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# WHY CAN CLASSICAL SCHWARZ METHODS APPLIED TO HYPERBOLIC SYSTEMS CONVERGE EVEN WITHOUT OVERLAP?

V.DOLEAN\*, M.J.GANDER† AND L.GERARDO-GIORDA‡

**Abstract.** Over the last two decades, classical Schwarz methods have been extended to systems of hyperbolic partial differential equations, and it was observed that the classical Schwarz method can be convergent even without overlap in certain cases. This is in strong contrast to the behavior of classical Schwarz methods applied to elliptic problems, for which overlap is essential for convergence. Over the last decade, optimized Schwarz methods have been developed for elliptic partial differential equations. These methods use more effective transmission conditions between subdomains, and are also convergent without overlap for elliptic problems.

We show here why the classical Schwarz method applied to the hyperbolic problem converges without overlap for the Cauchy-Riemann equations and Maxwell's equations. The reason is that the method is equivalent to a simple optimized Schwarz method for an equivalent elliptic problem. Using this link, we show how to develop more efficient Schwarz methods than the classical ones for the Cauchy-Riemann and Maxwell's equations. We illustrate our findings with numerical results.

**Key words.** Schwarz algorithms, optimized interface conditions, Maxwell equations

**AMS subject classifications.**

**1. Introduction.** Schwarz algorithms experienced a second youth over the last decades, when distributed computers became more and more performant and available. Fundamental convergence results for the classical Schwarz methods were derived for many partial differential equations, and can now be found in several authoritative reviews, see [3, 40, 39], and books, see [34, 32, 38]. The Schwarz methods were also extended to systems of partial differential equations, such as the time harmonic Maxwell equations, see [7, 11, 1], or to linear elasticity [17, 18], but much less is known about the behavior of the Schwarz methods applied to systems of equations. This is true in particular for the Euler equations, to which the Schwarz algorithm was first applied in [30, 31], where classical (characteristic) transmission conditions are used at the interfaces, or with more general interface conditions in [6]. The analysis of such algorithms applied to systems proved to be very different from the scalar case, see [14, 15].

Over the last decade, a new class of Schwarz methods was developed for scalar partial differential equations, namely the optimized Schwarz methods. These methods use more effective transmission conditions than the classical Dirichlet conditions at the interfaces between subdomains. New transmission conditions were originally proposed for three different reasons: first, to obtain Schwarz algorithms that are convergent without overlap, see [26] for Robin conditions. The second motivation for changing the transmission conditions was to obtain a convergent Schwarz method for the Helmholtz equation, where the classical Schwarz algorithm is not convergent, even with overlap. Approximate radiation conditions were introduced in [9, 12]. The third motivation was that the convergence rate of the classical Schwarz method is rather slow and very much dependent on the size of the overlap. In a short note on non-linear problems [24], Hagstrom et al. introduced Robin transmission conditions between subdomains

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\*UNIV. DE NICE SOPHIA-ANTIPOLIS, LABORATOIRE J.-A. DIEUDONNÉ, NICE, FRANCE. DOLEAN@MATH.UNICE.FR

†SECTION DE MATHÉMATIQUES, UNIVERSITÉ DE GENÈVE, CP 240, 1211 GENÈVE, MARTIN.GANDER@MATH.UNIGE.CH

‡UNIVERSITY OF TRENTO, ITALY. GERARDO@SCIENCE.UNITN.IT

and suggested nonlocal operators for best performance. In [4], these optimal, non-local transmission conditions were developed for advection-diffusion problems, with local approximations for small viscosity, and low order frequency approximations in [27, 8]. In [35] we can find the low-frequency approximation of the absorbing boundary conditions for Euler equations. Independently, at the algebraic level, generalized coupling conditions were introduced in [37, 36]. Optimized transmission conditions for the best performance of the Schwarz algorithm were introduced for advection diffusion problems in [25], for the Helmholtz equation in [5, 23], and for Laplace's equation in [16]. For complete results and attainable performance for symmetric, positive definite problems, see [20], and for time dependant problems, see [22, 21].

We show in this paper that the classical Schwarz method, which uses characteristic Dirichlet transmission conditions between subdomains, applied to the Cauchy Riemann and Maxwell's equations, is equivalent to an optimized Schwarz method applied to well known equivalent elliptic problems. This explains why the classical Schwarz method in that case can be convergent even without overlap, and it allows us to develop more effective Schwarz methods for these systems of partial differential equations. In Section 2, we first recall the equivalence of the Cauchy-Riemann equations with a scalar elliptic problem, and then prove that a classical Schwarz algorithm using characteristic Dirichlet transmission conditions between subdomains is equivalent to an optimized Schwarz method for the scalar equivalent, and hence is convergent even without overlap, a result that was observed for the Euler equations already in [15]. We then show that one can construct better conditions than the characteristic ones for the Cauchy-Riemann equations, and we show to which optimized Schwarz methods those conditions correspond in the elliptic setting, giving an entire hierarchy of algorithms. We finally show that the equivalence at the discretized level only holds up to small order terms in the discretization parameters. Our numerical experiments show that the new transmission conditions lead to much faster Schwarz methods for the Cauchy Riemann equations than the classical characteristic ones. In Section 3, we show that the results obtained for the model problem of the Cauchy Riemann equations can be generalized to the Maxwell equations. We prove again that using classical, characteristic conditions in a Schwarz algorithm applied to Maxwell's equations corresponds to an optimized Schwarz method for an equivalent scalar problem, both for the time harmonic and the time discretized case. We then derive better transmission conditions than the characteristic ones for the Schwarz algorithm applied to the Maxwell system, and illustrate our results with numerical experiments. We conclude in Section 4 with a summary of our findings, and an outlook on future work.

**2. Cauchy-Riemann Equations.** To analyze the relationship between Schwarz methods for scalar partial differential equations (PDEs) and systems of PDEs, we first use the Cauchy-Riemann equations as our guiding example. We write the Cauchy-Riemann equations in the form

$$\mathcal{L}\mathbf{u} \equiv \sqrt{\eta}\mathbf{u} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \partial_x \mathbf{u} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_y \mathbf{u} = \mathbf{f} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (2.1)$$

on  $\Omega = [0, 1] \times \mathbb{R}$ , and consider the associated boundary value problem with the boundary conditions

$$v(0, y) = r(y), \quad u(1, y) = s(y), \quad y \in \mathbb{R}. \quad (2.2)$$

The equations (2.1) can be interpreted as a hyperbolic system of conservation laws discretized in time, and therefore imposing the unknowns entering along the characteristics at the boundaries of the domain  $\Omega$  like in (2.2) leads to a well-posed problem.

**2.1. Relation to a Scalar Equation.** The scalar partial differential equations

$$\eta\tilde{u} - \Delta\tilde{u} = \tilde{f}, \quad \text{in } \Omega, \quad (2.3)$$

together with the boundary conditions

$$(\partial_x - \sqrt{\eta})\tilde{u}(0, y) = \tilde{r}(y), \quad \tilde{u}(1, y) = \tilde{s}(y), \quad y \in \mathbb{R} \quad (2.4)$$

is very much related to the Cauchy-Riemann equations. We summarize this well known relation in

**PROPOSITION 2.1.** *If  $\tilde{f} = (\sqrt{\eta} + \partial_x)f - \partial_y g$ ,  $\tilde{r} = \partial_y r - f(0, \cdot)$  and  $\tilde{s} = s$ , then the velocity component  $u$  of the Cauchy-Riemann equations (2.1) with boundary conditions (2.2) coincides with the solution  $\tilde{u}$  of the elliptic problem (2.3) with boundary conditions (2.4) for all  $x, y \in \Omega$ .*

*Proof.* We first note that by differentiating the first equation in (2.1) with respect to  $x$ , and the second equation with respect to  $y$ , taking the difference and substituting the resulting first order terms using again the first equation in (2.1), we obtain equation (2.3) and thus the differential equations coincide. In addition, while one of the boundary conditions stays the same, the boundary condition of Robin type in (2.4) is implied by differentiating the first boundary condition in (2.2) with respect to  $y$  and using the first Cauchy-Riemann equation (2.1). Since the solution of (2.3), (2.4) is unique (see for instance [29]), the result follows.  $\square$  A similar elliptic PDE can also be derived for  $v$ , but we will not need it in the sequel.

**2.2. Classical Schwarz Algorithm for the Cauchy-Riemann Equations.**

We decompose the domain  $\Omega$  into two overlapping or non-overlapping subdomains  $\Omega_1 = (0, b) \times \mathbb{R}$  and  $\Omega_2 = (a, 1) \times \mathbb{R}$ , and we denote the overlap by  $L := b - a \geq 0$ . A classical Schwarz algorithm for the Cauchy-Riemann equations (2.1) on these two subdomains is then defined by

$$\begin{aligned} \mathcal{L}u^{1,n} &= \mathbf{f}, & \text{in } \Omega_1, & \quad \mathcal{L}u^{2,n} &= \mathbf{f}, & \text{in } \Omega_2, \\ v^{1,n}(0, y) &= r(y), & y \in \mathbb{R}, & \quad u^{2,n}(1, y) &= s(y), & y \in \mathbb{R}, \\ u^{1,n}(b, y) &= u^{2,n-1}(b, y), & y \in \mathbb{R}, & \quad v^{2,n}(a, y) &= v^{1,n-1}(a, y), & y \in \mathbb{R}. \end{aligned} \quad (2.5)$$

Note that in this classical form of the Schwarz algorithm for the system of PDEs, we respected in the transmission conditions the information exchange along the characteristic directions, which is the most natural approach to follow when applying domain decomposition methods to hyperbolic problems, see for example [2, 31].

From the relation between the Cauchy-Riemann equations (2.1) and the associated elliptic problem (2.3) stated in Proposition 2.1, the related Schwarz algorithm for the elliptic problem would be (here  $\mathcal{B} = \partial_x - \sqrt{\eta}$ )

$$\begin{aligned} (\eta - \Delta)\tilde{u}^{1,n} &= \tilde{f}, & \text{in } \Omega_1 & \quad (\eta - \Delta)\tilde{u}^{2,n} &= \tilde{f}, & \text{in } \Omega_2 \\ \mathcal{B}\tilde{u}^{1,n}(0, y) &= \tilde{r}(y), & y \in \mathbb{R}, & \quad \tilde{u}^{2,n}(1, y) &= \tilde{s}(y), & y \in \mathbb{R}, \\ \tilde{u}^{1,n}(b, y) &= \tilde{u}^{2,n-1}(b, y), & y \in \mathbb{R}, & \quad \mathcal{B}\tilde{u}^{2,n}(a, y) &= \mathcal{B}\tilde{u}^{1,n-1}(a, y), & y \in \mathbb{R}. \end{aligned} \quad (2.6)$$

We now show that indeed algorithm (2.5) and algorithm (2.6) are equivalent, and hence using the classical Dirichlet characteristic transmission conditions for the hyperbolic system (2.1) corresponds to a simple optimized Schwarz method for the equivalent scalar problem (2.3).

**THEOREM 2.2.** *If algorithm (2.6) is started with the initial guess  $\tilde{u}^{1,0} = u^{1,0}$  and  $\tilde{u}^{2,0} = u^{2,0}$ , then the iterates of algorithm (2.6) and algorithm (2.5) coincide,  $u^{l,n}(x, y) = \tilde{u}^{l,n}(x, y)$  for all  $(x, y) \in \Omega_l$ ,  $l = 1, 2$  and  $n \geq 1$ .*

*Proof.* The proof is by induction. Proposition 2.1 entails the result for  $n = 1$ . Assume then that the result is true at iteration  $n - 1$ . Let  $u^{1,n}$ ,  $v^{1,n}$ ,  $u^{2,n}$ , and  $v^{2,n}$  be the iterates of the Schwarz algorithm applied to the Cauchy-Riemann equations. We then have, on the one hand

$$u^{1,n}(b, y) = u^{2,n-1}(b, y) = \tilde{u}^{2,n-1}(b, y) = \tilde{u}^{1,n}(b, y).$$

On the other hand, differentiating the interface condition on  $v$  in (2.5) with respect to  $y$  and using the first Cauchy-Riemann equation, we get

$$(\partial_x - \sqrt{\eta})u^{2,n} - f = \partial_y v^{2,n} = \partial_y u^{1,n-1} = (\partial_x - \sqrt{\eta})u^{1,n-1} - f.$$

When evaluating the above expression at  $x = a$ , the  $f$  terms cancel, and we obtain

$$(\partial_x - \sqrt{\eta})u^{2,n} = (\partial_x - \sqrt{\eta})u^{1,n-1} = (\partial_x - \sqrt{\eta})\tilde{u}^{1,n-1} = (\partial_x - \sqrt{\eta})\tilde{u}^{2,n}.$$

Since the boundary conditions at  $(0, y)$  and  $(1, y)$  stay the same, the result follows from Proposition 2.1.  $\square$  Theorem 2.2 indicates why the classical Schwarz algorithm (2.5) with characteristic Dirichlet transmission conditions for the Cauchy Riemann equations can converge even without overlap: it is equivalent to an optimized Schwarz method for a related elliptic PDE, and optimized Schwarz methods are also convergent without overlap, see [20]. It is therefore not surprising that for hyperbolic problems, the Schwarz algorithm does not necessarily need overlap in order to be convergent.

