(u) + a_1 \\ &= u^n + a_n u^{n-1} - (n-1)u^{n-2} + \dots \end{aligned}$$

using equation **2**. Therefore we get

$$\sum_{1 \leq i \leq n} u_i = -a_n, \quad \sum_{1 \leq i < j \leq n} u_i u_j = -(n-1),$$

and then

$$\sum_{i=1}^n u_i^2 = \left(\sum_{i=1}^n u_i \right)^2 - 2 \sum_{1 \leq i < j \leq n} u_i u_j = a_n^2 + 2(n-1) \geq 2(n-1).$$

According to lemma **B** we also have $\sum_{i=1}^n u_i^2 \leq n + 4$, we get a contradiction for $n > 6$.

▷ **Case $n \equiv 0 \pmod{3}$.** In this last case we have $n = 3 \pmod{6}$, so $\varepsilon_{n+1} = -1, \varepsilon_n = 0$ and $\varepsilon_{n-1} = 1$, so

$$-R(u) = V_n(u) - a_{n-1} V_{n-2}(u) - a_{n-2} V_{n-3}(u) + \dots - a_2 V_1(u) - a_1.$$

Let σ_i be the coefficients of

$$-R(u) = u^n + \sum_{k=1}^n (-1)^k \sigma_k u^{n-k}.$$

From the equation **2**, we see that

$$\sigma_1 = 0, \quad \sigma_2 = -(a_{n-1} + n - 1), \quad \sigma_4 = (n-3) a_{n-1} + \frac{(n-2)(n-3)}{2}.$$

Let S_k be the Newton sums $\sum_{i=1}^n u_i^k$ of the roots of the polynomial R . Using the classical Newton formulas ([FS]), we obtain

$$S_1 = \sigma_1 = 0, \quad S_2 = \sigma_1^2 - 2\sigma_2 = -2\sigma_2, \quad S_4 = 2\sigma_2^2 - 4\sigma_4,$$

and then

$$S_4 = 2(a_{n-1} + 2)^2 + 6n - 18 \geq 6n - 18.$$

By the lemma **B**, we deduce that $22 + n \geq 6n - 18$, i.e. $n \leq 8$ so $n = 3$. □

3.3 Proof of lemmas A and B

We shall use the following lemma

Lemma 3 (Lissajous ellipse) *Let $s \neq t$ be complex numbers such that*

$$T_3(t) = T_3(s).$$

There exists a complex number α such that

$$s = 2 \cos(\alpha + \pi/3), \quad t = 2 \cos(\alpha - \pi/3).$$

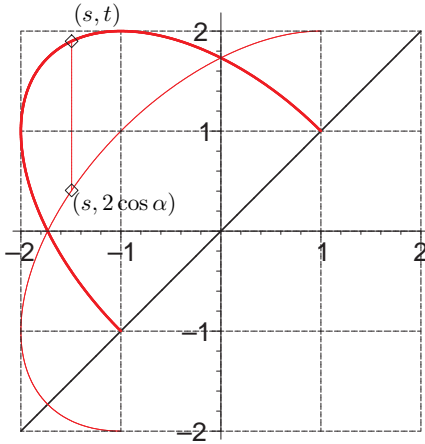
Furthermore, α is real if and only if s and t are both real, and then $t > s$ if and only if $\sin \alpha > 0$.

Proof. We have

$$\frac{T_3(t) - T_3(s)}{t - s} = t^2 + s^2 + st - 3. \quad (4)$$

Then, if $T_3(t) = T_3(s)$, $t \neq s$, we get

$$\frac{3}{2}(t+s)^2 + \frac{1}{2}(t-s)^2 = 2(t^2 + s^2 + st) = 6.$$



That means

$$\left[\frac{t+s}{2} \right]^2 + \left[\frac{t-s}{2\sqrt{3}} \right]^2 = 1.$$

Then there exists a complex number α such that

$$\cos \alpha = \frac{t+s}{2}, \quad \sin \alpha = \frac{t-s}{2\sqrt{3}},$$

that is

$$t = 2 \cos(\alpha - \pi/3), \quad s = 2 \cos(\alpha + \pi/3).$$

α is real if and only if $\cos \alpha$ and $\sin \alpha$ are both real that is to say, iff s and t are real. In this case: $t > s \Leftrightarrow \sin \alpha > 0$. \square

In order to prove lemma **B**, we shall use the following lemma which describes the geometrical configuration. Let us denote $s(\alpha) = 2 \cos(\alpha + \pi/3)$ and $t(\alpha) = 2 \cos(\alpha - \pi/3)$.

