Optimized Schwarz Methods for Maxwell equations

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OPTIMIZED SCHWARZ METHODS FOR MAXWELL EQUATIONS

V.DOLEAN*, M.J.GANDER† AND L.GERARDO-GIORDA‡

Abstract. Over the last two decades, classical Schwarz methods have been extended to systems of hyperbolic partial differential equations, and it was observed that the classical Schwarz method can be convergent even without overlap in certain cases. This is in strong contrast to the behavior of classical Schwarz methods applied to elliptic problems, for which overlap is essential for convergence. Over the last decade, optimized Schwarz methods have been developed for elliptic partial differential equations. These methods use more effective transmission conditions between subdomains, and are also convergent without overlap for elliptic problems.

We show here why the classical Schwarz method applied to the hyperbolic problem converges without overlap for the Maxwell's equations. The reason is that the method is equivalent to a simple optimized Schwarz method for an equivalent elliptic problem. Using this link, we show how to develop more efficient Schwarz methods than the classical ones for the Maxwell's equations. We illustrate our findings with numerical results.

Key words. Schwarz algorithms, optimized interface conditions, Maxwell equations

AMS subject classifications.

1. Introduction. Schwarz algorithms experienced a second youth over the last decades, when distributed computers became more and more performant and available. Fundamental convergence results for the classical Schwarz methods were derived for many partial differential equations, and can now be found in several authoritative reviews, see [2, 38, 37], and books, see [32, 30, 36]. The Schwarz methods were also extended to systems of partial differential equations, such as the time harmonic Maxwell equations, see [6, 10, 1], or to linear elasticity [16, 17], but much less is known about the behavior of the Schwarz methods applied to systems of equations. This is true in particular for the Euler equations, to which the Schwarz algorithm was first applied in [28, 29], where classical (characteristic) transmission conditions are used at the interfaces, or with more general interface conditions in [5]. The analysis of such algorithms applied to systems proved to be very different from the scalar case, see [13, 14].

Over the last decade, a new class of Schwarz methods was developed for scalar partial differential equations, namely the optimized Schwarz methods. These methods use more effective transmission conditions than the classical Dirichlet conditions at the interfaces between subdomains. New transmission conditions were originally proposed for three different reasons: first, to obtain Schwarz algorithms that are convergent without overlap, see [25] for Robin conditions. The second motivation for changing the transmission conditions was to obtain a convergent Schwarz method for the Helmholtz equation, where the classical Schwarz algorithm is not convergent, even with overlap. Approximate radiation conditions were introduced in [8, 11]. The third motivation was that the convergence rate of the classical Schwarz method is rather slow and very much dependent on the size of the overlap. In a short note on non-linear problems [23], Hagstrom et al. introduced Robin transmission conditions between subdomains

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and suggested nonlocal operators for best performance. In [3], these optimal, nonlocal transmission conditions were developed for advection-diffusion problems, with local approximations for small viscosity, and low order frequency approximations in [26, 7]. In [33] we can find the low-frequency approximation of the absorbing boundary conditions for Euler equations. Independantly, at the algebraic level, generalized coupling conditions were introduced in [35, 34]. Optimized transmission conditions for the best performance of the Schwarz algorithm were introduced for advection diffusion problems in [24], for the Helmholtz equation in [4, 22], and for Laplace's equation in [15]. For complete results and attainable performance for symmetric, positive definite problems, see [19], and for time dependant problems, see [21, 20].

We show in this paper that the classical Schwarz method, which uses characteristic Dirichlet transmission conditions between subdomains, applied to the Maxwell's equations, is equivalent to an optimized Schwarz method applied to well known equivalent elliptic problems. This explains why the classical Schwarz method in that case can be convergent even without overlap, and it allows us to develop more effective Schwarz methods for these systems of partial differential equations. Following the ideas first showed in [paperDD17], in Section 2, we show that the results obtained for the model problem of the Cauchy Riemann equations can be generalized to the Maxwell equations. We prove again that using classical, characteristic conditions in a Schwarz algorithm applied to Maxwell's equations corresponds to an optimized Schwarz method for an equivalent scalar problem, both for the time harmonic and the time discretized case. We then derive better transmission conditions than the characteristic ones for the Schwarz algorithm applied to the Maxwell system, and illustrate our results with numerical experiments. We conclude in Section 3 with a summary of our findings, and an outlook on future work.

2. Maxwell System. We now turn our attention to the Maxwell system, which describes the propagation of electromagnetic waves,

$$-\varepsilon\frac{\partial \boldsymbol{E}}{\partial t} + \text{curl } \boldsymbol{H} = \boldsymbol{J}, \qquad \mu\frac{\partial \boldsymbol{H}}{\partial t} + \text{curl } \boldsymbol{E} = 0, \tag{2.1}$$

where $\boldsymbol{E} = (E_1, E_2, E_3)^T$ and $\boldsymbol{H} = (H_1, H_2, H_3)^T$ denote the electric and magnetic fields, respectively, ε is the *electric permittivity*, μ is the *magnetic permeability*, and $\boldsymbol{J}$ is the applied current density. In the following, to simplify the notation, and without loss of generality, we normalize the parameters $\varepsilon = \mu = 1$, which corresponds to a scaling of time and the vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$, and we assume the applied current density to be divergence free, that is div$\boldsymbol{J} = 0$. Denoting the vector of physical unknowns by

$$\boldsymbol{u} = (E_1, E_2, E_3, H_1, H_2, H_3)^T, \tag{2.2}$$

the Maxwell system (2.1) can be rewritten in conservative form,

$$\partial_t \boldsymbol{u} + G_x\partial_x \boldsymbol{u} + G_y\partial_y \boldsymbol{u} + G_z\partial_z \boldsymbol{u} = \boldsymbol{f}, \tag{2.3}$$

where the right hand side is given by $\boldsymbol{f} = (J_1, J_2, J_3, 0, 0, 0)^T$, and the coefficient matrices are of the form

$$G_l = \begin{bmatrix} & N_l \\ -N_l & \end{bmatrix}, \qquad l = x, y, z,$$

where the 3×3 matrices N_l, $l = x, y, z$ are given by

$$N_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad N_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad N_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For any unit vector $\boldsymbol{v} = (v_1, v_2, v_3)$, $\|\boldsymbol{v}\| = 1$, we can define the characteristic matrix of system (2.3) by

$$C(\boldsymbol{v}) = v_1 \begin{bmatrix} & N_x \\ -N_x & \end{bmatrix} + v_2 \begin{bmatrix} & N_y \\ -N_y & \end{bmatrix} + v_3 \begin{bmatrix} & N_z \\ -N_z & \end{bmatrix} = \begin{bmatrix} & N_{\boldsymbol{v}} \\ -N_{\boldsymbol{v}} & \end{bmatrix},$$

whose eigenvalues are the characteristic speed of propagation along the direction $\boldsymbol{v}$. By the structure of the matrices N_l, $l = x, y, z$, the matrix $C(\boldsymbol{v})$ is symmetric, and hence has real eigenvalues, which implies that the Maxwell system is hyperbolic [31].

If we consider the Maxwell system (2.1) on the domain $\Omega = [0, 1] \times \mathbb{R}$, the characteristic matrix for the unit normal vector to the boundaries at $x = 0$ and $x = 1$, $\tilde{\boldsymbol{n}} = (1, 0, 0)$, is

$$C(\tilde{\boldsymbol{n}}) = \begin{pmatrix} & N_x \\ -N_x & \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_{1,2} = -1, \qquad \lambda_{3,4} = 0, \qquad \lambda_{5,6} = 1,$$

and since the eigenvalues are not distinct, the Maxwell system is not strictly hyperbolic [31]. The matrix of the left eigenvectors of $C(\tilde{\boldsymbol{n}})$ is given by

$$L = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

which leads to the characteristic variables $\boldsymbol{w} = (w_1, w_2, w_3, w_4, w_5, w_6)^T$ associated with the direction $\tilde{\boldsymbol{n}}$, where

$$\begin{array}{lll} w_1 = -\frac{1}{2}(E_2 - H_3), & w_2 = \frac{1}{2}(E_3 + H_2), & w_3 = H_1, \\ w_4 = E_1, & w_5 = \frac{1}{2}(E_2 + H_3), & w_6 = -\frac{1}{2}(E_3 - H_2). \end{array} \tag{2.4}$$

In the following, we will denote by $\boldsymbol{w}_+$, $\boldsymbol{w}_0$ and $\boldsymbol{w}_-$ the characteristic variables associated with the positive, null, and negative eigenvalues respectively, that is

$$\boldsymbol{w}_- = (w_1, w_2)^T, \qquad \boldsymbol{w}_0 = (w_3, w_4)^T, \qquad \boldsymbol{w}_+ = (w_5, w_6)^T. \tag{2.5}$$

The boundary value problem in the characteristic variables, associated with the Maxwell system (2.1) on the domain $\Omega = [0, 1] \times \mathbb{R}^2$,

$$\begin{array}{rcl} (\partial_t - \partial_x)w_1 + \frac{1}{2}\partial_z w_3 - \frac{1}{2}\partial_y w_4 & = & \frac{1}{2}J_2 \\ (\partial_t - \partial_x)w_2 + \frac{1}{2}\partial_y w_3 + \frac{1}{2}\partial_z w_4 & = & -\frac{1}{2}J_3 \\ \partial_t w_3 + \partial_z w_1 + \partial_y w_2 - \partial_z w_5 - \partial_y w_6 & = & 0 \\ \partial_t w_4 - \partial_y w_1 + \partial_z w_2 - \partial_y w_5 + \partial_z w_6 & = & -J_1 \\ (\partial_t + \partial_x)w_5 - \frac{1}{2}\partial_z w_3 - \frac{1}{2}\partial_y w_4 & = & -\frac{1}{2}J_2 \\ (\partial_t + \partial_x)w_6 - \frac{1}{2}\partial_y w_3 + \frac{1}{2}\partial_z w_4 & = & \frac{1}{2}J_3 \end{array} \tag{2.6}$$

together with the characteristic boundary conditions

$$\boldsymbol{w}_+(0,y,z)=\boldsymbol{r}(y,z),\quad \boldsymbol{w}_-(1,y,z)=\boldsymbol{s}(y,z),\quad (y,z)\in\mathbb{R}^2, \tag{2.7}$$

and with the radiation condition on the unbounded part of the domain

$$|u_j(x,y,z)|\le\frac{C}{\sqrt{y^2+z^2}}, \tag{2.8}$$

where C is a constant, is well-posed ([27].

