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ABSTRACT. This is an expository account of Katz's middle convolution operation on local systems over $\mathbf{P}^1 - \{q_1, \ldots, q_n\}$. We describe the Betti and de Rham versions, and point out that they give isomorphisms between different moduli spaces of local systems, following Völklein, Dettweiler-Reiter, Haraoka-Yokoyama. Kostov's program for applying the Katz algorithm is to say that in the range where middle convolution no longer reduces the rank, one should give a direct construction of local systems. This has been done by Kostov and Crawley-Boevey. We describe here an alternative construction using the notion of cyclotomic harmonic bundles: these are like variations of Hodge structure except that the Hodge decomposition can go around in a circle.

1. INTRODUCTION

There is a growing body of literature about Katz's "middle convolution" algorithm on local systems on $\mathbb{P}^1 - \{q_1, \ldots, q_n\}$. The purpose of the present paper is expository: we would like to describe two versions of Katz's construction in complex geometry, the Betti version involving complex local systems, and the de Rham version involving vector bundles with logarithmic connection. Katz's book [86] was written in the framework of ℓ-adic sheaves, which at first made it difficult to understand for complex geometers including myself. Subsequently, Völklein and Dettweiler-Reiter recast the construction in complex geometry and algebra. In §§2.7-2.9 of Katz's book, the convolution was defined in a geometric way which is applicable in any context where one has a Grothendieck formalism and a category of perverse sheaves. Thus, the translation into complex geometry may be viewed as coming directly from there [86, 5.9]. Katz then interpreted the convolution as conjugate to a tensor product, via Fourier transform, and used that to obtain some of the main properties of his construction. The complex analogy for this would *a priori* bring into play the notion of irregular connections on a 2-dimensional variety, a theory which remains poorly understood (see however [15] and the recent preprint [3]). It is possible to do a full treatment of middle convolution staying within the realm of complex geometry but without using Fourier transform, as has been shown and exploited by the works of Strambach, Völklein, Dettweiler, Reiter, Kostov, Crawley-Boevey, Haraoka, Yokoyama.

Many applications of Katz's theory concern the case of rigid local systems. For example, Gleizer has studied explicit solutions [55], and Roberts' preprint [123] includes an extensive discussion of how to apply the algorithm to determine which rigid local systems exist. Völklein, Dettweiler and Reiter have done extensive work on using Katz's existence results

in the rigid case to construct motivic local systems with interesting monodromy groups, obtaining results on the inverse Galois problem.

The middle convolution transformation was first applied in the non-rigid case by Kostov. An important invariant which we denote by $\delta(\vec{g})$ is the change in rank induced by Katz's transformation. As long as $\delta(\vec{g}) < 0$ we can apply middle convolution to reduce the rank (or otherwise, conclude that the local system couldn't exist). Kostov made the fundamental observation that when we get into the range $\delta(\vec{g}) \geq 0$, we should expect that the local system always exists and look for a direct construction. Kostov applied this to solve the existence problem in many cases [89]–[95], such as when $\vec{g}$ is *simple* i.e. the multiplicities of eigenvalues are not all divisible by the same integer $d > 1$, or for generic eigenvalues even if $\vec{g}$ is not simple [92]. Crawley-Boevey looked at the existence question from a point of view of root systems in [34] where Katz's algorithm plays a role. In that language, the transformation on local monodromy data is considered as a reflection in a root system, and a sequence of reflections is used to move up to the positive chamber. Once we are in the positive chamber, analogous to the condition $\delta(\vec{g}) \geq 0$, Crawley-Boevey found a direct construction of indecomposable parabolic bundles, and applied a parabolic variant of Weil's theorem to construct flat connections.

At the end of the present paper, we propose a technique for constructing local systems in the range $\delta(\vec{g}) \geq 0$ by using the correspondence between Higgs bundles and local systems in the parabolic case. This construction is heavily inspired by Kostov's program, and is obviously a variant on Crawley-Boevey's indecomposable parabolic bundles. So, it is not really very new but might present some advantages such as making clear the role played by the condition $\delta(\vec{g}) \geq 0$. The objects we introduce, cyclotomic harmonic bundles, might be interesting in their own right such as for studying the behavior of everything near infinity in the moduli spaces.