We analyze now the convergence rate of Algorithm (2.5) when the domain is the entire plane,  $\Omega = \mathbb{R}^2$ , and the subdomains are  $\Omega_1 = (-\infty, L)$  and  $\Omega_2 = (0, \infty)$ ,  $L \geq 0$ . Let  $e^{j,n}(x, y) = (e_u^{j,n}(x, y), e_v^{j,n}(x, y))^t := \mathbf{u}(x, y) - \mathbf{u}^{j,n}(x, y)$ ,  $j = 1, 2$  denote the error at iteration  $n$ . Then the  $e^{j,n}$  satisfy the homogeneous version of Algorithm (2.5), which after a Fourier transform  $\mathcal{F}$  in  $y$  with parameter  $k$ ,  $\hat{e}^{j,n} := \mathcal{F}(e^{j,n})$ , gives

$$\begin{aligned} \hat{\mathcal{L}}\hat{e}^{1,n} &= \mathbf{0}, & \text{in } (-\infty, L) \times \mathbb{R}, & \quad \hat{\mathcal{L}}\hat{e}^{2,n} = \mathbf{0}, & \text{in } (0, \infty) \times \mathbb{R}, \\ \hat{e}_v^{1,n}(0, k) &= 0, & y \in \mathbb{R}, & \quad \hat{e}_u^{2,n}(1, k) = 0, & y \in \mathbb{R}, \\ \hat{e}_u^{1,n}(b, k) &= \hat{e}_u^{2,n-1}(b, k), \quad k \in \mathbb{R}, & & \quad \hat{e}_v^{2,n}(a, k) = \hat{e}_v^{1,n-1}(a, k), \quad k \in \mathbb{R}, \end{aligned} \tag{2.7}$$

and  $\hat{\mathcal{L}}$  denotes the action of the operator  $\mathcal{L}$  after the Fourier transform in  $y$ , i.e.

$$\hat{\mathcal{L}}\hat{\mathbf{u}} := \mathcal{F}(\mathcal{L}\mathbf{u}) = \sqrt{\eta}\hat{\mathbf{u}} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \partial_x \hat{\mathbf{u}} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ik\hat{\mathbf{u}}.$$

**THEOREM 2.3.** *If the initial error on the interfaces contains the Fourier components  $\hat{e}^{1,0}(L, k)$  and  $\hat{e}^{2,0}(0, k)$ ,  $k \in \mathbb{R}$ , then for any overlap  $L \geq 0$ , algorithm (2.5) converges for all  $k$ ,*

$$|\hat{e}^{1,2n}(L, k)| + |\hat{e}^{2,2n}(0, k)| \leq (\rho(\eta, L, k))^{2n} |\hat{e}^{1,0}(L, k)| + |\hat{e}^{2,0}(0, k)|, \tag{2.8}$$

and the convergence factor is given by

$$\rho(\eta, L, k) = \sqrt{\frac{\sqrt{\eta + k^2} - \sqrt{\eta}}{\sqrt{\eta + k^2} + \sqrt{\eta}}} e^{-L\sqrt{\eta + k^2}} < 1, \quad \forall k \in \mathbb{R}. \tag{2.9}$$

*Proof.* Solving (2.7) at iteration  $n + 1$ , we obtain

$$\hat{e}^{1,n+1} = \alpha^{n+1} e^{\sqrt{\eta+k^2}(x-L)} \begin{pmatrix} \sqrt{\eta+k^2} + \sqrt{\eta} \\ -ik \end{pmatrix}, \hat{e}^{2,n+1} = \beta^{n+1} e^{-\sqrt{\eta+k^2}x} \begin{pmatrix} -ik \\ \sqrt{\eta+k^2} + \sqrt{\eta} \end{pmatrix}, \quad (2.10)$$

where  $\alpha^{n+1}$  and  $\beta^{n+1}$  are determined by the interface conditions to be

$$\alpha^{n+1} = \beta^n \frac{-ik}{\sqrt{\eta+k^2} + \sqrt{\eta}} e^{-\sqrt{\eta+k^2}L}, \quad \beta^{n+1} = \alpha^n \frac{-ik}{\sqrt{\eta+k^2} + \sqrt{\eta}} e^{-\sqrt{\eta+k^2}L}.$$

Performing a double step, this leads to the square of the convergence factor

$$\rho(\eta, L, k)^2 := \frac{\alpha^{n+1}}{\alpha^{n-1}} = \frac{\beta^{n+1}}{\beta^{n-1}} = -\frac{\sqrt{\eta+k^2} - \sqrt{\eta}}{\sqrt{\eta+k^2} + \sqrt{\eta}} e^{-2L\sqrt{\eta+k^2}},$$

which implies the result by induction on  $n$ .  $\square$

### 2.3. Optimized Schwarz Method for the Cauchy-Riemann Equations.

Algorithm (2.6) is a rather unusual optimized Schwarz algorithm for the elliptic problem (2.3), since it still uses Dirichlet transmission conditions at one of the interfaces. The guiding principle behind optimized Schwarz methods is to use approximations of transparent boundary conditions at the interfaces between subdomains. The Robin transmission condition on one of the interfaces in (2.6) can be interpreted as a zeroth order low frequency approximation of a transparent condition, see [19]. In order to find better transmission conditions for the Cauchy-Riemann equations, we now derive their associated transparent boundary conditions.

**2.3.1. Transparent Boundary Conditions.** We consider now the Cauchy-Riemann equations (2.1) on the domain  $\Omega = (0, 1) \times \mathbb{R}$ , with  $f$  compactly supported in  $\Omega$ , but with the new boundary conditions

$$(v + \mathcal{S}_1 u)(0, y) = 0, \quad (u + \mathcal{S}_2 v)(1, y) = 0, \quad y \in \mathbb{R}, \quad (2.11)$$

where the operators  $\mathcal{S}_l$ ,  $l = 1, 2$  are general, pseudo-differential operators acting in the  $y$  direction.

LEMMA 2.4. *If the operators  $\mathcal{S}_l$ ,  $l = 1, 2$ , have the Fourier symbol*

$$\sigma_l := \mathcal{F}(\mathcal{S}_l) = \frac{ik}{\sqrt{\eta} + \sqrt{\eta+k^2}}, \quad l = 1, 2, \quad (2.12)$$

*then the solution of the Cauchy-Riemann equations (2.1) on the domain  $\Omega = (0, 1) \times \mathbb{R}$  with boundary conditions (2.11) coincides with the restriction to the domain  $\Omega$  of the solution of the Cauchy-Riemann equations (2.1) posed on  $\mathbb{R}^2$ .*

*Proof.* It suffices to show that the difference between the solution of the global problem and the solution of the restricted problem vanishes. This difference, denoted by  $e$ , satisfies the homogeneous counterpart of (2.1) with boundary conditions (2.11), and its Fourier transform is

$$\hat{e}(x, k) = \alpha e^{\sqrt{\eta+k^2}x} \begin{pmatrix} \sqrt{\eta+k^2} + \sqrt{\eta} \\ -ik \end{pmatrix} + \beta e^{-\sqrt{\eta+k^2}x} \begin{pmatrix} -ik \\ \sqrt{\eta+k^2} + \sqrt{\eta} \end{pmatrix}. \quad (2.13)$$

Now the first boundary condition in (2.11) implies  $\beta\sqrt{\eta+k^2} = 0$ , and hence  $\beta = 0$ , and the second one implies  $\alpha\sqrt{\eta+k^2}e^{-\sqrt{\eta+k^2}} = 0$ , which gives  $\alpha = 0$ , and hence  $\hat{e} \equiv 0$ .  $\square$

REMARK 1. *The symbols (2.12) can be written in several mathematically equivalent forms,*

$$\sigma_l = \frac{ik}{\sqrt{\eta} + \sqrt{\eta+k^2}} = \frac{\sqrt{\eta} - \sqrt{\eta+k^2}}{ik} = \sqrt{\frac{\sqrt{\eta} - \sqrt{\eta+k^2}}{\sqrt{\eta} + \sqrt{\eta+k^2}}}. \quad (2.14)$$

The first form contains a local and a non-local term in  $k$ , since multiplication with  $ik$  corresponds to derivation in  $y$ , which is a local operation (as the application of any polynomial in  $ik$  would be), whereas the term containing the square-root of  $k^2$  is a non-local operation. The second form contains two non-local operations, since the division by  $ik$  corresponds to an integration. This integration can however be passed to the other variable in (2.11) by multiplication with  $ik$ . The last form contains only non-local terms. These different forms motivate different local approximations of the transparent boundary conditions, and thus lead to different optimized Schwarz methods, as we will show in the sequel. We now consider the associated elliptic equations (2.3) on the domain  $\Omega = (0, 1) \times \mathbb{R}$ , with  $f$  compactly supported in  $\Omega$ , but with the new boundary conditions

$$(\partial_x - \tilde{\mathcal{S}}_1)u(0, y) = 0, \quad (\partial_x + \tilde{\mathcal{S}}_2)u(1, y) = 0, \quad y \in \mathbb{R}, \quad (2.15)$$

where the operators  $\tilde{\mathcal{S}}_l$ ,  $l = 1, 2$  are general, pseudo-differential operators acting in the  $y$  direction.

LEMMA 2.5. *If the operators  $\tilde{\mathcal{S}}_l$ ,  $l = 1, 2$ , have the Fourier symbol*

$$\tilde{\sigma}_l := \mathcal{F}(\tilde{\mathcal{S}}_l) = \sqrt{\eta+k^2}, \quad l = 1, 2, \quad (2.16)$$

then the solution of (2.3) on the domain  $\Omega = (0, 1) \times \mathbb{R}$  with boundary conditions (2.15) coincides with the restriction to  $\Omega$  of the solution of the (2.3) on  $\mathbb{R}^2$ .

*Proof.* The proof follows as in Lemma 2.4 using Fourier analysis.  $\square$

PROPOSITION 2.6. *The velocity component  $u$  of the solution of the Cauchy-Riemann equations (2.1) with boundary conditions (2.11), (2.12) coincides with the solution  $\tilde{u}$  of the elliptic problem (2.3) with boundary conditions (2.15), (2.16) for all  $x, y \in \Omega = (0, 1) \times \mathbb{R}$ .*

*Proof.* We have already seen in Proposition 2.1 that the equations inside the domain coincide. It therefore suffices to show that the boundary conditions are also equivalent. By using the first Fourier transformed equation inside the domain, i.e.  $ik\hat{v} = (\partial_x - \sqrt{\eta})\hat{u}$ , the boundary condition at  $x = 1$ , i.e.  $(\sqrt{\eta} + \sqrt{\eta+k^2})\hat{u} + ik\hat{v} = 0$ , becomes  $(\partial_x + \sqrt{\eta+k^2})\hat{u} = 0$ , which is the transparent boundary condition for the elliptic equation. The same argument applies to the other boundary condition: using the first Fourier transformed equation, the boundary condition at  $x = 0$  becomes  $(\sqrt{\eta} + \sqrt{\eta+k^2})(\partial_x - \sqrt{\eta})\hat{u} - k^2\hat{u} = 0$ . Taking into account that  $k^2 = (\sqrt{\eta+k^2} + \sqrt{\eta})(\sqrt{\eta+k^2} - \sqrt{\eta})$ , we further obtain  $(\sqrt{\eta+k^2} + \sqrt{\eta})(\sqrt{\eta+k^2} - \partial_x)\hat{u} = 0$ , which is equivalent to the transparent boundary condition for the scalar equation at  $x = 0$ .  $\square$

**2.3.2. Schwarz Algorithm with General Interface Conditions.** Proposition 2.6 shows again the equivalence between the two problems when transparent

boundary conditions are used. In the case of  $\eta - \Delta$ , using transparent transmission conditions at the interfaces between subdomains leads to convergence of the Schwarz algorithm in a finite number of steps, identical to the number of subdomains, for subdomains arranged in a linear sequence, see [19] and references therein. By the equivalence shown in Proposition 2.6, the same result also holds for the Cauchy-Riemann equations, when on the subdomains  $\Omega_1 = (-\infty, L) \times \mathbb{R}$  and  $\Omega_2 = (0, \infty) \times \mathbb{R}$  the new Schwarz algorithm

$$\begin{aligned} \mathcal{L}u^{1,n} &= 0, & \text{in } \Omega_1, \\ u^{1,n}(L, y) + \mathcal{S}_1 v^{1,n}(L, y) &= u^{2,n-1}(L, y) + \mathcal{S}_1 v^{2,n-1}(L, y), \\ \mathcal{L}u^{2,n} &= 0, & \text{in } \Omega_2, \\ v^{2,n}(0, y) + \mathcal{S}_2 u^{2,n}(0, y) &= v^{1,n-1}(0, y) + \mathcal{S}_2 u^{1,n}(0, y), \end{aligned} \quad (2.17)$$

is used, provided the symbols of  $\mathcal{S}_l$  are the  $\sigma_l$  given in (2.12),  $l = 1, 2$ . As explained in Remark 1, these are non-local operators and hence not as easy to use as the characteristic Dirichlet transmission conditions. They are therefore often approximated. Using approximate operators in the transmission conditions leads to the class of optimized Schwarz methods.