2.1. Time-harmonic Solutions. As in the case of the second order wave equation, $\partial_{tt}u-\Delta u=f$, it is also suitable for the Maxwell equations to assume the wave to be periodic in time. In this case, the time derivative becomes an algebraic term, and only the spatial domain needs to be discretized for a numerical approximation of the solution. The harmonic solutions of the Maxwell equations are complex valued static vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$ such that the dynamic fields

$$\mathcal{E}(\boldsymbol{x},t)=\mathcal{R}e(\boldsymbol{E}(\boldsymbol{x})\exp(i\omega t)),\qquad \mathcal{H}(\boldsymbol{x},t)=\mathcal{R}e(\boldsymbol{H}(\boldsymbol{x})\exp(i\omega t))$$

satisfy the Maxwell system (2.1). The positive real parameter ω is called the pulsation of the harmonic wave. The harmonic solutions $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the time-harmonic equations

$$\operatorname{curl}\boldsymbol{E}+i\omega\boldsymbol{H}=\boldsymbol{0},\qquad \operatorname{curl}\boldsymbol{H}-i\omega\boldsymbol{E}=\boldsymbol{J}, \tag{2.9}$$

or, written in component form, and owing to (2.2)

$$\begin{array}{lll} -i\omega u_1+\partial_y u_6-\partial_z u_5=J_1, & -i\omega u_2+\partial_z u_4-\partial_x u_6=J_2, & -i\omega u_3+\partial_x u_5-\partial_y u_4=J_3,\\ i\omega u_4+\partial_y u_3-\partial_z u_2=0, & i\omega u_5+\partial_z u_1-\partial_x u_3=0, & i\omega u_6+\partial_x u_2-\partial_y u_1=0. \end{array} \tag{2.10}$$

The time-harmonic problem in characteristic variables reads thus

$$\begin{array}{ll} (i\omega-\partial_x)w_1+\frac12\partial_z w_3-\frac12\partial_y w_4=\frac12 J_2 & i\omega w_4-\partial_y w_1+\partial_z w_2-\partial_y w_5+\partial_z w_6=-J_1\\ (i\omega-\partial_x)w_2+\frac12\partial_y w_3+\frac12\partial_z w_4=-\frac12 J_3 & (i\omega+\partial_x)w_5-\frac12\partial_z w_3-\frac12\partial_y w_4=-\frac12 J_2\\ i\omega w_3+\partial_z w_1+\partial_y w_2-\partial_z w_5-\partial_y w_6=0 & (i\omega+\partial_x)w_6-\frac12\partial_y w_3+\frac12\partial_z w_4=\frac12 J_3 \end{array} \tag{2.11}$$

2.2. Relation to a Scalar Equation. We investigate now the relation between the solution of the time harmonic Maxwell system and the solution of some suitable scalar equation. We consider the equation (2.9) in $\Omega=[0,1]\times\mathbb{R}^2$ and the associated boundary value problem with the boundary conditions given by (2.7).

Proposition 2.1.

Let $\boldsymbol{u}$ be as defined in (2.2) and $\boldsymbol{w}$ be the characteristic variables defined in (2.4). Any component $\tilde{w}_j$, $j=1,\dots,6$, of the characteristic variables of the Maxwell system (2.10) satisfies, in the interior of $\Omega=[0,1]\times\mathbb{R}^2$, the Helmholtz equation,

$$(\omega^2+\Delta)w_j=\tilde{f}_j,\quad j=1,2,\dots,6, \tag{2.12}$$

with right hand side given by

$$\begin{array}{ll} \tilde{f}_1=\frac12\left(-(\partial_x+i\omega)J_2+\partial_y J_1\right) & \tilde{f}_4=i\omega J_1\\ \tilde{f}_2=\frac12\left((\partial_x+i\omega)J_3-\partial_z J_1\right) & \tilde{f}_5=\frac12\left(-(\partial_x-i\omega)J_2+\partial_y J_1\right)\\ \tilde{f}_3=-\partial_y J_3+\partial_z J_2 & \tilde{f}_6=\frac12\left((\partial_x-i\omega)J_3-\partial_z J_1\right) \end{array} \tag{2.13}$$

Proof. From the last three equations in (2.10), we obtain u_j, $j = 4,5,6$ as functions of u_j, $j = 1,2,3$ only. Substituting these expressions for u_j, $j = 4,5,6$, into the first three equations in (2.10), we obtain a system for u_j,$j = 1,2,3$

$$\begin{array}{rcl} \omega^2 u_1 + (\partial_{yy} + \partial_{zz})\, u_1 - \partial_{xy} u_2 - \partial_{xz} u_3 &=& i\omega J_1, \\ \omega^2 u_2 + (\partial_{xx} + \partial_{zz})\, u_2 - \partial_{yz} u_3 - \partial_{xy} u_1 &=& i\omega J_2, \\ \omega^2 u_3 + (\partial_{xx} + \partial_{yy})\, u_3 - \partial_{xz} u_1 - \partial_{yz} u_2 &=& i\omega J_3. \end{array} \tag{2.14}$$

We now eliminate the variable u_3 from the first two equations by differentiating the first one with respect to y and subtracting it from the second one differentiated with respect to x. We then eliminate u_3 also from the second and third equations by applying the operator $[\omega^2 + (\partial_{xx} + \partial_{yy})]$ to the second one and adding it the third one differentiated with respect to y and z. After some simplifications, we obtain a new system for u_1 and u_2,

$$\begin{array}{rcl} \partial_y((\omega^2+\Delta)u_1) - \partial_x((\omega^2+\Delta)u_2) &=& i\omega[\partial_y J_1 - \partial_x J_2], \\ -\partial_{xy}((\omega^2+\Delta)u_1) + (\omega^2 + \partial_{xx})((\omega^2+\Delta)u_2) &=& i\omega((\omega^2 + \partial_{xx} + \partial_{yy})J_2. + \partial_{yz} J_3). \end{array}$$

Applying ∂_x to the first equation and adding it to the second one, we finally obtain, after a division by ω^2,

$$(\omega^2 + \Delta)u_2 = i\omega J_2,$$

Different manipulations in (2.14) allow to reduce to a single equation in u_1 and u_3. In a similar way, we can eliminate from the first three equations in (2.10), the variables u_j, $j = 1,2,3$ as functions of u_j, $j = 4,5,6$ only. We then proceed as above, and it can be easily seen that we get

$$\begin{array}{lll} (\omega^2+\Delta)u_1 = i\omega J_1, & (\omega^2+\Delta)u_3 = i\omega^2 J_3, & (\omega^2+\Delta)u_4 = -\partial_y J_3 + \partial_z J_2, \\ (\omega^2+\Delta)u_5 = -\partial_z J_1 + \partial_x J_3, & (\omega^2+\Delta)u_6 = -\partial_x J_2 + \partial_y J_1. & \end{array}$$

So far, the thesis follows immediately by linear combinations. □

The above proposition states that the characteristic variables of the Maxwell system satisfy an Helmholtz equation in the interior of the domain $\Omega = [0,1] \times \mathbb{R}^2$. So far, let us consider the following partial differential equation

$$(\omega^2 + \Delta)\tilde{w}_1 = f_1 \qquad \text{in } \Omega \tag{2.15}$$

together with the boundary conditions

$$(\partial_x - i\omega)\tilde{w}_1(0,y,z) = \tilde{r}_1(y,z), \qquad \tilde{w}_1(1,y,z) = \tilde{s}_1(1,y,z), \qquad (y,z) \in \mathbb{R}^2. \tag{2.16}$$

which is very much related to the Maxwell system, as stated in the following proposition.

