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Abstract

Let $\mathbf{X} = (X_1, X_2, ...)$ be a non-deterministic infinite exchangeable sequence with values in $\{0, 1\}$. We show that $\mathbf{X}$ is Hoeffding-decomposable if, and only if, $\mathbf{X}$ is either an i.i.d. sequence or a Pólya sequence. This completes the results established in Peccati [2004]. The proof uses several combinatorial implications of the correspondence between Hoeffding decomposability and weak independence. Our results must be compared with previous characterizations of i.i.d. and Pólya sequences given by Hill *et al.* [1987] and Diaconis and Yilvisaker [1979] .

Key words. Exchangeable Sequences; Hoeffding Decompositions; Pólya Urns; Weak Independence.

AMS 2000 classifications: 60G09; 60G99

1 Introduction, notation and statements

Let $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$ be an exchangeable sequence of random observations, with values in some finite set D. We say that $\mathbf{X}_{[1,\infty)}$ is *Hoeffding decomposable* if, for every $n \geq 2$, every symmetric statistic $T(X_1, ..., X_n)$ admits a unique representation as an orthogonal sum of uncorrelated U-statistics with degenerate kernels of increasing order. Hoeffding decompositions (also known as *ANOVA decompositions*) have been extensively studied for i.i.d. sequences (see for instance Hoeffding [1948], Hajek [1968], Karlin and Rinott [1982], Vitale [1991] and Bentkus *et al.* [1997]) and for extractions without replacement from a finite population (see Zhao and Chen [1990], Bloznelis and Götze [2001, 2002] and Bloznelis [2005]). In Peccati [2003, 2004, 2005], the second author of this paper has extended the theory of Hoeffding decompositions to the framework of general exchangeable random sequences. In particular, it was shown therein that the class of Hoeffding decomposable exchangeable sequences coincides with the collection of weakly independent sequences.

The aim of this note is to complete the results established in Peccati [2004] by proving that *a (non deterministic) infinite exchangeable sequence with values in* $\{0, 1\}$ *is Hoeffding decomposable if, and only if, it is either a Pólya sequence or i.i.d.*. As shown in Section 4, this result links the seemingly unrelated notions of Hoeffding decomposable sequence and *urn process*, a concept studied e.g. in Hill *et al.* [1987].

Before stating our main theorem, we recall some basic definitions and facts concerning Hoeffding decompositions and exchangeable sequences. We focus on sequences with values in a finite set. The reader is referred to Peccati [2004] for any unexplained concept or notation, as well as for general statements concerning sequences with values in arbitrary Polish spaces.

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1.1 Preliminaries

Let D be a finite set, and consider an infinite exchangeable sequence $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$ of D-valued random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{F} = \sigma(\mathbf{X}_{[1,\infty)})$. We recall that, according to the well-known *de Finetti Theorem* (see e.g. Aldous [1983]), the assumption of exchangeability implies that $\mathbf{X}_{[1,\infty)}$ is necessarily a *mixture* of i.i.d. sequences with values in D.

For every $n \geq 1$ and every $1 \leq u \leq n$, we write $[n] = \{1, ..., n\}$ and $[u, n] = \{u, u+1, ..., n\}$, and set $\mathbf{X}_{[u,n]} \triangleq (X_u, X_{u+1}..., X_n)$ and $\mathbf{X}_{[n]} \triangleq \mathbf{X}_{[1,n]} = (X_1, X_2, ..., X_n)$. For every $n \geq 2$, we define the sequence of spaces

$$\left\{SU_k\left(\mathbf{X}_{[n]}\right) : k = 0, ..., n\right\},$$

generated by symmetric U-statistics of increasing order, as follows: $SU_0\left(\mathbf{X}_{[n]}\right) \triangleq \Re$ and, for $k = 1, ..., n$, $SU_k\left(\mathbf{X}_{[n]}\right)$ is the collection of all random variables of the type

$$F\left(\mathbf{X}_{[n]}\right) = \sum_{1 \leq j_1 < ... < j_k \leq n} \varphi\left(X_{j_1}, ..., X_{j_k}\right), \tag{1}$$

where φ is a real-valued symmetric function from D^k to $\Re$. A random variable such as F in (1) is called a *U-statistic with symmetric kernel of order k*. It is easily seen that the kernel φ appearing in (1) is unique, in the sense that if φ' is another symmetric kernel satisfying (1), then $\varphi\left(\mathbf{X}_{[k]}\right) = \varphi'\left(\mathbf{X}_{[k]}\right)$, a.s.-$\mathbb{P}$. The following facts are immediately checked: (i) for every $k = 0, ..., n$, $SU_k\left(\mathbf{X}_{[n]}\right)$ is a vector space, (ii) $SU_{k-1}\left(\mathbf{X}_{[n]}\right) \subset SU_k\left(\mathbf{X}_{[n]}\right)$, (iii) $SU_n\left(\mathbf{X}_{[n]}\right) = L_s\left(\mathbf{X}_{[n]}\right)$, where (for $n \geq 1$) $L_s\left(\mathbf{X}_{[n]}\right)$ is defined as the set of all random variables of the type $T\left(\mathbf{X}_{[n]}\right) = T\left(X_1, ..., X_n\right)$, where T is a *symmetric function* from D^n to $\Re$. The class of all symmetric functions, from D^n to $\Re$, will be denoted by $\mathcal{S}\left(D^n\right)$. Note that $L_s\left(\mathbf{X}_{[n]}\right)$ is a Hilbert space with respect to the inner product $\langle T_1, T_2\rangle \triangleq \mathbb{E}\left[T_1\left(\mathbf{X}_{[n]}\right) T_2\left(\mathbf{X}_{[n]}\right)\right]$, so that each $SU_k\left(\mathbf{X}_{[n]}\right)$ is a closed subspace of $L_s\left(\mathbf{X}_{[n]}\right)$. Finally, the sequence of *symmetric Hoeffding spaces* $\left\{SH_k\left(\mathbf{X}_{[n]}\right) : k = 0, ..., n\right\}$ associated to $\mathbf{X}_{[n]}$ is defined as $SH_0\left(\mathbf{X}_{[n]}\right) \triangleq SU_0\left(\mathbf{X}_{[n]}\right) = \Re$, and

$$SH_k\left(\mathbf{X}_{[n]}\right) \triangleq SU_k\left(\mathbf{X}_{[n]}\right) \cap SU_{k-1}\left(\mathbf{X}_{[n]}\right)^{\perp}, \quad k = 1, ..., n, \tag{2}$$

where all orthogonals (here and in the sequel) are taken in $L_s\left(\mathbf{X}_{[n]}\right)$. Observe that $SH_k\left(\mathbf{X}_{[n]}\right) \subset SU_k\left(\mathbf{X}_{[n]}\right)$ for every k, so that each $F \in SH_k\left(\mathbf{X}_{[n]}\right)$ has necessarily the form (1) for some well-chosen symmetric kernel φ. Moreover, since $SU_n\left(\mathbf{X}_{[n]}\right) = L_s\left(\mathbf{X}_{[n]}\right)$, one has the following orthogonal decomposition:

$$L_s\left(\mathbf{X}_{[n]}\right) = \bigoplus_{k=0}^{n} SH_k\left(\mathbf{X}_{[n]}\right), \tag{3}$$

where " $\oplus$ " stands for an orthogonal sum. In particular, (3) implies that every symmetric random variable $T\left(\mathbf{X}_{[n]}\right) \in L_s\left(\mathbf{X}_{[n]}\right)$ admits a unique representation as a non-correlated sum of $n+1$ terms, with the kth summand ($k = 0, ..., n$) equal to an element of $SH_k\left(\mathbf{X}_{[n]}\right)$.