We will look at Katz's operation as giving an isomorphism between moduli spaces for different local monodromy data. Of course this applies to the rigid case too, but as Katz pointed out long ago, for local systems on $\mathbb{P}^1$ with specified singularities, local rigidity implies the stronger rigidity statement that there is at most one irreducible representation with the given local data. Thus, in the rigid case the moduli spaces are single points so even the cardinality is not an interesting invariant. Instead, we are motivated by looking for low-dimensional moduli spaces, for which things like Hitchin's hyperkähler structure, or the Riemann-Hilbert correspondence, could be viewed explicitly. The phrase "toy example" was coined by T. Hausel in [65] to refer to this kind of low-dimensional case arising from a punctured projective line. He looked at a space of parabolic Higgs bundles of rank 2 on $\mathbb{P}^1 - \{q_1, \ldots, q_4\}$. Boalch looked at an example of the middle convolution relating this space to a space of rank 3 representations in [14] [15], and considered the Painlevé equations for these cases. It seems like a good idea to pursue the philosophy of looking at low-dimensional cases, and to get started we need to have a thorough understanding of how the classification based on Katz's algorithm works. That's the motivation for this paper.

Conceptually, the middle convolution operation is pretty easy to understand. Let Y and Z denote two copies of the projective line $\mathbb{P}^1$, with reduced effective divisors $Q_Y \subset Y$ and

$Q_Z \subset Z$ (which we will often denote just by Q), both given by the same finite collection of n points $Q = \{q_1, \ldots, q_n\}$. Let $D \subset Z \times Y$ denote the "diagonal configuration" consisting of the diagonal Δ plus the vertical and horizontal divisors given by preprojections of Q.

A *convoluter* is a rank one local system on $Z \times Y$ with singularities along D. In Katz's original setup this would be a rank one ℓ-adic sheaf. In the complex geometric "Betti" and "de Rham" situations we consider here, the convolution object is respectively a rank one representation of $\pi_1(Z \times Y - D)$, or a logarithmic connection on the trivial bundle given by a logarithmic one-form. In either case we denote the convoluter by β. Let

$$\xi : Z \times Y - D \longrightarrow Z - Q_Z, \ \ \eta : Z \times Y - D \longrightarrow Y - Q_Y$$

denote the two projections. Given an irreducible rank r local system L on $Y - Q$, we can form the "raw convolution" defined as the higher direct image

$$RC_\beta(L) := \mathbb{R}^1 \xi_*(\eta^*(L) \otimes \beta),$$

a local system on $Z - Q_Z$. Unfortunately, the raw convolution will not in general be an irreducible local system, because there are some contributions whenever the tensor product $\eta^*(L) \otimes \beta$ has trivial eigenvalues along the "horizontal" piece $H := \eta^*(Q_Y)$ of the divisor D. This is remedied by defining the "middle convolution" to be the *middle direct image*

$$MC_\beta(L) := M\mathbb{R}^1 \xi_*(\eta^*(L) \otimes \beta),$$

heuristically defined as the kernel of the map to the quotient systems corresponding to the unwanted local cohomolgy groups.

When studying local systems on $\mathbb{P}^1$ with singularities, we are interested in fixing the local type of the singularities. For the present paper, we will simplify things considerably by making the convention that the local monodromy transformations be semisimple (Convention 2.1) or the corresponding statement for the residues of a logarithmic connection (Convention 2.2). This allows us to avoid complicated discussions of Jordan normal forms. The reader who is interested may refer to the original references for discussions of this aspect.

When we discuss moduli spaces we will consider the moduli spaces of local systems with fixed conjugacy classes of local monodromy. The notion of rigidity considered by Katz takes into account the fixing of the local conjugacy classes. So, the first and in some sense main question about Katz's construction is to understand what is its effect on the local monodromy transformations.

We will try to explain the answer, and how to see why it works that way. This will occupy most of the paper, and is the main subject of our exposition. Of course it has already been treated by Katz in the ℓ-adic case, and by Strambach, Völklein, Dettweiler, Reiter, Kostov, Crawley-Boevey, Haraoka, Yokoyama and others in the complex case. Thus there is nothing new in our exposition. We hope it will be useful as an explanation allowing readers more easily to consult the original references.