PROPOSITION 2.2. *Let $\boldsymbol{w}$ be the solution of (2.11) with boundary conditions (2.7), and let $\tilde{w}_1$ be the solution of (2.15) with boundary conditions (2.16). If $f_1(x,y,z) = \tilde{f}_1(x,y,z)$, $\tilde{s}_1(y,z) = s_1(y,z)$, and*

$$\tilde{r}_1(y,z) = \Delta_{yz}^{-1}\Big[(\partial_x + i\omega)\left[(\partial_{yy} - \partial_{zz})w_5 - 2\partial_{yz}w_6\right] + \partial_y\left(\partial_z J_3 - \partial_y J_2\right)\Big](0,y,z)$$

where Δ_{yz} denotes the Laplace operator in the y and z variables, then

$$\tilde{w}_1(x,y,z) = w_1(x,y,z) \qquad \text{in } \bar{\Omega}.$$

Proof. The foregoing proposition states that the characteristic variable w_1 satisfies an Helmholtz equation inside Ω with right hand side $\tilde{f}_1$. Thus the differential equations coincide and we need to verify only the equivalence between the boundary conditions. In that order, notice that the boundary condition of Dirichlet type in $(1, y, z)$ stays the same. Consider then the first two and the last two equations in (2.11): this is a 4×6 linear system, and we can express the components of the solution w_j, $j = 1, .., 4$ in terms of the sole components w_5 and w_6. For any $\boldsymbol{x} = (x, y, z) \in \bar{\Omega}$, it can be easily seen that we have

$$(\partial_x - i\omega)w_1 = \Delta_{yz}^{-1}\big[(\partial_{yy} - \partial_{zz})(\partial_x + i\omega)w_5 - 2\partial_{yz}(\partial_x + i\omega)w_6 + \partial_{yz}J_3 + \partial_{yy}J_2\big]$$

In particular, the above relation holds true for $(0, y, z)$, thus, w_1 satisfies (2.15) with boundary conditions (2.16), and the thesis then follows by uniqueness. ∎

Remark 1. *Notice that a similar result can be obtained for any other propagating component of the characteristic variables w_2, w_5, and w_6, with boundary conditions given in (2.7) and by*

$$(\partial_x - i\omega)w_2 = \Delta_{yz}^{-1}\Big[(\partial_x + i\omega)\left[(\partial_{yy} - \partial_{zz})w_5 - 2\partial_{yz}w_6\right] + \partial_y\left(\partial_z J_3 + \partial_y J_2\right)\Big], \quad \textit{in } (0, y, z),$$

$$(\partial_x + i\omega)w_5 = \Delta_{yz}^{-1}\Big[(\partial_x - i\omega)\left[(\partial_{yy} - \partial_{zz})w_1 - 2\partial_{yz}w_2\right] + \partial_y\left(\partial_y J_3 + \partial_z J_2\right)\Big], \quad \textit{in } (1, y, z),$$

$$(\partial_x + i\omega)w_6 = \Delta_{yz}^{-1}\Big[(\partial_x - i\omega)\left[2\partial_{yz}w_1 + (\partial_{yy} - \partial_{zz})w_2\right] + \partial_y\left(\partial_y J_3 + \partial_z J_2\right)\Big], \quad \textit{in } (1, y, z),$$

2.3. Classical Schwarz Algorithm for the Maxwell System. We consider now the problem (2.9) in $\Omega = [0, 1] \times \mathbb{R}^2$, with boundary conditions on $(0, y, z)$ and $(1, y, z)$ given by (2.7),and with the radiation conditions (2.8). We decompose the domain into two subdomains $\Omega_1 := (0, b) \times \mathbb{R}^2$ and $\Omega_2 := (a, 1) \times \mathbb{R}^2$, and we denote the overlap with $L := b - a \geq 0$. We solve system (2.3) in both subdomains and we enforce on the subdomain interfaces the continuity of the incoming characteristic variables. There are two incoming characteristics on both interfaces. Hence, to have a well-posed problem, we have to impose two conditions on each subdomain. The classical Schwarz algorithm, using subscript to denote components, and superscript to denote the subdomain and the iteration count, is given by

$$\begin{aligned}
i\omega\,\boldsymbol{u}^{1,n} + \sum_{l=x,y,z} G_l\partial_l\boldsymbol{u}^{1,n} &= \boldsymbol{f},\ \text{in } \Omega_1, & \qquad i\omega\,\boldsymbol{u}^{2,n} + \sum_{l=x,y,z} \boldsymbol{G}_l\partial_l\boldsymbol{u}^{2,n} &= \boldsymbol{f},\ \text{in } \Omega_2,\\
\boldsymbol{w}_+^{1,n}(0, y, z) &= \boldsymbol{r}(y, z), & \boldsymbol{w}_-^{2,n}(1, y, z) &= \boldsymbol{w}(y, z),\\
\boldsymbol{w}_-^{1,n}(b, y, z) &= \boldsymbol{w}_-^{2,n-1}(b, y, z), & \boldsymbol{w}_+^{2,n}(a, y, z) &= \boldsymbol{w}_+^{1,n-1}(a, y, z),
\end{aligned} \tag{2.17}$$

where in the transmission conditions $(y, z) \in \mathbb{R}^2$. From the previous equivalence result, a Schwarz algorithm for the equivalent Helmholtz problem would read

$$\begin{aligned}
(\omega^2 + \Delta)\tilde{w}_1^{1,n} &= \tilde{f}_1,\ \text{in } \Omega_1, & \qquad (\omega^2 + \Delta)\tilde{w}_1^{2,n} &= \tilde{f}_1,\ \text{in } \Omega_2,\\
(\partial_x - i\omega)\tilde{w}_1^{1,n}(0, y, z) &= \tilde{r}_1(y, z), & \tilde{w}_1^{2,n}(1, y, z) &= \tilde{s}_1(y, z),\\
\tilde{w}_1^{1,n}(b, y, z) &= \tilde{w}_1^{2,n-1}(b, y, z), & (\partial_x - i\omega)\tilde{w}_1^{2,n}(b, y, z) &= (\partial_x - i\omega)\tilde{w}_1^{1,n-1}(b, y, z).
\end{aligned} \tag{2.18}$$

We now show that the Schwarz algorithm with Dirichlet transmission conditions applied to the time harmonic Maxwell system (2.10) is equivalent to a simple optimized Schwarz method for a related scalar partial differential equation, as stated in the following proposition.

PROPOSITION 2.3. *Let $w_1^{2,0}$, $w_2^{2,0}$, $w_5^{1,0}$, $w_6^{1,0}$ being given. If algorithm (2.18) is started with initial data $\tilde{w}_1^{1,0}$ and $\tilde{w}_1^{2,0}$ such that*

$$(\partial_x - i\omega)\tilde{w}_1^{1,0} = \Delta_{yz}^{-1}\left[(\partial_{yy} - \partial_{zz})(\partial_x + i\omega)w_5^{1,0} - 2\partial_{yz}(\partial_x - i\omega)w_6^{1,0} + \partial_{yz}J_3 - \partial_{yy}J_2\right]$$

and $\tilde{w}_1^{2,0} = w_1^{2,0}$, then for any $n \geq 1$ the first characteristic variable w_1 of the iterates of (2.17) and the iterates of (2.18) coincide, i.e.

$$w_1^{1,n} = \tilde{w}_1^{1,n} \qquad w_1^{2,n} = \tilde{w}_1^{2,n}.$$

Proof. The proof proceeds by induction. Proposition 2.2 entails the result for $n = 1$. Assume then that the result is true at iteration $n - 1$. We then have

$$(\partial_x - i\omega)\tilde{w}_1^{1,n-1} = \Delta_{yz}^{-1}\left[(\partial_x + i\omega)\left[(\partial_{yy} - \partial_{zz})w_5^{1,n-1} - 2\partial_{yz}w_6^{1,n-1}\right] + \partial_y\left(\partial_z J_3 + \partial_y J_2\right)\right]$$

which holds in particular at (a, y, z). By uniqueness, the boundary condition in $(1, y, z)$ entails then $w_1^{2,n} = \tilde{w}_1^{2,n}$. In a similar way, we have $w_1^{1,n-1}(b, y, z) = \tilde{w}_1^{1,n-1}(b, y, z)$, and from the boundary condition in $(0, y, z)$ the thesis follows. ∎

From the foregoing proposition, the Schwarz algorithm with Dirichlet transmission conditions applied to the time harmonic Maxwell system (2.10) is equivalent to a simple optimized Schwarz method for a related scalar partial differential equation. This implies in particular the equivalent convergence behavior we show in the following proposition for an infinite domain $\Omega = \mathbb{R}^3$. In what follows we denote by k_x, k_y and k_z the Fourier variables corresponding to a transform with respect to x, y and z, respectively. **MAH**

PROPOSITION 2.4. *Let $\Omega = \mathbb{R}^3$, and consider the Maxwell system (2.10) in Ω with the radiation condition*

$$\lim_{r\to\infty} r(\boldsymbol{n} \times \boldsymbol{E} + \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{H})) = 0, \tag{2.19}$$

where $r = |\boldsymbol{x}|$, $\boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}|$. Let Ω be decomposed into $\Omega_1 := (-\infty, L) \times \mathbb{R}^2$ and $\Omega_2 := (0, +\infty) \times \mathbb{R}^2$, ($L \geq 0$). For any given initial guess $\boldsymbol{u}^{1,0} \in (L^2(\Omega_1))^6$, $\boldsymbol{u}^{2,0} \in (L^2(\Omega_2))^6$, the Schwarz algorithm (2.17) applied to system (2.10) converges for all Fourier modes such that $k_y^2 + k_z^2 \neq \omega^2$. The convergence factor is

$$R_{th} = \begin{cases} \left|\frac{\sqrt{\omega^2 - (k_y^2 + k_z^2)} - \omega}{\sqrt{\omega^2 - (k_y^2 + k_z^2)} + \omega}\right|, & \text{for } k_y^2 + k_z^2 < \omega^2, \\ e^{-\sqrt{k_y^2 + k_z^2 - \omega^2}L}, & \text{for } k_y^2 + k_z^2 > \omega^2. \end{cases} \tag{2.20}$$

