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# Dynkin's isomorphism without symmetry

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## 1 Introduction:

The purpose of this note is to extend Dynkin isomorphism involving functionals of the occupation field of a symmetric Markov processes with non polar points and of the associated Gaussian field to a suitable class of non symmetric Markov processes. This was briefly proposed in [5] using Grassmann variables, extending to the non symmetric case some results of [4]. Here we propose an alternative approach, not relying on Grassmann variables that can be applied to the study of local times, in the spirit of [6]. It works in general on a finite space and on an infinite space under some assumption on the skew symmetric part of the generator which is checked on two examples.

## 2 The finite case

In this section, we will prove two formulas given in Proposition 1 and Corollary 2 which relate the local time field of a non symmetric Markov process on a finite space to the square of the associated complex "twisted" Gaussian field. The result and the proof appear to be a direct extension of Dynkin's isomorphism.

### 2.1 Dual processes

Let us first consider the case of an irreducible Markov process on a finite space  $X$ , with finite lifetime  $\zeta$ , generator  $L$  and potential  $V = (-L)^{-1}$ .

Let  $m = \mu V$  for any nonnegative probability  $\mu$  on  $X$ . Recall that  $L$  can be written in the form  $L = q(I - \Pi)$  with  $q$  positive and  $\Pi$  submarkovian. Then the  $m$ -adjoint  $\hat{L}$  can be expressed similarly with the same  $q$  and a possibly different submarkovian matrix  $\hat{\Pi}$ . Moreover,  $m = \hat{\mu} \hat{V}$ , with  $\hat{\mu}$  the law of  $x_{\zeta-}$  under  $\mathbb{P}_\mu$ .

Note that for any  $z = x + iy \in \mathbb{C}^X$ , the "energy"  $\langle Lz, \bar{z} \rangle_m = \sum_x -(Lz)_x \bar{z}_x m_x$  is nonnegative as it can be written  $\frac{1}{2}(\sum C_{x,y}(z_x - z_y)(\bar{z}_x - \bar{z}_y) + \langle \Pi 1 + \hat{\Pi} 1 - 2, z\bar{z} \rangle_m)$ , with  $C_{x,y} = C_{y,x} = m_x q_x \Pi_{x,y}$ . The highest eigenvector of  $\frac{1}{2}(\Pi + \hat{\Pi})$  is nonnegative by the well known argument which shows that the module contraction lowers the energy. and it follows from the strict submarkovianity that

the corresponding eigenvalue is strictly smaller than 1. Hence there is a "mass gap": For some positive  $\varepsilon$ , the "energy"  $\langle -Lz, \bar{z} \rangle_m$  dominates  $\varepsilon \langle z, \bar{z} \rangle_m$  for all  $z$ .

## 2.2 A twisted Gaussian measure

Then, although  $L$  is not symmetric, an elementary computation (given in a more general context in the following section) shows that for any  $\chi \in \mathbb{R}_+^X$ , denoting  $M_\chi$  the diagonal matrix with coefficients given by  $\chi$ ,

$$\frac{1}{(2\pi)^{|X|}} \int (e^{-\langle z\bar{z}, \chi \rangle_m} e^{\frac{1}{2}\langle Lz, \bar{z} \rangle_m} \Pi dx_u dy_u = \det(-M_m L + M_{\chi m})^{-1}$$

. As a consequence, differentiating with respect to  $\chi_x$ ,

$$\frac{1}{(2\pi)^{|X|}} \int z_x \bar{z}_x (e^{-\langle z\bar{z}, \chi \rangle_m} e^{\frac{1}{2}\langle Lz, \bar{z} \rangle_m} \Pi dx_u dy_u = \det(-M_m L + M_{\chi m})^{-1} \frac{1}{m_x} (-L + M_\chi)_{xx}^{-1}$$

In a similar way by perturbation using a non diagonal matrix, one obtains

$$\frac{1}{(2\pi)^{|X|}} \int z_x \bar{z}_y (e^{-\langle z\bar{z}, \chi \rangle_m} e^{\frac{1}{2}\langle Lz, \bar{z} \rangle_m} \Pi dx_u dy_u = \det(-M_m L + M_{\chi m})^{-1} \frac{1}{m_y} (-L + M_\chi)_{xy}^{-1}$$