We have already seen that the classical Schwarz algorithm applied to the Cauchy-Riemann equations (2.5), i.e.  $\mathcal{S}_l \equiv 0$ ,  $l = 1, 2$ , corresponds to an optimized Schwarz algorithm for the equivalent scalar elliptic problem. To find algorithms with better performance, we now derive the convergence factor for algorithm (2.17) with general transmission operators  $\mathcal{S}_l$ , and corresponding symbols  $\sigma_l$ ,  $l = 1, 2$ . Proceeding as in Theorem 2.3, we obtain the convergence factor for a double iteration,

$$\rho_{opt}(\eta, L, k, \sigma_1, \sigma_2) = \left| \frac{-ik + \sigma_1(\sqrt{\eta + k^2} + \sqrt{\eta})}{\sqrt{\eta + k^2} + \sqrt{\eta} - ik\sigma_1} \frac{-ik + \sigma_2(\sqrt{\eta + k^2} + \sqrt{\eta})}{\sqrt{\eta + k^2} + \sqrt{\eta} - ik\sigma_2} e^{-2\sqrt{\eta + k^2}L} \right|^{\frac{1}{2}}. \quad (2.18)$$

A good choice of  $\sigma_l$ ,  $l = 1, 2$  is a choice that makes the convergence factor  $\rho_{opt}$  small for all values of  $k$ , and from (2.18), we see that the choice (2.12) is optimal, since then  $\rho_{opt} \equiv 0$  for all  $k$ . But a good choice should also lead to transmission conditions which are as easy and inexpensive to use, as the classical characteristic Dirichlet conditions. We use in the next subsection again the relation of the Cauchy-Riemann equations to the equivalent elliptic problem to find more effective transmission conditions.

**2.3.3. Relation to a Schwarz Algorithm for the Scalar Equation.** We now look at several particular choices of the transmission operators  $\mathcal{S}_l$  with Fourier symbol  $\sigma_l$ ,  $l = 1, 2$ , in algorithm (2.17), and show the corresponding convergence factor. In each case, we also explain to which choice of transmission conditions this corresponds in the equivalent elliptic problem setting.

**Case 1:**  $\sigma_1 = \sigma_2 = 0$ , which corresponds to the characteristic Dirichlet transmission conditions or algorithm (2.5). The resulting convergence factor is

$$\rho_1(\eta, L, k) = \left| \frac{\sqrt{\eta + k^2} - \sqrt{\eta}}{\sqrt{\eta + k^2} + \sqrt{\eta}} e^{-2\sqrt{\eta + k^2}L} \right|^{\frac{1}{2}}, \quad (2.19)$$

which shows that the algorithm is convergent for all  $k$ , even without overlap, and the convergence factor is smaller than for the classical Schwarz algorithm with Dirichlet transmission conditions applied to the equivalent elliptic problem, by the factor in front of the exponential, see [19].



**Case 2:**  $\sigma_1 = \frac{ik}{\sqrt{\eta+p}}$ ,  $\sigma_2 = \frac{\sqrt{\eta-p}}{ik}$ ,  $p > 0$ , a mixed case, where the first form of the exact symbol in (2.14) is used to approximate  $\sigma_1$  and the second form is used to approximate  $\sigma_2$ . This corresponds to first order transmission conditions, since  $ik$  corresponds to a derivative in  $y$  and the division by  $ik$  can be avoided by multiplying the entire transmission condition by  $ik$ . The convergence factor is

$$\rho_2(\eta, L, k, p) = \left| \left( \frac{\sqrt{\eta+k^2-p}}{\sqrt{\eta+k^2+p}} \right)^2 e^{-2\sqrt{\eta+k^2}L} \right|^{\frac{1}{2}}, \quad (2.20)$$

which is equivalent to the algorithm in the elliptic case with Robin transmission conditions  $\partial_x \pm p$  are used as transmission conditions, see [19].

**Case 3:**  $\sigma_1 = \sigma_2 = \sigma = \frac{ik}{\sqrt{\eta+p}}$ ,  $p > 0$ , where only the first form of the exact symbol (2.14) has been used to approximate both  $\sigma_1$  and  $\sigma_2$ . The resulting convergence factor is

$$\rho_3(\eta, L, k, p) = \left| \frac{\sqrt{\eta+k^2} - \sqrt{\eta}}{\sqrt{\eta+k^2} + \sqrt{\eta}} \right|^{\frac{1}{2}} \rho_2(\eta, L, k, p) < \rho_2(\eta, L, k, p), \quad (2.21)$$

and thus the convergence factor is again smaller by the same factor as in Case 1 than the corresponding convergence factor for the equivalent elliptic problem when the Robin transmission conditions, see [19].

Choosing the second form of the symbol (2.14) to approximate both  $\sigma_1$  and  $\sigma_2$  is not a good idea since it inverts the additional low frequency factor which is less than one in (2.21).

**Case 4:**  $\sigma_1 = \frac{ik}{\sqrt{\eta+p_1}}$ ,  $\sigma_2 = \frac{\sqrt{\eta-p_2}}{ik}$ ,  $p_{1,2} > 0$ , a choice motivated by Remark 1, which leads to the convergence factor

$$\rho_4(\eta, L, k, p_1, p_2) = \left| \frac{\sqrt{\eta+k^2} - p_1}{\sqrt{\eta+k^2} + p_1} \cdot \frac{\sqrt{\eta+k^2} - p_2}{\sqrt{\eta+k^2} + p_2} e^{-2\sqrt{\eta+k^2}L} \right|^{\frac{1}{2}}. \quad (2.22)$$

This corresponds to the two-sided Robin transmission conditions in the elliptic case in [19], which are of the form  $\partial_x - p_1$  for the first subdomain and  $\partial_x + p_2$  for the second one.

**Case 5:**  $\sigma_1 = \frac{ik}{\sqrt{\eta+p_1}}$ ,  $\sigma_2 = \frac{ik}{\sqrt{\eta+p_2}}$ ,  $p_{1,2} > 0$ , which gives the even better convergence factor

$$\rho_5(\eta, L, k, p_1, p_2) = \left| \frac{\sqrt{\eta+k^2} - \sqrt{\eta}}{\sqrt{\eta+k^2} + \sqrt{\eta}} \right|^{\frac{1}{2}} \rho_4(\eta, L, k, p_1, p_2) < \rho_4(\eta, L, k, p_1, p_2).$$

In the cases with parameters, the best choice for the parameters is in general the one that minimizes the convergence factor for all  $k \in K = [k_{\min}, k_{\max}]$ , where  $K$  denotes the set of relevant numerical frequencies. One therefore needs to solve the min-max problems

$$\min_{p>0} \max_{k \in K} \rho_j(\eta, L, k, p), \quad j = 2, 3, \quad \min_{p_1, p_2 > 0} \max_{k \in K} \rho_j(\eta, L, k, p_1, p_2) \quad j = 4, 5. \quad (2.23)$$

In Case 2 and 4, the solution of the problem is already given in [19] for the equivalent elliptic case, and can therefore directly be used for the Cauchy-Riemann equations.

Case	with overlap, $L = h$		without overlap, $L = 0$	
	$\rho$	parameters	$\rho$	parameters
1	$1 - 2\eta^{\frac{1}{4}}\sqrt{h}$	none	$1 - \frac{\sqrt{\eta}}{C}h$	none
2	$1 - 2^{\frac{13}{6}}\eta^{\frac{1}{6}}h^{\frac{1}{3}}$	$p = \frac{2^{-\frac{1}{3}}\eta^{\frac{1}{3}}}{h^{\frac{1}{3}}}$	$1 - \frac{4\eta^{\frac{1}{4}}\sqrt{h}}{\sqrt{C}}$	$p = \frac{\sqrt{C}\eta^{\frac{1}{4}}}{\sqrt{h}}$
3	$1 - 2^{\frac{3}{2}}\eta^{\frac{1}{8}}h^{\frac{1}{4}}$	$p = \frac{\eta^{\frac{1}{4}}}{\sqrt{h}}$	$1 - \frac{2^{\frac{4}{3}}\eta^{\frac{1}{6}}h^{\frac{1}{3}}}{C^{\frac{1}{3}}}$	$p = \frac{2^{\frac{1}{3}}C^{\frac{2}{3}}\eta^{\frac{1}{6}}}{h^{\frac{2}{3}}}$
4	$1 - 2^{\frac{4}{5}}\eta^{\frac{1}{10}}h^{\frac{1}{5}}$	$p_1 = \frac{\eta^{\frac{1}{5}}}{2^{\frac{2}{5}}h^{\frac{3}{5}}}, p_2 = \frac{\eta^{\frac{2}{5}}}{16^{\frac{1}{5}}h^{\frac{1}{5}}}$	$1 - \frac{\sqrt{2}\eta^{\frac{1}{8}}h^{\frac{1}{4}}}{C^{\frac{1}{4}}}$	$p_1 = \frac{\sqrt{2}C^{\frac{3}{4}}\eta^{\frac{1}{8}}}{h^{\frac{3}{4}}}, p_2 = \frac{C^{\frac{1}{4}}\eta^{\frac{3}{8}}}{\sqrt{2}h^{\frac{1}{4}}}$
5	$1 - 2\eta^{\frac{1}{12}}h^{\frac{1}{6}}$	$p_1 = \frac{\eta^{\frac{1}{3}}}{h^{\frac{1}{3}}}, p_2 = \frac{\eta^{\frac{1}{2}}}{h^{\frac{2}{3}}}$	$1 - \frac{2^{\frac{4}{5}}\eta^{\frac{1}{10}}h^{\frac{1}{5}}}{C^{\frac{1}{5}}}$	$p_1 = \frac{(2C)^{\frac{4}{5}}\eta^{\frac{1}{10}}}{h^{\frac{4}{5}}}, p_2 = \frac{(2C)^{\frac{2}{5}}\eta^{\frac{3}{10}}}{h^{\frac{2}{5}}}$

TABLE 2.1

Asymptotic convergence rate and optimal choice of the parameters in the transmission conditions for the five variants of the optimized Schwarz method applied to the Cauchy-Riemann equations, when the mesh parameter  $h$  is small, and the maximum numerical frequency is estimated by  $k_{\max} = \frac{C}{h}$ .

Cauchy-Riemann equations	$\eta - \Delta$
Case 1: Dirichlet/Dirichlet	Dirichlet/Taylor-Robin
Case 2: Robin optimized/Robin optimized	Robin optimized/Robin optimized
Case 4: Two sided Optimized-Robin	Two sided Optimized-Robin
Transparent	Transparent

TABLE 2.2

Algorithm equivalences between the Schwarz methods for Cauchy-Riemann and the optimized Schwarz methods for the corresponding scalar elliptic problem.

The other cases are specific to the Cauchy-Riemann equations and an asymptotic analysis similar to the one shown in [19] leads to the results given in Table 2.1, where the estimate  $k_{\max} = \frac{C}{h}$ ,  $C > 0$  a constant, was used (a reasonable value would be  $C = \pi$ ). One can clearly see in this table that there are much better transmission conditions than the characteristic ones for the Cauchy-Riemann equations: for a Schwarz algorithm with overlap of the order of the mesh parameter,  $L = h$ , the characteristic transmission conditions lead to a convergence factor  $1 - O(\sqrt{h})$ , which depends strongly on  $h$ , whereas with better transmission conditions, one can achieve the convergence factor  $1 - O(h^{\frac{1}{6}})$ , which now depends only very weakly on  $h$ , at the same cost per iteration. Similar results also hold for the Schwarz algorithm without overlap, as shown in Table 2.1.

To conclude, we show in Table 2.2 a summary of the different choices of transmission conditions in the optimized Schwarz algorithm applied to the Cauchy-Riemann equations and to the equivalent scalar elliptic problem, together with the equivalences we have shown.

**3. Maxwell System.** We now turn our attention to the Maxwell system, which describes the propagation of electromagnetic waves,

$$-\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \text{curl } \mathbf{H} = \mathbf{J}, \quad \mu \frac{\partial \mathbf{H}}{\partial t} + \text{curl } \mathbf{E} = 0, \quad (3.1)$$

where  $\mathbf{E} = (E_1, E_2, E_3)^T$  and  $\mathbf{H} = (H_1, H_2, H_3)^T$  denote the electric and magnetic fields, respectively,  $\varepsilon$  is the *electric permittivity*,  $\mu$  is the *magnetic permeability*, and  $\mathbf{J}$  is the applied current density. In the following, to simplify the notation, and without

loss of generality, we normalize the parameters  $\varepsilon = \mu = 1$ , which corresponds to a scaling of time and the vector fields  $\mathbf{E}$  and  $\mathbf{H}$ , and we assume the applied current density to be divergence free, that is  $\operatorname{div} \mathbf{J} = 0$ . Denoting the vector of physical unknowns by

$$\mathbf{u} = (E_1, E_2, E_3, H_1, H_2, H_3)^T, \quad (3.2)$$

the Maxwell system (3.1) can be rewritten in conservative form,

$$\partial_t \mathbf{u} + G_x \partial_x \mathbf{u} + G_y \partial_y \mathbf{u} + G_z \partial_z \mathbf{u} = \mathbf{f}, \quad (3.3)$$

where the right hand side is given by  $\mathbf{f} = (J_1, J_2, J_3, 0, 0, 0)^T$ , and the coefficient matrices are of the form

$$G_l = \begin{bmatrix} & N_l \\ -N_l & \end{bmatrix}, \quad l = x, y, z,$$

where the  $3 \times 3$  matrices  $N_l$ ,  $l = x, y, z$  are given by

$$N_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad N_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad N_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For any unit vector  $\mathbf{v} = (v_1, v_2, v_3)$ ,  $\|\mathbf{v}\| = 1$ , we can define the characteristic matrix of system (3.3) by

$$C(\mathbf{v}) = v_1 \begin{bmatrix} & N_x \\ -N_x & \end{bmatrix} + v_2 \begin{bmatrix} & N_y \\ -N_y & \end{bmatrix} + v_3 \begin{bmatrix} & N_z \\ -N_z & \end{bmatrix} = \begin{bmatrix} & N\mathbf{v} \\ -N\mathbf{v} & \end{bmatrix},$$

whose eigenvalues are the characteristic speed of propagation along the direction  $\mathbf{v}$ . By the structure of the matrices  $N_l$ ,  $l = x, y, z$ , the matrix  $C(\mathbf{v})$  is symmetric, and hence has real eigenvalues, which implies that the Maxwell system is hyperbolic [33].