Proof. Because of linearity, it suffices to analyze the convergence to the zero solution when the right hand side vanishes. Performing a Fourier transform of system (2.10) in the y and z direction, the first and the fourth equation provide an algebraic expression for $\hat{u}_1$ and $\hat{u}_4$, which is in agreement with the fact that these latter are the characteristic variables associated with the null eigenvalue. Inserting these expressions

into the remaining Fourier transformed equations, we obtain the first order system

$$\partial_x \begin{pmatrix} \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_5 \\ \hat{u}_6 \end{pmatrix} + \frac{i}{\omega} \begin{bmatrix} 0 & 0 & k_y k_z & \omega^2 - k_y^2 \\ 0 & 0 & k_z^2 - \omega^2 & -k_y k_z \\ -k_y k_z & k_y^2 - \omega^2 & 0 & 0 \\ \omega^2 - k_z^2 & k_y k_z & 0 & 0 \end{bmatrix} \begin{pmatrix} \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_5 \\ \hat{u}_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.21}$$

The eigenvalues of the matrix in (2.21) and their corresponding eigenvectors are

$$\lambda_{1,2}^{TH} = -\sqrt{|\boldsymbol{k}|^2 - \omega^2}, \qquad \boldsymbol{v}_1 = \begin{pmatrix} \frac{k_y k_z}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ \frac{k_z^2-\omega^2}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} \frac{\omega^2-k_y^2}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ -\frac{k_y k_z}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ 0 \\ 1 \end{pmatrix},$$

and

$$\lambda_{3,4}^{TH} = \sqrt{|\boldsymbol{k}|^2 - \omega^2}, \qquad \boldsymbol{v}_3 = \begin{pmatrix} -\frac{k_y k_z}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ \frac{\omega^2-k_z^2}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{v}_4 = \begin{pmatrix} \frac{k_y^2-\omega^2}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ \frac{k_y k_z}{i\omega\sqrt{|\boldsymbol{k}|^2-\omega^2}} \\ 0 \\ 1 \end{pmatrix},$$

where $|\boldsymbol{k}|^2 = k_y^2 + k_z^2$.
Because of the radiation condition, the solutions u^l of system (2.21) in Ω_l $(l = 1, 2)$ are given by

$$\boldsymbol{u}^1 = (\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2) e^{\sqrt{|\boldsymbol{k}|^2-\omega^2}(x-L)}, \qquad \boldsymbol{u}^2 = (\beta_1 \boldsymbol{v}_3 + \beta_2 \boldsymbol{v}_4) e^{-\sqrt{|\boldsymbol{k}|^2-\omega^2}x}, \tag{2.22}$$

where the coefficients α_j and β_j $(j = 1, 2)$ are uniquely determined by the interface conditions. At the n-th step of the Schwarz algorithm, the coefficients $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ satisfy the system

$$\boldsymbol{\alpha}^n = A_1^{-1} A_2 e^{-\sqrt{|\boldsymbol{k}|^2-\omega^2}L} \boldsymbol{\beta}^{n-1}, \qquad \boldsymbol{\beta}^n = B_1^{-1} B_2 e^{-\sqrt{|\boldsymbol{k}|^2-\omega^2}L} \boldsymbol{\alpha}^{n-1},$$

where the matrices in the iteration are given by

$$A_1 = \begin{bmatrix} -k_y k_z & k_y^2 - \omega^2 + i\omega\lambda \\ k_z^2 - \omega^2 + i\omega\lambda & -k_y k_z \end{bmatrix}, \qquad A_2 = \begin{bmatrix} k_y k_z & -(k_y^2 - \omega^2 - i\omega\lambda) \\ -(k_z^2 - \omega^2 - i\omega\lambda) & k_y k_z \end{bmatrix}, \tag{2.23}$$

and where $B_l = A_l, l = 1, 2$ and we have set $\lambda := \sqrt{|\boldsymbol{k}|^2 - \omega^2}$. A complete iteration over two steps of the Schwarz algorithm leads then to

$$\boldsymbol{\alpha}^{n+1} = A_1^{-1} A_2 B_1^{-1} B_2 e^{-2\lambda L} \boldsymbol{\alpha}^{n-1}, \qquad \boldsymbol{\beta}^{n+1} = B_1^{-1} B_2 A_1^{-1} A_2 e^{-2\lambda L} \boldsymbol{\beta}^{n-1},$$

and we finally obtain

$$A_1^{-1} A_2 B_1^{-1} B_2 = B_1^{-1} B_2 A_1^{-1} A_2 = \left(\frac{|\boldsymbol{k}|^2}{(\lambda + i\omega)^2} \right)^2 Id.$$

Now by the definition of λ, we have $|\boldsymbol{k}|^2 = (\lambda - i\omega)(\lambda + i\omega)$, and thus the convergence factor of the algorithm is

$$\rho(|\boldsymbol{k}|) = \left| \frac{\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega}{\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega} e^{-\sqrt{|\boldsymbol{k}|^2-\omega^2}L} \right|.$$

Separating the two cases $|\boldsymbol{k}|^2 < \omega^2$ and $|\boldsymbol{k}|^2 > \omega^2$ then concludes the proof. □
Notice that for $|\boldsymbol{k}|^2 = \omega^2$, the convergence factor equals 1, independently of the overlap, which indicates that the algorithm is not convergent in general when used in the iterative form described here. This precise result was also observed for the equivalent optimized Schwarz method applied to the Helmholtz equation, see [22]. In practice, the Schwarz methods are however used as preconditioners for Krylov methods, and then a few non-convergent modes in the iterative form can easily be handled by the Krylov method.

We also see from the convergence factor (2.20) that the overlap is necessary for the convergence of the evanescent modes, $|\boldsymbol{k}|^2 > \omega^2$. Without overlap, $L = 0$, we have $\rho(|\boldsymbol{k}|) < 1$ only for the propagative modes, $|\boldsymbol{k}|^2 < \omega^2$, and $\rho(|\boldsymbol{k}|) = 1$ when $|\boldsymbol{k}|^2 \geq \omega^2$. In the time-harmonic case, the classical Schwarz algorithm without overlap is thus convergent only for propagative modes, and corresponds to the algorithm proposed by B. Després *et al.* in [9] for the Helmholtz equation.

2.4. Optimized Schwarz Method for the Maxwell Equations.

2.4.1. Transparent Boundary Conditions. To design optimized Schwarz algorithms for the Maxwell system, we derive now the transparent boundary conditions for those equations. We consider the time harmonic Maxwell equations (2.10) on the domain $\Omega = (0,1) \times \mathbb{R}^2$, with right hand side $\boldsymbol{J}$ compactly supported in Ω, together with the boundary conditions

$$(\boldsymbol{w}_+ + \mathcal{S}_1 \boldsymbol{w}_-)(0,y,z) = 0, \quad (\boldsymbol{w}_- + \mathcal{S}_2 \boldsymbol{w}_+)(1,y,z) = 0, \qquad (y,z) \in \mathbb{R}^2, \qquad (2.24)$$

where $\boldsymbol{w}_-$ and $\boldsymbol{w}_+$ are defined in (2.5), and the operators $\mathcal{S}_l$, $l = 1,2$, are general, pseudodifferential operators acting in the y and z directions.

LEMMA 2.5. *If the operators $\mathcal{S}_l$, $l = 1,2$ have the Fourier symbol*

$$\mathcal{F}(\mathcal{S}_l) = \frac{1}{(\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2} \begin{bmatrix} k_y^2 - k_z^2 & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 \end{bmatrix}, \qquad j = 1,2, \qquad (2.25)$$

then the solution of the Maxwell equations (2.10) in Ω with boundary conditions (2.24) coincides with the restriction on Ω of the solution of the Maxwell system (2.10) on $\mathbb{R}^3$.

Proof. As in Section **??**, we prove that the difference between the solution of the global problem and that of the restricted problem (which we will denote by $\boldsymbol{e}$), which satisfies in Ω the homogeneous counterpart of (2.10) with boundary conditions (2.24) vanishes. Proceeding as in the previous section, the solution in Fourier is given by

$$\hat{\boldsymbol{e}} = (\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2) e^{\sqrt{|\boldsymbol{k}|^2 - \omega^2} x} + (\alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4) e^{-\sqrt{|\boldsymbol{k}|^2 - \omega^2} x},$$

where the vectors $\boldsymbol{v}_j$ $(j = 1,..,4)$ are defined in Section 2.3. Using the boundary condition (2.24) in $(0,y,z)$, we obtain that the coefficients α_j, $j = 3,4$ satisfy the system of equations

$$\begin{bmatrix} -k_y k_z & k_y^2 - \omega^2 + i\omega\sqrt{|\boldsymbol{k}|^2 - \omega^2} \\ k_z^2 - \omega^2 + i\omega\sqrt{|\boldsymbol{k}|^2 - \omega^2} & -k_y k_z \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies $\alpha_3 = \alpha_4 = 0$. Now using the boundary condition at $(1,y,z)$, we obtain for the coefficients α_j, $j = 1,2$, the same system of equations as for α_j, $j = 3,4$, which implies $\alpha_1 = \alpha_2 = 0$. Thus $\widehat{\boldsymbol{e}} = \boldsymbol{0}$, which concludes the proof. □

REMARK 2. *As in the case of the Cauchy-Riemann equations in Remark* **??**, *the symbols in (2.25) can be written in several, mathematically equivalent forms:*

$$\mathcal{F}(\mathcal{S}_l) = \frac{1}{(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2}M = \frac{1}{|\boldsymbol{k}|^2}\frac{\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega}{\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega}M = (\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2M^{-1},$$

where $M = \begin{bmatrix} k_y^2-k_z^2 & -2k_yk_z \\ -2k_yk_z & k_z^2-k_y^2 \end{bmatrix}$.

which also will lead to different approximations of the transparent conditions in the context of optimized Schwarz methods. The first form contains a local and a non-local term, since multiplication with the matrix M corresponds to second order derivation operations in y and z, which is a local operation, whereas the term containing the square-root of $|\boldsymbol{k}|^2$ is a non-local operation. The last form contains two non-local operations, since the inversion of the matrix M corresponds to an integration. This integration can however be passed to the other variable by multiplication with this matrix. The second form contains also two non-local terms and a local one. These different forms motivate different local approximations of the transparent boundary conditions.