The next definition, which is essentially borrowed from Peccati [2004], formalizes the notion of "Hoeffding decomposability" evoked at the beginning of the section.

Definition A. The random sequence $\mathbf{X}_{[1,\infty)}$ is *Hoeffding decomposable* if, for every $n \geq 2$ and every $k = 1, ..., n$, the following double implication holds: $F \in SH_k\left(\mathbf{X}_{[n]}\right)$ if, and only if, the kernel φ appearing in its representation (1) satisfies the degeneracy condition

$$\mathbb{E}\left[\varphi\left(\mathbf{X}_{[k]}\right) \mid \mathbf{X}_{[2,k]}\right] = 0, \quad \text{a.s.-}\mathbb{P}. \tag{4}$$

When a U-statistic F as in (1) is such that φ verifies (4), one says that F is a *completely degenerate* symmetric U-statistic of order k, and that φ is a *completely degenerate symmetric kernel* of order k.

For instance, when $k = 3$, one has $\mathbf{X}_{[2,k]} = (X_2, X_3)$, and condition (4) becomes:

$$\mathbb{E}\left[\varphi\left(X_1, X_2, X_3\right) \mid X_2, X_3\right] = 0.$$

Of course, by exchangeability, (4) holds if, and only if, $\mathbb{E}\left[\varphi\left(\mathbf{X}_{[k]}\right) \mid \mathbf{X}_{[k-1]}\right] = 0$, a.s.-$\mathbb{P}$.

For every infinite non-deterministic exchangeable sequence $\mathbf{X}_{[1,\infty)}$ (not necessarily Hoeffding decomposable) and every $k \geq 1$, the class of all kernels $\varphi : D^k \mapsto \Re$, such that (4) is verified, is noted $\Xi_k\left(\mathbf{X}_{[1,\infty)}\right)$.

It is well known (see e.g. Hoeffding [1948], Hajek [1968] or Karlin and Rinott [1983]) that each i.i.d. sequence is decomposable in the sense of Definition A. In Peccati [2004], the second author established a complete characterization of Hoeffding decomposable sequences (with values in arbitrary Polish spaces), in terms of *weak independence*. To introduce this concept, we need some more notation. Fix $n \geq 2$, and consider a symmetric function $T \in \mathcal{S}(D^n)$. We define the function $[T]_{n,n-1}^{(n-1)}$ as the unique application from D^{n-1} to $\Re$ such that

$$[T]_{n,n-1}^{(n-1)}\left(\mathbf{X}_{[2,n]}\right) = \mathbb{E}\left(T\left(\mathbf{X}_{[n]}\right) \mid \mathbf{X}_{[2,n]}\right), \quad \text{a.s.-}\mathbb{P}. \tag{5}$$

For instance, if $n = 2$, then $\mathbf{X}_{[2]} = (X_1, X_2)$, $\mathbf{X}_{[2,2]} = X_2$ and $[T]_{2,1}^{(1)}(X_2) = \mathbb{E}(T(X_1, X_2) \mid X_2)$. Note that the exchangeability assumption and the symmetry of T imply that the application $D^{n-1} \mapsto \Re : \mathbf{x} \mapsto [T]_{n,n-1}^{(n-1)}(\mathbf{x})$ is symmetric. Also, with this notation, $T \in \Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$ if, and only if, $[T]_{n,n-1}^{(n-1)}\left(\mathbf{X}_{[2,n]}\right) = 0$, a.s.-$\mathbb{P}$.

Analogously, for $u = 2, ..., n$ we define the function $[T]_{n,n-1}^{(n-u)} : D^{n-1} \mapsto \Re$ through the relation:

$$[T]_{n,n-1}^{(n-u)}\left(\mathbf{X}_{[u+1,u+n-1]}\right) = \mathbb{E}\left(T\left(\mathbf{X}_{[n]}\right) \mid \mathbf{X}_{[u+1,u+n-1]}\right), \quad \text{a.s.-}\mathbb{P}. \tag{6}$$

To understand our notation, observe that, for $u = 2, ..., n$, the two sets $[n]$ and $[u+1, u+n-1]$ have exactly $n-u$ elements in common. For instance, if $n = 3$ and $u = 2$, then $[u+1, u+n-1] = \{3, 4\}$, and $[T]_{3,2}^{(1)}(X_3, X_4) = \mathbb{E}(T(X_1, X_2, X_3) \mid X_3, X_4)$. Again, exchangeability and symmetry yield that the function $\mathbf{x} \mapsto [T]_{n,n-1}^{(0)}(\mathbf{x})$ (corresponding to the case $u = n$) is symmetric on D^{n-1}. On the other hand, for $u = 2, ..., n-1$, the application $(x_1, ..., x_{n-1}) \mapsto [T]_{n,n-1}^{(n-u)}(x_1, ..., x_{n-1})$ is (separately) symmetric in the variables $(x_1, ..., x_{n-u})$ and $(x_{n-u+1}, ..., x_{n-1})$, and not necessarily symmetric as a function on D^{n-1}. When $u = 2, ..., n-1$, we note $\widetilde{[T]}_{n,n-1}^{(n-u)}$ the canonical symmetrization of $[T]_{n,n-1}^{(n-u)}$.

Finally, for $u = 2, ..., n$, set

$$\widetilde{\Xi}_{n,n-u}\left(\mathbf{X}_{[1,\infty)}\right) \triangleq \left\{T \in \mathcal{S}(D^n) : \widetilde{[T]}_{n,n-1}^{(n-u)}\left(\mathbf{X}_{[u+1,u+n-1]}\right) = 0, \quad \text{a.s.-}\mathbb{P}\right\} \tag{7}$$

(recall that $\mathcal{S}(D^n)$ denotes the class of symmetric functions on D^n). Note that, by exchangeability, $\widetilde{[T]}_{n,n-1}^{(n-u)}\left(\mathbf{X}_{[u+1,u+n-1]}\right) = 0$, a.s.-$\mathbb{P}$, if, and only if, $\widetilde{[T]}_{n,n-1}^{(n-u)}\left(\mathbf{X}_{[n-1]}\right) = 0$, a.s.-$\mathbb{P}$. The following technical definition is taken from Peccati [2004].

Definition B. The exchangeable sequence $\mathbf{X}_{[1,\infty)}$ is *weakly independent* if, for every $n \geq 2$,

$$\Xi_n\left(\mathbf{X}_{[1,\infty)}\right) \subset \bigcap_{u=2}^{n} \widetilde{\Xi}_{n,n-u}\left(\mathbf{X}_{[1,\infty)}\right). \tag{8}$$

In other words, $\mathbf{X}_{[1,\infty)}$ is weakly independent if, for every $n \geq 2$ and every $T \in \mathcal{S}(D^n)$, the following implication holds: if $[T]_{n,n-1}^{(n-1)}\left(\mathbf{X}_{[n-1]}\right) = 0$, then $\widetilde{[T]}_{n,n-1}^{(n-u)}\left(\mathbf{X}_{[n-1]}\right) = 0$ for every $u = 2, ..., n$.

