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CONFORMALLY EINSTEIN PRODUCTS AND NEARLY KÄHLER MANIFOLDS

ANDREI MOROIANU AND LIVIU ORNEA

ABSTRACT. In the first part of this note we study compact Riemannian manifolds (M, g) whose Riemannian product with \mathbb{R} is conformally Einstein. We then consider 6-dimensional almost Hermitian manifolds of type $W_1 + W_4$ in the Gray–Hervella classification admitting a parallel vector field and show that (under some mild assumption) they are obtained as Riemannian cylinders over compact Sasaki–Einstein 5-dimensional manifolds.

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1. INTRODUCTION

A. Gray was led to define nearly Kähler manifolds (also known as almost Tachibana spaces) by his research on weak holonomy of U_n -structures. If (N, h, J) denotes an almost Hermitian manifold, with fundamental two-form Ω and Levi–Civita connection ∇ , we have the following

Definition 1.1. [8] *An almost Hermitian manifold (N, h, J) is nearly Kähler if $\nabla\Omega$ is totally skew-symmetric.*

From the viewpoint of the representations of U_n on the space of tensors with the same symmetries as $\nabla\Omega$, nearly Kähler manifolds appear in the class W_1 of the Gray–Hervella classification (see [10] and §3 below). It is also known that nearly Kähler manifolds with integrable almost complex structure are necessarily Kähler. The specific, non-trivial, case is then the so-called *strict* nearly Kähler, when $\nabla J \neq 0$ at every point of M .

The actual interest in (strict) nearly Kähler geometry is partially motivated by its links with the theory of connections with (skew-symmetric) torsion and with the theory of SU_3 -structures, subjects that appear to be very important in contemporary theoretical physics (cf., for example, [7], [16]).

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The local structure of nearly Kähler manifolds was first discussed by Gray in [9] and was recently completely understood by P.A. Nagy in [13]: any nearly Kähler manifold is locally a product of 6–dimensional strict nearly Kähler manifolds, locally homogeneous manifolds and twistor spaces of positive quaternionic Kähler manifolds. According to this result, what remains to be studied are strict nearly Kähler structures in dimension 6. This is, in fact, the first interesting case, since 4–dimensional nearly Kähler manifolds are automatically Kähler. On the other hand, this dimension is particularly important also because of the following result:

Proposition 1.2. [12] *A strict 6–dimensional nearly Kähler manifold is Einstein with positive scalar curvature.*

As such, one may look for (connected) strict nearly Kähler manifolds of dimension 6 among the manifolds admitting real Killing spinors. In fact, in this dimension, a strict nearly Kähler structure on a simply connected manifold is equivalent with the existence of a non–trivial real Killing spinor, cf. [11].

In dimension 6, nearly Kähler structures are closely related to G_2 structures *via* the cone construction (see [1] for example):

Proposition 1.3. *A simply connected Riemannian manifold (N^6, h) carries a nearly Kähler structure if and only if its cone $(N^6 \times \mathbb{R}^+, t^2h + dt^2)$ has holonomy contained in G_2 .*

For later use, let us also recall the following related result:

Proposition 1.4. [1] *A simply connected Riemannian manifold (M^{2m+1}, g) carries a Sasaki–Einstein structure if and only if its cone $(M \times \mathbb{R}^+, t^2g + dt^2)$ has holonomy contained in SU_{m+1} .*

Returning to nearly Kähler geometry, the only known examples in dimension 6 are homogeneous: the sphere S^6 , the 3–symmetric space $S^3 \times S^3$ and the twistor spaces, $\mathbb{C}P^3$ and $F(1, 2)$, of the 4–dimensional self–dual Einstein manifolds S^4 and $\mathbb{C}P^2$. On the other hand, Butruille proved in [3] that every compact homogeneous strictly nearly Kähler manifold must be one of these.