If we consider the Maxwell system (3.1) on the domain  $\Omega = [0, 1] \times \mathbb{R}$ , the characteristic matrix for the unit normal vector to the boundaries at  $x = 0$  and  $x = 1$ ,  $\tilde{\mathbf{n}} = (1, 0, 0)$ , is

$$C(\tilde{\mathbf{n}}) = \begin{pmatrix} & N_x \\ -N_x & \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_{1,2} = -1, \quad \lambda_{3,4} = 0, \quad \lambda_{5,6} = 1,$$

and since the eigenvalues are not distinct, the Maxwell system is not strictly hyperbolic [33]. The matrix of the left eigenvectors of  $C(\tilde{\mathbf{n}})$  is given by

$$L = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

which leads to the characteristic variables  $\mathbf{w} = (w_1, w_2, w_3, w_4, w_5, w_6)^T$  associated with the direction  $\tilde{\mathbf{n}}$ , where

$$\begin{aligned} w_1 &= -\frac{1}{2}(E_2 - H_3), & w_2 &= \frac{1}{2}(E_3 + H_2), & w_3 &= H_1, \\ w_4 &= E_1, & w_5 &= \frac{1}{2}(E_2 + H_3), & w_6 &= -\frac{1}{2}(E_3 - H_2). \end{aligned} \quad (3.4)$$

In the following, we will denote by  $\mathbf{w}_+$ ,  $\mathbf{w}_0$  and  $\mathbf{w}_-$  the characteristic variables associated with the positive, null, and negative eigenvalues respectively, that is

$$\mathbf{w}_- = (w_1, w_2)^T, \quad \mathbf{w}_0 = (w_3, w_4)^T, \quad \mathbf{w}_+ = (w_5, w_6)^T. \quad (3.5)$$

The boundary value problem in the characteristic variables, associated with the Maxwell system (3.1) on the domain  $\Omega = [0, 1] \times \mathbb{R}^2$ ,

$$\begin{aligned} (i\omega - \partial_x)w_1 + \frac{1}{2}\partial_z w_3 - \frac{1}{2}\partial_y w_4 &= \frac{1}{2}J_2 \\ (i\omega - \partial_x)w_2 + \frac{1}{2}\partial_y w_3 + \frac{1}{2}\partial_z w_4 &= -\frac{1}{2}J_3 \\ i\omega w_3 + \partial_z w_1 + \partial_y w_2 - \partial_z w_5 - \partial_y w_6 &= 0 \\ i\omega w_1 - \partial_y w_1 + \partial_z w_2 - \partial_y w_5 + \partial_z w_6 &= -J_1 \\ (i\omega + \partial_x)w_5 - \frac{1}{2}\partial_z w_3 - \frac{1}{2}\partial_y w_4 &= -\frac{1}{2}J_2 \\ (i\omega + \partial_x)w_6 - \frac{1}{2}\partial_y w_3 + \frac{1}{2}\partial_z w_4 &= \frac{1}{2}J_3 \end{aligned} \quad (3.6)$$

together with the characteristic boundary conditions

$$\mathbf{w}_+(0, y, z) = \mathbf{r}(y, z), \quad \mathbf{w}_-(1, y, z) = \mathbf{s}(y, z), \quad (y, z) \in \mathbb{R}^2, \quad (3.7)$$

and with the radiation condition on the unbounded part of the domain

$$|u_j(x, y, z)| \leq \frac{C}{\sqrt{y^2 + z^2}}, \quad (3.8)$$

where  $C$  is a constant, is well-posed (see [28]).

**3.1. Time-harmonic Solutions.** As in the case of the second order wave equation,  $\partial_{tt}u - \Delta u = f$ , it is also suitable for the Maxwell equations to assume the wave to be periodic in time. In this case, the time derivative becomes an algebraic term, and only the spatial domain needs to be discretized for a numerical approximation of the solution. The harmonic solutions of the Maxwell equations are complex valued static vector fields  $\mathbf{E}$  and  $\mathbf{H}$  such that the dynamic fields

$$\mathcal{E}(\mathbf{x}, t) = \mathcal{R}e(\mathbf{E}(\mathbf{x}) \exp(i\omega t)), \quad \mathcal{H}(\mathbf{x}, t) = \mathcal{R}e(\mathbf{H}(\mathbf{x}) \exp(i\omega t))$$

satisfy the Maxwell system (3.1). The positive real parameter  $\omega$  is called the pulsation of the harmonic wave. The harmonic solutions  $\mathbf{E}$  and  $\mathbf{H}$  satisfy the time-harmonic equations

$$\text{curl } \mathbf{E} + i\omega \mathbf{H} = \mathbf{0}, \quad \text{curl } \mathbf{H} - i\omega \mathbf{E} = \mathbf{J}, \quad (3.9)$$

or, written in component form, and owing to (3.2)

$$\begin{aligned} -i\omega u_1 + \partial_y u_6 - \partial_z u_5 &= J_1, & -i\omega u_2 + \partial_z u_4 - \partial_x u_6 &= J_2, & -i\omega u_3 + \partial_x u_5 - \partial_y u_4 &= J_3, \\ i\omega u_4 + \partial_y u_3 - \partial_z u_2 &= 0, & i\omega u_5 + \partial_z u_1 - \partial_x u_3 &= 0, & i\omega u_6 + \partial_x u_2 - \partial_y u_1 &= 0. \end{aligned} \quad (3.10)$$

**3.2. Relation to a Scalar Equation.** We investigate now the relation between the solution of the time harmonic Maxwell system and the solution of some suitable scalar equation. We consider the equation (3.9) in  $\Omega = [0, 1] \times \mathbb{R}^2$  and the associated boundary value problem with the boundary conditions given by (3.7).

PROPOSITION 3.1.

Let  $\mathbf{u}$  be as defined in (3.2) and  $\mathbf{w}$  be the characteristic variables defined in (3.4). Any component  $\tilde{w}_j$ ,  $j = 1, \dots, 6$ , of the characteristic variables of the Maxwell system (3.10) satisfies, in the interior of  $\Omega = [0, 1] \times \mathbb{R}^2$ , the Helmholtz equation,

$$(\omega^2 + \Delta)\tilde{w}_j = \tilde{f}_j, \quad j = 1, 2, \dots, 6, \quad (3.11)$$

with right hand side given by

$$\begin{aligned} \tilde{f}_1 &= \frac{1}{2}(-(\partial_x + i\omega)J_2 + \partial_y J_1) & \tilde{f}_4 &= i\omega J_1 \\ \tilde{f}_2 &= \frac{1}{2}((\partial_x + i\omega)J_3 - \partial_z J_1) & \tilde{f}_5 &= \frac{1}{2}(-(\partial_x - i\omega)J_2 + \partial_y J_1) \\ \tilde{f}_3 &= -\partial_y J_3 + \partial_z J_2 & \tilde{f}_6 &= \frac{1}{2}((\partial_x - i\omega)J_3 - \partial_z J_1) \end{aligned} \quad (3.12)$$

*Proof.* From the last three equations in (3.10), we obtain  $u_j$ ,  $j = 4, 5, 6$  as functions of  $u_j$ ,  $j = 1, 2, 3$  only. Substituting these expressions for  $u_j$ ,  $j = 4, 5, 6$ , into the first three equations in (3.10), we obtain a system for  $u_j$ ,  $j = 1, 2, 3$

$$\begin{aligned} \omega^2 u_1 + (\partial_{yy} + \partial_{zz})u_1 - \partial_{xy}u_2 - \partial_{xz}u_3 &= i\omega J_1, \\ \omega^2 u_2 + (\partial_{xx} + \partial_{zz})u_2 - \partial_{yz}u_3 - \partial_{xy}u_1 &= i\omega J_2, \\ \omega^2 u_3 + (\partial_{xx} + \partial_{yy})u_3 - \partial_{xz}u_1 - \partial_{yz}u_2 &= i\omega J_3. \end{aligned} \quad (3.13)$$

We now eliminate the variable  $u_3$  from the first two equations by differentiating the first one with respect to  $y$  and subtracting it from the second one differentiated with respect to  $x$ . We then eliminate  $u_3$  also from the second and third equations by applying the operator  $[\omega^2 + (\partial_{xx} + \partial_{yy})]$  to the second one and adding it the third one differentiated with respect to  $y$  and  $z$ . After some simplifications, we obtain a new system for  $u_1$  and  $u_2$ ,

$$\begin{aligned} \partial_y((\omega^2 + \Delta)u_1) - \partial_x((\omega^2 + \Delta)u_2) &= i\omega[\partial_y J_1 - \partial_x J_2], \\ -\partial_{xy}((\omega^2 + \Delta)u_1) + (\omega^2 + \partial_{xx})((\omega^2 + \Delta)u_2) &= i\omega((\omega^2 + \partial_{xx} + \partial_{yy})J_2 + \partial_{yz}J_3). \end{aligned}$$

Applying  $\partial_x$  to the first equation and adding it to the second one, we finally obtain, after a division by  $\omega^2$ ,

$$(\omega^2 + \Delta)u_2 = i\omega J_2,$$

Different manipulations in (3.13) allow to reduce to a single equation in  $u_1$  and  $u_3$ . In a similar way, we can eliminate from the first three equations in (3.10), the variables  $u_j$ ,  $j = 1, 2, 3$  as functions of  $u_j$ ,  $j = 4, 5, 6$  only. We then proceed as above, and it can be easily seen that we get

$$\begin{aligned} (\omega^2 + \Delta)u_1 &= i\omega J_1, & (\omega^2 + \Delta)u_3 &= i\omega^2 J_3, & (\omega^2 + \Delta)u_4 &= -\partial_y J_3 + \partial_z J_2, \\ (\omega^2 + \Delta)u_5 &= -\partial_z J_1 + \partial_x J_3, & (\omega^2 + \Delta)u_6 &= -\partial_x J_2 + \partial_y J_1. \end{aligned}$$

So far, the thesis follows immediately by linear combinations.  $\square$

The above proposition states that the characteristic variables of the Maxwell system satisfy an Helmholtz equation in the interior of the domain  $\Omega = [0, 1] \times \mathbb{R}^2$ . So far, let us consider the following partial differential equation

$$(\omega^2 + \Delta)\tilde{w}_1 = f_1 \quad \text{in } \Omega \quad (3.14)$$

together with the boundary conditions

$$(\partial_x - i\omega)\tilde{w}_1(0, y, z) = \tilde{r}_1(y, z), \quad \tilde{w}_1(1, y, z) = \tilde{s}_1(1, y, z), \quad (y, z) \in \mathbb{R}^2. \quad (3.15)$$

which is very much related to the Maxwell system, as stated in the following proposition.