But with the usual notations for Markov processes, setting  $l_t^x = \int_0^{t \wedge \zeta} 1_{\{x_s=x\}} \frac{1}{m_{x_s}} ds$  and  $l_\zeta^x = l^x$ , we have

$$\frac{1}{m_y} (-L + M_\chi)_{xy}^{-1} = \mathbb{E}_x \left( \int_0^{\zeta} e^{-\langle \chi, l_t^y \rangle_m} dl_t^y \right)$$

Defining the path measure  $\mathbb{E}_{x,y}$  by:  $\mathbb{E}_x \left( \int_0^{\zeta} G(x_s, s \leq t) dl_t^y \right) = \mathbb{E}_{x,y}(G)$  the above relation writes

$$\frac{1}{m_y} (-L + M_\chi)_{xy}^{-1} = \mathbb{E}_{x,y}(e^{-\langle \chi, l^y \rangle_m})$$

It follows that we have proved the following:

**Proposition 1** For any continuous function  $F$  on  $\mathbb{R}_+^X$

$$\begin{aligned} (*) \quad & \int z_x \bar{z}_y F(z_u \bar{z}_u, u \in X) e^{\frac{1}{2}\langle Lz, \bar{z} \rangle_m} \Pi dx_u dy_u \\ & = \int \mathbb{E}_{x,y}(F(l^u + z_u \bar{z}_u, u \in X)) e^{\frac{1}{2}\langle Lz, \bar{z} \rangle_m} \Pi dx_u dy_u \end{aligned}$$

### 2.3 Positivity

It should be noted that setting  $\rho_u = \frac{1}{2}z_u\bar{z}_u$  and  $z_x = \sqrt{\rho_x/2}e^{i\theta_x}$  the image on  $\mathbb{R}_+^X$  of the normalized complex measure  $\nu_X = \frac{1}{(2\pi)^{|X|}} \det(-M_m L) e^{\frac{1}{2}\langle Lz, \bar{z} \rangle_m} \prod dx_u dy_u$  by the map  $z_u \rightarrow \rho_u$  is an infinitely divisible probability distribution  $Q$  on  $\mathbb{R}_+^X$  with density  $\frac{1}{(2\pi)^{|X|}} \det(-M_m L) \int e^{\langle L\sqrt{\rho}e^{i\theta}, \sqrt{\rho}e^{-i\theta} \rangle_m} \prod d\theta_u$ . Note that the positivity is not a priori obvious when  $L$  is not  $m$ -symmetric. This important fact follows easily by considering the moment generating function  $\Phi(s) = \frac{\det(-L)}{\det(-L+M_s)} = \det(I + (-L)^{-1}M_s)^{-1}$  defined for all  $s$  with non negative coordinates, positive and analytic. The expansion in power series around any  $s$  (which appears for example in [8]) is explicit:

$$\begin{aligned} \frac{\Phi(s+h)}{\Phi(s)} &= \det(I + (-L+M_s)^{-1}M_h)^{-1} = \exp(-\log(\det(I + (-L+M_s)^{-1}M_h))) \\ &= \exp(\sum \frac{(-1)^k}{k} Tr[(-L+M_s)^{-1}M_h]^k). \end{aligned}$$

As  $(-L+M_s)^{-1}$  is nonnegative, it implies that  $\Phi$  is completely monotone as in this last expression, all coefficients of  $h$ -monomials of order  $n$  are of the sign of  $(-1)^n$ .<sup>1</sup>

Note that the same argument works for fractional powers of  $\Phi(s)$  which shows the infinite divisibility. Let us incidentally mention it has been known for a long time (cf [8]) that this expansion can be simplified further in terms of permanents.

For  $x = y$ , the above proposition then yields the following:

**Corollary 2** *For any continuous function  $F$  on  $\mathbb{R}_+^X$*

$$(**) \int \rho_x F(\rho_u, u \in X) Q(d\rho) = \int \mathbb{E}_{x,x}(F(l^u + \rho_u, u \in X)) Q(d\rho)$$

Note that this last formula is also obtained in [3] after a direct definition of the measure  $Q$ .

Another interpretation of this positivity and of infinite divisibility can be given in terms of a Poisson process of loops. It will be developed in a forthcoming paper but let us simply mention that  $Q$  appears to be equal to the distribution of the occupation field associated with the Poisson process of loops canonically defined by the Markov chain.