The next theorem, which is one of the main results of Peccati [2004], shows that the notions of weak independence and Hoeffding decomposability are equivalent for infinite exchangeable sequences.

THEOREM 0 (Peccati [2004, Th. 6]). *Suppose that the infinite exchangeable sequence* $\mathbf{X}_{[1,\infty)}$ *is such that, for every* $n \geq 2$,

$$SH_k\left(\mathbf{X}_{[n]}\right) \neq \{0\}, \quad \forall k = 1, ..., n. \tag{9}$$

Then, $\mathbf{X}_{[1,\infty)}$ *is Hoeffding decomposable if, and only if, it is weakly independent.*

Remark. Condition (9) excludes for instance the case: $X_n = X_1$, for each $n \geq 1$.

Note that Theorem 0 also holds for exchangeable sequences with values in general Polish spaces. In Peccati [2004] Theorem 0 has been used to show the following two facts:

(F1) There are infinite exchangeable sequences which are Hoeffding decomposable and not i.i.d., as for instance the *Generalized Urn Sequences* analyzed in Section 5 of Peccati [2004].

(F2) There exist infinite exchangeable sequences that *are not* Hoeffding decomposable. For instance, one can consider a $\{0,1\}$-valued exchangeable sequence $\mathbf{X}^Y_{[1,\infty)}$ such that, conditioned on the realization of a random variable Y uniformly distributed on $(0,\varepsilon)$ $(0 < \varepsilon < 1)$, $\mathbf{X}^Y_{[1,\infty)}$ is composed of independent Bernoulli trials with random parameter Y. See Peccati [2004, p. 1807-1808] for more details.

Although the combination of Theorem 0, (F1) and (F2) gives several insights into the structure of Hoeffding decomposable sequences, the analysis contained in Peccati [2004] left open a crucial question: *can one characterize the laws of Hoeffding decomposable sequences, in terms of their de Finetti representation as mixtures of i.i.d. sequences?* In the following sections, we will provide a complete answer when $D = \{0,1\}$, by proving that in this case the class of Hoeffding decomposable sequences contains exclusively i.i.d. and Pólya sequences. The extension of our results to spaces D with more than two elements is an open problem.

1.2 Main results

For the rest of the paper, we will focus on the case $D = \{0,1\}$. According to the de Finetti Theorem, in this case the exchangeability of $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$ yields the existence of a probability measure γ on $[0,1]$ such that, for every $n \geq 1$ and every vector $(j_1, ..., j_n) \in \{0,1\}^n$,

$$\mathbb{P}\{X_1 = j_1, ..., X_n = j_n\} = \int_{[0,1]} \theta^{\Sigma_k j_k} (1-\theta)^{n-\Sigma_k j_k} \gamma(\mathrm{d}\theta). \tag{10}$$

The measure γ, appearing in (10), is called the *de Finetti measure* associated with $\mathbf{X}_{[1,\infty)}$. In what follows, we shall systematically suppose that $\mathbf{X}_{[1,\infty)}$ is *non-deterministic*, that is, that the support of the measure γ *is not contained* in $\{0\} \cup \{1\}$. In particular, it is easy to prove that, when $D = \{0,1\}$, condition (9) holds if, and only if, $\mathbf{X}_{[1,\infty)}$ is non-deterministic.

Definition C. The exchangeable sequence $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$ is called a *Pólya sequence* if there exist two real numbers $\alpha, \beta > 0$ such that

$$\gamma(\mathrm{d}\theta) = \frac{1}{B(\alpha,\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \mathrm{d}\theta, \tag{11}$$

where γ is the de Finetti measure associated to $\mathbf{X}_{[1,\infty)}$ through formula (10), and

$$B(\alpha,\beta) = \int_0^1 \theta^{\alpha-1} (1-\theta)^{\beta-1} \mathrm{d}\theta$$

is the usual Beta function. The numbers α and β are the *parameters* of the Pólya sequence $\mathbf{X}_{[1,\infty)}$. A random variable ξ, with values in $[0,1]$ and with law γ as in (11), is called a *Beta random variable* of parameters α and β.

Classic references for the theory of Pólya sequences are Blackwell [1973] and Blackwell and MacQueen [1973] (see also Pitman [1996, 2006] for a state of the art review). Thanks to Peccati [2004, Corollary 9], we already know that Pólya and i.i.d. sequences are Hoeffding decomposable. The next result, which is the main achievement of our paper, shows that those are the only exchangeable and Hoeffding decomposable sequences with values in $\{0,1\}$. The proof is deferred to Section 3.

Theorem 1 *Let $\mathbf{X}_{[1,\infty)}$ be a non-deterministic infinite exchangeable sequence of $\{0,1\}$-valued random variables. Then, the following two assertions are equivalent:*

1. *$\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable;*
2. *$\mathbf{X}_{[1,\infty)}$ is either an i.i.d. sequence or a Pólya sequence.*

In Section 4 we will discuss some connections between Theorem 1 and the concept of *urn process*, as defined in Hill *et al.* [1987].

Remarks. We state two projection formulae, concerning respectively i.i.d. and Pólya sequences.

(I) Let $\mathbf{X}_{[1,\infty)}$ be an i.i.d. sequence with values in $\{0,1\}$, and fix $n \geq 2$ and $T \in L_s\left(\mathbf{X}_{[n]}\right)$. Then, for $k = 1, ..., n$, the projection of T on the k-th Hoeffding space $SH_k\left(\mathbf{X}_{[n]}\right)$, denoted by $\pi\left[T, SH_k\right]$, is

$$\pi\left[T, SH_k\right] = \sum_{a=1}^{k} (-1)^{k-a} \sum_{1 \leq j_1 < \cdots < j_a \leq n} \left[T - \mathbb{E}\left(T\right)\right]_{n,a}^{(a)} \left(X_{j_1}, ..., X_{j_a}\right). \tag{12}$$

Formula (12) is classic (see e.g. Hoeffding [1948], or Vitale [1991]), and can be easily deduced by an application of the inclusion-exclusion principle.

(II) Let $\mathbf{X}_{[1,\infty)}$ be a Pólya sequence of parameters $\alpha, \beta > 0$, and fix $n \geq 2$ and $T \in L_s\left(\mathbf{X}_{[n]}\right)$. Then, for $k = 1, ..., n$, the projection of T on the k-th Hoeffding space associated with $\mathbf{X}_{[n]}$ is of the form

$$\pi\left[T, SH_k\right] = \sum_{a=1}^{k} \theta_n^{(k,a)} \sum_{1 \leq j_1 < \cdots < j_a \leq n} \left[T - \mathbb{E}\left(T\right)\right]_{n,a}^{(a)} \left(X_{j_1}, ..., X_{j_a}\right).$$

The explicit formulae describing the real coefficients $\theta_n^{(k,a)}$ are given recursively in Peccati [2004, formula (23)]. For instance, when $n = 3$, then

$$\begin{cases} \theta_3^{(1,1)} = \frac{\alpha+\beta+1}{\alpha+\beta+2}, \\ \theta_3^{(2,1)} = -\frac{(\alpha+\beta+1)(\alpha+\beta+4)}{(\alpha+\beta+3)(\alpha+\beta+2)} - \frac{\alpha+\beta+1}{\alpha+\beta+2}, \\ \theta_3^{(2,2)} = \frac{\alpha+\beta+4}{\alpha+\beta+2}. \end{cases}$$

The rest of the paper is organized as follows: in Section 2 we collect several technical results, leading to a new characterization of Hoeffding decomposability in terms of conditional probabilities (see Proposition 4 below); the proof of Theorem 1 is contained in Section 3; in Section 4, a brief discussion is presented, relating Theorem 1 with several notions associated with $\{0,1\}$-valued exchangeable sequences.