**PROPOSITION 3.2.** *Let  $\mathbf{w}$  be the solution of (3.6) with boundary conditions (3.7), and let  $\tilde{w}_1$  be the solution of (3.14) with boundary conditions (3.15). If  $f_1(x, y, z) = \tilde{f}_1(x, y, z)$ ,  $\tilde{s}_1(y, z) = s_1(y, z)$ , and*

$$\tilde{r}_1(y, z) = \Delta_{yz}^{-1} \left[ (\partial_x + i\omega) [(\partial_{yy} - \partial_{zz})\check{r}_1 - 2\partial_{yz}\check{r}_2] + \partial_y (\partial_z J_3 - \partial_y J_2) \right] (0, y, z)$$

where  $\Delta_{yz}$  denotes the Laplace operator in the  $y$  and  $z$  variables, then

$$\tilde{w}_1(x, y, z) = w_1(x, y, z) \quad \text{in } \bar{\Omega}.$$

*Proof.* The foregoing proposition states that the characteristic variable  $w_1$  satisfies an Helmholtz equation inside  $\Omega$  with right hand side  $\tilde{f}_1$ . Thus the differential equations coincide and we need to verify only the equivalence between the boundary conditions. In that order, notice that the boundary condition of Dirichlet type in  $(1, y, z)$  stays the same. Consider then the first two and the last two equations in (3.6): this is a  $4 \times 6$  linear system, and we can express the components of the solution  $w_j$ ,  $j = 1, \dots, 4$  in terms of the sole components  $w_5$  and  $w_6$ . For any  $\mathbf{x} = (x, y, z) \in \bar{\Omega}$ , it can be easily seen that we have

$$(\partial_x - i\omega)w_1 = \Delta_{yz}^{-1} [(\partial_{yy} - \partial_{zz})(\partial_x + i\omega)w_5 - 2\partial_{yz}(\partial_x + i\omega)w_6 + \partial_{yz}J_3 + \partial_{yy}J_2]$$

In particular, the above relation holds true for  $(0, y, z)$ , and, owing to the boundary conditions (3.7), we get

$$(\partial_x - i\omega)w_1|_{x=0} = \Delta_{yz}^{-1} \left[ (\partial_x + i\omega) [(\partial_{yy} - \partial_{zz})r_1 - 2\partial_{yz}r_2] + \partial_y (\partial_z J_3 + \partial_y J_2) \right]_{|x=0}.$$

Thus,  $w_1$  satisfies (3.14) with boundary conditions (3.15), and the thesis then follows by uniqueness.  $\square$

**REMARK 2.** *Notice that a similar result can be obtained for any other propagating component of the characteristic variables  $w_2$ ,  $w_5$ , and  $w_6$ , with boundary conditions given in (3.7) and by*

$$(\partial_x - i\omega)w_2 = \Delta_{yz}^{-1} \left[ (\partial_x + i\omega) [(\partial_{yy} - \partial_{zz})\check{r}_1 - 2\partial_{yz}\check{r}_2] + \partial_y (\partial_z J_3 + \partial_y J_2) \right], \quad \text{in } (0, y, z),$$

$$(\partial_x + i\omega)w_5 = \Delta_{yz}^{-1} \left[ (\partial_x - i\omega) [(\partial_{yy} - \partial_{zz})\check{s}_1 - 2\partial_{yz}\check{s}_2] + \partial_y (\partial_y J_3 + \partial_z J_2) \right], \quad \text{in } (1, y, z),$$

$$(\partial_x + i\omega)w_6 = \Delta_{yz}^{-1} \left[ (\partial_x - i\omega) [2\partial_{yz}\check{s}_1 + (\partial_{yy} - \partial_{zz})\check{s}_2] + \partial_y (\partial_y J_3 + \partial_z J_2) \right], \quad \text{in } (1, y, z),$$

**3.3. Classical Schwarz Algorithm for the Maxwell System.** We consider now the problem (3.9) in  $\Omega = [0, 1] \times \mathbb{R}^2$ , with boundary conditions on  $(0, y, z)$  and  $(1, y, z)$  given by (3.7), and with the radiation conditions (3.8). We decompose the domain into two subdomains  $\Omega_1 := (0, b) \times \mathbb{R}^2$  and  $\Omega_2 := (a, 1) \times \mathbb{R}^2$ , and we denote the overlap with  $L := b - a \geq 0$ . We solve system (3.3) in both subdomains and we enforce on the subdomain interfaces the continuity of the incoming characteristic variables. There are two incoming characteristics on both interfaces. Hence, to have a well-posed problem, we have to impose two conditions on each subdomain. The classical Schwarz algorithm, using subscript to denote components, and superscript to denote the subdomain and the iteration count, is given by

$$\begin{aligned} i\omega \mathbf{u}^{1,n} + \sum_{l=x,y,z} G_l \partial_l \mathbf{u}^{1,n} &= \mathbf{f}, \text{ in } \Omega_1, & i\omega \mathbf{u}^{2,n} + \sum_{l=x,y,z} G_l \partial_l \mathbf{u}^{2,n} &= \mathbf{f}, \text{ in } \Omega_2, \\ \mathbf{w}_+^{1,n}(0, y, z) &= \mathbf{r}(y, z), & \mathbf{w}_-^{2,n}(1, y, z) &= \mathbf{w}(y, z), \\ \mathbf{w}_-^{1,n}(b, y, z) &= \mathbf{w}_-^{2,n-1}(b, y, z), & \mathbf{w}_+^{2,n}(a, y, z) &= \mathbf{w}_+^{1,n-1}(a, y, z), \end{aligned} \quad (3.16)$$

where in the transmission conditions  $(y, z) \in \mathbb{R}^2$ . From the previous equivalence result, a Schwarz algorithm for the equivalent Helmholtz problem would read

$$\begin{aligned} (\omega^2 + \Delta) \tilde{w}_1^{1,n} &= \tilde{f}_1, \text{ in } \Omega_1, & (\omega^2 + \Delta) \tilde{w}_1^{2,n} &= \tilde{f}_1, \text{ in } \Omega_2, \\ (\partial_x - i\omega) \tilde{w}_1^{1,n}(0, y, z) &= \tilde{r}_1(y, z), & \tilde{w}_1^{2,n}(1, y, z) &= \tilde{s}_1(y, z), \\ \tilde{w}_1^{1,n}(b, y, z) &= \tilde{w}_1^{2,n-1}(b, y, z), & (\partial_x - i\omega) \tilde{w}_1^{2,n}(b, y, z) &= (\partial_x - i\omega) \tilde{w}_1^{1,n-1}(b, y, z). \end{aligned} \quad (3.17)$$

We now show that the Schwarz algorithm with Dirichlet transmission conditions applied to the time harmonic Maxwell system (3.10) is equivalent to a simple optimized Schwarz method for a related scalar partial differential equation, as stated in the following proposition.

**PROPOSITION 3.3.** *Let  $w_1^{2,0}, w_2^{2,0}, w_5^{1,0}, w_6^{1,0}$  being given. If algorithm (3.17) is started with initial data  $\tilde{w}_1^{1,0}$  and  $\tilde{w}_1^{2,0}$  such that*

$$(\partial_x - i\omega) \tilde{w}_1^{1,0} = \Delta_{yz}^{-1} [(\partial_{yy} - \partial_{zz})(\partial_x + i\omega)w_5^{1,0} - 2\partial_{yz}(\partial_x - i\omega)w_6^{1,0} + \partial_{yz}J_3 - \partial_{yy}J_2]$$

and  $\tilde{w}_1^{2,0} = w_1^{2,0}$ , then for any  $n \geq 1$  the first characteristic variable  $w_1$  of the iterates of (3.16) and the iterates of (3.17) coincide, i.e.

$$w_1^{1,n} = \tilde{w}_1^{1,n} \quad w_1^{2,n} = \tilde{w}_1^{2,n}.$$

*Proof.* The proof proceeds by induction. Proposition 3.2 entails the result for  $n = 1$ . Assume then that the result is true at iteration  $n - 1$ . We then have

$$(\partial_x - i\omega) \tilde{w}_1^{1,n-1} = \Delta_{yz}^{-1} \left[ (\partial_x + i\omega) \left[ (\partial_{yy} - \partial_{zz})w_5^{1,n-1} - 2\partial_{yz}w_6^{1,n-1} \right] + \partial_y (\partial_z J_3 + \partial_y J_2) \right]$$

which holds in particular at  $(a, y, z)$ . By uniqueness, the boundary condition in  $(1, y, z)$  entails then  $w_1^{2,n} = \tilde{w}_1^{2,n}$ . In a similar way, we have  $w_1^{1,n-1}(b, y, z) = \tilde{w}_1^{1,n-1}(b, y, z)$ , and from the boundary condition in  $(0, y, z)$  the thesis follows.  $\square$  From the foregoing proposition, the Schwarz algorithm with Dirichlet transmission conditions applied to the time harmonic Maxwell system (3.10) is equivalent to a simple optimized Schwarz method for a related scalar partial differential equation. This implies in particular the

equivalent convergence behavior we show in the following proposition for an infinite domain  $\Omega = \mathbb{R}^3$ .

PROPOSITION 3.4. *Let  $\Omega = \mathbb{R}^3$ , and consider the Maxwell system (3.10) in  $\Omega$  with the radiation condition*

$$\lim_{r \rightarrow \infty} r(\mathbf{n} \times \mathbf{E} + \mathbf{n} \times (\mathbf{n} \times \mathbf{H})) = 0, \quad (3.18)$$

where  $r = |\mathbf{x}|$ ,  $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$ . Let  $\Omega$  be decomposed into  $\Omega_1 := (-\infty, L) \times \mathbb{R}^2$  and  $\Omega_2 := (0, +\infty) \times \mathbb{R}^2$ , ( $L \geq 0$ ). For any given initial guess  $\mathbf{u}^{1,0} \in (L^2(\Omega_1))^6$ ,  $\mathbf{u}^{2,0} \in (L^2(\Omega_2))^6$ , the Schwarz algorithm (3.16) applied to system (3.10) converges for all Fourier modes such that  $k_y^2 + k_z^2 \neq \omega^2$ . The convergence factor is

$$R_{th} = \begin{cases} \left| \frac{\sqrt{\omega^2 - (k_y^2 + k_z^2)} - \omega}{\sqrt{\omega^2 - (k_y^2 + k_z^2)} + \omega} \right|, & \text{for } k_y^2 + k_z^2 < \omega^2, \\ e^{-\sqrt{k_y^2 + k_z^2 - \omega^2}L}, & \text{for } k_y^2 + k_z^2 > \omega^2. \end{cases} \quad (3.19)$$

*Proof.* Because of linearity, it suffices to analyze the convergence to the zero solution when the right hand side vanishes. Performing a Fourier transform of system (3.10) in the  $y$  and  $z$  direction, the first and the fourth equation provide an algebraic expression for  $\hat{u}_1$  and  $\hat{u}_4$ , which is in agreement with the fact that these latter are the characteristic variables associated with the null eigenvalue. Inserting these expressions into the remaining Fourier transformed equations, we obtain the first order system

$$\partial_x \begin{pmatrix} \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_5 \\ \hat{u}_6 \end{pmatrix} + \frac{i}{\omega} \begin{bmatrix} 0 & 0 & k_y k_z & \omega^2 - k_y^2 \\ 0 & 0 & k_z^2 - \omega^2 & -k_y k_z \\ -k_y k_z & k_y^2 - \omega^2 & 0 & 0 \\ \omega^2 - k_z^2 & k_y k_z & 0 & 0 \end{bmatrix} \begin{pmatrix} \hat{u}_2 \\ \hat{u}_3 \\ \hat{u}_5 \\ \hat{u}_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.20)$$

The eigenvalues of the matrix in (3.20) and their corresponding eigenvectors are

$$\lambda_{1,2}^{TH} = -\sqrt{|\mathbf{k}|^2 - \omega^2}, \quad \mathbf{v}_1 = \begin{pmatrix} \frac{k_y k_z}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ \frac{k_z^2 - \omega^2}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \frac{\omega^2 - k_y^2}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ -\frac{k_y k_z}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ 0 \\ 1 \end{pmatrix},$$

and

$$\lambda_{3,4}^{TH} = \sqrt{|\mathbf{k}|^2 - \omega^2}, \quad \mathbf{v}_3 = \begin{pmatrix} -\frac{k_y k_z}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ \frac{\omega^2 - k_z^2}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} \frac{k_y^2 - \omega^2}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ \frac{k_y k_z}{i\omega\sqrt{|\mathbf{k}|^2 - \omega^2}} \\ 0 \\ 1 \end{pmatrix}.$$

Because of the radiation condition, the solutions  $u^j$  of system (3.20) in  $\Omega_l$  ( $l = 1, 2$ ) are given by

$$\mathbf{u}^1 = (\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) e^{\sqrt{|\mathbf{k}|^2 - \omega^2}(x-L)}, \quad \mathbf{u}^2 = (\beta_1 \mathbf{v}_3 + \beta_2 \mathbf{v}_4) e^{-\sqrt{|\mathbf{k}|^2 - \omega^2}x}, \quad (3.21)$$

where the coefficients  $\alpha_j$  and  $\beta_j$  ( $j = 1, 2$ ) are uniquely determined by the interface conditions. At the  $n$ -th step of the Schwarz algorithm, the coefficients  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$



and  $\boldsymbol{\beta} = (\beta_1, \beta_2)$  satisfy the system

$$\boldsymbol{\alpha}^n = A_1^{-1} A_2 e^{-\sqrt{|\mathbf{k}|^2 - \omega^2} L} \boldsymbol{\beta}^{n-1}, \quad \boldsymbol{\beta}^n = B_1^{-1} B_2 e^{-\sqrt{|\mathbf{k}|^2 - \omega^2} L} \boldsymbol{\alpha}^{n-1},$$

where the matrices in the iteration are given by

$$A_1 = \begin{bmatrix} -k_y k_z & k_y^2 - \omega^2 + i\omega\lambda \\ k_z^2 - \omega^2 + i\omega\lambda & -k_y k_z \end{bmatrix}, \quad A_2 = \begin{bmatrix} k_y k_z & -(k_y^2 - \omega^2 - i\omega\lambda) \\ -(k_z^2 - \omega^2 - i\omega\lambda) & k_y k_z \end{bmatrix}, \quad (3.22)$$

and where  $B_l = A_l, l = 1, 2$  and we have set  $\lambda := \sqrt{|\mathbf{k}|^2 - \omega^2}$ . A complete iteration over two steps of the Schwarz algorithm leads then to

$$\boldsymbol{\alpha}^{n+1} = A_1^{-1} A_2 B_1^{-1} B_2 e^{-2\lambda L} \boldsymbol{\alpha}^{n-1}, \quad \boldsymbol{\beta}^{n+1} = B_1^{-1} B_2 A_1^{-1} A_2 e^{-2\lambda L} \boldsymbol{\beta}^{n-1},$$

and we finally obtain

$$A_1^{-1} A_2 B_1^{-1} B_2 = B_1^{-1} B_2 A_1^{-1} A_2 = \left( \frac{|\mathbf{k}|^2}{(\lambda + i\omega)^2} \right)^2 Id.$$

Now by the definition of  $\lambda$ , we have  $|\mathbf{k}|^2 = (\lambda - i\omega)(\lambda + i\omega)$ , and thus the convergence factor of the algorithm is

$$\rho(|\mathbf{k}|) = \left| \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega}{\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega} e^{-\sqrt{|\mathbf{k}|^2 - \omega^2} L} \right|.$$

Separating the two cases  $|\mathbf{k}|^2 < \omega^2$  and  $|\mathbf{k}|^2 > \omega^2$  then concludes the proof.  $\square$  Notice that for  $|\mathbf{k}|^2 = \omega^2$ , the convergence factor equals 1, independently of the overlap, which indicates that the algorithm is not convergent in general when used in the iterative form described here. This precise result was also observed for the equivalent optimized Schwarz method applied to the Helmholtz equation, see [23]. In practice, the Schwarz methods are however used as preconditioners for Krylov methods, and then a few non-convergent modes in the iterative form can easily be handled by the Krylov method.