Similarly, we can consider the associated Helmholtz equation 2.12 in $\Omega = (0,1)\times\mathbb{R}^2$, with right hand side compactly supported in Ω, and with boundary conditions

$$(\partial_x - \tilde{\mathcal{S}}_1)\boldsymbol{u}(0,y,z) = 0, \quad (\partial_x + \tilde{\mathcal{S}}_2)\boldsymbol{u}(1,y,z) = 0, \qquad (y,z)\in\mathbb{R}^2, \tag{2.26}$$

where $\tilde{\mathcal{S}}_j$ $(j=1,2)$ are general, pseudodifferential operators acting in the y and z directions.

LEMMA 2.6. *If the operators $\tilde{\mathcal{S}}_l$ $(l=1,2)$ have the Fourier symbol*

$$\tilde{\sigma}_l = \mathcal{F}(\tilde{\mathcal{S}}_l) = \sqrt{|\boldsymbol{k}|^2-\omega^2} \tag{2.27}$$

then the solution of (2.12) in Ω with boundary conditions (2.26) coincides with the restriction on Ω of the solution of the Helmholtz equation (2.12) on $\mathbb{R}^3$.

Proof. The proof follows along the same lines as in the previous Lemma. Performing a Fourier transform in the x_2 and x_3 directions, the symbol of the difference between the solution of the global problem and the solution of the restricted one, that we denote with $\tilde{\boldsymbol{e}}$, is given by

$$\tilde{\boldsymbol{e}} = \alpha e^{\sqrt{|\boldsymbol{k}|^2-\omega^2}} + \beta e^{-\sqrt{|\boldsymbol{k}|^2-\omega^2}}.$$

The boundary condition at $(0,y,z)$ implies then $\alpha=0$, whereas the boundary condition at $(1,y,z)$ implies $\beta=0$, which concludes the proof. ∎

2.4.2. Schwarz Algorithm with General Interface Conditions. The operators $\mathcal{S}_l$, $l=1,2$, introduced in the previous section and leading to the optimal performance, are unfortunately non-local operators, hence difficult to be used in practice. They must therefore be approximated in some suitable way. If one is willing to use second order transmission conditions, then the only parts of the symbols in (2.25) that need to be approximated are the multiplication by $(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^{-2}$, because the entries of the matrices are polynomials in the Fourier variables, which correspond to derivatives in the y and z direction. We apply now to (2.10) a Schwarz algorithm with more general interface conditions, which are given for Ω_1 by

$$[(\boldsymbol{w}_- + \mathcal{S}_1\boldsymbol{w}_+)(L,y,z)]^{1,n} = [(\boldsymbol{w}_- + \mathcal{S}_1\boldsymbol{w}_+)(L,y,z)]^{2,n-1}, \quad (y,z)\in\mathbb{R}^2, \tag{2.28}$$

and for Ω_2 by

$$[(\boldsymbol{w}_+ + \mathcal{S}_2\boldsymbol{w}_-)(0,y,z)]^{2,n} = [(\boldsymbol{w}_+ + \mathcal{S}_2\boldsymbol{w}_-)(0,y,z)]^{1,n-1}, \quad (y,z) \in \mathbb{R}^2. \tag{2.29}$$

PROPOSITION 2.7. **a)** *If the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ have the Fourier symbol*

$$\sigma_l := \mathcal{F}(\mathcal{S}_l) = \gamma_l \begin{bmatrix} k_y^2 - k_z^2 & -2k_yk_z \\ -2k_yk_z & k_z^2 - k_y^2 \end{bmatrix}, \qquad \gamma_l \in \mathbb{C}(k_z,k_y) \quad (l=1,2), \tag{2.30}$$

then the convergence factor of the Schwarz algorithm with interface conditions (2.28)-(2.29) is

$$\rho(\omega,L,|\boldsymbol{k}|,\gamma_1,\gamma_2) = \left| \frac{(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2}{(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2} \; \frac{1-\gamma_1(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2}{1-\gamma_1(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2} \; \frac{1-\gamma_2(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2}{1-\gamma_2(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2} \; e^{-2\sqrt{|\boldsymbol{k}|^2-\omega^2}L} \right|^{1/2}. \tag{2.31}$$

b) *If the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ have the Fourier symbol*

$$\sigma_l := \mathcal{F}(\mathcal{S}_l) = \delta_l \begin{bmatrix} k_y^2 - k_z^2 & -2k_yk_z \\ -2k_yk_z & k_z^2 - k_y^2 \end{bmatrix}^{-1}, \qquad \gamma_l \in \mathbb{C}(k_z,k_y) \quad (l=1,2), \tag{2.32}$$

then the convergence factor of the Schwarz algorithm with interface conditions (2.28)-(2.29) is

$$\rho(\omega,L,|\boldsymbol{k}|,\delta_1,\delta_2) = \left| \frac{(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2}{(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2} \; \frac{\delta_1-(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2}{\delta_1-(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2} \; \frac{\delta_2-(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2}{\delta_2-(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2} \; e^{-2\sqrt{|\boldsymbol{k}|^2-\omega^2}L} \right|^{1/2}. \tag{2.33}$$

c) *If the operator $\mathcal{S}_1$ has the Fourier symbol (2.30) and $\mathcal{S}_2$ have the Fourier symbol (2.32) then the convergence factor of the Schwarz algorithm with interface conditions (2.28)-(2.29) is*

$$\rho(\omega,L,|\boldsymbol{k}|,\gamma_1,\delta_2) = \left| \frac{1-\gamma_1(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2}{1-\gamma_1(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2} \; \frac{\delta_2-(\sqrt{|\boldsymbol{k}|^2-\omega^2}-i\omega)^2}{\delta_2-(\sqrt{|\boldsymbol{k}|^2-\omega^2}+i\omega)^2} \; e^{-2\sqrt{|\boldsymbol{k}|^2-\omega^2}L} \right|^{1/2}. \tag{2.34}$$

Proof. The convergence result is again based on Fourier analysis, as in Section 2.3. At the n-th step of the Schwarz algorithm, the coefficients $\boldsymbol{\alpha}^n = (\alpha_1^n, \alpha_2^n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ in (2.22) satisfy

$$\boldsymbol{\alpha}^n = \bar{A}_1^{-1}\bar{A}_2 e^{-\lambda L}\boldsymbol{\beta}^{n-1}, \qquad \boldsymbol{\beta}^n = \bar{B}_1^{-1}\bar{B}_2 \; e^{-\lambda L}\boldsymbol{\alpha}^{n-1}, \tag{2.35}$$

where $\lambda = \sqrt{|\boldsymbol{k}|^2-\omega^2}$. A complete double iteration of the Schwarz algorithm leads to

$$\boldsymbol{\alpha}^{n+1} = \bar{A}_1^{-1}\bar{A}_2\bar{B}_1^{-1}\bar{B}_2 e^{-2\lambda L}\boldsymbol{\alpha}^{n-1}, \qquad \boldsymbol{\beta}^{n+1} = \bar{B}_1^{-1}\bar{B}_2\bar{A}_1^{-1}\bar{A}_2 e^{-2\lambda L}\boldsymbol{\beta}^{n-1},$$

where the matrices in (2.35) are given by

$$\bar{A}_1 = A_1 + \sigma_1 A_2, \quad \bar{A}_2 = A_2 + \sigma_1 A_1, \quad \bar{B}_1 = A_1 + \sigma_2 A_2, \quad \bar{B}_2 = A_2 + \sigma_2 A_1.$$

where A_l $(l=1,2)$ are defined in (2.23).
a) In the first case this leads to the iteration matrix

$$\bar{A}_1^{-1}\bar{A}_2\bar{B}_1^{-1}\bar{B}_2 = \bar{B}_1^{-1}\bar{B}_2\bar{A}_1^{-1}\bar{A}_2 = \left(\frac{|\boldsymbol{k}|^2}{(\lambda+i\omega)^2}\right)^2 \frac{(1-\gamma_1(\lambda+i\omega)^2)(1-\gamma_2(\lambda+i\omega)^2)}{(1-\gamma_1(\lambda-i\omega)^2)(1-\gamma_2(\lambda-i\omega)^2)} \, Id.$$

Therefore, since $|\boldsymbol{k}|^2 = (\lambda - i\omega)(\lambda + i\omega)$, the convergence factor is given by

$$\rho(\omega, L, |\boldsymbol{k}|, \gamma_1, \gamma_2) = \left| \frac{(\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega)^2}{(\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2} \, \frac{1 - \gamma_1(\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2}{1 - \gamma_1(\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega)^2} \, \frac{1 - \gamma_2(\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2}{1 - \gamma_2(\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega)^2} \, e^{-2\sqrt{|\boldsymbol{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

b) In the second case this leads to the iteration matrix

$$\bar{A}_1^{-1}\bar{A}_2\bar{B}_1^{-1}\bar{B}_2 = \bar{B}_1^{-1}\bar{B}_2\bar{A}_1^{-1}\bar{A}_2 = \left(\frac{|\boldsymbol{k}|^2}{(\lambda - i\omega)^2}\right)^2 \frac{(\delta_1 - (\lambda - i\omega)^2)(\delta_2 - (\lambda - i\omega)^2)}{(\delta_1 - (\lambda + i\omega)^2)(\delta_2 - (\lambda + i\omega)^2)} \, Id.$$