2 Ancillary lemmas

From now on, $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$ will be a non-deterministic exchangeable sequence with values in $D = \{0,1\}$. For $n \geq 2$, we write $\mathcal{S}\left(\{0,1\}^n\right)$ to indicate the vector space of symmetric functions on $\{0,1\}^n$, while $\mathfrak{S}_n$ stands for the group of permutations of the set $[n] = \{1, ..., n\}$. Given a vector $\mathbf{x}_n = (x_1, ..., x_n) \in \{0,1\}^n$ and a permutation $\pi \in \mathfrak{S}_n$, we note $\mathbf{x}_{\pi(n)}$ the action of π on $\mathbf{x}_n$, that is, $\mathbf{x}_{\pi(n)} = \left(x_{\pi(1)}, ..., x_{\pi(n)}\right)$. By exchangeability, we have of course that

$$\mathbb{P}\left(\mathbf{X}_{[n]} = \mathbf{x}_n\right) = \mathbb{P}\left(\mathbf{X}_{[n]} = \mathbf{x}_{\pi(n)}\right), \quad \forall n \geq 2, \forall \pi \in \mathfrak{S}_n,$$

yielding that, for $n \geq 2$, the value of the probability $\mathbb{P}\left(\mathbf{X}_{[n]} = \mathbf{x}_n\right)$ depends exclusively on n and on the number of zeros contained in the vector $\mathbf{x}_n$. For $n \geq 1$ and $j = 0, ..., n$, we shall denote by $\mathbb{P}_n\left(0^{(j)}\right)$ the *common value* taken by the quantity $\mathbb{P}\left(\mathbf{X}_{[n]} = \mathbf{x}_n\right)$ for all $\mathbf{x}_n = (x_1, ..., x_n) \in \{0,1\}^n$ such that $\mathbf{x}_n$ contains exactly j zeros. For instance, when $n = 3$ and $j = 1$, one has that $\mathbb{P}_3\left(0^{(1)}\right) = \mathbb{P}\left(\mathbf{X}_{[3]} = (0,1,1)\right)$ $= \mathbb{P}\left(\mathbf{X}_{[3]} = (1,0,1)\right) = \mathbb{P}\left(\mathbf{X}_{[3]} = (1,1,0)\right)$. Note that, since $\mathbf{X}_{[1,\infty)}$ is non-deterministic, $\mathbb{P}_n\left(0^{(j)}\right) > 0$ for every $n \geq 1$ and every $j = 0, ..., n$. Analogously, for every $n \geq 2$, every $j = 0, ..., n$, and every symmetric function $\varphi \in \mathcal{S}\left(\{0,1\}^n\right)$, we will write $\varphi\left(0^{(j)}\right)$ to indicate the common value taken by $\varphi(\mathbf{x}_n)$ for all $\mathbf{x}_n \in \{0,1\}^n$ containing exactly j zeros.

The following result gives a complete characterization of the spaces

$$\Xi_n\left(\mathbf{X}_{[1,\infty)}\right), \qquad n \geq 2,$$

defined through relation (4) (note that, to define the spaces Ξ_n we do not need $\mathbf{X}_{[1,\infty)}$ to be Hoeffding decomposable).

Lemma 2 *With the assumptions and notation of this section, the set $\Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$ is the 1-dimensional vector space spanned by the symmetric kernel $\varphi_n^{(0)} : \{0,1\}^n \mapsto \Re$ defined by*

$$\varphi_n^{(0)}\left(0^{(k)}\right) = (-1)^k \frac{\mathbb{P}_n\left(0^{(0)}\right)}{\mathbb{P}_n\left(0^{(k)}\right)}, \qquad k = 0, ..., n. \tag{13}$$

Proof. Consider $\varphi_n \in \Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$. By the definition of $\Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$, for any fixed $j = 0, ..., n-1$ and any fixed $\mathbf{x}_{n-1} \in \{0,1\}^{n-1}$ such that $\sum_{i=1}^{n-1}(1 - x_i) = j$, we have

$$\begin{aligned} 0 &= \mathbb{E}\left[\varphi_n\left(\mathbf{X}_{[n]}\right) \mid \mathbf{X}_{[2,n]} = \mathbf{x}_{n-1}\right] \\ &= \varphi_n\left(0^{(j+1)}\right) \frac{\mathbb{P}_n\left(0^{(j+1)}\right)}{\mathbb{P}_{n-1}\left(0^{(j)}\right)} + \varphi_n\left(0^{(j)}\right) \frac{\mathbb{P}_n\left(0^{(j)}\right)}{\mathbb{P}_{n-1}\left(0^{(j)}\right)}, \end{aligned}$$

and therefore $\varphi_n\left(0^{(j+1)}\right) = -\left(\mathbb{P}_n\left(0^{(j)}\right)/\mathbb{P}_n\left(0^{(j+1)}\right)\right) \times \varphi_n\left(0^{(j)}\right)$. Arguing recursively on j, one has

$$\varphi_n\left(0^{(j+1)}\right) = (-1)^{j+1} \frac{\mathbb{P}_n\left(0^{(0)}\right)}{\mathbb{P}_n\left(0^{(j+1)}\right)} \varphi_n\left(0^{(0)}\right), \quad j = 0, ..., n-1, \tag{14}$$

showing that any symmetric kernel $\varphi_n \in \Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$ is completely determined by the quantity $\varphi_n\left(0^{(0)}\right)$. Now define a kernel $\varphi_n^{(0)} \in \Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$ by using (14) and by setting $\varphi_n^{(0)}\left(0^{(0)}\right) = \mathbb{P}_n\left(0^{(0)}\right)/\mathbb{P}_n\left(0^{(0)}\right) = 1$. It is easily seen that $\varphi_n^{(0)}$ must coincide with the function defined in (13). To conclude, consider another element φ_n of $\Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$. Since there exists a constant $K \in \Re$ such that $\varphi_n\left(0^{(0)}\right) = K = K\varphi_n^{(0)}\left(0^{(0)}\right)$, and since φ_n has to satisfy (14), we deduce that $\varphi_n = K\varphi_n^{(0)}$, thus completing the proof. ■

Given a function $f : \{0,1\}^m \to \Re$, we note $\widetilde{f}$ its canonical symmetrization, that is: for every $\mathbf{x}_m \in \{0,1\}^m$

$$\widetilde{f}(\mathbf{x}_m) = \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} f\left(\mathbf{x}_{\pi(m)}\right).$$

The following result will prove very useful.