We also see from the convergence factor (3.19) that the overlap is necessary for the convergence of the evanescent modes,  $|\mathbf{k}|^2 > \omega^2$ . Without overlap,  $L = 0$ , we have  $\rho(|\mathbf{k}|) < 1$  only for the propagative modes,  $|\mathbf{k}|^2 < \omega^2$ , and  $\rho(|\mathbf{k}|) = 1$  when  $|\mathbf{k}|^2 \geq \omega^2$ . In the time-harmonic case, the classical Schwarz algorithm without overlap is thus convergent only for propagative modes, and corresponds to the algorithm proposed by B. Després *et al.* in [10] for the Helmholtz equation.

### 3.4. Optimized Schwarz Method for the Maxwell Equations.

**3.4.1. Transparent Boundary Conditions.** To design optimized Schwarz algorithms for the Maxwell system, we derive now the transparent boundary conditions for those equations. We consider the time harmonic Maxwell equations (3.10) on the domain  $\Omega = (0, 1) \times \mathbb{R}^2$ , with right hand side  $\mathbf{J}$  compactly supported in  $\Omega$ , together with the boundary conditions

$$(\mathbf{w}_+ + \mathcal{S}_1 \mathbf{w}_-)(0, y, z) = 0, \quad (\mathbf{w}_- + \mathcal{S}_2 \mathbf{w}_+)(1, y, z) = 0, \quad (y, z) \in \mathbb{R}^2, \quad (3.23)$$

where  $\mathbf{w}_-$  and  $\mathbf{w}_+$  are defined in (3.5), and the operators  $\mathcal{S}_l, l = 1, 2$ , are general, pseudodifferential operators acting in the  $y$  and  $z$  directions.

LEMMA 3.5. *If the operators  $\mathcal{S}_l, l = 1, 2$  have the Fourier symbol*

$$\mathcal{F}(\mathcal{S}_l) = \frac{1}{(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} \begin{bmatrix} k_y^2 - k_z^2 & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 \end{bmatrix}, \quad j = 1, 2, \quad (3.24)$$

then the solution of the Maxwell equations (3.10) in  $\Omega$  with boundary conditions (3.23) coincides with the restriction on  $\Omega$  of the solution of the Maxwell system (3.10) on  $\mathbb{R}^3$ .

*Proof.* As in Section 2, we prove that the difference between the solution of the global problem and that of the restricted problem (which we will denote by  $\mathbf{e}$ ), which satisfies in  $\Omega$  the homogeneous counterpart of (3.10) with boundary conditions (3.23) vanishes. Proceeding as in the previous section, the solution in Fourier is given by

$$\hat{\mathbf{e}} = (\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) e^{\sqrt{|\mathbf{k}|^2 - \omega^2} x} + (\alpha_3 \mathbf{v}_3 + \alpha_4 \mathbf{v}_4) e^{-\sqrt{|\mathbf{k}|^2 - \omega^2} x},$$

where the vectors  $\mathbf{v}_j$  ( $j = 1, \dots, 4$ ) are defined in Section 3.3. Using the boundary condition (3.23) in  $(0, y, z)$ , we obtain that the coefficients  $\alpha_j$ ,  $j = 3, 4$  satisfy the system of equations

$$\begin{bmatrix} -k_y k_z & k_y^2 - \omega^2 + i\omega \sqrt{|\mathbf{k}|^2 - \omega^2} \\ k_z^2 - \omega^2 + i\omega \sqrt{|\mathbf{k}|^2 - \omega^2} & -k_y k_z \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies  $\alpha_3 = \alpha_4 = 0$ . Now using the boundary condition at  $(1, y, z)$ , we obtain for the coefficients  $\alpha_j$ ,  $j = 1, 2$ , the same system of equations as for  $\alpha_j$ ,  $j = 3, 4$ , which implies  $\alpha_1 = \alpha_2 = 0$ . Thus  $\hat{\mathbf{e}} = \mathbf{0}$ , which concludes the proof.  $\square$

REMARK 3. As in the case of the Cauchy-Riemann equations in Remark 1, the symbols in (3.24) can be written in several, mathematically equivalent forms:

$$\mathcal{F}(\mathcal{S}_l) = \frac{1}{(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} M = \frac{1}{|\mathbf{k}|^2} \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega}{\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega} M = (\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2 M^{-1},$$

$$\text{where } M = \begin{bmatrix} k_y^2 - k_z^2 & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 \end{bmatrix}.$$

which also will lead to different approximations of the transparent conditions in the context of optimized Schwarz methods. The first form contains a local and a non-local term, since multiplication with the matrix  $M$  corresponds to second order derivation operations in  $y$  and  $z$ , which is a local operation, whereas the term containing the square-root of  $|\mathbf{k}|^2$  is a non-local operation. The last form contains two non-local operations, since the inversion of the matrix  $M$  corresponds to an integration. This integration can however be passed to the other variable by multiplication with this matrix. The second form contains also two non-local terms and a local one. These different forms motivate different local approximations of the transparent boundary conditions.

Similarly, we can consider the associated Helmholtz equation 3.11 in  $\Omega = (0, 1) \times \mathbb{R}^2$ , with right hand side compactly supported in  $\Omega$ , and with boundary conditions

$$(\partial_x - \tilde{\mathcal{S}}_1) \mathbf{u}(0, y, z) = 0, \quad (\partial_x + \tilde{\mathcal{S}}_2) \mathbf{u}(1, y, z) = 0, \quad (y, z) \in \mathbb{R}^2, \quad (3.25)$$

where  $\tilde{\mathcal{S}}_j$  ( $j = 1, 2$ ) are general, pseudodifferential operators acting in the  $y$  and  $z$  directions.

LEMMA 3.6. If the operators  $\tilde{\mathcal{S}}_l$  ( $l = 1, 2$ ) have the Fourier symbol

$$\tilde{\sigma}_l = \mathcal{F}(\tilde{\mathcal{S}}_l) = \sqrt{|\mathbf{k}|^2 - \omega^2} \quad (3.26)$$

then the solution of (3.11) in  $\Omega$  with boundary conditions (3.25) coincides with the restriction on  $\Omega$  of the solution of the Helmholtz equation (3.11) on  $\mathbb{R}^3$ .

*Proof.*

The proof follows along the same lines as in the previous Lemma. Performing a Fourier transform in the  $x_2$  and  $x_3$  directions, the symbol of the difference between the solution of the global problem and the solution of the restricted one, that we denote with  $\tilde{\mathbf{e}}$ , is given by

$$\tilde{\mathbf{e}} = \alpha e^{\sqrt{|\mathbf{k}|^2 - \omega^2}} + \beta e^{-\sqrt{|\mathbf{k}|^2 - \omega^2}}.$$

The boundary condition at  $(0, y, z)$  implies then  $\alpha = 0$ , whereas the boundary condition at  $(1, y, z)$  implies  $\beta = 0$ , which concludes the proof.  $\square$

**3.4.2. Schwarz Algorithm with General Interface Conditions.** The operators  $\mathcal{S}_l$ ,  $l = 1, 2$ , introduced in the previous section and leading to the optimal performance, are unfortunately non-local operators, hence difficult to be used in practice. They must therefore be approximated in some suitable way. If one is willing to use second order transmission conditions, then the only parts of the symbols in (3.24) that need to be approximated are the multiplication by  $(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^{-2}$ , because the entries of the matrices are polynomials in the Fourier variables, which correspond to derivatives in the  $y$  and  $z$  direction. We apply now to (3.10) a Schwarz algorithm with more general interface conditions, which are given for  $\Omega_1$  by

$$[(\mathbf{w}_- + \mathcal{S}_1 \mathbf{w}_+)(L, y, z)]^{1,n} = [(\mathbf{w}_- + \mathcal{S}_1 \mathbf{w}_+)(L, y, z)]^{2,n-1}, \quad (y, z) \in \mathbb{R}^2, \quad (3.27)$$

and for  $\Omega_2$  by

$$[(\mathbf{w}_+ + \mathcal{S}_2 \mathbf{w}_-)(0, y, z)]^{2,n} = [(\mathbf{w}_+ + \mathcal{S}_2 \mathbf{w}_-)(0, y, z)]^{1,n-1}, \quad (y, z) \in \mathbb{R}^2. \quad (3.28)$$

PROPOSITION 3.7. **a)** *If the operators  $\mathcal{S}_1$  and  $\mathcal{S}_2$  have the Fourier symbol*

$$\sigma_l := \mathcal{F}(\mathcal{S}_l) = \gamma_l \begin{bmatrix} k_y^2 - k_z^2 & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 \end{bmatrix}, \quad \gamma_l \in \mathbb{C}(k_z, k_y) \quad (l = 1, 2), \quad (3.29)$$

*then the convergence factor of the Schwarz algorithm with interface conditions (3.27)-(3.28) is*

$$\rho(\omega, L, |\mathbf{k}|, \gamma_1, \gamma_2) = \left| \frac{(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2}{(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} \frac{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2}{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2} \frac{1 - \gamma_2(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2}{1 - \gamma_2(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2} e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{1/2}. \quad (3.30)$$

**b)** *If the operators  $\mathcal{S}_1$  and  $\mathcal{S}_2$  have the Fourier symbol*

$$\sigma_l := \mathcal{F}(\mathcal{S}_l) = \delta_l \begin{bmatrix} k_y^2 - k_z^2 & -2k_y k_z \\ -2k_y k_z & k_z^2 - k_y^2 \end{bmatrix}^{-1}, \quad \delta_l \in \mathbb{C}(k_z, k_y) \quad (l = 1, 2), \quad (3.31)$$

*then the convergence factor of the Schwarz algorithm with interface conditions (3.27)-(3.28) is*

$$\rho(\omega, L, |\mathbf{k}|, \delta_1, \delta_2) = \left| \frac{(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2}{(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2} \frac{\delta_1 - (\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2}{\delta_1 - (\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} \frac{\delta_2 - (\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2}{\delta_2 - (\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{1/2}. \quad (3.32)$$

c) If the operator  $\mathcal{S}_1$  has the Fourier symbol (3.29) and  $\mathcal{S}_2$  have the Fourier symbol (3.31) then the convergence factor of the Schwarz algorithm with interface conditions (3.27)-(3.28) is

$$\rho(\omega, L, |\mathbf{k}|, \gamma_1, \delta_2) = \left| \frac{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2}{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2} \frac{\delta_2 - (\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2}{\delta_2 - (\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{1/2}. \quad (3.33)$$

*Proof.* The convergence result is again based on Fourier analysis, as in Section 3.3. At the  $n$ -th step of the Schwarz algorithm, the coefficients  $\boldsymbol{\alpha}^n = (\alpha_1^n, \alpha_2^n)$  and  $\boldsymbol{\beta} = (\beta_1, \beta_2)$  in (3.21) satisfy

$$\boldsymbol{\alpha}^n = \bar{A}_1^{-1} \bar{A}_2 e^{-\lambda L} \boldsymbol{\beta}^{n-1}, \quad \boldsymbol{\beta}^n = \bar{B}_1^{-1} \bar{B}_2 e^{-\lambda L} \boldsymbol{\alpha}^{n-1}, \quad (3.34)$$

where  $\lambda = \sqrt{|\mathbf{k}|^2 - \omega^2}$ . A complete double iteration of the Schwarz algorithm leads to

$$\boldsymbol{\alpha}^{n+1} = \bar{A}_1^{-1} \bar{A}_2 \bar{B}_1^{-1} \bar{B}_2 e^{-2\lambda L} \boldsymbol{\alpha}^{n-1}, \quad \boldsymbol{\beta}^{n+1} = \bar{B}_1^{-1} \bar{B}_2 \bar{A}_1^{-1} \bar{A}_2 e^{-2\lambda L} \boldsymbol{\beta}^{n-1},$$

where the matrices in (3.34) are given by

$$\bar{A}_1 = A_1 + \sigma_1 A_2, \quad \bar{A}_2 = A_2 + \sigma_1 A_1, \quad \bar{B}_1 = A_1 + \sigma_2 A_2, \quad \bar{B}_2 = A_2 + \sigma_2 A_1.$$

where  $A_l$  ( $l = 1, 2$ ) are defined in (3.22).