At the end of the paper, we consider the question of how to construct local systems in the range $\delta(\vec{g}) \geq 0$. We propose a construction which is based on Donaldson-style Yang-Mills theory [49] [70], in which we construct a polystable Higgs bundle with parabolic structure corresponding to the local monodromy. This is similar to the construction of systems of Hodge bundles which was used in [130]. Here we introduce a new notion which makes the

problem much easier: *cyclotomic harmonic bundles.* These are harmonic bundles which are fixed under the action of a finite cyclic subgroup $\mu_m \subset \mathbb{C}^*$, for the usual action of $\mathbb{C}^*$ on the space of Higgs bundles. These are related to the Higgs bundles considered by Hitchin in [72]. A cyclotomic harmonic bundle is very much like a variation of Hodge structure, in that the bundle decomposes $E = \bigoplus_{p=0}^{m-1} E^p$. The only difference is that the indexation is really by $p \in \mu_m^* \cong \mathbb{Z}/m\mathbb{Z}$, and the Kodaira-Spencer components of the Higgs field go between E^p and $E^{p-1} \otimes \Omega_X^1$ where $p-1$ is taken modulo m. Thus θ is no longer necessarily nilpotent. Our construction takes place in the maximal case when $m = r$ is the rank of E, and θ is not nilpotent. This means that the E^p are line bundles. Thus the description of (E, θ) is elementary. It turns out that incorporating parabolic structures into the picture in order to obtain a required local monodromy type, the condition $\delta(\vec{g}) \geq 0$ is exactly what is needed for the degrees of the line bundles to work out correctly and enable the construction. Unfortunately it doesn't work when the dimension of the moduli space is 2.

In the last section we discuss some questions and directions for further study.

Acknowledgements: I would specially like to thank V. Kostov for many helpful discussions, and for a particularly illuminating talk many years ago in which he explained his utilisation of Katz's algorithm. Also I would like to thank O. Gleizer for some interesting discussions a while ago. At Princeton last year, Deligne and Katz raised the question of how to understand what is going on, which prompted the present write-up. Many aspects we consider here, such as moduli spaces with fixed conjugacy classes, and logarithmic connections, showed up in the course of recent joint works with K. Corlette and J. Iyer.

2. CONNECTIONS AND LOCAL SYSTEMS

Denote by Y a smooth projective curve, with $K \subset Y$ a reduced divisor. Write $K = k_1 + \ldots + k_n$, with points $k_i \in Y$. We consider local systems L over $Y - K$. If $x \in Y - K$ is a choice of basepoint then a local system corresponds to the *monodromy representation*

$$\rho : \pi_1(Y - K, x) \to GL(L_x).$$

The *local monodromy transformations* are the $\rho(\alpha_i)$ where α_i is a loop in standard form going from x to near k_i, once around clockwise, then back to x. In this paper, we will systematically make the convention :

Convention 2.1. *The local monodromy transformations are semisimple, i.e. diagonalizable matrices.*

A *logarithmic connection* on (Y, K) is a vector bundle E on Y, with a connection operator

$$\nabla : E \to E \otimes \Omega_Y^1(\log K).$$

The *monodromy* of (E, ∇) is a local system on $Y - K$, described for example as the sheaf $L = E^\nabla$ of analytic holomorphic sections e of E with $\nabla(e) = 0$. Over $Y - K$ this is a locally constant sheaf or a local system, and corresponds to a monodromy representation $\rho_{E,\nabla}$.

The *residue* of (E, ∇) at a point $k_i \in K$ is the pair $(E_{k_i}, \mathrm{res}(\nabla, k_i))$ consisting of the fiber of E over k_i, and the residue of the connection which is an endomorphism of the fiber.

We say that "the residues of ∇ are semisimple" if these endomorphisms are semisimple i.e. diagonalizable. Furthermore, in order to insure that the monodromy transformation satisfies

Convention 2.1, it is convenient to ask that the eigenvalues of the residues never differ by integers. Indeed, if there are pairs of eigenvalues differing by integers even in a semisimple residue, this can typically lead to Jordan blocks of size > 1 in the monodromy. This gives the analogous convention for the de Rham case.

Convention 2.2. *The residues of ∇ are semisimple and their eigenvalues don't differ by nonzero integers.*

These conventions greatly reduce the complexity of the notation and arguments required to understand Katz's constructions. Of course Katz and subsequent authors all considered the more general case of arbitrary Jordan normal forms, and we refer the reader to those references for a more in-depth look at this aspect.

2.1. **Middle cohomology.** If the residues don't have integer eigenvalues, then Deligne's theory gives an easy description of the cohomology of the monodromy local system. We denote by $DR(Y, E)$ the *logarithmic de Rham complex*

$$DR(Y, E) := [E \xrightarrow{\nabla} E \otimes \Omega^1_Y(\log K)].$$

The connection ∇ and the divisor K are missing and implicit in this notation. This abuse allows us to shorten most displays below. Unless otherwise stated, all de Rham complexes are supposed to be logarithmic with respect to the relevant divisor.