Therefore, since $|\boldsymbol{k}|^2 = (\lambda - i\omega)(\lambda + i\omega)$, the convergence factor is given by

$$\rho(\omega, L, |\boldsymbol{k}|, \delta_1, \delta_2) = \left| \frac{(\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2}{(\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega)^2} \, \frac{\delta_1 - (\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega)^2}{\delta_1 - (\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2} \, \frac{\delta_2 - (\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega)^2}{\delta_2 - (\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2} \, e^{-2\sqrt{|\boldsymbol{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

c) The conclusion follows as in the first two cases. □

REMARK 3. *From (2.31), we see that the choice* $\gamma_1 = \gamma_2 = 1/(\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega)^2$ *is optimal, since then* $\rho_{th}(|\boldsymbol{k}|) \equiv 0$*, respectively, for all frequencies* $|\boldsymbol{k}|$*. With this choice of* γ_1 *and* γ_2*, the matrices* $\bar{A}_2$ *and* $\bar{B}_2$ *actually vanish.*

2.4.3. Relation to a Schwarz Algorithm for a Scalar Equation. We present here several particular choices of the transmission operator $\mathcal{S}_l$ with Fourier symbol σ_l $(l = 1, 2)$ in the interface conditions (2.28) and (2.29).

Case 1: taking $\gamma_1 = \gamma_2 = 0$ in (2.30), which amounts to enforce the classical Dirichlet transmission conditions, the convergence factor is

$$\rho_1(\omega, L, |\boldsymbol{k}|) = \left| \left(\frac{\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega}{\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega} \right)^2 e^{-2\sqrt{|\boldsymbol{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

In the non-overlapping case, $L = 0$, this choice ensures convergence only for propagative modes, and corresponds to the Taylor interface conditions of order zero proposed in [9] for the Helmholtz equation.

Case 2: taking $\gamma_1 = \gamma_2 = \dfrac{1}{|\boldsymbol{k}|^2} \dfrac{p - i\omega}{p + i\omega}$ in (2.30) or $\gamma_1 = \dfrac{1}{|\boldsymbol{k}|^2 - 2\omega^2 + 2i\omega p}$ in (2.30) and $\delta_2 = |\boldsymbol{k}|^2 - 2\omega^2 - 2i\omega p$ in (2.32) with $p \in \mathbb{C}$, the convergence factor is

$$\rho_2(\omega, L, |\boldsymbol{k}|, p) = \left| \left(\frac{\sqrt{|\boldsymbol{k}|^2 - \omega^2} - p}{\sqrt{|\boldsymbol{k}|^2 - \omega^2} + p} \right)^2 e^{-2\sqrt{|\boldsymbol{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

Case 3: taking $\gamma_1 = \gamma_2 = \dfrac{1}{|\boldsymbol{k}|^2 - 2\omega^2 + 2i\omega p}$ in (2.30) with $p \in \mathbb{C}$, the convergence factor is

$$\rho_3(\omega, L, |\boldsymbol{k}|, p) = \left| \frac{\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega}{\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega} \right| \rho_2(\omega, L, |\boldsymbol{k}|, p) \leq \rho_2(\omega, L, |\boldsymbol{k}|, p).$$

Case 4: taking $\gamma_l = \dfrac{1}{|\boldsymbol{k}|^2}\dfrac{p_l - i\omega}{p_l + i\omega}$, $l = 1, 2$ in (2.30) or $\gamma_1 = \dfrac{1}{|\boldsymbol{k}|^2 - 2\omega^2 + 2i\omega p_1}$ in (2.30) and $\delta_2 = |\boldsymbol{k}|^2 - 2\omega^2 - 2i\omega p_2$ in (2.32) with $p_l \in \mathbb{C}$, $l = 1, 2$, the convergence factor is

$$\rho_4(\omega, L, |\boldsymbol{k}|, p_1, p_2) = \left| \frac{\sqrt{|\boldsymbol{k}|^2 - \omega^2} - p_1}{\sqrt{|\boldsymbol{k}|^2 - \omega^2} + p_1} \frac{\sqrt{|\boldsymbol{k}|^2 - \omega^2} - p_2}{\sqrt{|\boldsymbol{k}|^2 - \omega^2} + p_2} e^{-2\sqrt{|\boldsymbol{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

Case 5: taking $\gamma_l = \dfrac{1}{|\boldsymbol{k}|^2 - 2\omega^2 + 2i\omega p_l}$ in (2.30) with $p_l \in \mathbb{C}$, $l = 1, 2$, the convergence factor is

$$\rho_5(\omega, L, |\boldsymbol{k}|, p_1, p_2) = \left| \frac{\sqrt{|\boldsymbol{k}|^2 - \omega^2} - i\omega}{\sqrt{|\boldsymbol{k}|^2 - \omega^2} + i\omega} \right| \rho_4(\omega, L, |\boldsymbol{k}|, p_1, p_2) \leq \rho_4(\omega, L, |\boldsymbol{k}|, p_1, p_2).$$

Except in the Case 1, all the other cases provide second order transmission conditions even if we use only a zero order approximation of the non-local operator $\sqrt{|\boldsymbol{k}|^2 - \omega^2}$. Note also that in the Case 2 and Case 4 the convergence rate is the same as the one obtained in the case of the Helmholtz equations in [22]. In the cases with parameters, the best choice for the parameters is in general the one that minimizes the convergence factor for all $|\boldsymbol{k}| \in K$, where K denotes the set of relevant numerical frequencies. One therefore needs to solve the min-max problems

$$\min_{p>0} \max_{|\boldsymbol{k}| \in K} \rho_j(\omega, L, |\boldsymbol{k}|, p), \quad j = 2, 3, \qquad \min_{p_1, p_2 > 0} \max_{|\boldsymbol{k}| \in K} \rho_j(\omega, L, |\boldsymbol{k}|, p_1, p_2) \quad j = 4, 5. \tag{2.36}$$

where $K = [(\boldsymbol{k}_{min}, \boldsymbol{k}_-) \cup (\boldsymbol{k}_+, \boldsymbol{k}_{max})]^2$, $\boldsymbol{k}_\pm$ are parameters to be chosen, $\boldsymbol{k}_{min}$ denotes the smallest frequency relevant to the subdomain and $\boldsymbol{k}_{max} = \frac{C}{h}$ denotes the largest frequency supported by the numerical grid (a reasonable choice for C would be π). If the domain Ω is a cilindrical conductor with homogeneous Dirichlet conditions on the lateral surface, the solution is the sum of the transverse electric (TE) and transverse magnetic (TM) fields. If the transverse section of the conductor is a rectangle with sides of length a and b, TE and TM are expanded in Fourier series with the harmonics $\sin(\frac{n\pi y}{a})\sin(\frac{m\pi z}{b})$, where the relevant frequencies are $k = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$. The lowest one is $\boldsymbol{k}_{min} = \pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$, while their spatial distribution is more complicated than in the case treated in [22, 18]. However, if ω falls on one of the relevant frequencies, say $\omega = \pi\sqrt{\frac{m_0^2}{a^2} + \frac{n_0^2}{b^2}}$ for some m_0 and n_0, choosing for instance $\boldsymbol{k}_\pm = \pi\sqrt{\frac{(m_0 \pm 1)^2}{a^2} + \frac{n_0^2}{b^2}}$ leaves precisely one frequency $k = \omega$ (that can be easily handled by a Krylov method when the Schwarz algorithm is used as a preconditioner) and treats all the other frequencies by optimization. If ω falls between the relevant frequencies, say $\pi\sqrt{\frac{m_1^2}{a^2} + \frac{n_1^2}{b^2}} \leq \omega \leq \pi\sqrt{\frac{m_2^2}{a^2} + \frac{n_2^2}{b^2}}$, the we get the iterative method to converge by choosing $\boldsymbol{k}_- = \pi\sqrt{\frac{m_1^2}{a^2} + \frac{n_1^2}{b^2}}$ and $\boldsymbol{k}_+ = \pi\sqrt{\frac{m_2^2}{a^2} + \frac{n_2^2}{b^2}}$, which will allow us to directly verify the asymptotic analysis without the use of a Krylov method. Using the same techniques as in [18], we obtain the asymptotically optimized parameters for the other cases as shown in Table 2.1.