a) In the first case this leads to the iteration matrix

$$\bar{A}_1^{-1} \bar{A}_2 \bar{B}_1^{-1} \bar{B}_2 = \bar{B}_1^{-1} \bar{B}_2 \bar{A}_1^{-1} \bar{A}_2 = \left( \frac{|\mathbf{k}|^2}{(\lambda + i\omega)^2} \right)^2 \frac{(1 - \gamma_1(\lambda + i\omega)^2)(1 - \gamma_2(\lambda + i\omega)^2)}{(1 - \gamma_1(\lambda - i\omega)^2)(1 - \gamma_2(\lambda - i\omega)^2)} Id.$$

Therefore, since  $|\mathbf{k}|^2 = (\lambda - i\omega)(\lambda + i\omega)$ , the convergence factor is given by

$$\rho(\omega, L, |\mathbf{k}|, \gamma_1, \gamma_2) = \left| \frac{(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2}{(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} \frac{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2}{1 - \gamma_1(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2} \frac{1 - \gamma_2(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2}{1 - \gamma_2(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2} e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

b) In the second case this leads to the iteration matrix

$$\bar{A}_1^{-1} \bar{A}_2 \bar{B}_1^{-1} \bar{B}_2 = \bar{B}_1^{-1} \bar{B}_2 \bar{A}_1^{-1} \bar{A}_2 = \left( \frac{|\mathbf{k}|^2}{(\lambda - i\omega)^2} \right)^2 \frac{(\delta_1 - (\lambda - i\omega)^2)(\delta_2 - (\lambda - i\omega)^2)}{(\delta_1 - (\lambda + i\omega)^2)(\delta_2 - (\lambda + i\omega)^2)} Id.$$

Therefore, since  $|\mathbf{k}|^2 = (\lambda - i\omega)(\lambda + i\omega)$ , the convergence factor is given by

$$\rho(\omega, L, |\mathbf{k}|, \delta_1, \delta_2) = \left| \frac{(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2}{(\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2} \frac{\delta_1 - (\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2}{\delta_1 - (\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} \frac{\delta_2 - (\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega)^2}{\delta_2 - (\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2} e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

c) The conclusion follows as in the first two cases.  $\square$

REMARK 4. From (3.30), we see that the choice  $\gamma_1 = \gamma_2 = 1/(\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega)^2$  is optimal, since then  $\rho_{th}(|\mathbf{k}|) \equiv 0$ , respectively, for all frequencies  $|\mathbf{k}|$ . With this choice of  $\gamma_1$  and  $\gamma_2$ , the matrices  $\bar{A}_2$  and  $\bar{B}_2$  actually vanish.

**3.4.3. Relation to a Schwarz Algorithm for a Scalar Equation.** We present here several particular choices of the transmission operator  $\mathcal{S}_l$  with Fourier symbol  $\sigma_l$  ( $l = 1, 2$ ) in the interface conditions (3.27) and (3.28).

**Case 1:** taking  $\gamma_1 = \gamma_2 = 0$  in (3.29), which amounts to enforce the classical Dirichlet transmission conditions, the convergence factor is

$$\rho_1(\omega, L, |\mathbf{k}|) = \left| \left( \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega}{\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega} \right)^2 e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

In the non-overlapping case,  $L = 0$ , this choice ensures convergence only for propagative modes, and corresponds to the Taylor interface conditions of order zero proposed in [10] for the Helmholtz equation.

**Case 2:** taking  $\gamma_1 = \gamma_2 = \frac{1}{|\mathbf{k}|^2} \frac{p - i\omega}{p + i\omega}$  in (3.29) or  $\gamma_1 = \frac{1}{|\mathbf{k}|^2 - 2\omega^2 + 2i\omega p}$  in (3.29) and  $\delta_2 = |\mathbf{k}|^2 - 2\omega^2 - 2i\omega p$  in (3.31) with  $p \in \mathbb{C}$ , the convergence factor is

$$\rho_2(\omega, L, |\mathbf{k}|, p) = \left| \left( \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - p}{\sqrt{|\mathbf{k}|^2 - \omega^2} + p} \right)^2 e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

**Case 3:** taking  $\gamma_1 = \gamma_2 = \frac{1}{|\mathbf{k}|^2 - 2\omega^2 + 2i\omega p}$  in (3.29) with  $p \in \mathbb{C}$ , the convergence factor is

$$\rho_3(\omega, L, |\mathbf{k}|, p) = \left| \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega}{\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega} \right| \rho_2(\omega, L, |\mathbf{k}|, p) \leq \rho_2(\omega, L, |\mathbf{k}|, p).$$

**Case 4:** taking  $\gamma_l = \frac{1}{|\mathbf{k}|^2} \frac{p_l - i\omega}{p_l + i\omega}$ ,  $l = 1, 2$  in (3.29) or  $\gamma_1 = \frac{1}{|\mathbf{k}|^2 - 2\omega^2 + 2i\omega p_1}$  in (3.29) and  $\delta_2 = |\mathbf{k}|^2 - 2\omega^2 - 2i\omega p_2$  in (3.31) with  $p_l \in \mathbb{C}$ ,  $l = 1, 2$ , the convergence factor is

$$\rho_4(\omega, L, |\mathbf{k}|, p_1, p_2) = \left| \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - p_1}{\sqrt{|\mathbf{k}|^2 - \omega^2} + p_1} \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - p_2}{\sqrt{|\mathbf{k}|^2 - \omega^2} + p_2} e^{-2\sqrt{|\mathbf{k}|^2 - \omega^2}L} \right|^{\frac{1}{2}}.$$

**Case 5:** taking  $\gamma_l = \frac{1}{|\mathbf{k}|^2 - 2\omega^2 + 2i\omega p_l}$  in (3.29) with  $p_l \in \mathbb{C}$ ,  $l = 1, 2$ , the convergence factor is

$$\rho_5(\omega, L, |\mathbf{k}|, p_1, p_2) = \left| \frac{\sqrt{|\mathbf{k}|^2 - \omega^2} - i\omega}{\sqrt{|\mathbf{k}|^2 - \omega^2} + i\omega} \right| \rho_4(\omega, L, |\mathbf{k}|, p_1, p_2) \leq \rho_4(\omega, L, |\mathbf{k}|, p_1, p_2).$$

Except in the Case 1, all the other cases provide second order transmission conditions even if we use only a zero order approximation of the non-local operator  $\sqrt{|\mathbf{k}|^2 - \omega^2}$ . Note also that in the Case 2 and Case 4 the convergence rate is the same as one obtained in the case of the Helmholtz equations in [23]. In the cases with parameters, the best choice for the parameters is in general the one that minimizes the convergence factor for all  $\mathbf{k} \in K$ , where  $K$  denotes the set of relevant numerical frequencies. One therefore needs to solve the min-max problems

$$\min_{p>0} \max_{\mathbf{k} \in K} \rho_j(\omega, L, |\mathbf{k}|, p), \quad j = 2, 3, \quad \min_{p_1, p_2 > 0} \max_{\mathbf{k} \in K} \rho_j(\omega, L, |\mathbf{k}|, p_1, p_2) \quad j = 4, 5. \quad (3.35)$$

Case	with overlap, $L = h$		without overlap, $L = 0$	
	$\rho$	parameters	$\rho$	parameters
1	$1 - \sqrt{k_+ - \omega^2}h$	none	1	none
2	$1 - 2C_\omega^{\frac{1}{6}}h^{\frac{1}{3}}$	$p = \frac{C_\omega^{\frac{1}{3}}}{2 \cdot h^{\frac{1}{3}}}$	$1 - \frac{\sqrt{2}C_\omega^{\frac{1}{4}}}{\sqrt{C}}\sqrt{h}$	$p = \frac{\sqrt{C}C_\omega^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
3	$1 - 2(k_+^2 - \omega^2)^{\frac{1}{6}}h^{\frac{1}{3}}$	$p = \frac{(k_+^2 - \omega^2)^{\frac{1}{3}}}{2 \cdot h^{\frac{1}{3}}}$	$1 - \frac{\sqrt{2}(k_+^2 - \omega^2)^{\frac{1}{4}}}{\sqrt{C}}\sqrt{h}$	$p = \frac{\sqrt{C}(k_+^2 - \omega^2)^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
4	$1 - 2^{\frac{2}{5}}C_\omega^{\frac{1}{10}}h^{\frac{1}{5}}$	$\begin{cases} p_1 = \frac{C_\omega^{\frac{2}{5}}}{2^{\frac{7}{5}} \cdot h^{\frac{1}{5}}}, \\ p_2 = \frac{C_\omega^{\frac{1}{5}}}{2^{\frac{6}{5}} \cdot h^{\frac{4}{5}}} \end{cases}$	$1 - \frac{C_\omega^{\frac{1}{8}}}{C^{\frac{1}{4}}}h^{\frac{1}{4}}$	$\begin{cases} p_1 = \frac{C_\omega^{\frac{3}{8}} \cdot C^{\frac{1}{4}}}{2 \cdot h^{\frac{1}{4}}}, \\ p_2 = \frac{C_\omega^{\frac{3}{8}} \cdot C^{\frac{3}{4}}}{h^{\frac{3}{4}}} \end{cases}$
5	$1 - 2^{\frac{2}{5}}(k_+^2 - \omega^2)^{\frac{1}{10}}h^{\frac{1}{5}}$	$\begin{cases} p_1 = \frac{(k_+^2 - \omega^2)^{\frac{2}{5}}}{2^{\frac{7}{5}} \cdot h^{\frac{1}{5}}}, \\ p_2 = \frac{(k_+^2 - \omega^2)^{\frac{1}{5}}}{2^{\frac{6}{5}} \cdot h^{\frac{4}{5}}} \end{cases}$	$1 - \frac{(k_+^2 - \omega^2)^{\frac{1}{8}}}{C^{\frac{1}{4}}}h^{\frac{1}{4}}$	$\begin{cases} p_1 = \frac{(k_+^2 - \omega^2)^{\frac{3}{8}} \cdot C^{\frac{1}{4}}}{2 \cdot h^{\frac{1}{4}}}, \\ p_2 = \frac{(k_+^2 - \omega^2)^{\frac{3}{8}} \cdot C^{\frac{3}{4}}}{h^{\frac{3}{4}}} \end{cases}$

TABLE 3.1

Asymptotic convergence rate and optimal choice of the parameters in the transmission conditions for the five variants of the optimized Schwarz method applied to the Cauchy-Riemann equations, when the mesh parameter  $h$  is small, and the maximum numerical frequency is estimated by  $k_{\max} = \frac{C}{h}$ .

where  $K = [(k_{\min}, k_-) \cup (k_+, k_{\max})]^2$ ,  $k_\pm$  are parameters to be chosen,  $k_{\min}$  denotes the smallest frequency relevant to the subdomain and  $k_{\max} = \frac{C}{h}$  denotes the largest frequency supported by the numerical grid (a reasonable choice for  $C$  would be  $\pi$ ). As already explained in [23], if the domain  $\Omega$  is a strip of height  $L$  with homogeneous Dirichlet conditions on the top and bottom, the solution can be expanded in Fourier series with the harmonics  $\sin\left(\frac{j\pi y}{L}\right)$ ,  $j \in \mathbb{N}$ . Hence the relevant frequencies are  $k = \frac{j\pi}{L}$ . They are equally distributed with a spacing  $\frac{\pi}{L}$  (the lowest one is  $k_{\min} = \frac{\pi}{L}$ ) and choosing  $k_\pm = \omega \pm \frac{\pi}{L}$  leaves precisely one frequency  $k = \omega$  for the Krylov method and treats all the others by optimization. If  $\omega$  falls between the relevant frequencies, say  $\frac{j\pi}{L} \leq \omega \leq \frac{(j+1)\pi}{L}$ , then we get the iterative method to converge by choosing  $\omega_- = \frac{j\pi}{L}$  and  $\omega_+ = \frac{(j+1)\pi}{L}$ , which will allow us to directly verify the asymptotic analysis without the use of a Krylov method. Using the same techniques we obtain the asymptotically optimized parameters for the other cases as shown in Table 3.1.

**3.5. Numerical Experiments for the time-harmonic equations.** We now show numerical experiments for the simplified two-dimensional model (transverse electric waves) time harmonic Maxwell equations solved on the unit square  $\Omega = (0, 1)^2$ . The frequency  $\omega = 2\pi$  is chosen such that the thumb rule (at least 10 discretization point per wavelength) should be respected. We decompose the domain into two subdomains  $\Omega_1 = (0, \beta) \times (0, 1)$  and  $\Omega_2 = (\alpha, 1) \times (0, 1)$ , where  $0 < \alpha \leq \beta < 1$ , and therefore the overlap is  $L = \beta - \alpha$ , and we consider both decompositions with and without overlap. We discretize the equations using a finite volume method, on a uniform mesh with mesh parameter  $h$ . In all comparisons that follow, we simulate directly the error equations,  $f = 0$ , and we use a random initial guess to ensure that all the frequency components are present in the iteration.