Proposition 2.3. *Suppose that the eigenvalues of the residues of (E, ∇) are never integers. Let $L_\rho := E^\nabla$ be the monodromy local system on $Y - K$. Then hypercohomology of the logarithmic de Rham complex $DR(Y, E)$ calculates the cohomology of $Y - K$ with coefficients in L_ρ:*

$$\mathbb{H}^\cdot DR(Y, E) \sim H^\cdot(Y - K, L_\rho).$$

In the case where some eigenvalues are integers, the situation is more complicated. The same monodromy representation can come from several different logarithmic connections, whose residual eigenvalues will differ by integers. The cohomology which is calculated by the de Rham complex will in principle depend on which lift we have chosen. A canonical choice, somewhat different from the choices coming from lifts, is given by the notion of "middle cohomology" and the "middle de Rham complex". The reason for the word "middle" is that it corresponds to the middle perversity in intersection cohomology. In the one-dimensional case, as was well understood by Katz, the notion of intersection cohomology corresponds to the more classical construction j_*, as opposed to the derived $\mathbb{R}j_*$. In this context the word "middle" is more notation than notion.

The Betti version is as follows. Let $j : Y - K \hookrightarrow Y$ denote the inclusion. Assume K is nonempty. If ρ is a representation of the fundamental group of $Y - K$ corresponding to a local system L_ρ on $Y - J$, then we define the *middle cohomology*

$$MH^i(Y, L_\rho) := H^i(Y, j_*(L_\rho)).$$

The non-derived $j_*(L_\rho)$ is the degree zero part of the total derived $\mathbb{R}j_*(L_\rho)$; the other piece is $R^1j_*(L_\rho)[-1]$. Which gives an exact triangle in the derived category

$$j_*(L_\rho) \to \mathbb{R}j_*(L_\rho) \to R^1j_*(L_\rho)[-1] \to j_*(L_\rho)[1] \dots.$$

Look at the long exact sequence of hypercohomology for this triangle. The hypercohomology of the total $\mathbb{R}j_*$ gives the cohomology of $Y - K$. Also $R^1 j_*(L_\rho)$ is concentrated at K, so it only contributes for global sections. In particular we have

$$MH^0(Y, L_\rho) = H^0(Y - K, L_\rho)$$

and there is a long exact sequence

(2.1) $0 \to MH^1(Y, L_\rho) \to H^1(Y - K, L_\rho) \to \oplus_{k \in K} R^1 j_*(L_\rho)_k \to MH^2(Y, L_\rho) \to 0.$

Since K is nonempty, $Y - K$ is homotopic to a one dimensional complex so its H^2 with local coefficients vanishes.

An observation which is important for defining the middle convolution is the following.

Proposition 2.4. *Suppose that for at least one point $k_i \in K$, the local monodromy has no fixed vectors. Then the middle cohomology in degrees zero and two vanishes.*

Proof: It is clear that $H^0(Y, L_\rho) = 0$ because there are no flat sections near the point k_i, so there can be no global flat sections. This proves that $MH^0 = 0$. There is a Poincaré-Verdier duality between $MH^i(Y, L)$ and $MH^{2-i}(Y, L^*)$, and L^* also has no fixed vectors at k_i. This gives $MH^2(Y, L) = 0$. One can also prove the vanishing by a direct topological argument. $\square$

As a corollary we obtain the dimension of the middle cohomology group in this case:

Corollary 2.5. *Suppose L_ρ is a local system of rank r on $Y - K$, and suppose that for at least one point $k_i \in K$ the local monodromy has no fixed vectors. Then we have*

$$\dim MH^1(Y, L_\rho) = r(n - 2) - \sum_{j=1}^{n} \mathrm{cofix}(L_\rho, k_j).$$

where n is the number of points in K and $\mathrm{cofix}(L_\rho, k_j)$ is the dimension of the space of cofixed vectors of the local monodromy at k_i.

Proof: Note that $H^0(Y - K, L_\rho) = 0$ as pointed out in the proof of 2.4, and $H^2(Y - K, L_\rho) = 0$ because $Y - K$ is homotopically a 1-dimensional complex. Thus by calculating the Euler characteristic we have

$$\dim H^1(Y - K, L_\rho) = r(n - 2).$$

On the other hand,

$$\dim H^1(B_{k_j}^*, L_\rho) = \mathrm{cofix}(L_\rho, k_j).$$

The exact sequence 2.1 gives the dimension of MH^1. $\square$

2.2. **Middle homology.** The *middle homology* is obtained by duality with the middle co-homology:

$$MH_i(Y - K, L) := MH^i(Y - K, L^*)^*.$$

This is interesting only if the monodromy of L has some eigenvalues equal to 1 around a point k_i. The loop around that point, with the eigenvector as coefficient, gives a cycle in $H_1(Y - K, L)$.