REMARK: If  $Y \subset X$ , it is well known that the trace of the process on  $Y$  is a Markov process the potential of which is the restriction of  $V$  to  $Y \times Y$ . The distribution  $\nu_X$  induces  $\nu_Y$ . Therefore the formulas (\*) and (\*\*) on  $X$  and  $Y$  are consistent.

EXAMPLE: let us consider, as an example, the case where  $X = \{1, 2, \dots, N\}$ ,  $q_i = 1$ ,  $\Pi_{i,j} = \mathbf{1}_{i < N, j = i+1}$ ,  $\mu_i = \mathbf{1}_{i=1}$ ,  $m_i = 1$  and  $(-L)_{i,j}^{-1} = \mathbf{1}_{i \leq j}$ .

The characteristic polynomial of  $\frac{1}{2}(\Pi + \widehat{\Pi})$  is  $(-\lambda + \sqrt{\lambda^2 - 1})^N + (-\lambda - \sqrt{\lambda^2 - 1})^N$  hence one gets easily that the mass gap equals  $2 \sin^2(\frac{\pi}{2N})$

<sup>1</sup>Completely monotone functions in several variables were already used in [1].

Under  $\mathbb{E}_{x,x}$  all local times vanish except  $l^x$  which follows an exponential distribution. Moreover an easy calculation shows that  $Q$  reduces to a product of exponential distributions. The formula (\*\*\*) reduces to the convolution of two exponentials.

### 3 The infinite case

We now explain how in certain situations, the above can be extended to a Markov process on an infinite space  $X$ . Of course, a Markov process for which points are not polar can always be viewed elementarily as a consistent system of processes on finite subspaces but we aim at a stronger representation allowing to consider any functional of the occupation field. There are some obvious obstructions to a generalization. The mass gap property does not always hold: consider for example the case of a constant drift on an interval, analogous to the above example. Some conditions have to be assumed in order that the energy controls the skew-symmetric part of the generator.

#### 3.1 Some calculations in Gaussian space

Let  $H$  be a real Hilbert space with scalar product  $\langle, \rangle$ . At first the reader may suppose it finite dimensional and then check that the assumptions we will make allow to extend the results to the infinite dimensional case.

Let  $\phi$  be the canonical Gaussian field indexed by  $H$ . Given any ONB  $e_k$  of  $H$ ,  $w_k = \phi(e_k)$  are independent normal variables. Recall that for all  $f \in H$ ,  $\phi(f) = \sum_k \langle f, e_k \rangle w_k$  and  $E(e^{i\phi(f)}) = e^{-\frac{\|f\|^2}{2}}$ .

In the following,  $\phi(f)$  can be denoted by  $\langle \phi, f \rangle$  though of course  $\phi$  does not belong to  $H$  in general.

Let  $K$  be any Hilbert-Schmidt operator on  $H$ . Note that  $K\phi = \sum_k w_k K e_k$  is well defined as a  $H$ -valued random variable, and that  $E(\|K\phi\|^2) = \text{Tr}(KK^*)$ .

Let  $C$  be a symmetric non negative trace-class linear operator on  $H$ . Recall that the positive integrable random variable  $\langle C\phi, \phi \rangle \in \mathbb{L}^1$  can be defined by  $\sum_k \langle C e_k, e_k \rangle w_k^2$  for any ONB diagonalizing  $C$ .

Moreover,  $E(e^{-\frac{1}{2}\langle C\phi, \phi \rangle + i\phi(f)}) = \det(I + C)^{-\frac{1}{2}} e^{-\frac{\langle (I+C)^{-1} f, f \rangle}{2}}$  (the determinant can be defined as  $\prod(1 + \lambda_i)$ , where the  $\lambda_i$  are the eigenvalues of  $C$ ).

In fact  $\det(I + T)$  is well defined for any trace class operator  $T$  as  $1 + \sum_{n=1}^{\infty} \text{Tr}(T^n)$  (Cf [7] Chapter 3). It extends continuously the determinant defined with finite ranks operators and it verifies the identity:

$\det(I + T_1 + T_2 + T_1 T_2) = \det(I + T_1) \det(I + T_2)$ . By Lidskii's theorem, it is also given by the product  $\prod(1 + \lambda_i)$  defined by the eigenvalues of the trace class (hence compact) operator  $T$ , counted with their algebraic multiplicity.