Lemma 4 *Let $\alpha, \alpha' \in [0, \pi]$ be such that $s(\alpha) < s(\alpha')$, and $t(\alpha) < t(\alpha')$. Then $\alpha > \alpha'$ and $\frac{2\pi}{3} > \frac{\alpha + \alpha'}{2} > \frac{\pi}{3}$.*

Proof. We have $2 \cos \alpha = s(\alpha) + t(\alpha) < s(\alpha') + t(\alpha') = 2 \cos \alpha'$

so $\alpha > \alpha'$.

$$t(\alpha') - t(\alpha) = 4 \sin\left(\frac{\alpha + \alpha'}{2} - \frac{\pi}{3}\right) \cdot \sin\left(\frac{\alpha - \alpha'}{2}\right) > 0,$$

$$s(\alpha') - s(\alpha) = 4 \sin\left(\frac{\alpha + \alpha'}{2} + \frac{\pi}{3}\right) \cdot \sin\left(\frac{\alpha - \alpha'}{2}\right) > 0.$$

From $\alpha > \alpha'$ we get $0 < \frac{\alpha + \alpha'}{2} - \frac{\pi}{3}$ and $\frac{\alpha + \alpha'}{2} + \frac{\pi}{3} < \pi$, that is to say

$$\frac{\pi}{3} < \frac{\alpha + \alpha'}{2} < \frac{2\pi}{3}.$$

□

Proof of lemma B.

Let $s_1 < \dots < s_n$ and $t_1 < \dots < t_n$ be such that $T_3(s_i) = T_3(t_i)$. Using lemmas **3** and **4** there are $\alpha_1 > \dots > \alpha_n \in]0, \pi[$ such that $s_i = s(\alpha_i)$, $t_i = t(\alpha_i)$ and we have

$$\frac{2\pi}{3} > \frac{\alpha_1 + \alpha_2}{2} > \alpha_2 > \dots > \alpha_{n-1} > \frac{\alpha_{n-1} + \alpha_n}{2} > \frac{\pi}{3}.$$

At least two of the α_i 's lie in the intervals $]0, \pi/2]$ or $[\pi/2, \pi[$. We have only two cases to consider: $\pi > \alpha_1 > \alpha_2 \geq \frac{\pi}{2}$, or $\frac{\pi}{2} \geq \alpha_{n-1} > \alpha_n > 0$.

On the other hand, we get the equality

$$\cos^2 x + \cos^2 y = 1 - \cos^2(x + y) + 2 \cos x \cos y \cos(x + y). \quad (5)$$

▷ **Case 1.** $\frac{\pi}{2} \geq \alpha_{n-1} > \alpha_n > 0$.

We get $\cos \alpha_n \geq 0$, $\cos \alpha_{n-1} \geq 0$ and $\cos(\alpha_{n-1} + \alpha_n) < -\frac{1}{2}$ so eq. **5** becomes

$$\cos^2 \alpha_{n-1} + \cos^2 \alpha_n \leq 1 - \cos^2(\alpha_{n-1} + \alpha_n) \leq \frac{3}{4}$$

and

$$\begin{aligned} \sum_{i=1}^n \cos^2 \alpha_i &= \cos^2 \alpha_1 + \sum_{i=2}^{n-2} \cos^2 \alpha_i + (\cos^2 \alpha_{n-1} + \cos^2 \alpha_n) \\ &\leq 1 + (n-3) \cdot \frac{1}{4} + \frac{3}{4} = \frac{1}{4}(n+4), \end{aligned}$$

that is

$$S_2 = \sum_{i=1}^n u_i^2 = \sum_{i=1}^n (2 \cos \alpha_i)^2 \leq n + 4.$$

▷ **Case 2.** $\pi > \alpha_1 > \alpha_2 \geq \frac{\pi}{2}$.