Let $F_i \subset L_z$ denote the subspace of vectors fixed by the monodromy transformation $\rho_L(\alpha_i)$. Since we are assuming that the local monodromy transformations are unipotent, the dimension of F_i is equal to the multiplicity of 1 as eigenvalue of $\rho_L(\alpha_i)$. We get a map

$$\phi : \bigoplus_{i=1}^{k} F_i \to H_1(Y - K, L).$$

Lemma 2.6. *Suppose that the monodromy around at least one of the points k_i has no fixed vector. Then ϕ is injective and the first middle homology is the cokernel of the map ϕ:*

$$MH_1(\Gamma, L) := \frac{H_1(Y - K, L)}{\phi \bigoplus_{i=1}^{k} F_i}.$$

Proof: This is dual to Proposition 2.4. □

2.3. Middle de Rham cohomology.
For the de Rham version of middle cohomology, if (E, ∇) is a vector bundle with logarithmic connection on (Y, K), we define the *middle de Rham complex*

$$MDR(Y, E, \nabla) = \left[MDR^0(Y, E, \nabla) \to MDR^1(Y, E, \nabla) \right]$$

with

$$MDR^0(Y, E, \nabla) := E,$$

and

$$MDR^1(E, \nabla) := \ker \left(E \otimes \Omega_Y^1(\log K) \to E_K^0 \right)$$

where E_K^0 is the quotient of the fiber E_K over K, corresponding to the 0-eigenspaces of $\mathrm{res}(\nabla, k_i)$ at the points $k_i \in K$. The differential is given by ∇ as for the usual de Rham complex.

We should stress here that this definition is the right one only under our convention and assumption that the local monodromy, and the residues of ∇, are semisimple.

Define the *middle de Rham cohomology* to be the hypercohomology $\mathbb{H}^i MDR(Y, E, \nabla)$.

By definition we have a short exact sequence of complexes of sheaves on Y,

$$0 \to MDR(Y, E, \nabla) \to DR(Y, E, \nabla) \to E_K^0[-1] \to 0.$$

This gives the same kind of long exact sequence as before.

Lemma 2.7. *Suppose that the residues of ∇ have no nonzero integer eigenvalues. Then the above short exact sequence for the middle de Rham cohomology coincides after Riemann-Hilbert correspondence, with the previous exact triangle for the middle Betti cohomology. In particular, if L_ρ is the monodromy local system E^∇ then we have a natural isomorphism*

$$\mathbb{H}^i MDR(Y, E, \nabla) \cong MH^i(Y, L_\rho).$$

□

If there are nonzero integer eigenvalues, on the other hand, then the corresponding subspaces are fixed for the monodromy transformation but don't appear in the quotient E_K^0. In this case the middle de Rham cohomology will be different from the middle Betti cohomology.

Exercise 1. *Describe what cohomology is calculated by $DR(Y, E, \nabla)$ and $MDR(Y, E, \nabla)$ when the residues of ∇ may have some nonzero integer eigenvalues.*

Hint: It depends on the sign of the eigenvalues.

Proposition 2.4 above thus has the corresponding corollary in the de Rham case.

Corollary 2.8. *Suppose that the residues of ∇ are semisimple and have no nonzero integer eigenvalues, and suppose that for at least one point $k_i \in K$, the residue $\mathrm{res}(\nabla, k_i)$ has all eigenvalues different from 0. Then the middle cohomology in degrees zero and two vanishes:*

$$\mathbb{H}^0 MDR(Y, E, \nabla) = 0, \quad \mathbb{H}^2 MDR(Y, E, \nabla) = 0.$$

The dimension of the middle cohomology in degree 1 is given by

$$\dim MH^1(Y, L_\rho) = r(n-2) - \sum_{j=1}^{n} \mathrm{rk}(E_{k_j}^0).$$

Proof: This is immediate from 2.4 and 2.7, and with Corollary 2.5 or its proof we get the dimension count. □

2.4. The Betti moduli spaces.

We are interested in the moduli of representations with fixed conjugacy classes at the singularities. The first version to look at is the "Betti" moduli space. See [32].