2.5. Numerical Experiments for the time-harmonic equations. We now show numerical experiments for the time harmonic Maxwell equations solved on the

	with overlap, $L = h$		without overlap, $L = 0$	
Case	ρ	parameters	ρ	parameters
1	$1 - \sqrt{k_+ - \omega^2 h}$	none	1	none
2	$1 - 2C_\omega^{\frac{1}{6}} h^{\frac{1}{3}}$	$p = \frac{C_\omega^{\frac{1}{3}}}{2 \cdot h^{\frac{1}{3}}}$	$1 - \frac{\sqrt{2} C_\omega^{\frac{1}{4}}}{\sqrt{C}} \sqrt{h}$	$p = \frac{\sqrt{C} C_\omega^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
3	$1 - 2(k_+^2 - \omega^2)^{\frac{1}{6}} h^{\frac{1}{3}}$	$p = \frac{(k_+^2 - \omega^2)^{\frac{1}{3}}}{2 \cdot h^{\frac{1}{3}}}$	$1 - \frac{\sqrt{2}(k_+^2 - \omega^2)^{\frac{1}{4}}}{\sqrt{C}} \sqrt{h}$	$p = \frac{\sqrt{C}(k_+^2 - \omega^2)^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
4	$1 - 2^{\frac{2}{5}} C_\omega^{\frac{1}{10}} h^{\frac{1}{5}}$	$\begin{cases} p_1 = \frac{C_\omega^{\frac{2}{5}}}{2^{\frac{7}{5}} \cdot h^{\frac{1}{5}}}, \\ p_2 = \frac{C_\omega^{\frac{1}{5}}}{2^{\frac{6}{5}} \cdot h^{\frac{3}{5}}} \end{cases}$	$1 - \frac{C_\omega^{\frac{1}{8}}}{C^{\frac{1}{4}}} h^{\frac{1}{4}}$	$\begin{cases} p_1 = \frac{C_\omega^{\frac{3}{8}} \cdot C^{\frac{1}{4}}}{2 \cdot h^{\frac{1}{4}}}, \\ p_2 = \frac{C_\omega^{\frac{1}{8}} \cdot C^{\frac{3}{4}}}{h^{\frac{3}{4}}} \end{cases}$
5	$1 - 2^{\frac{2}{5}} (k_+^2 - \omega^2)^{\frac{1}{10}} h^{\frac{1}{5}}$	$\begin{cases} p_1 = \frac{(k_+^2 - \omega^2)^{\frac{2}{5}}}{2^{\frac{7}{5}} \cdot h^{\frac{1}{5}}}, \\ p_2 = \frac{(k_+^2 - \omega^2)^{\frac{1}{5}}}{2^{\frac{6}{5}} \cdot h^{\frac{3}{5}}} \end{cases}$	$1 - \frac{(k_+^2 - \omega^2)^{\frac{1}{8}}}{C^{\frac{1}{4}}} h^{\frac{1}{4}}$	$\begin{cases} p_1 = \frac{(k_+^2 - \omega^2)^{\frac{3}{8}} \cdot C^{\frac{1}{4}}}{2 \cdot h^{\frac{1}{4}}}, \\ p_2 = \frac{(k_+^2 - \omega^2)^{\frac{1}{8}} \cdot C^{\frac{3}{4}}}{h^{\frac{3}{4}}} \end{cases}$

TABLE 2.1

Asymptotic convergence rate and optimal choice of the parameters in the transmission conditions for the five variants of the optimized Schwarz method applied to the Maxwell equations, when the mesh parameter h is small, and the maximum numerical frequency is estimated by $k_{\max} = \frac{C}{h}$, and where $C_\omega = \min\left(\boldsymbol{k}_+^2 - \omega^2, \omega^2 - \boldsymbol{k}_-^2\right)$.

	with overlap, $L = h$				without overlap, $L = 0$			
h	1/16	1/32	1/64	1/128	1/16	1/32	1/64	1/128
Case 1	18	27	46	71	-	-	-	-
Case 2	16	16	17	20	28	36	50	68
Case 3	10	12	14	16	31	40	56	81
Case 4	17	17	20	22	26	28	33	38
Case 5	10	12	14	17	41	53	63	73

TABLE 2.2

Number of iterations to attain convergence for different interface conditions and different mesh sizes in the overlapping and non-overlapping case. The tolerance is fixed at $\varepsilon = 10^{-6}$.

unit square $\Omega = (0,1)^2$. The frequency $\omega = 2\pi$ is choosen such that the thumb rule (at least 10 discretization point per wavelength) shoult be respected. We decompose the domain into two subdomains $\Omega_1 = (0,\beta) \times (0,1)$ and $\Omega_2 = (\alpha,1) \times (0,1)$, where $0 < \alpha \leq \beta < 1$, and therefore the overlap is $L = \beta - \alpha$, and we consider both decompositions with and without overlap. We discretize the equations using a finite volume method, on a uniform mesh with mesh parameter h. In all comparisons that follow, we simulate directly the error equations, $f = 0$, and we use a random initial guess to ensure that all the frequency components are present in the iteration.

Table 2.2 shows the iteration count for all Schwarz algorithms considered previously, in the overlapping and non-overlapping case. We have to note that for the non-overlapping algorithm the iterative version doesn't converge therefore, applying a Krylov method is necessary.

2.6. Time Discretized Solutions. If we do not assume the wave to be periodic in time, the time domain also needs to be discretized. We consider a uniform time grid with time step Δt, and use a semi-implicit time integration scheme for the time

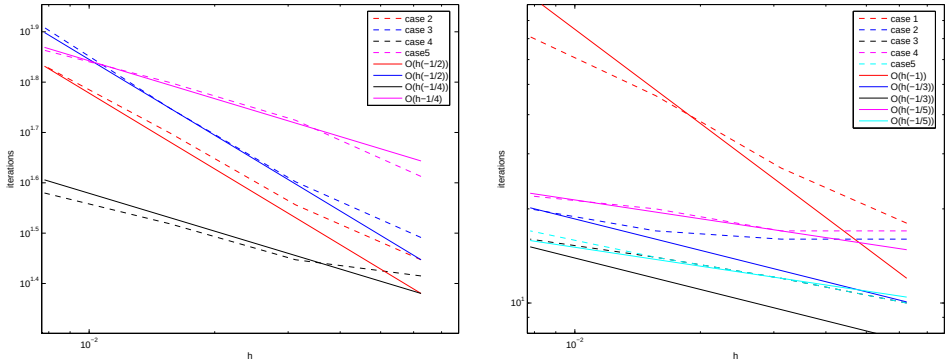

FIG. 2.1. *Asymptotics for the Non Overlapping (left) and Overlapping (right) cases for the Time harmonic equations*

derivative in (2.1) of the form

$$-\frac{\boldsymbol{E}^{n+1}-\boldsymbol{E}^n}{\Delta t}+\text{curl}\left(\frac{\boldsymbol{H}^{n+1}+\boldsymbol{H}^n}{2}\right)=\boldsymbol{J},\qquad \frac{\boldsymbol{H}^{n+1}-\boldsymbol{H}^n}{\Delta t}+\text{curl}\left(\frac{\boldsymbol{E}^{n+1}+\boldsymbol{E}^n}{2}\right)=\boldsymbol{0},$$

where the mean value is introduced to ensure the energy conservation (see [12]). With this time discretization, the Maxwell system (2.3) can be rewritten as a system that needs to be solved in each time step,

$$\begin{array}{rclrcl} -\sqrt{\eta}E_1+\partial_y H_3-\partial_z H_2 &=& \widetilde{J}_1, & \sqrt{\eta}H_1+\partial_y E_3-\partial_z E_2 &=& g_1,\\ -\sqrt{\eta}E_2+\partial_z H_1-\partial_x H_3 &=& \widetilde{J}_2, & \sqrt{\eta}H_2+\partial_z E_1-\partial_x E_3 &=& g_2,\\ -\sqrt{\eta}E_3+\partial_x H_2-\partial_y H_1 &=& \widetilde{J}_3, & \sqrt{\eta}H_3+\partial_x E_2-\partial_y E_1 &=& g_3, \end{array} \tag{2.37}$$

where we have set $(\boldsymbol{E},\boldsymbol{H})=(\boldsymbol{E}^{n+1},\boldsymbol{H}^{n+1})$ $\sqrt{\eta}=\frac{2}{\Delta t}$, $\widetilde{\boldsymbol{J}}=\boldsymbol{J}-\frac{2}{\Delta t}\boldsymbol{E}^n-\text{curl }\boldsymbol{H}^n$, $\boldsymbol{g}=\frac{2}{\Delta t}\boldsymbol{H}^n-\text{curl }\boldsymbol{E}^n$, and E_j and H_j denote the new fields at the time step $n+1$. As in the time harmonic case, we have the equivalent of Proposition 2.1:

PROPOSITION 2.8. *Let $\boldsymbol{u}$ be as defined in (2.2). At time step n+1, any component u_i of the solution of the Maxwell system (2.37) satisfies the elliptic equation*

$$(\eta-\Delta)u_j=f_j, \tag{2.38}$$

where the right hand side depends on $\boldsymbol{J}$, η, and the solution at the previous time step $\boldsymbol{u}^n$.

Proof. The result follows like in the time harmonic case. ∎

In contrast to the time harmonic case however, (2.38) is a positive definite Helmholtz equation, which is much easier to solve numerically than the Helmholtz equation (2.12).

There is also an equivalence result including boundary conditions, as in Proposition 2.2, for which we omit the details here. Instead, we state directly the equivalent of Proposition 2.4, i.e a convergence result for the classical Schwarz algorithm applied to the time discretized Maxwell equations.