Let  $\phi_1$  and  $\phi_2$  be two independent copies of the canonical Gaussian process indexed by  $H$ . Let  $B$  be a skew-symmetric Hilbert-Schmidt operator on  $H$ . Note that  $\langle B\phi_1, \phi_2 \rangle = \sum \langle B e_k, e_l \rangle w_k^1 w_l^2 = -\langle B\phi_2, \phi_1 \rangle$  is well defined in  $\mathbb{L}^2$  and that  $E(e^{i\langle B\phi_1, \phi_2 \rangle}) = E(e^{-\frac{1}{2}\|B\phi_1\|^2}) = \det(I + BB^*)^{-\frac{1}{2}}$ .

As  $B$  is Hilbert-Schmidt,  $BB^*$  is trace-class. The renormalized determinant  $\det_2(I+B) = \det((I+B)e^{-B})$  is well defined (Cf [7]), and as the eigenvalues of  $B$  are purely imaginary and pairwise conjugated, it is strictly positive. Moreover since  $B^* = -B$ ,  $\det_2(I+B) = \det_2(I-B) = \det(I+BB^*)^{\frac{1}{2}}$

Finally, it comes that  $E(e^{i\langle B\phi_1, \phi_2 \rangle}) = \det_2(I+B)^{-1}$

More generally, setting  $\psi = \phi_1 + i\phi_2$ ,

$$E(e^{-\frac{1}{2}\langle (C-B)\psi, \bar{\psi} \rangle}) = \det_2(I+C+B)^{-1} \exp(-Tr(C))$$

(Recall that when  $T$  is trace class,  $\det_2(I+T) \exp(Tr(T)) = \det(I+T)$ ).

Indeed,  $E(e^{-\frac{1}{2}\langle (C-B)\psi, \bar{\psi} \rangle}) = E(e^{-\frac{1}{2}\langle C\phi_1, \phi_1 \rangle - \frac{1}{2}\langle C\phi_2, \phi_2 \rangle + i\langle B\phi_1, \phi_2 \rangle})$

$$= \det(I+C)^{-\frac{1}{2}} E(e^{-\frac{1}{2}\langle (I+C)^{-1}B\phi_1, B\phi_1 \rangle + \langle C\phi_1, \phi_1 \rangle}) \text{ (by integration in } \phi_2)$$

$$= \det(I+C)^{-\frac{1}{2}} \det(I+C-B(I+C)^{-1}B)^{-\frac{1}{2}}$$

$$= \det(I-(I+C)^{-\frac{1}{2}}B(I+C)^{-1}B(I+C)^{-\frac{1}{2}})^{-\frac{1}{2}} \det(I+C)^{-1}$$

$$= \det((I+(I+C)^{-\frac{1}{2}}B(I+C)^{-\frac{1}{2}})(I-(I+C)^{-\frac{1}{2}}B(I+C)^{-\frac{1}{2}}))^{-\frac{1}{2}} \det(I+C)^{-1}$$

$$= \det_2(I+(I+C)^{-\frac{1}{2}}B(I+C)^{-\frac{1}{2}})^{-1} \det(I+C)^{-1} \text{ (since } \det_2(I+(I+C)^{-\frac{1}{2}}B(I+C)^{-\frac{1}{2}}) = \det_2(I-(I+C)^{-\frac{1}{2}}B(I+C)^{-\frac{1}{2}}) \text{ by skew symmetry as before)}$$

$$= \det_2(I+C+B)^{-1} \exp(-Tr(C)).$$

Note that  $I+C+B$  is always invertible, as  $C+B$  is a compact operator and  $-1$  is not an eigenvalue.

Let  $f_1$  and  $f_2$  be two elements of  $H$ . Set  $D(f) = \langle f, f_1 \rangle f_2$ . For small enough  $\varepsilon$ ,

$$E(e^{-\frac{1}{2}\langle (C-B)\psi, \bar{\psi} \rangle + \varepsilon\psi(f_1)\bar{\psi}(f_2)})$$

$$= \det_2(I+C+B+\varepsilon D)^{-1} \exp(-Tr(C+\varepsilon D))$$

$$= \det_2(I+C+B)^{-1} \exp(-Tr(C)) \det(I+\varepsilon(I+C+B)^{-1}D)^{-1}.$$

Hence, differentiating both members at  $\varepsilon = 0$ ,

$$E(\psi(f_1)\bar{\psi}(f_2)e^{-\frac{1}{2}\langle (C-B)\psi, \bar{\psi} \rangle}) = \det_2(I+C+B)^{-1} \exp(-Tr(C)) Tr((I+C+B)^{-1}D)$$