We get $\cos \alpha_1 \leq 0$, $\cos \alpha_2 \leq 0$ and $\cos(\alpha_1 + \alpha_2) < -\frac{1}{2}$ so eq. **5** becomes

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 \leq 1 - \cos^2(\alpha_1 + \alpha_2) \leq \frac{3}{4}$$

and similarly, we get

$$S_2 = \sum_{i=1}^n (2 \cos \alpha_i)^2 \leq n + 4.$$

Analogously, we get $\cos^4 x + \cos^4 y \leq (\cos^2 x + \cos^2 y)^2$, and we deduce:

$$S_4 = \sum_{i=1}^n (2 \cos \alpha_i)^4 \leq n + 22.$$

□

Proof of lemma A.

Let $s < t$ be real numbers such that $T_3(s) = T_3(t)$. According to the ellipse lemma **3**, there exists a real number α such that

$$t = 2 \cos(\alpha - \pi/3), \quad s = 2 \cos(\alpha + \pi/3).$$

We have $s + t = 2 \cos \alpha$ and $t - s = 4 \sin \frac{\pi}{3} \sin \alpha$, so

$$\begin{aligned} \frac{T_k(t) - T_k(s)}{t - s} &= \frac{2(\cos k(\alpha - \pi/3) - \cos k(\alpha + \pi/3))}{4 \sin \frac{\pi}{3} \sin \alpha} = \frac{\sin k\alpha \cdot \sin \frac{k\pi}{3}}{\sin \alpha \cdot \sin \frac{\pi}{3}} \\ &= \frac{2}{\sqrt{3}} \sin \frac{k\pi}{3} V_{k-1}(2 \cos \alpha). \end{aligned}$$

□

4 Parametrized models of K_3 , K_5 , K_7 and K_9

We get parametrizations of K_n : $\mathcal{C} = (x(t), y(t), z(t))$, with n crossings obtained for parameter values satisfying the hypothesis of lemma **1**. According to lemma **A**, we choose n distinct points $-1 \leq u_1 < \dots < u_n \leq 1$. We look for Q_1 and Q_2 of minimal degrees, such that

$$R_1(t + s) = \frac{Q_1(t) - Q_1(s)}{t - s}, \quad R_2(t + s) = \frac{Q_2(t) - Q_2(s)}{t - s}$$

satisfy, for $i = 1, \dots, n$,

$$R_1(u_i) = 0, \quad R_2(u_i) = (-1)^i.$$

We then choose $y(t) = Q_1(t)$ and $z(t) = Q_2(t)$. We also add some linear combinations of T_{6i} efficiently. We then obtain a knot whose projection is alternating, when R_1 has no more roots in $[-2, 2]$. As we have chosen symmetric u_i 's, all of our curves are symmetric with respect to the y -axis.

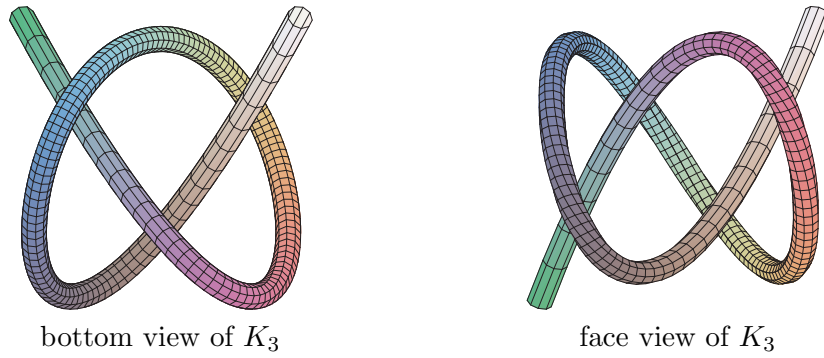
4.1 Parametrization of K_3

We can parametrize K_3 by $x = T_3(t), y = T_4(t), z = T_5(t)$. It is a Lissajous space curve (compare [Sh]). The plane curve $(T_3(t), T_4(t))$ has 3 crossing points. The plane curve $(T_3(t), T_5(t))$ has 4 crossing points corresponding to parameters (s_i, t_i) with