Let $q_1, \ldots, q_n$ be n distinct points in $Y := \mathbb{P}^1$ and fix a basepoint z different from these. Put $\Gamma := \pi_1(Y - \{q_1, \ldots, q_n\}, z)$. Let $\gamma_1, \ldots, \gamma_n$ denote standard loops based at z going around the points $q_1, \ldots, q_n$ respectively.

Fix closed subsets $C_1, \ldots, C_n \subset GL(r) = GL(r, \mathbb{C})$ invariant under the conjugation action. In the present paper in keeping with Convention 2.1 these will be semisimple conjugacy classes (see below). However the definition can be made with more general closed subsets which would then have to contain many different conjugacy classes including semisimple ones. In this case the structure of the moduli space is more complicated, for example it can be nonempty even when the moduli space for the semisimple conjugacy classes in the closure might be empty.

Let

$$\mathrm{Rep}(\Gamma, GL(r); C_1 \ldots C_n) \subset \mathrm{Rep}(\Gamma, GL(r))$$

be the closed subset of representations $\rho : \Gamma \to GL(r)$ such that

$$\rho(\gamma_i) \in C_i.$$

Since it is a closed subset of an affine variety, it is also affine. The group $GL(r)$ acts on the representation variety and it preserves our closed subset because we have assumed that the C_i are conjugation-invariant. Thus we get an action of $GL(r)$ on the affine variety $\mathrm{Rep}(\Gamma, GL(r); C_1 \ldots C_n)$ so we can take the universal categorical quotient

$$M_B(C_1, \ldots, C_n) := \mathrm{Rep}(\Gamma, GL(r); C_1 \ldots C_n)/GL(r).$$

This has the following usual description on the level of points. Two points ρ, ρ' of the representation variety are *S-equivalent* if the closures of their orbits intersect. In the preimage of any point of M_B there is a unique closed orbit, which shows that this relation is an equivalence relation and the points of $M_B(C_1, \ldots, C_n)$ are the S-equivalence classes. We have the

same description for the action of $Gl(r)$ on $\mathrm{Rep}(\Gamma, GL(r))$, and since $\mathrm{Rep}(\Gamma, GL(r); C_1 \ldots C_n)$ is a closed $GL(r)$-invariant subvariety, the closure of an orbit of $\rho \in \mathrm{Rep}(\Gamma, GL(r); C_1 \ldots C_n)$ is the same when taken in the bigger representation variety or the closed subset. Therefore, the relation of S-equivalence when we restrict the conjugacy classes, is the restriction of this relation on the full variety.[*] The relation of S-equivalence for the full representation variety is well-understood, see Lubotsky-Magid [103] for example. In particular, two points ρ, ρ' are S-equivalent if and only if their semisimplifications are isomorphic. The semisimplification is again a representation in $\mathrm{Rep}(\Gamma, GL(r); C_1 \ldots C_n)$, and the points of $M_B(C_1, \ldots, C_n)$ represent the isomorphism classes of semisimple representations.

Now restrict our attention to the case of semisimple conjugacy classes $C_1, \ldots, C_n$, that is to say the conjugacy classes of diagonalizable matrices. The $C_i \subset GL(r)$ are closed subsets, so the above discussion applies.

It is convenient to think of a semisimple class as being determined by a divisor on $\mathbb{G}_m$. Write a divisor as $g = \sum_{a \in \mathbb{G}_m} m(a)[a]$ where $[a]$ is the point a considered as a reduced effective divisor and the sum is finite. If $r = \deg(g) := \sum_a m(a)$, then the divisor g corresponds to the conjugacy class $C(g) \subset GL(r)$ of diagonalizable matrices having eigenvalues a with multiplicities $m(a)$. A sequence of semisimple conjugacy classes is then represented by a *local monodromy vector* of n divisors $\vec{g} = (g_1, \ldots, g_n) \in Div(\mathbb{G}_m)^n$. We come to our main notation for the Betti moduli spaces:

$$M_B(\vec{g}) := M_B(C(g_1), \ldots, C(g_n)),$$

where the collection of points $Q = q_1 + \ldots + q_n$ is implicit but not mentionned.

The vector or partition consisting of the $m(a)$ is a partition of r. Kostov calls this the *multiplicity vector*, and the vector of multiplicity vectors corresponding to $g_1, \ldots, g_n$ is called by Kostov the *polymultiplicity vector* or PMV. To obtain a geographic understanding one should look only at the PMV, see Roberts [123].