Therefore

$$\frac{E(\psi(f_1)\bar{\psi}(f_2)e^{-\frac{1}{2}\langle (C-B)\psi, \bar{\psi} \rangle})}{E(e^{-\frac{1}{2}\langle (C-B)\psi, \bar{\psi} \rangle})} = \langle (I+C+B)^{-1}(f_1), f_2 \rangle.$$

If  $C$  is only Hilbert-Schmidt, we can consider only the renormalized "Wick square"  $:\langle C\phi, \phi \rangle := \sum_k \langle Ce_k, e_k \rangle (w_k^2 - 1)$  for any ONB diagonalizing  $C$  and  $E(e^{-\frac{1}{2}:\langle C\phi, \phi \rangle + i\phi(f)}) = \det_2(I+C)^{-\frac{1}{2}} e^{-\frac{\langle (I+C)^{-1}f, f \rangle}{2}}$

The results given above extend immediately as follows:

$$E(e^{-\frac{1}{2}:\langle (C-B)\psi, \bar{\psi} \rangle :}) = \det_2(I+C+B)^{-1}$$

$$\frac{E(\psi(f_1)\bar{\psi}(f_2)e^{-\frac{1}{2}:\langle (C-B)\psi, \bar{\psi} \rangle :})}{E(e^{-\frac{1}{2}:\langle (C-B)\psi, \bar{\psi} \rangle :})} = \langle (I+C+B)^{-1}(f_1), f_2 \rangle.$$

### 3.2 A class of Markov processes in duality

Let  $(V_\alpha, \alpha \geq 0)$  and  $\widehat{V}_\alpha$  be two Markovian or submarkovian resolvents in duality in a space  $\mathbb{L}^2(X, \mathcal{B}, m)$  with generators  $L$  and  $\widehat{L}$ , such that:

1) Denoting  $\mathcal{D} = \mathcal{D}(L) \cap \mathcal{D}(\widehat{L})$ ,  $L(\mathcal{D})$  is dense in  $\mathbb{L}^2(m)$

2)  $\langle -Lf, f \rangle_m \geq \varepsilon \langle f, f \rangle_m$  for some  $\varepsilon > 0$  and any  $f \in \mathcal{D}$  (i.e. we assume the existence of a spectral gap: it can always be obtained by adding a negative constant to  $L$ ).

Let  $H$  be the completion of  $\mathcal{D}$  with respect to the energy norm. It is a functional space imbedded in  $\mathbb{L}^2(X, \mathcal{B}, m)$ . Let  $A$  be the associated self adjoint generator so that  $H = \mathcal{D}(\sqrt{-A})$ . On  $\mathcal{D}$ ,  $\frac{1}{2}(L + \widehat{L}) = A$ .

The final assumption is crucial to allow the control of the antisymmetric part:

3)  $B = (-A)^{-1} \frac{L - \widehat{L}}{2}$  is a Hilbert-Schmidt operator on  $H$ .

Equivalently,  $(-A)^{-\frac{1}{2}} \frac{L - \widehat{L}}{2} (-A)^{-\frac{1}{2}}$  is an antisymmetric Hilbert Schmidt operator on  $\mathbb{L}^2(m)$  since for any ONB  $e_k$  of  $H$ ,  $(-A)^{\frac{1}{2}} e_k$  is an ONB of  $\mathbb{L}^2(m)$ . Note that  $I - B$  is bounded and invertible on  $H$  and that  $V_0 = (I - B)^{-1} (-A)^{-1}$  maps  $\mathbb{L}^2(m)$  into  $H$ . Indeed, one can see first that on  $\mathcal{D}$ ,  $A(I - B) = L$  so that on  $L(\mathcal{D})$ ,  $V_0 = (I - B)^{-1} (-A)^{-1}$

#### EXAMPLES

This applies to the case of the finite space considered above.

Let us mention other examples:

1) Diffusion with drift on the circle:  $X = S^1$ ,  $A = \frac{\partial^2}{\partial \theta^2} - \varepsilon$ ,  $L - \widehat{L} = b(\theta) \frac{\partial}{\partial \theta}$ , where  $b$  is a bounded function on  $S^1$ .