Lemma 3 *Fix $m \geq 2$ and $v \in \{1, ..., m-1\}$ and let the application*

$$f_{v,m-v} : \{0,1\}^m \mapsto \Re : (x_1, ..., x_m) \mapsto f(x_1, ..., x_m),$$

be separately symmetric in the variables $(x_1, ..., x_v)$ and $(x_{v+1}, ..., x_m)$ (and not necessarily symmetric as a function on $\{0,1\}^m$). Then, for any $\mathbf{x}_m = (x_1, ..., x_m) \in \{0,1\}^m$ such that $\sum_{j=1}^m (1 - x_j) = z$ for some $z = 0, ..., m$,

$$\widetilde{f}_{v,m-v}(\mathbf{x}_m) = \frac{\sum_{k=0\vee(z-(m-v))}^{z\wedge v} \binom{v}{k}\binom{m-v}{z-k} f_{v,m-v}\left(0^{(k)}, 0^{(z-k)}\right)}{\sum_{k=0\vee(z-(m-v))}^{z\wedge v} \binom{v}{k}\binom{m-v}{z-k}}. \tag{15}$$

where $f_{v,m-v}\left(0^{(k)},0^{(z-k)}\right)$ *denotes the common value of* $f_{v,m-v}\left(\mathbf{y}_m\right)$ *when* $\mathbf{y}_m=(y_1,...,y_m)$ *is such that the vector* $(y_1,...,y_v)$ *contains exactly* k *zeros, and the vector* $(y_{v+1},...,y_m)$ *contains exactly* $(z-k)$ *zeros.*

As a consequence, $\widetilde{f}_{v,m-v}\left(\mathbf{x}_m\right)=0$ *for every* $\mathbf{x}_m\in\{0,1\}^m$ *if, and only if, for all* $z=0,...,m$*,*

$$\sum_{k=0\vee(z-(m-v))}^{z\wedge v}\binom{v}{k}\binom{m-v}{z-k}f_{v,m-v}\left(0^{(k)},0^{(z-k)}\right)=0. \tag{16}$$

Proof. Fix $\mathbf{x}_m\in\{0,1\}^m$ such that $\sum_{j=1}^m(1-x_j)=z$ for some $z=0,...,m$. Without loss of generality, we can assume

$$\mathbf{x}_m=(\underbrace{0,0,...,0}_{z\text{ times}},\underbrace{1,1,...,1}_{m-z\text{ times}}).$$

Observe that, for all $k=\max\{0,z-(m-v)\},...,\min\{z,v\}$, there are exactly $z!\,(m-z)!\binom{v}{k}\binom{m-v}{z-k}$ permutations $\pi\in\mathfrak{S}_m$ such that $\sum_{j=1}^v\left(1-x_{\pi(j)}\right)=k$ and $\sum_{j=v+1}^m\left(1-x_{\pi(j)}\right)=z-k$. The set of all such permutations will be denoted by $\mathfrak{S}_m^{(k)}$. It is immediately seen that

$$\begin{aligned}\widetilde{f}_{v,m-v}\left(\mathbf{x}_m\right) &= \frac{1}{m!}\sum_{k=0\vee(z-(m-v))}^{z\wedge v}\sum_{\pi\in\mathfrak{S}_m^{(k)}}f_{v,m-v}\left(0^{(k)},0^{(z-k)}\right)\\ &= \frac{1}{m!}\sum_{k=0\vee(z-(m-v))}^{z\wedge v}f_{v,m-v}\left(0^{(k)},0^{(z-k)}\right)\times\operatorname{card}\left(\mathfrak{S}_m^{(k)}\right).\end{aligned}$$

Formula (15) now follows by observing that

$$\frac{m!}{z!\,(m-z)!}=\binom{m}{z}=\sum_{k=0\vee(z-(m-v))}^{z\wedge v}\binom{v}{k}\binom{m-v}{z-k}.$$

The last assertion in the statement of this lemma is an easy consequence of (15). ■

We shall conclude the section by obtaining a full characterization of $\{0,1\}$-valued Hoeffding decomposable sequences (stated in Proposition 4 below).

To do this, recall that, for any symmetric $\varphi:\{0,1\}^n\mapsto\Re$, every $u=2,...,n$ and every $\mathbf{x}_{n-1}\in\{0,1\}^{n-1}$,

$$[\varphi]_{n,n-1}^{(n-u)}\left(\mathbf{x}_{n-1}\right)=\mathbb{E}\left(\varphi\left(\mathbf{X}_{[n]}\right)\mid\mathbf{X}_{[u+1,u+n-1]}=\mathbf{x}_{n-1}\right).$$

Observe that the function $[\varphi]_{n,n-1}^{(n-u)}:\{0,1\}^{n-1}\mapsto\Re$ clearly meets the symmetry properties of Lemma 3 with $m=n-1$ and $v=n-u$. Now fix $z\in\{0,...,n-1\}$, and suppose that $\mathbf{x}_{n-1}\in\{0,1\}^{n-1}$ is such that $\sum_{j=1}^{n-1}(1-x_j)=z$ and $\sum_{j=1}^{n-u}(1-x_j)=k$. Then,

$$[\varphi]_{n,n-1}^{(n-u)}\left(\mathbf{x}_{n-1}\right)=\sum_{m=0}^{u}\binom{u}{m}\varphi\left(0^{(k+m)}\right)\frac{\mathbb{P}_{n-1+u}\left(0^{(z+m)}\right)}{\mathbb{P}_{n-1}\left(0^{(z)}\right)}. \tag{17}$$

By applying (16) in the case $m=n-1$ and $v=n-u$, we deduce that $\widetilde{[\varphi]}_{n,n-1}^{(n-u)}\left(0^{(z)}\right)=0$ if, and only if,

$$\sum_{k=0\vee(z-(u-1))}^{z\wedge(n-u)}\binom{n-u}{k}\binom{u-1}{z-k}[\varphi]_{n,n-1}^{(n-u)}\left(0^{(k)},0^{(z-k)}\right)=0, \tag{18}$$

where the notation $\widetilde{[\varphi]}_{n,n-1}^{(n-u)}\left(0^{(z)}\right)$ and $[\varphi]_{n,n-1}^{(n-u)}\left(0^{(k)},0^{(z-k)}\right)$ has been introduced to indicate the value of $\widetilde{[\varphi]}_{n,n-1}^{(n-u)}\left(\mathbf{y}_{n-1}\right)$ (resp. $[\varphi]_{n,n-1}^{(n-u)}\left(\mathbf{w}_{n-1}\right)$), where $\mathbf{y}_{n-1}=(y_1,...,y_{n-1})\in\{0,1\}^{n-1}$ is any vector containing

exactly z zeros (resp. $\mathbf{w}_{n-1} = (w_1, ..., w_{n-1}) \in \{0,1\}^{n-1}$ is any vector containing exactly k zeros in $(w_1, ..., w_{n-u})$ and $z-k$ zeros in $(w_{n-u+1}, ..., w_{n-1})$).