It was tempting to try to obtain examples of (non–complete) nearly Kähler manifolds on cones over some almost contact metric manifolds. But an easy calculation shows that conic nearly Kähler structures cannot exist. Instead, we here prove in Theorem 3.2 that nearly Kähler structures do exist on cylinders over 5–dimensional Sasaki–Einstein manifolds. In fact the product structure of the cylinder is not itself nearly Kähler, but it is globally conformal with a nearly Kähler one, the conformal factor depending only on the coordinate of the generator. Conversely, we show that a cylindrical globally conformal strict nearly Kähler structure with simply connected base can only exist for Sasaki–Einstein basis.

The proof relies on a result which may be important in itself (we thus present it separately). This result is the classification of compact Riemannian manifolds, the cylinders over which are globally conformal Einstein with positive scalar curvature. These manifolds are themselves Einstein with positive scalar curvature. Moreover, the conformal factor can be explicitly determined and depends only on the coordinate of the generator, cf. Theorem 2.1 below.

2. CONFORMALLY EINSTEIN PRODUCTS

Theorem 2.1. *Let (M^n, g) be a compact Riemannian manifold not isometric to a round sphere. Suppose that the Riemannian cylinder over M is conformally Einstein with positive scalar curvature, that is, there exists a smooth function f on $M \times \mathbb{R}$, such that the Ricci tensor of the Riemannian manifold $(M \times \mathbb{R}, e^{2f}(g + dt^2))$ is a positive multiple of the metric. Then (M, g) is an Einstein manifold with positive scalar curvature and the conformal factor satisfies $e^{2f(x,t)} = \alpha^2 \cosh^{-2}(\beta t + \gamma)$ for some real constants α, β, γ .*

Proof. Let us denote by N and \tilde{N} the Riemannian manifolds $(M \times \mathbb{R}, (g + dt^2))$ and $(M \times \mathbb{R}, e^{2f}(g + dt^2))$ respectively. We view the function f on N as a smooth 1-parameter family of functions on M by $f_t(x) := f(x, t)$. In this way, the exterior derivative df satisfies $df = df_t + f'_t dt$, where df_t denotes the derivative of f_t on M and $f'_t = \frac{\partial f}{\partial t}$. Similarly the Laplace operators of M and N are related by $\Delta^N f = -f''_t + \Delta^M f_t$. In the sequel we will denote by $\partial_t := \frac{\partial}{\partial t}$ the "vertical" vector field on N and by X, Y , vector fields on M , identified with their canonical extension to N commuting with ∂_t . The formula for the conformal change of the Ricci tensor (see e.g. [2], 1.159),

$$\text{Ric}^{\tilde{N}} = \text{Ric}^N - (n-1)(\nabla^N df - df \otimes df) + (\Delta^N f - (n-1)|df|^2)(g + dt^2), \quad (1)$$

yields in particular

$$\text{Ric}^{\tilde{N}}(X, \partial_t) = -(n-1)(X(f'_t) - X(f_t)f'_t), \quad \forall X \in TM. \quad (2)$$

Since \tilde{N} is Einstein, (2) shows that $X(f'_t) = X(f_t)f'_t$, which can be rewritten as $X(\partial_t(e^{-f})) = 0$ for every $X \in TM$. Consequently, there exist smooth functions $a \in \mathcal{C}^\infty(\mathbb{R})$ and $b \in \mathcal{C}^\infty(M)$ such that $e^{-f(x,t)} = a(t) + b(x)$, in other words

$$f(x, t) = -\ln(a(t) + b(x)), \quad \forall x \in M, \forall t \in \mathbb{R}.$$