$$s_1 < s_2 < s_3 < s_4 < t_2 < t_1 < t_4 < t_3$$

so there do not exist real numbers $s_1 < s_2 < s_3$, and $t_1 < t_2 < t_3$ such that $x(s_i) = x(t_i)$, $z(s_i) = z(t_i)$.

This example shows that our method cannot be generalized when the projections of K_n have at least $n + 1$ crossing points.

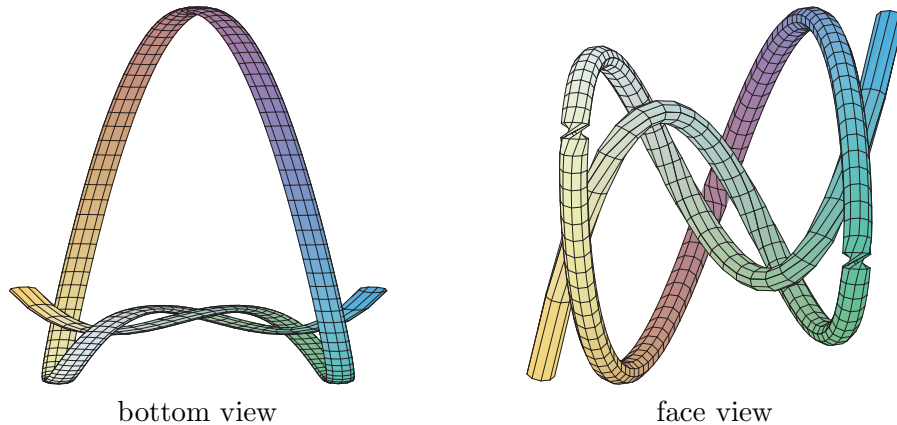
Figure 4: The trefoil knot K_3

4.2 Parametrization of K_5

Let us consider the curve of degree $(3, 7, 8)$:

$$\begin{aligned} x &= T_3(t), \\ y &= T_8(t) - 2T_6(t) + 2.189T_4(t) - 2.170T_2(t), \\ z &= T_7(t) - 0.56T_5(t) - 0.01348T_1(t). \end{aligned}$$

The curve $(x(t), y(t))$ has exactly 5 double points when the projection $(x(t), z(t))$ has exactly 6. Note here that $\deg z(t) < \deg y(t)$. In conclusion we have found a curve of degree $(3, 7, 8)$. Using

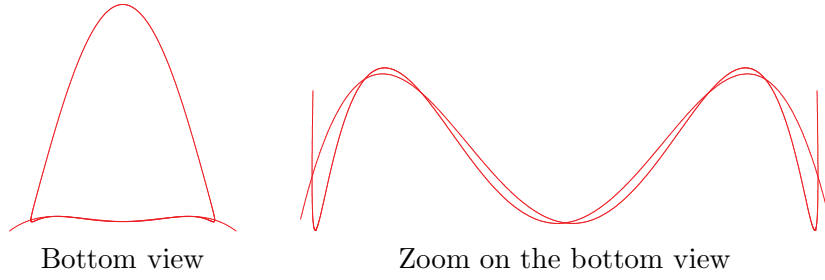
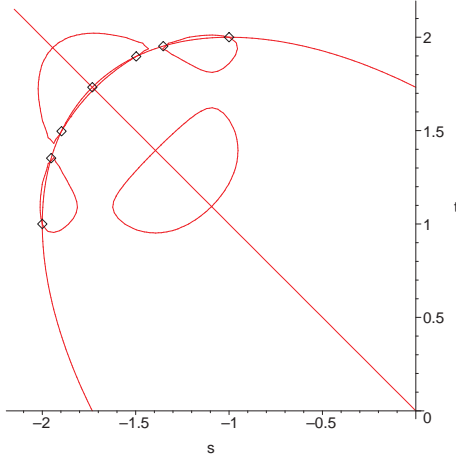
Figure 5: Knot K_5

our theorem, we see that this curve has minimal degree. A. Ranjan and R. Mishra showed the existence of such an example ([RS, Mi]).