Now recall that, by Theorem 0, $\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable if, and only if, it is weakly independent, and that $\mathbf{X}_{[1,\infty)}$ is weakly independent if, and only if, for all $n \geq 2$ and for any $\varphi \in \Xi_n\left(\mathbf{X}_{[1,\infty)}\right)$, one has $\varphi \in \widetilde{\Xi}_{n,u}\left(\mathbf{X}_{[1,\infty)}\right)$ for all $u = 2, ..., n$. By Lemma 2, we deduce that the sequence $\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable if, and only if, for every $n \geq 2$ and every $u = 2, ..., n$, $\varphi_n^{(0)} \in \widetilde{\Xi}_{n,u}\left(\mathbf{X}_{[1,\infty)}\right)$, where $\varphi_n^{(0)}$ is defined in (13). By (18), this last relation is true if, and only if, for every $n \geq 2$, every $z = 0, ..., n-1$ and every $u = 2, ..., n$,

$$\sum_{k=0\vee(z-(u-1))}^{z\wedge(n-u)} \binom{n-u}{k}\binom{u-1}{z-k}\left[\varphi_n^{(0)}\right]_{n,n-1}^{(n-u)}\left(0^{(k)}, 0^{(z-k)}\right) = 0. \tag{19}$$

Substituting (13) and (17) in (18), we obtain that (19) is true if, and only if,

$$0 = \frac{\mathbb{P}_n\left(0^{(0)}\right)}{\mathbb{P}_{n-1}\left(0^{(z)}\right)} \sum_{k=0\vee(z-(u-1))}^{z\wedge(n-u)} (-1)^k \binom{n-u}{k}\binom{u-1}{z-k} \times \tag{20}$$
$$\times \sum_{m=0}^{u} (-1)^m \binom{u}{m} \frac{\mathbb{P}_{n-1+u}\left(0^{(m+z)}\right)}{\mathbb{P}_n\left(0^{(m+k)}\right)}.$$

Note that

$$\frac{\mathbb{P}_{n-1+u}\left(0^{(m+z)}\right)}{\mathbb{P}_n\left(0^{(m+k)}\right)} = \frac{1}{\binom{u-1}{z-k}} \mathbb{P}_{n+u-1}^n\left(0^{(m+z)} \mid 0^{(m+k)}\right), \tag{21}$$

where $\mathbb{P}_{n+u-1}^n\left(0^{(m+z)} \mid 0^{(m+k)}\right)$ denotes the conditional probability that the vector $\mathbf{X}_{[n+u-1]}$ contains exactly $m+z$ zeros, given that the subvector $\mathbf{X}_{[n]}$ contains exactly $m+k$ zeros.

Remark. For every $n \geq 1$, $0 \leq a \leq b$, every $v \geq 1$, the quantity $\mathbb{P}_{n+v}^n\left(0^{(b)} \mid 0^{(a)}\right)$ is equal to

$$\mathbb{P}(\mathbf{X}_{[n+1,n+v]} \text{ contains exactly } b-a \text{ zeros} \mid \mathbf{X}_{[n]} \text{ contains exactly } a \text{ zeros}).$$

By plugging (21) into (20), we obtain the announced characterization of weak independence.

Proposition 4 *Let $\mathbf{X}_{[1,\infty)}$ be a non-deterministic infinite sequence of exchangeable $\{0,1\}$-valued random variables. For $\mathbf{X}_{[1,\infty)}$ to be Hoeffding decomposable, it is necessary and sufficient that, for every $n \geq 2$, every $u = 2, ..., n$ and every $z = 0, ..., n-1$,*

$$0 = \sum_{k=0\vee(z-(u-1))}^{z\wedge(n-u)} (-1)^k \binom{n-u}{k} \times \tag{22}$$
$$\times \sum_{m=0}^{u} (-1)^m \binom{u}{m} \mathbb{P}_{n+u-1}^n\left(0^{(m+z)} \mid 0^{(m+k)}\right)$$

As shown in the next section, Proposition 4 is the key tool to prove Theorem 1.

3 Proof of Theorem 1

Here is an outline of the proof. We already know (thanks to Peccati [2004, Corollary 9]) that, if $\mathbf{X}_{[1,\infty)}$ is either i.i.d. or Pólya, then it is also Hoeffding decomposable, thus proving the implication $2 \Rightarrow 1$. We shall therefore show that Hoeffding decomposability implies necessarily that $\mathbf{X}_{[1,\infty)}$ is either i.i.d.

or Pólya. The proof of this last implication is divided in four steps. By using some easy remarks (Step 1) and Proposition 4, we will prove that (22) implies a universal relation linking the moments of the de Finetti measure γ underlying any Hoeffding decomposable exchangeable sequence (Step 2). After a discussion concerning the moments of Beta random variables (Step 3), we conclude the proof in Step 4.

Step 1. We start with an easy remark. Define

$$S \triangleq \{(x,y,z) : 0 < x < y < z < 1\}, \tag{23}$$

as well as the two functions

$$\begin{aligned} f(x,y,z) &= 2x^2z - xy^2 - x^2y, \quad \text{and} & (24)\\ g(x,y,z) &= zx - 2y^2 + yz. & (25) \end{aligned}$$

Then, for any $(x,y,z) \in S$ one cannot have $f(x,y,z) = 0$ and $g(x,y,z) = 0$ simultaneously.

Step 2. Let $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$ be a non-deterministic exchangeable sequence with values in $\{0,1\}$, and let γ be the de Finetti measure uniquely associated to $\mathbf{X}_{[1,\infty)}$ through formula (10). We denote by

$$\mu_n = \mu_n(\gamma) = \int_{[0,1]} \theta^n \gamma(\mathrm{d}\theta), \quad n \geq 0, \tag{26}$$

the sequence of moments of γ (the dependence on γ is dropped when there is no risk of confusion). We shall prove the following statement: *if $\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable, then*

$$\mu_{n+1} g(\mu_n, \mu_{n-1}, \mu_{n-2}) = f(\mu_n, \mu_{n-1}, \mu_{n-2}), \quad n \geq 2, \tag{27}$$

where f and g are respectively defined by (24) and (25).

To prove (27), first recall that, due to Proposition 4, if $\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable, then formula (22) must hold for every $n \geq 2$, every $u = 2, ..., n$ and every $z = 0, ..., n-1$. In particular, it has to hold true for $u = 2$, i.e., for all $n \geq 2$ and all $z = 0, ..., n-1$, one must have that

$$\sum_{k=0\vee(z-1)}^{z\wedge(n-2)} (-1)^k \binom{n-2}{k} \sum_{m=0}^{2} (-1)^m \binom{2}{m} \mathbb{P}^n_{n+1}\left(0^{(m+z)} \mid 0^{(m+k)}\right) = 0, \tag{28}$$

for every $n \geq 2$ and every $z = 0, ..., n-1$. For $z = 0$, formula (28) becomes

$$\mathbb{P}^n_{n+1}\left(0^{(2)} \mid 0^{(2)}\right) - 2\mathbb{P}^n_{n+1}\left(0^{(1)} \mid 0^{(1)}\right) + \mathbb{P}^n_{n+1}\left(0^{(0)} \mid 0^{(0)}\right) = 0. \tag{29}$$

For $z = n-1$, (28) is equivalent to

$$\mathbb{P}^n_{n+1}\left(0^{(n)} \mid 0^{(n)}\right) - 2\mathbb{P}^n_{n+1}\left(0^{(n-1)} \mid 0^{(n-1)}\right) + \mathbb{P}^n_{n+1}\left(0^{(n-2)} \mid 0^{(n-2)}\right) = 0. \tag{30}$$

For $1 \leq z \leq n-2$, (28) becomes

$$\begin{aligned} 0 = \; & \binom{n-2}{z-1}\Big[\mathbb{P}^n_{n+1}\left(0^{(z+2)} \mid 0^{(z+1)}\right) \\ & \qquad - 2\mathbb{P}^n_{n+1}\left(0^{(z+1)} \mid 0^{(z)}\right) + \mathbb{P}^n_{n+1}\left(0^{(z)} \mid 0^{(z-1)}\right)\Big] \\ & - \binom{n-2}{z}\Big[\mathbb{P}^n_{n+1}\left(0^{(z+2)} \mid 0^{(z+2)}\right) \\ & \qquad - 2\mathbb{P}^n_{n+1}\left(0^{(z+1)} \mid 0^{(z+1)}\right) + \mathbb{P}^n_{n+1}\left(0^{(z)} \mid 0^{(z)}\right)\Big]. \end{aligned} \tag{31}$$