We now readily compute

$$df = df_t + f'_t dt = -\frac{db + a' dt}{a + b}, \quad f''_t = -\frac{a''(a + b) - (a')^2}{(a + b)^2}, \quad (3)$$

$$\Delta^N f = -f''_t + \Delta^M f_t = \frac{a''(a + b) - (a')^2}{(a + b)^2} - \frac{\Delta^M b}{a + b} - \frac{|db|^2}{(a + b)^2}, \quad (4)$$

and

$$\begin{aligned}
\nabla^N df &= -\frac{\nabla^N(db + a'dt)}{a+b} + \frac{1}{(a+b)^2}d(a+b) \otimes (db + a'dt) \\
&= -\frac{\nabla^M(db) + a''dt^2}{a+b} + \frac{1}{(a+b)^2}(db + a'dt) \otimes (db + a'dt) \\
&\stackrel{(3)}{=} -\frac{H(b) + a''dt^2}{a+b} + df \otimes df,
\end{aligned}$$

where $H(b)$ denotes the Hessian of b on M . Let r denote the Einstein constant of \tilde{N} . Plugging the relations above back into (1) yields

$$\begin{aligned}
\frac{r(g + dt^2)}{(a+b)^2} = \text{Ric}^{\tilde{N}} &= \text{Ric}^M + \frac{n-1}{a+b}(H(b) + a''dt^2) \\
&+ \frac{(a'' - \Delta^M b)(a+b) - n(a')^2 - n|db|^2}{(a+b)^2}(g + dt^2),
\end{aligned}$$

which is equivalent to the system

$$\begin{cases} r = (na'' - \Delta^M b)(a+b) - n(a')^2 - n|db|^2 \\ rg = ((a'' - \Delta^M b)(a+b) - n(a')^2 - n|db|^2)g + (a+b)^2\text{Ric}^M + (n-1)(a+b)H(b). \end{cases}$$

Subtracting the first equation from the second one, the system becomes

$$\begin{cases} r = (na'' - \Delta^M b)(a+b) - n(a')^2 - n|db|^2 \\ (n-1)a''g = (a+b)\text{Ric}^M + (n-1)H(b). \end{cases} \quad (5)$$

We distinguish three cases:

CASE 1: b is constant on M . Replacing a by $a-b$, we may assume that $b=0$, so by the first equation in (5), a satisfies the ODE

$$a''a - (a')^2 = \frac{r}{n}.$$

The general solution of this equation is

$$a(t) = \sqrt{\frac{r}{n\beta^2}} \cosh(\beta t + \gamma).$$

Thus $e^{2f(t)} = \frac{1}{a^2} = \alpha^2 \cosh^{-2}(\beta t + \gamma)$, with $\alpha := \frac{n\beta^2}{r}$. The second relation in (5) shows that M is Einstein, with positive Einstein constant β^2 .

CASE 2: a is constant on \mathbb{R} . Again, replacing b by $b-a$, we may assume that $a=0$, so the first equation of (5) becomes

$$n|db|^2 + b\Delta^M b + r = 0.$$

Integrating over M yields

$$0 = \int_M (n|db|^2 + b\Delta^M b + r)dv = \int_M ((n+1)|db|^2 + r)dv > 0,$$

showing that this case is impossible.

CASE 3: Neither a nor b are constant functions. We differentiate the second relation of (5) twice, first with respect to t , then with respect to some arbitrary vector $X \in TM$ and obtain

$$a'''X(b) = \frac{1}{n}a'X(\Delta^M b). \quad (6)$$

Taking some $x \in M$ and $X \in T_x M$ such that $X_x(b) \neq 0$, this relation shows that $a''' = \delta a'$ for some $\delta \in \mathbb{R}$. Similarly, taking some $t \in \mathbb{R}$ such that $a'(t) \neq 0$ gives some $\delta' \in \mathbb{R}$ such that $X(\Delta^M b) = \delta'X(b)$ for all $X \in TM$. Plugging these two relations back into (6) yields $\delta' = n\delta$. Summarizing, we have

$$\begin{cases} a'' = \delta a + \varepsilon \\ \Delta^M b = n\delta b + \varepsilon' \end{cases} \quad (7)$$