Indeed, considering the orthonormal basis  $e^{ik\theta}$  in  $\mathbb{L}^2(d\theta)$ ,  $\frac{1}{\sqrt{k^2 + \varepsilon}} e^{ik\theta}$  is an orthonormal basis in  $H = H^1$ , and

$$\sum_k \left\| (-A)^{-\frac{1}{2}} b(\theta) \frac{\partial}{\partial \theta} \frac{1}{\sqrt{k^2 + \varepsilon}} e^{ik\theta} \right\|_{\mathbb{L}^2(d\theta)}^2 = \sum_{k,l} \frac{k^2}{k^2 + \varepsilon} (\widehat{b}(l - k))^2 \frac{1}{l^2 + \varepsilon} < \infty$$

2) Levy processes on the circle: The Fourier coefficients  $a_k + ib_k$  of  $L$  should verify  $\sum_k \left(\frac{b_k}{a_k}\right)^2 < \infty$

### 3.3 An extension of Dynkin's isomorphism

Assume  $X$  is locally compact and separable, and that functions of  $H$  are continuous. By the Banach-Steinhaus theorem, given any point  $x \in X$ , there exists an element of  $H$ , denoted  $\eta_x$  defined by the identity:  $f(x) = \langle \eta_x, f \rangle_H$ . Note that  $\eta_x = \sum e_k(x) e_k$  for any orthonormal basis of  $H$ .

The resolvent  $V_\lambda$  is necessarily Fellerian and induces a strong Markov process. Denote by  $l_t^x$  the local time at  $x$  of this Markov process. Let  $x$  and  $y$  be two points of  $X$ . Set:  $\langle \eta_x, \eta_y \rangle_H = \sum e_k(x) e_k(y) = K(x, y)$  so that  $A^{-1} f(x) = \int K(x, y) f(y) m(dy)$  or  $A^{-1} f = \int \eta_y f(y) m(dy)$ .

Set  $V_0(x, y) = \langle I + B \rangle^{-1} \eta_x, \eta_y \rangle_H$  and note that  $V_0(x, y)$  is a kernel for  $V_0$ . Indeed, for any  $f, g \in \mathbb{L}^2(m)$ ,  $\langle V_0 f, g \rangle_{\mathbb{L}^2(m)} = \langle (I - B)^{-1} (-A)^{-1} f, g \rangle_{\mathbb{L}^2(m)} = \langle (I - B)^{-1} (-A)^{-1} f, A^{-1} g \rangle_H = \int f(x) g(y) V_0(x, y) m(dx) m(dy)$

As a consequence,  $V_0(x, y) = \mathbb{E}_x(l_\zeta^y)$ .

Applying the construction of the section 3-1, we see the kernel  $K(x, y)$  is the covariance of a Gaussian process  $(Z_x = \psi(\eta_x) = \sum e_k(x)w_k, x \in X)$ , for any ONB  $e_k$  of  $H$ .

More generally, for any non-negative finitely supported measure  $\chi = \sum_1^N p_j \delta_{u_j}$  on  $X$ , letting  $C$  be the finite rank operator:  $C = \sum_1^N p_j \eta_{u_j} \otimes \eta_{u_j}$ , set  $V_\chi(x, y) = \langle (I - B + C)^{-1} \eta_x, \eta_y \rangle_H$

In a similar way as above for  $V_0$ , we have  $V_\chi(x, y) = \mathbb{E}_x(e^{-\int l_i^z \chi(dz)} dl_i^y)$ .

Then, from section 3-1

$$\begin{aligned} E(Z_x \bar{Z}_y) e^{-\frac{1}{2} \langle (C-B)\psi, \bar{\psi} \rangle_H} &= E(e^{-\frac{1}{2} \langle (C-B)\psi, \bar{\psi} \rangle_H}) \langle (I-B+C)^{-1} \eta_x, \eta_y \rangle_H \\ &= E(e^{-\frac{1}{2} \langle (C-B)\psi, \bar{\psi} \rangle_H}) V_\chi(x, y) \end{aligned}$$

But  $\langle C\psi, \bar{\psi} \rangle_H = \int \psi(\eta_u) \bar{\psi}(\eta_u) \chi(du) = \int Z_u \bar{Z}_u \chi(du)$  and  $\langle B\psi, \bar{\psi} \rangle_H = \langle (-A)^{-1} \frac{L-\hat{L}}{2} \psi, \bar{\psi} \rangle_H = \sum \langle (-A)^{-1} \frac{L-\hat{L}}{2} e_k, e_l \rangle_H w_k^1 w_l^2$ .