4.3 Parametrization of K_7

We choose

$$\begin{aligned} x &= T_3(t), \\ y &= T_{10}(t) - 2.360T_8(t) + 4.108T_6(t) - 6.037T_4(t) + 7.397T_2(t), \\ z &= T_{11}(t) + 3.580T_7(t) - 3.739T_5(t) - T_1(t). \end{aligned}$$

Figure 6: Knot K_7 

The values of the parameters corresponding to the double points are obtained as intersection points between the ellipse

$$t^2 + s^2 + st - 3 = 0$$

and the curve of degree 9:

$$(y(t) - y(s))/(t - s) = 0$$

The curve $(x(t), y(t))$ has exactly 7 double points corresponding to $\cos(\alpha) = \{\pm 1/2, \pm 3/10, \pm 2/10, 0\}$.

In conclusion we have found a curve of degree $(3, 10, 11)$. Using our theorem, this curve has minimal degree.

4.4 Parametrization of K_9

We choose polynomials of degree $(3, 13, 14)$.

$$\begin{aligned} x &= T_3(t), \\ y &= T_{14}(t) - 4.516 T_{12}(t) + 12.16 T_{10}(t) - 24.46 T_8(t) + 39.92 T_6(t) \\ &\quad - 55.30 T_4(t) + 66.60 T_2(t), \\ z &= T_{13}(t) - 2.389 T_{11}(t) - 5.161 T_7(t) + 5.161 T_5(t) + 1.397 T_1(t). \end{aligned}$$

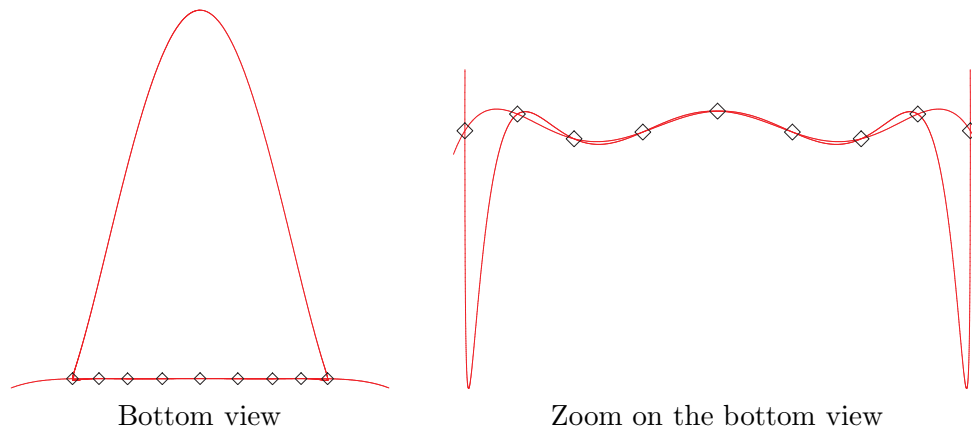
The curve $(x(t), y(t))$ has exactly 9 double points corresponding to $\cos(\alpha) = \{\pm 1/2, \pm 3/10, \pm 2/10, \pm 1/10, 0\}$. One can prove that it is minimal under the assumption that the projection $(x(t), y(t))$ has exactly 9 double points.

Conclusion

We have found minimal degree polynomial curves for torus knots K_n , $n = 3, 5, 7$. For degree 9, one can prove that it is minimal under the assumption that the projection $(x(t), y(t))$ has exactly 9 double points. We have similar constructions for higher degrees.

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Figure 7: Knot K_9

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