2.5. **Nitsure's de Rham moduli space.** We can define the following 2-functor $\mathcal{M}_{DR}(r, d)$ of $\mathbb{C}$-schemes of finite type T. Put $\mathcal{M}_{DR}(r)[T]$ equal to the groupoid of (E, ∇) where E is a vector bundle of rank r and degree d on $P \times T$ and

$$\nabla : E \to E \otimes_{\mathcal{O}_{P \times T}} \Omega^1_{P \times T/T}(\log Q \times T)$$

is a relative logarithmic connection. Standard moduli theory shows that it is an Artin algebraic stack locally of finite type.

Say that a logarithmic connection (E, ∇) is *semistable* if for any subbundle $F \subset E$ preserved by ∇, we have

$$\frac{\deg(F)}{\mathrm{rk}(F)} \leq \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Define stability using a strict inequality for strict nonzero subbundles. Semistability and stability are open conditions [116], and the open substack of semistable objects

$$\mathcal{M}^{\mathrm{se}}_{DR}(r, d) \subset \mathcal{M}_{DR}(r, d)$$

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[*]Notice that for this statement, we have used the condition that the C_i are closed subsets. If we tried to do this with locally closed subsets, for example corresponding to nonsemisimple conjugacy classes but not their closures, it wouldn't work the same way.

is an Artin stack of finite type (it follows from the boundedness in Nitsure's construction [116]).

Semistability of a logarithmic connection would be a consequence of semistability of the underlying bundle, but doesn't imply it in general. Esnault with Viehweg [53] and Hertling [51], and also Bolibruch [17] have studied the problem of realization of monodromy representations as logarithmic connections on semistable bundles in the higher genus case, generalizing Bolibruch's well-known work on $\mathbb{P}^1$. Our present notion of semistability of the pair (E, ∇) is somehow less subtle.

Nitsure constructs in [116] the *moduli space* which is a universal categorical quotient

$$\mathcal{M}_{DR}^{se}(r, d) \to M_{DR}(r, d).$$

The points represent S-equivalence classes of semistable logarithmic connections, and there is a unique polystable object in each S-equivalence class.

Suppose $\mathbf{c}_1, \ldots, \mathbf{c}_n \subset \mathbf{gl}(r)$ are closed subsets invariant under the adjoint action of $GL(r)$ on its Lie algebra $\mathbf{gl}(r)$. Then, as before, we obtain a closed substack

$$\mathcal{M}_{DR}^{se}(r, d; \mathbf{c}_1, \ldots, \mathbf{c}_n) \subset \mathcal{M}_{DR}^{se}(r, d)$$

consisting of logarithmic connections (E, ∇) such that up to choice of basis of E_{q_i}, the residue $\mathrm{res}(\nabla, q_i)$ lies in $\mathbf{c}_i$.

Again, here we will concentrate on the case where each $\mathbf{c}_i$ is the conjugacy class of a semisimple matrix, which is closed and $GL(r)$-invariant. As above, such a conjugacy class may be parametrized by an effective divisor $g_i \in Div(\mathbb{A}^1)$, with $\deg(g_i) = r$.

We denote by $\vec{g} = (g_1, \ldots, g_n)$ a vector of divisors parametrizing semisimple conjugacy classes, either $C_i = C(g_i) \subset GL(r)$ in the "multiplicative case" $g_i \in Div(\mathbb{G}_m)$ or $\mathbf{c}_i = \mathbf{c}(g_i) \subset \mathbf{gl}(r)$ in the "additive case" $g_i \in Div(\mathbb{A}^1)$.

The rank r is recovered from $\vec{g}$ as the degree of any one of the divisors g_i (they all have to have the same degree). We can also define the *trace* of a divisor $g = \sum m(\alpha)[\alpha]$ to be the sum

$$\mathrm{tr}(g) := \sum m(\alpha)\alpha \in \mathbb{C}.$$

If $A \in \mathbf{c}(g)$ is a matrix in the corresponding conjugacy class then $Tr(A) = \mathrm{tr}(g)$. The residue formula for the logarithmic connection on the determinant line bundle of E provides the formula

$$d = \deg(E) = \sum_{i=1}^{n} \mathrm{tr}(g_i).$$

This obviously has to be an integer, otherwise the moduli space will be empty. The degree d of the bundle E may be recovered from $\vec{g}$. Thus, we are justified in writing

$$M_{DR}(\vec{g}) := M_{DR}(r, d; \mathbf{c}(g_1), \ldots, \mathbf{c}(g_n)).$$

2.6. **Deformations and obstructions.** Suppose (E, ∇) represents a point in $M_{DR}(\vec{g})$. We compare the deformation and obstruction theory of (E, ∇) as logarithmic connection, with that of (E, ∇) as a point in the moduli stack $\mathcal{M}_{DR}(\vec{g})$. The following result was pointed out by N. Katz in the late 1980's.