On the other hand, at least formally in general but exactly in the finite dimensional case,

$$\int \frac{L-\hat{L}}{2} Z_u \bar{Z}_u m(du) = \langle (-A)^{-1} \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_H = \sum w_k^1 w_l^2 \langle (-A)^{-1} \frac{L-\hat{L}}{2} e_k, e_l \rangle_H$$

Hence we can denote:  $\langle B\psi, \bar{\psi} \rangle_H$  by  $\langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)}$ . Therefore

$$\begin{aligned} E(Z_x \bar{Z}_y) e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)} - \int Z_u \bar{Z}_u \chi(du)} \\ = E \otimes \mathbb{E}_x \left( \int_0^\zeta e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)}} e^{-\int l_i^z \chi(dz) - \int Z_u \bar{Z}_u \chi(du)} dl_i^y \right) \end{aligned}$$

Let  $\mathbb{E}_{x,y}^t$  denote the non-normalized law of the bridge of duration  $t$  from  $x$  to  $y$ . Set  $\mu_{x,y} = \int_0^\infty \mathbb{E}_{x,y}^t dt$ , so that  $\mu_{x,y}(1) = \mathbb{E}_x(l_\zeta^y) = V_0(x, y)$  and denote  $l^z$  the local time at  $z$  for the duration of the bridge.

$$\begin{aligned} \text{Then } E(Z_x \bar{Z}_y) e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)} - \int Z_u \bar{Z}_u \chi(du)} \\ = E \left( e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)} - \int Z_u \bar{Z}_u \chi(du)} \int e^{-\int l^z \chi(dz)} d\mu_{x,y} \right) \end{aligned}$$

Finally, we get that for any continuous bounded function  $F$  of  $N$  non negative real coordinates, and any  $N$ -uple of points  $u_j$  in  $X$ ,

$$E(Z_x \bar{Z}_y) e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)}} F(Z_{u_j} \bar{Z}_{u_j}) = E \otimes \mu_{x,y} \left( e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)}} F(l^{u_j} + Z_{u_j} \bar{Z}_{u_j}) \right)$$

and the formula finally extends to

**Proposition 3** For any bounded measurable function of a real field on  $X$  :

$$(*bis) E(Z_x \bar{Z}_y) e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)}} F(Z \bar{Z}) = E \otimes \mu_{x,y} \left( e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)}} F(l + Z \bar{Z}) \right)$$

It induces the formula (\*) obtained in the finite case if we consider the trace of the process on any finite subset.

We see also that the restriction of the twisted Gaussian measure

$$e^{\frac{1}{2} \langle \frac{L-\hat{L}}{2} Z, \bar{Z} \rangle_{\mathbb{L}^2(m)}} P(dZ)$$

to  $\sigma(Z_u \bar{Z}_u, u \in X)$  is a probability measure  $Q$  under which the distribution of the process  $(Z_u \bar{Z}_u, u \in X)$  is infinitely divisible. Moreover, the important fact



is that this probability is absolutely continuous with respect to the restriction of  $P$  to  $\sigma(Z_u \overline{Z}_u, u \in X)$ . It is clear that that formula (\*\*) of the corollary extends in the same way to yield the following

**Corollary 4** *For any bounded measurable function of a nonnegative field on  $X$  :*

$$(** \text{ bis}) \int \rho_x F(\rho_u, u \in X) Q(d\rho) = \int \mu_{x,x}(F(l^u + \rho_u, u \in X)) Q(d\rho)$$

Hence it follows for example that the continuity of the Gaussian field  $Z$  implies the continuity of the local time field  $l$  under all loop measures  $\mu_{xx}$ .

Note finally that these results, as in the symmetric case, can be extended to some situations where the local time does not exist (like the two dimensional Brownian motion), by considering the centered occupation field and the "Wick square" :  $Z_x \overline{Z}_x$  : (formally given by  $Z_x \overline{Z}_x - K(x, x)$ ) as generalized random fields. This makes sense for the Wick square provided  $K$  is a Hilbert-Schmidt operator.

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