for some real constants $\varepsilon, \varepsilon'$. If $\delta = 0$, the second relation yields (by integration over M) $\varepsilon' = 0$, so b is constant, a contradiction. Thus $\delta \neq 0$. Replacing a by $a + \frac{\varepsilon}{\delta}$ (and correspondingly replacing b by $b - \frac{\varepsilon'}{n\delta}$), we may assume $\varepsilon = 0$. The second relation in (7) also shows that $n\delta$ is an eigenvalue of the Laplace operator (corresponding to the eigenfunction $b + \frac{\varepsilon'}{n\delta}$), whence $\delta > 0$. The second equation of the system (5) now becomes

$$\begin{cases} \text{Ric}^M = (n-1)\delta g \\ H(b) = -b\delta g \end{cases} \quad (8)$$

Since b is non-zero, the Obata theorem (see [14, Theorem 3]) implies that M is isometric to a round sphere, a contradiction, which shows that this case is impossible as well. \square

3. MANIFOLDS OF TYPE $W_1 + W_4$

The aim of this section is to classify all compact 5-dimensional Riemannian manifolds (M, g) with the property that the Riemannian cylinder $M \times \mathbb{R}$ carries an almost Hermitian structure of type $W_1 + W_4$. We start with some necessary explanations.

For each almost Hermitian manifold (N^{2m}, h, J) , with fundamental form $\Omega := h(J, \cdot)$, the Nijenhuis tensor, viewed as a tensor of type $(3, 0)$ via the metric, splits in two components $N = N_1 + N_2$, where N_1 is totally skew-symmetric and N_2 satisfies the Bianchi identity. Similarly, the covariant derivative of J with respect to the Levi-Civita connection of h splits in four components under the action of the structure group U_m (see [10]):

$$\nabla J = (\nabla J)_1 + (\nabla J)_2 + (\nabla J)_3 + (\nabla J)_4.$$

The first component corresponds to N_1 , or to the $(3, 0) + (0, 3)$ -part of $d\Omega$. The second component can be identified with N_2 , while the two other components correspond respectively to the primitive part of $d\Omega^{(2,1)+(1,2)}$ and to the contraction $\Omega \lrcorner d\Omega$ which is a 1-form called the *Lee form*. The manifold N is called of type $W_1 + W_4$ if $(\nabla J)_2$ and $(\nabla J)_3$ vanish identically. Similar definitions apply for every subset of subscripts in

$\{1, 2, 3, 4\}$. For example a manifold of type $W_3 + W_4$ is Hermitian, a manifold of type W_2 is symplectic, and a manifold of type W_1 is nearly Kähler.

From the definition it is more or less obvious that if the metric h is replaced by a conformally equivalent metric $\tilde{h} := e^{2f}h$, the first three components of ∇J are invariant and the fourth component satisfies $(\tilde{\nabla}J)_4 = (\nabla J)_4 + df$. Therefore, the Lee form of a manifold of type $W_1 + W_4$ is closed (resp. exact), if and only if the manifold is locally (resp. globally) conformal nearly Kähler. In dimension 6 we have the following result due to Butruille:

Theorem 3.1. [4] *Let M be a 6-dimensional almost Hermitian manifold of type $W_1 + W_4$. Then its Lee form is closed, so M is locally conformal nearly Kähler.*

This result generalizes the well-known fact that for $m \geq 3$, every almost Hermitian manifold of type W_4 is locally conformal Kähler (lck).

Using Theorem 3.1, we obtain the following classification of "cylindrical" structures of type $W_1 + W_4$ in dimension 6:

Theorem 3.2. *If the Riemannian cylinder $N := (M \times \mathbb{R}, g + dt^2)$ over a compact 5-dimensional simply connected Riemannian manifold (M, g) carries an almost Hermitian structure of type $W_1 + W_4$ which is not of type W_4 , then (M, g) is Sasaki-Einstein. Conversely, if (M, g) is Sasaki-Einstein, then its cylinder N carries a structure of type $W_1 + W_4$, besides its canonical Vaisman structure (locally conformal Kähler, i.e. W_4 , with parallel Lee form, cf. [15]).*

Proof. Assume first that N carries a structure of type $W_1 + W_4$. By Theorem 3.1, N (which is simply connected) is globally conformal nearly Kähler. Since we assumed that N is not lck, there exists a function f on N such that $(N, e^{2f}(g + dt^2))$ is a strict nearly Kähler manifold. By Proposition 1.2, every such manifold in dimension 6 is Einstein with positive scalar curvature. We apply Theorem 2.1 and obtain that $e^{2f} = \alpha^2 \cosh^{-2}(\beta t + \gamma)$ for some real constants α, β, γ .