PROPOSITION 2.9. *Let $\Omega=\mathbb{R}^3$, and consider the Maxwell system (2.10) in Ω with the radiation condition*

$$\lim_{r\to\infty} r(\boldsymbol{n}\times\boldsymbol{E}+\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{H}))=0, \tag{2.39}$$

	with overlap, $L = h$		without overlap, $L = 0$	
Case	ρ	parameters	ρ	parameters
1	$1 - 2^{\frac{3}{2}}\eta^{\frac{1}{4}}\sqrt{h}$	none	$1 - 2\frac{\sqrt{\eta}}{C}h$	none
2	$1 - 2^{\frac{13}{6}}\eta^{\frac{1}{6}}h^{\frac{1}{3}}$	$p = \frac{2^{-\frac{1}{3}}\eta^{\frac{1}{3}}}{h^{\frac{1}{3}}}$	$1 - \frac{4\eta^{\frac{1}{4}}\sqrt{h}}{\sqrt{C}}$	$p = \frac{\sqrt{C}\eta^{\frac{1}{4}}}{\sqrt{h}}$
3	$1 - 2^{\frac{7}{4}}\eta^{\frac{1}{8}}h^{\frac{1}{4}}$	$p = \frac{\sqrt{2}\eta^{\frac{1}{4}}}{\sqrt{h}}$	$1 - \frac{2^{\frac{5}{3}}\eta^{\frac{1}{6}}}{C^{\frac{1}{3}}}h^{\frac{1}{3}}$	$p = \frac{2^{\frac{2}{3}}C^{\frac{2}{3}}\eta^{\frac{1}{6}}}{h^{\frac{2}{3}}}$
4	$1 - 2^{\frac{4}{5}}\eta^{\frac{1}{10}}h^{\frac{1}{5}}$	$p_1 = \frac{\eta^{\frac{1}{5}}}{2^{\frac{2}{5}}h^{\frac{3}{5}}}, p_2 = \frac{\eta^{\frac{2}{5}}}{16^{\frac{1}{5}}h^{\frac{1}{5}}}$	$1 - \frac{\sqrt{2}\eta^{\frac{1}{8}}}{C^{\frac{1}{4}}}h^{\frac{1}{4}}$	$p_1 = \frac{\sqrt{2}C^{\frac{3}{4}}\eta^{\frac{1}{8}}}{h^{\frac{3}{4}}}, p_2 = \frac{C^{\frac{1}{4}}\eta^{\frac{3}{8}}}{\sqrt{2}h^{\frac{1}{4}}}$
5	$1 - 2^{\frac{7}{6}}\eta^{\frac{1}{12}}h^{\frac{1}{6}}$	$p_1 = \frac{2^{\frac{2}{3}}\eta^{\frac{1}{3}}}{h^{\frac{1}{3}}}$, $p_2 = \frac{2^{\frac{1}{3}}\eta^{\frac{1}{6}}}{h^{\frac{2}{3}}}$	$1 - \frac{2\eta^{\frac{1}{10}}}{C^{\frac{1}{5}}}h^{\frac{1}{5}}$	$p_1 = \frac{2C^{\frac{4}{5}}\eta^{\frac{1}{10}}}{h^{\frac{4}{5}}}$, $p_2 = \frac{2C^{\frac{2}{5}}\eta^{\frac{3}{10}}}{h^{\frac{2}{5}}}$

TABLE 2.3

Asymptotic convergence rate and optimal choice of the parameters in the transmission conditions for the five variants of the optimized Schwarz method applied to the time domain Maxwell equations, when the mesh parameter h is small, and the maximum numerical frequency is estimated by $k_{\max} = \frac{C}{h}$.

where $r = |\boldsymbol{x}|$, $\boldsymbol{n} = \boldsymbol{x}/|\boldsymbol{x}|$. Let Ω be decomposed into $\Omega_1 := (-\infty, L) \times \mathbb{R}^2$ and $\Omega_2 := (0, +\infty) \times \mathbb{R}^2$, ($L \geq 0$). For any given initial guess $\boldsymbol{u}^{1,0} \in (L^2(\Omega_1))^6$, $\boldsymbol{u}^{2,0} \in (L^2(\Omega_2))^6$, the Schwarz algorithm (2.17) applied to system (2.37) converges for all Fourier modes to the solution of (2.37) in the following sense

$$||\boldsymbol{u}(L,\cdot) - \boldsymbol{u}_1^{2n}(L,\cdot)||_2 + ||\boldsymbol{u}(0,\cdot) - \boldsymbol{u}_2^{2n}(0,\cdot)||_2 \leq R^n(||\boldsymbol{u}(L,\cdot) - \boldsymbol{u}_1^0(L,\cdot)||_2 + ||\boldsymbol{u}(0,\cdot) - \boldsymbol{u}_2^0(0,\cdot)||_2), \tag{2.40}$$

The convergence factor is

$$R_{td} = \frac{\sqrt{L\eta + 2} - \sqrt{L\eta}}{\sqrt{L\eta + 2} + \sqrt{L\eta}} e^{-\sqrt{L\eta}\sqrt{L\eta+2}} < 1. \tag{2.41}$$

Proof. This result follows like in the time harmonic case. □

The preceding theorem shows that the classical Schwarz algorithm with Dirichlet transmission conditions applied to the time-discretized Maxwell system is convergent for all frequencies $|\boldsymbol{k}|$, and that the overlap is not necessary to ensure convergence. The classical Schwarz algorithm corresponds to a simple optimized Schwarz algorithm for the associated positive definite Helmholtz equation (2.38). With this equivalence, Lemma 2.5, Remark 2, Proposition 2.7 and all the cases in subsection 2.4.3 hold unchanged upon replacing $i\omega$ by η, so we do not restate these results here. We show however in Table 2.3 the asymptotically optimal parameters to use in the time domain case. It is interesting to note the relationship of the optimized parameters for the Maxwell time domain case with the one for Cauchy-Riemann: Case 2 and 4 are identical, since the corresponding convergence rates in the two cases are the same, while for Case 1, 3 and 5 there is a small difference in the constants, which is due to the additional low frequency term in the Maxwell case. The difference appears to be systematic, the convergence factor of the Maxwell case is obtained from the convergence factor of the Cauchy-Riemann case by replacing h by $2h$, while for the optimized parameters one has to multiply by 2 in addition to the replacement of h by $2h$.

2.7. Numerical Experiments for the time discretized equations. We now show numerical experiments for the time discretized Maxwell equations solved on the unit square $\Omega = (0,1)^2$. Here $\eta = 1$. We decompose the domain into two subdomains

	with overlap, $L = h$				without overlap, $L = 0$			
h	1/16	1/32	1/64	1/128	1/16	1/32	1/64	1/128
Case 1	17	24	33	45	280	559	1310	2630
Case 2	13	15	19	24	39	56	77	111
Case 3	12	14	16	18	13	16	20	26
Case 4	12	13	15	17	21	25	30	36
Case 5	12	14	16	18	13	17	19	22

TABLE 2.4

Number of iterations to attain convergence for different interface conditions and different mesh sizes in the overlapping and non-overlapping case. The tolerance is fixed at $\varepsilon = 10^{-6}$.

$\Omega_1 = (0, \beta) \times (0, 1)$ and $\Omega_2 = (\alpha, 1) \times (0, 1)$, where $0 < \alpha \leq \beta < 1$, and therefore the overlap is $L = \beta - \alpha$, and we consider both decompositions with and without overlap. We discretize the equations using a finite volume method, on a uniform mesh with mesh parameter h. In all comparisons that follow, we simulate directly the error equations, $f = 0$, and we use a random initial guess to ensure that all the frequency components are present in the iteration.

Table 2.4 shows the iteration count for all Schwarz algorithms considered previously, in the overlapping and non-overlapping case.

We can see from the table 2.4, that the classical non-overlapping algorithm converges but very slowly, the need of optimized methods is very obvious in this case. We should note that numerically the optimized methods behave as we expected even if the difference between them is not very important especially in the overlapping case.

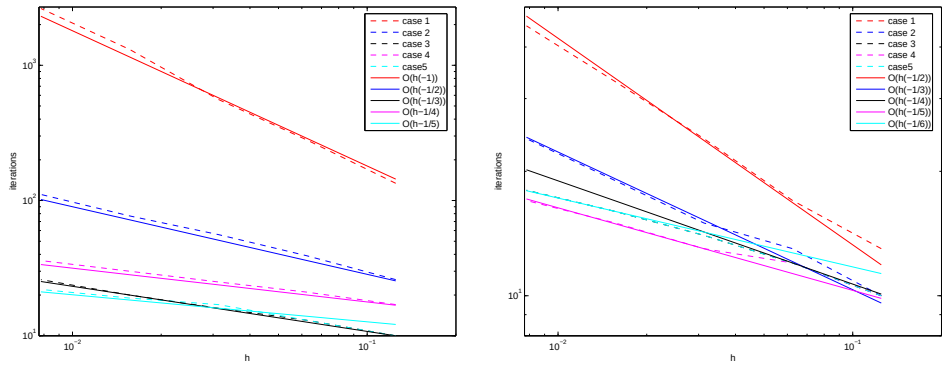

FIG. 2.2. *Asymptotics for the Non Overlapping (left) and Overlapping (right) cases for the Time harmonic equations*

3. Conclusions. We have shown in this paper, that for the Cauchy-Riemann equations and the Maxwell's equations, a Schwarz algorithm using characteristic information at the interfaces between subdomains is equivalent to an optimized Schwarz method applied to the corresponding scalar equation, with a low frequency approximation of the optimal transmission conditions. This equivalence shows that the Schwarz algorithms with characteristic conditions for those systems of partial differential equations are convergent even without overlap. We then used the equivalence to develop better transmission conditions than the characteristic ones for the Cauchy-Riemann

and the Maxwell equations. We illustrated with numerical experiments that the new algorithm converge much more rapidly than the classical ones. Although we have shown the equivalence only for two model problems, our results indicate that it also holds for other systems of partial differential equations. In particular for the Euler equations, it was already observed in [Victorita], that the classical Schwarz algorithm with characteristic information exchange at the interfaces is convergent, even without overlap. We currently focus on the optimization of the new transmission conditions introduced here, and also on optimized transmission conditions for the Euler equations and for linear elasticity.

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