Table 3.2 shows the iteration count for all Schwarz algorithms considered previously, in the overlapping and non-overlapping case when we accelerate the convergence by a GMRES method. We have to note that for the non-overlapping algorithm the

h	with overlap, $L = h$				without overlap, $L = 0$			
	1/16	1/32	1/64	1/128	1/16	1/32	1/64	1/128
Case 1	18	27	46	71	-	-	-	-
Case 2	16	16	17	20	28	36	50	68
Case 3	10	12	14	16				
Case 4	17	17	20	22				
Case 5	10	12	14	17				

TABLE 3.2

Number of iterations to attain convergence for different interface conditions and different mesh sizes in the overlapping and non-overlapping case. The tolerance is fixed at  $\varepsilon = 10^{-6}$ .

iterative version doesn't converge therefore, applying a Krylov method is necessary.

**3.6. Time Discretized Solutions.** If we do not assume the wave to be periodic in time, the time domain also needs to be discretized. We consider a uniform time grid with time step  $\Delta t$ , and use a semi-implicit time integration scheme for the time derivative in (3.1) of the form

$$-\frac{\mathbf{E}^{n+1} - \mathbf{E}^n}{\Delta t} + \text{curl} \left( \frac{\mathbf{H}^{n+1} + \mathbf{H}^n}{2} \right) = \mathbf{J}, \quad \frac{\mathbf{H}^{n+1} - \mathbf{H}^n}{\Delta t} + \text{curl} \left( \frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2} \right) = \mathbf{0},$$

where the mean value is introduced to ensure the energy conservation (see [13]). With this time discretization, the Maxwell system (3.3) can be rewritten as a system that needs to be solved in each time step,

$$\begin{aligned} -\sqrt{\eta}E_1 + \partial_y H_3 - \partial_z H_2 &= \tilde{J}_1, & \sqrt{\eta}H_1 + \partial_y E_3 - \partial_z E_2 &= g_1, \\ -\sqrt{\eta}E_2 + \partial_z H_1 - \partial_x H_3 &= \tilde{J}_2, & \sqrt{\eta}H_2 + \partial_z E_1 - \partial_x E_3 &= g_2, \\ -\sqrt{\eta}E_3 + \partial_x H_2 - \partial_y H_1 &= \tilde{J}_3, & \sqrt{\eta}H_3 + \partial_x E_2 - \partial_y E_1 &= g_3, \end{aligned} \quad (3.36)$$

where we have set  $(\mathbf{E}, \mathbf{H}) = (\mathbf{E}^{n+1}, \mathbf{H}^{n+1})$ ,  $\sqrt{\eta} = \frac{2}{\Delta t}$ ,  $\tilde{\mathbf{J}} = \mathbf{J} - \frac{2}{\Delta t} \mathbf{E}^n - \text{curl} \mathbf{H}^n$ ,  $\mathbf{g} = \frac{2}{\Delta t} \mathbf{H}^n - \text{curl} \mathbf{E}^n$ , and  $E_j$  and  $H_j$  denote the new fields at the time step  $n+1$ . As in the time harmonic case, we have the equivalent of Proposition 3.1:

**PROPOSITION 3.8.** *Let  $\mathbf{u}$  be as defined in (3.2). At time step  $n+1$ , any component  $u_i$  of the solution of the Maxwell system (3.36) satisfies the elliptic equation*

$$(\eta - \Delta)u_j = f_j, \quad (3.37)$$

where the right hand side depends on  $\mathbf{J}$ ,  $\eta$ , and the solution at the previous time step  $\mathbf{u}^n$ .

*Proof.* The result follows like in the time harmonic case.  $\square$  In contrast to the time harmonic case however, (3.37) is a positive definite Helmholtz equation, which is much easier to solve numerically than the Helmholtz equation (3.11).

There is also an equivalence result including boundary conditions, as in Proposition 3.2, for which we omit the details here. Instead, we state directly the equivalent of Proposition 3.4, i.e a convergence result for the classical Schwarz algorithm applied to the time discretized Maxwell equations.

**PROPOSITION 3.9.** *Let  $\Omega = \mathbb{R}^3$ , and consider the Maxwell system (3.10) in  $\Omega$  with the radiation condition*

$$\lim_{r \rightarrow \infty} r(\mathbf{n} \times \mathbf{E} + \mathbf{n} \times (\mathbf{n} \times \mathbf{H})) = 0, \quad (3.38)$$

Case	with overlap, $L = h$		without overlap, $L = 0$	
	$\rho$	parameters	$\rho$	parameters
1	$1 - 2^{\frac{3}{2}}\eta^{\frac{1}{4}}\sqrt{h}$	none	$1 - 2\frac{\sqrt{\eta}}{C}h$	none
2	$1 - 2^{\frac{13}{6}}\eta^{\frac{1}{6}}h^{\frac{1}{3}}$	$p = \frac{2^{-\frac{1}{3}}\eta^{\frac{1}{3}}}{h^{\frac{1}{3}}}$	$1 - \frac{4\eta^{\frac{1}{4}}\sqrt{h}}{\sqrt{C}}$	$p = \frac{\sqrt{C}\eta^{\frac{1}{4}}}{\sqrt{h}}$
3	$1 - 2^{\frac{7}{4}}\eta^{\frac{1}{8}}h^{\frac{1}{4}}$	$p = \frac{\sqrt{2}\eta^{\frac{1}{4}}}{\sqrt{h}}$	$1 - \frac{2^{\frac{5}{3}}\eta^{\frac{1}{6}}}{C^{\frac{1}{3}}}h^{\frac{1}{3}}$	$p = \frac{2^{\frac{2}{3}}C^{\frac{2}{3}}\eta^{\frac{1}{6}}}{h^{\frac{2}{3}}}$
4	$1 - 2^{\frac{4}{5}}\eta^{\frac{1}{10}}h^{\frac{1}{5}}$	$p_1 = \frac{\eta^{\frac{1}{5}}}{2^{\frac{2}{5}}h^{\frac{3}{5}}}, p_2 = \frac{\eta^{\frac{2}{5}}}{16^{\frac{1}{5}}h^{\frac{1}{5}}}$	$1 - \frac{\sqrt{2}\eta^{\frac{1}{8}}}{C^{\frac{1}{4}}}h^{\frac{1}{4}}$	$p_1 = \frac{\sqrt{2}C^{\frac{3}{4}}\eta^{\frac{1}{8}}}{h^{\frac{3}{4}}}, p_2 = \frac{C^{\frac{1}{4}}\eta^{\frac{3}{8}}}{\sqrt{2}h^{\frac{1}{4}}}$
5	$1 - 2^{\frac{7}{6}}\eta^{\frac{1}{12}}h^{\frac{1}{6}}$	$p_1 = \frac{2^{\frac{1}{3}}\eta^{\frac{1}{3}}}{h^{\frac{1}{3}}}, p_2 = \frac{2^{\frac{1}{3}}\eta^{\frac{1}{6}}}{h^{\frac{2}{3}}}$	$1 - \frac{2\eta^{\frac{1}{10}}}{C^{\frac{1}{5}}}h^{\frac{1}{5}}$	$p_1 = \frac{2C^{\frac{4}{5}}\eta^{\frac{1}{10}}}{h^{\frac{4}{5}}}, p_2 = \frac{2C^{\frac{2}{5}}\eta^{\frac{3}{10}}}{h^{\frac{2}{5}}}$

TABLE 3.3

Asymptotic convergence rate and optimal choice of the parameters in the transmission conditions for the five variants of the optimized Schwarz method applied to the time domain Maxwell equations, when the mesh parameter  $h$  is small, and the maximum numerical frequency is estimated by  $k_{\max} = \frac{C}{h}$ .

where  $r = |\mathbf{x}|$ ,  $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$ . Let  $\Omega$  be decomposed into  $\Omega_1 := (-\infty, L) \times \mathbb{R}^2$  and  $\Omega_2 := (0, +\infty) \times \mathbb{R}^2$ , ( $L \geq 0$ ). For any given initial guess  $\mathbf{u}^{1,0} \in (L^2(\Omega_1))^6$ ,  $\mathbf{u}^{2,0} \in (L^2(\Omega_2))^6$ , the Schwarz algorithm (3.16) applied to system (3.36) converges for all Fourier modes to the solution of (3.36) in the following sense

$$\|\mathbf{u}(L, \cdot) - \mathbf{u}_1^{2n}(L, \cdot)\|_2 + \|\mathbf{u}(0, \cdot) - \mathbf{u}_2^{2n}(0, \cdot)\|_2 \leq R^n (\|\mathbf{u}(L, \cdot) - \mathbf{u}_1^0(L, \cdot)\|_2 + \|\mathbf{u}(0, \cdot) - \mathbf{u}_2^0(0, \cdot)\|_2), \quad (3.39)$$

The convergence factor is

$$R_{td} = \frac{\sqrt{L\eta + 2} - \sqrt{L\eta}}{\sqrt{L\eta + 2} + \sqrt{L\eta}} e^{-\sqrt{L\eta}\sqrt{L\eta+2}} < 1. \quad (3.40)$$

*Proof.* This result follows like in the time harmonic case.  $\square$  The preceding theorem shows that the classical Schwarz algorithm with Dirichlet transmission conditions applied to the time-discretized Maxwell system is convergent for all frequencies  $|\mathbf{k}|$ , and that the overlap is not necessary to ensure convergence. The classical Schwarz algorithm corresponds to a simple optimized Schwarz algorithm for the associated positive definite Helmholtz equation (3.37). With this equivalence, Lemma 3.5, Remark 3, Proposition 3.7 and all the cases in subsection 3.4.3 hold unchanged upon replacing  $i\omega$  by  $\eta$ , so we do not restate these results here. We show however in Table 3.3 the asymptotically optimal parameters to use in the time domain case. It is interesting to note the relationship of the optimized parameters for the Maxwell time domain case with the one for Cauchy-Riemann: Case 2 and 4 are identical, since the corresponding convergence rates in the two cases are the same, while for Case 1, 3 and 5 there is a small difference in the constants, which is due to the additional low frequency term in the Maxwell case. The difference appears to be systematic, the convergence factor of the Maxwell case is obtained from the convergence factor of the Cauchy-Riemann case by replacing  $h$  by  $2h$ , while for the optimized parameters one has to multiply by 2 in addition to the replacement of  $h$  by  $2h$ .

**3.7. Numerical Experiments for the time discretized equations.** We now show numerical experiments for the time discretized Maxwell equations solved on the unit square  $\Omega = (0, 1)^2$ . Here the parameter  $\eta = 1$ . We decompose the domain into two subdomains  $\Omega_1 = (0, \beta) \times (0, 1)$  and  $\Omega_2 = (\alpha, 1) \times (0, 1)$ , where  $0 < \alpha \leq \beta < 1$ ,



h	with overlap, $L = h$				without overlap, $L = 0$			
	1/16	1/32	1/64	1/128	1/16	1/32	1/64	1/128
Case 1	17	24	33	45	280	559	1310	2630
Case 2	13	15	19	24	39	56	77	111
Case 3	12	14	16	18	13	16	20	26
Case 4	12	13	15	17	21	25	30	36
Case 5	12	14	16	18	13	17	19	22

TABLE 3.4

Number of iterations to attain convergence for different interface conditions and different mesh sizes in the overlapping and non-overlapping case. The tolerance is fixed at  $\varepsilon = 10^{-6}$ .

and therefore the overlap is  $L = \beta - \alpha$ , and we consider both decompositions with and without overlap. We discretize the equations using a finite volume method, on a uniform mesh with mesh parameter  $h$ . In all comparisons that follow, we simulate directly the error equations,  $f = 0$ , and we use a random initial guess to ensure that all the frequency components are present in the iteration. Table 3.4 shows the iteration count for all Schwarz algorithms considered previously, in the overlapping and non-overlapping case when we accelerate the convergence by a GMRES method.

We can see from the table 3.4, that the classical non-overlapping algorithm converges but very slowly, the need of optimized methods is very obvious in this case. We should note that numerically the optimized methods behave as we expected even if the difference between them is not very important especially in the overlapping case.

**4. Conclusions.** We have shown in this paper, that for the Cauchy-Riemann equations and the Maxwell's equations, a Schwarz algorithm using characteristic information at the interfaces between subdomains is equivalent to an optimized Schwarz method applied to the corresponding scalar equation, with a low frequency approximation of the optimal transmission conditions. This equivalence shows that the Schwarz algorithms with characteristic conditions for those systems of partial differential equations are convergent even without overlap. We then used the equivalence to develop better transmission conditions than the characteristic ones for the Cauchy-Riemann and the Maxwell equations. We illustrated with numerical experiments that the new algorithm converge much more rapidly than the classical ones. Although we have shown the equivalence only for two model problems, our results indicate that it also holds for other systems of partial differential equations. In particular for the Euler equations, it was already observed in [Victorita], that the classical Schwarz algorithm with characteristic information exchange at the interfaces is convergent, even without overlap. We currently focus on the optimization of the new transmission conditions introduced here, and also on optimized transmission conditions for the Euler equations and for linear elasticity.

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