Combining (29), (30) and (31), we deduce that (28) is true if, and only if, for all $p = 0, ..., n-2$,

$$\mathbb{P}^n_{n+1}\left(0^{(p+2)} \mid 0^{(p+2)}\right) - 2\mathbb{P}^n_{n+1}\left(0^{(p+1)} \mid 0^{(p+1)}\right) + \mathbb{P}^n_{n+1}\left(0^{(p)} \mid 0^{(p)}\right) = 0. \tag{32}$$

Now, for $p = 0, ..., n-2$, write Δ_p to indicate the (forward) difference operator of order p, given by: $\Delta_0 f(n) = f(n)$, $\Delta_1 f(n) = f(n+1) - f(n)$, and

$$\Delta_p = \underbrace{\Delta_1 \circ \cdots \circ \Delta_1}_{p \text{ times}}.$$

For $p = 0, ..., n-2$ one has

$$\begin{aligned}\mathbb{P}^n_{n+1}\left(0^{(p)} \mid 0^{(p)}\right) &= \mathbb{P}^n_{n+1}\left(X_{n+1} = 1 \mid \mathbf{X}_{[n]} \text{ contains } p \text{ zeros}\right) \\ &= \frac{\Delta_p \mu_{n+1-p}}{\Delta_p \mu_{n-p}},\end{aligned}$$

where the sequence of moments μ_n, $n \geq 1$, is given by (26). Since (32) must hold for $p = 0$, we deduce that

$$\frac{\Delta_2 \mu_{n-1}}{\Delta_2 \mu_{n-2}} - 2\frac{\Delta_1 \mu_n}{\Delta_1 \mu_{n-1}} + \frac{\mu_{n+1}}{\mu_n} = 0,$$

and straightforward calculations yield relation (27).

Remark. Suppose that $\mathbf{X}_{[1,\infty)}$ is exchangeable and non-deterministic, and define μ_n, $n \geq 0$, via (26). Then, we have that $\mu_{n+1} \in (0,1)$ for every $n \geq 0$, and that, for every $n \geq 2$, $(\mu_n, \mu_{n-1}, \mu_{n-2}) \in S$, where S is defined as in (23). As a consequence, the conclusions of Step 1 and (27) imply that, if $\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable, then $f(\mu_n, \mu_{n-1}, \mu_{n-2}) \neq 0$ and $g(\mu_n, \mu_{n-1}, \mu_{n-2}) \neq 0$ for every $n \geq 2$. Therefore,

$$\mu_{n+1} = \frac{f(\mu_n, \mu_{n-1}, \mu_{n-2})}{g(\mu_n, \mu_{n-1}, \mu_{n-2})}. \tag{33}$$

Step 3. We claim that, for any $(c_1, c_2) \in (0,1)^2$ such that $c_1^2 < c_2 < c_1$, there exists a unique pair $(\alpha^*, \beta^*) \in (0, +\infty) \times (0, +\infty)$ such that

$$\mathbb{E}[\xi] = c_1 \quad \text{and} \quad \mathbb{E}\left[\xi^2\right] = c_2,$$

where ξ is a Beta random variable of parameters α^* and β^*. To check this, just observe that, if ξ is Beta of parameters α and β then

$$\mathbb{E}(\xi) = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \mathbb{E}\left(\xi^2\right) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)},$$

and that, for every fixed $(c_1, c_2) \in (0,1)^2$ such that $c_1^2 < c_2 < c_1$, the system

$$\begin{cases} \frac{\alpha}{\alpha+\beta} = c_1 \\ \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} = c_2 \end{cases}, \tag{34}$$

admits a unique solution $(\alpha^*, \beta^*) \in (0, +\infty) \times (0, +\infty)$: namely

$$\begin{cases} \alpha^* = \frac{c_1(1-c_2)}{c_2 - c_1^2}, \\ \beta^* = \frac{(1-c_1)(c_1-c_2)}{c_2-c_1^2}. \end{cases} \tag{35}$$

We are now in a position to conclude the proof of the implication $1 \Rightarrow 2$ in the statement of Theorem 1.

Step 4. Let $\mathbf{X}_{[1,\infty)}$ be a non-deterministic exchangeable sequence, denote by γ its de Finetti measure and by $\{\mu_n(\gamma) : n \geq 0\}$ the sequence of moments appearing in (26). We suppose that $\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable. There are only two possible cases: either $\mu_1(\gamma)^2 = \mu_2(\gamma)$, or $\mu_1(\gamma)^2 < \mu_2(\gamma)$. If $\mu_1(\gamma)^2 = \mu_2(\gamma)$, then necessarily $\gamma = \delta_x$ for some $x \in (0,1)$, and therefore $\mathbf{X}_{[1,\infty)}$ is a sequence of i.i.d. Bernoulli trials with common parameter equal to x. If $\mu_1(\gamma)^2 < \mu_2(\gamma)$, then, thanks to the results contained in Step 3 (note that $\mu_2(\gamma) < \mu_1(\gamma)$, since $\mathbf{X}_{[1,\infty)}$ is non-deterministic), there exists a unique pair $(\alpha^*, \beta^*) \in (0,+\infty) \times (0,+\infty)$ such that

$$\mu_1(\gamma) = \mathbb{E}(\xi) = \frac{1}{B(\alpha^*, \beta^*)} \int_0^1 \theta \theta^{\alpha^*-1} (1-\theta)^{\beta^*-1} \, \mathrm{d}\theta \tag{36}$$

$$\mu_2(\gamma) = \mathbb{E}(\xi^2) = \frac{1}{B(\alpha^*, \beta^*)} \int_0^1 \theta^2 \theta^{\alpha^*-1} (1-\theta)^{\beta^*-1} \, \mathrm{d}\theta, \tag{37}$$

where ξ stands for a Beta random variable of parameters α^* and β^*. Moreover, (33) and the fact that Pólya sequences are Hoeffding decomposable imply that, for any $n \geq 2$,

$$\mu_{n+1}(\gamma) = \frac{f(\mu_n(\gamma), \mu_{n-1}(\gamma), \mu_{n-2}(\gamma))}{g(\mu_n(\gamma), \mu_{n-1}(\gamma), \mu_{n-2}(\gamma))}, \quad \text{and}$$

$$\mathbb{E}(\xi^{n+1}) = \frac{f(\mathbb{E}(\xi^n), \mathbb{E}(\xi^{n-1}), \mathbb{E}(\xi^{n-2}))}{g(\mathbb{E}(\xi^n), \mathbb{E}(\xi^{n-1}), \mathbb{E}(\xi^{n-2}))},$$

where f and g are given by (24) and (25). As (36) and (37) are in order, we deduce that, for every $n \geq 1$,

$$\mu_n(\gamma) = \mathbb{E}(\xi^n) = \frac{1}{B(\alpha^*, \beta^*)} \int_0^1 \theta^n \theta^{\alpha^*-1} (1-\theta)^{\beta^*-1} \, \mathrm{d}\theta. \tag{38}$$

Since probability measures on $[0,1]$ are determined by their moments, the combination of (36), (37) and (38) gives

$$\gamma(\mathrm{d}\theta) = \frac{1}{B(\alpha^*, \beta^*)} \theta^{\alpha^*-1} (1-\theta)^{\beta^*-1} \, \mathrm{d}\theta,$$

implying that $\mathbf{X}_{[1,\infty)}$ is a Pólya sequence of parameters α^* and β^*. This concludes the proof of Theorem 1.