Let us now consider the diffeomorphism

$$\varphi : M \times \mathbb{R} \rightarrow M \times (0, \pi), \quad (x, t) \mapsto (x, 2 \tan^{-1}(e^{\beta t + \gamma})).$$

A straightforward computation shows that

$$e^{2f}(g + dt^2) = \frac{\alpha^2}{\beta^2} \varphi^*(\beta^2 \sin^2 s g + ds^2). \quad (9)$$

We have obtained that the so-called *sine-cone* (see [6]) of $(M, \beta^2 g)$ has a nearly Kähler structure. The first part of the theorem then follows from the next lemma, which can be found (in a slightly different version) in [6].

Lemma 3.3. *Up to constant rescalings, the sine-cone $(M \times (0, \pi), \sin^2 s g + ds^2)$ of a simply connected 5-dimensional Riemannian manifold (M, g) has a nearly Kähler structure if and only if M is Sasaki-Einstein.*

Proof of the lemma. In [6] the authors prove the result by an explicit calculation, using so-called *hypo structures* on Sasaki–Einstein manifolds. We provide here a different argument. The key idea is the fact that the Riemannian product of two cone metrics is again a cone metric, as shown by the formula

$$(t^2g + dt^2) + (s^2h + ds^2) = r^2(\sin^2\theta g + \cos^2\theta h + d\theta^2) + dr^2, \quad (s, t) = (r \cos\theta, r \sin\theta).$$

In particular, taking $h = 0$ (the metric of a point), shows that the cylinder over the Riemannian cone of a metric g is isometric to the Riemannian cone over the sine–cone of g . By Lemma 1.4, if M is Sasaki–Einstein, its Riemannian cone \bar{M} has holonomy in SU_3 , so the cylinder $\bar{M} \times \mathbb{R}$ has holonomy in $SU_3 \times \{1\} \subset G_2$. The previous remark, together with Proposition 1.3, shows that the sine–cone of M is nearly Kähler. Conversely, if this holds, then the Riemannian cone of the sine–cone has holonomy in G_2 . Thus the holonomy of the cylinder $\bar{M} \times \mathbb{R}$ is a subgroup of G_2 . But since $G_2 \cap (O_6 \times \{1\}) = SU_3 \times \{1\} \subset O_7$, this means that \bar{M} has holonomy in SU_3 , so M is Sasaki–Einstein. The lemma, and the first part of the theorem are thus proved.

Conversely, let (M^5, g) be a simply connected Sasaki–Einstein manifold. We first notice that by Lemma 1.4, the Riemannian cone \bar{M} is Kähler, so the cylinder $M \times \mathbb{R}$, which is conformal to \bar{M} , is lck (and even Vaisman, see [15]).

On the other hand, Lemma 3.3 shows that the sine–cone of M is nearly Kähler, and therefore the cylinder over M , which by (9) is conformal to the sine–cone, has a structure of type $W_1 + W_4$. \square

Remark. The almost Hermitian structure on the cylinder can be easily made explicit, cf. [6, Theorem 3.6]. As it was pointed out by S. Ivanov, the almost complex structure is not invariant by any translation, so one can not obtain compact examples of structures of type $W_1 + W_4$ in this way. In fact, Theorem 3.1 and Lemma 8 in [5], show that every structure of (strict) type $W_1 + W_4$ in dimension 6 is *globally* conformal to a nearly Kähler structure.

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