4 Further remarks

(I) With the terminology of Hill *et al.* [1987], a random sequence $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$, with values in $\{0,1\}$, is called an *urn process* if there exists a measurable function $f : [0,1] \mapsto [0,1]$ and positive natural numbers $r, b > 0$, such that, for every $n \geq 1$,

$$\mathbb{P}(X_{n+1} = 1 \mid X_1, ..., X_n) = f\left(\frac{r + X_1 + \cdots + X_n}{r + b + n}\right). \tag{39}$$

According to Theorem 1 in Hill *et al.* [1987], the only exchangeable and non-deterministic urn processes are i.i.d. and Pólya sequences with integer parameters (for which f is, respectively, constant and equal to the identity map). This yields immediately the following consequence of Theorem 1, showing that the two (seemingly unrelated) notions of urn process and Hoeffding decomposable sequence are in many cases equivalent. The proof can be achieved by using the calculations performed in Step 4.

Corollary 5 *Let $\mathbf{X}_{[1,\infty)} = \{X_n : n \geq 1\}$ be a $\{0,1\}$-valued infinite exchangeable non-deterministic sequence such that*

$$\mathbb{P}(X_1 = 1) = c_1 \qquad \textit{and} \qquad \mathbb{P}(X_1 = X_2 = 1) = c_2, \tag{40}$$

for some constants c_1 and c_2 such that $0 < c_1^2 < c_2 < c_1 < 1$. If the system (34) admits integer solutions, then $\mathbf{X}_{[1,\infty)}$ is Hoeffding decomposable if, and only if, it is an urn process.

In general, a sequence $\mathbf{X}_{[1,\infty)}$ verifying (40) is Hoeffding decomposable if, and only if, it is a Pólya sequence with parameters α^* and β^* given by (35).

(II) The arguments rehearsed in the proof of Theorem 1 provide an alternative proof of Theorem 5 in Diaconis and Ylvisaker [1979]. Indeed, in this reference it is shown that, if an exchangeable sequence is such that its predictive probabilities

$$\mathbb{P}_{n+1}^n\left(0^{(p)} \mid 0^{(p)}\right) = \mathbb{P}\left(X_{n+1} = 1 \mid \mathbf{X}_{[n]} \text{ contains exactly } p \text{ zeros}\right)$$

depend linearly on p, then its de Finetti measure must be Beta or Dirac. To see how Diaconis and Ylvisaker's result can be recovered using our techniques, suppose that a given exchangeable random sequence is not i.i.d. and is such that its predictive probabilities verify the equation

$$\mathbb{P}_{n+1}^n\left(0^{(p)} \mid 0^{(p)}\right) = a_n p + b_n \tag{41}$$

for some positive sequences $\{a_n\}$ and $\{b_n\}$. Then, it is immediately seen that $\mathbb{P}_{n+1}^n\left(0^{(p)} \mid 0^{(p)}\right)$ also verifies (32), and one deduces from the previous discussion that the associated de Finetti measure must be Beta. Conversely, if one supposes that the predictive probabilities $\mathbb{P}_{n+1}^n\left(0^{(p)} \mid 0^{(p)}\right)$ of an exchangeable non-i.i.d. sequence verify the difference equation (32), then one must conclude that the $\mathbb{P}_{n+1}^n\left(0^{(p)} \mid 0^{(p)}\right)$ verifies (41) and that

$$a_n = \frac{1}{1 + a\,(n-1)} \quad \text{and} \quad b_n = \frac{b}{1 + a\,(n-1)},$$

for some $a > 0$, $b > 0$ such that $a + b < 1$.

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References

[1] D.J. Aldous (1983). Exchangeability and related topics. *École d'été de Probabilités de Saint-Flour XIII*, **LNM** 1117, Springer, New York

[2] V. Bentkus, F. Götze. and W.R. Van Zwet (1997). An Edgeworth expansion for symmetric statistics. *Ann. Statist.*, **25**, 851-896.

[3] D. Blackwell (1973). Discreteness of Ferguson selections. *Annals of Statistics* **1** (2), 356-358

[4] D. Blackwell and J. MacQueen (1973). Ferguson distribution via Pólya urn schemes. *Annals of Statistics* **1** (2), 353-355

[5] M. Bloznelis (2005). Orthogonal decomposition of symmetric functions defined on random permutations. *Combinatorics, Probability and Computing*, **14**, 249-268

[6] M. Bloznelis and F. Götze (2001). Orthogonal decomposition of finite population statistics and its applications to distributional asymptotics. *The Annals of Statistics* **29** (3), 353-365

[7] M. Bloznelis and F. Götze (2002). An Edgeworth expansion for finite population statistics. *The Annals of Probability*, **30**, 1238-1265

[8] P. Diaconis and D. Yilvisaker (1979). Conjugate priors for exponential families. *The Annals of Statistics*, **7**, 269-281.

[9] J. Hájek (1968). Asymptotic normality of simple linear rank statistics under alternatives. *The Annals of Mathematical Statistics* **39**, 325-346

[10] B.H. Hill, D. Lane and W. Sudderth (1987). Exchangeable urn processes. *The Annals of Probability*, **15**(4), 1586-1592.

[11] S. Karlin and Y. Rinott (1982). Applications of ANOVA type decompositions for comparisons of conditional variance statistics including jackknife estimates. *The Annals of Statistics*, **10** (2), 485-501

[12] V.S. Koroljuk and Yu. V. Borovskich (1994). *Theory of U-Statistics*. Kluwer Academic Publishers, London

[13] A.Y. Lo (1991). A characterization of the Dirichlet process. *Statistics and Probability Letters* **12**, 185-187.

[14] G. Peccati (2003). Hoeffding decompositions for exchangeable sequences and chaotic representation for functionals of Dirichlet processes. *C.R.A.S. - Mathématiques*, Vol. **336**/10, 845-850.

[15] G. Peccati (2004). Hoeffding-ANOVA decompositions for symmetric statistics of exchangeable observations. *The Annals of Probability*, **32** (3A), 1796-1829.

[16] G. Peccati (2005). Multiple integral representation for functionals of Dirichlet processes. Preprint.

[17] J. Pitman (1996). Some developments of the Blackwell-MacQueen urn scheme. Dans : *Statistics, Probability and Game Theory: papers in honor of David Blackwell.* Volume 30 of Lecture Notes-Monograph Series. Institute of Mathematical Statistics, Hayward, California.

[18] J. Pitman (2006). *Combinatorial Stochastic Processes.* Springer-Verlag.

[19] R.A. Vitale (1990). Covariances of symmetric statistics. *Journal of Multivariate Analysis*, **41**, 14-26

[20] L. Zhao and X. Chen (1990). Normal approximation for finite-population U-statistics. *Acta Mathematicae Applicatae Sinica* **6** (3), 263-272