Theorem 2.9. *The deformation and obstruction theory for the moduli stack $\mathcal{M}_{DR}(\vec{g})$ at a point (E, ∇) is governed by the middle cohomology groups of the endomorphism bundle $\mathbb{H}^i MDR(Y, End(E))$ for $i = 0$ (automorphisms), $i = 1$ (deformations) and $i = 2$ (obstructions).*

Proof: We give an heuristic but ultramodern explanation. Definitions and explicitations would need to be filled in, but this should convince the reader why it is true.

The deformation and obstruction theory of (E, ∇) as a logarithmic connection, that is as a point in $\mathcal{M}_{DR}(r, d)$, is given by the L_∞ algebra

$$\mathbf{D}_{E,\nabla}(\mathcal{M}_{DR}) := \mathbb{H}^{\cdot}(Y, End(E) \otimes \Omega_Y^{\cdot}(\log Q)).$$

Let $R := \mathbf{gl}(r)//GL(r)$ denote the moduli stack of conjugacy classes of matrices. It is a smooth Artin algebraic stack. At a point corresponding to a matrix A, its deformation theory is controlled by the L_∞-algebra concentrated in degrees 0 and 1

$$\mathbf{D}_A(R) := \mathbf{gl}(r) \xrightarrow{[-,A]} \mathbf{gl}(r).$$

If A is in the semisimple conjugacy class $\mathbf{c}_i$, then the deformation theory of $\langle \mathbf{c}_i \rangle := \mathbf{c}_i//GL(r)$ is controlled by

$$\mathbf{D}_A(\langle \mathbf{c}_i \rangle) = [\mathbf{gl}(r) \xrightarrow{[-,A]} (\mathrm{Im}(u \mapsto [u, A]))]$$

$$(2.2) \qquad\qquad = \ker(\mathbf{D}_A(R) \to \mathbf{gl}(r)^A)$$

where $\mathbf{gl}(r)^A$ is the space of cofixed or vectors of the adjoint action of A, which is isomorphic to the space of fixed vectors since A is semisimple. In practical terms, if A is in diagonal form then the degree one piece of $\mathbf{D}_A(\langle \mathbf{c}_i \rangle)$ is the space of off-block-diagonal matrices (for the blocks determined by the eigenvalues of A) and $\mathbf{gl}(r)^A$ is the space of block-diagonal matrices.

For any point $q_i \in Q$ the construction $(E, \nabla) \mapsto \mathrm{res}(\nabla, q_i)$ gives a morphism of moduli stacks $\mathcal{M}_{DR}(r, d) \to R$. Putting these together gives a morphism $\mathcal{M}_{DR}(r, d) \to R^n$. On the other hand, a vector of divisors $\vec{g}$ represents a collection of conjugacy classes $\mathbf{c}(g_i)$ which gives the substack

$$R(\vec{g}) := \prod_{i=1}^{n} \langle \mathbf{c}(g_i) \rangle \subset R^n,$$

and by definition

$$\mathcal{M}_{DR}(\vec{g}) = \mathcal{M}_{DR}(r, d) \times_{R^n} R(\vec{g}).$$

To get the deformation theory for this fiber product, we should take the homotopy fiber product of the L_∞-algebras. The one for $R(\vec{g})$ is the kernel of a map (2.2) on the one for R^n.

Hinich explains how to go between a sheaf of L_∞-algebras, and a global L_∞-algebra [68]. In our case, $\mathbf{D}_{E,\nabla}(\mathcal{M}_{DR})$ is the globalization of the sheaf of L_∞-algebras $DR(Y, End(E))$ (the logarithmic de Rham complex along Q). Going back and forth a few times we see that the deformation theory for $\mathcal{M}_{DR}(\vec{g})$ is controlled by the globalization, or hypercohomology, of

the kernel of the map corresponding to (2.2) on the sheaf of L_∞-algebras, this map expressed in local terms at the singularities as

$$(2.3) \qquad DR(Y, End(E)) \to \bigoplus_{i=1}^{n} End(E)_{q_i}^0$$

where the superscript 0 means the trivial eigenspace for the action of $\mathrm{res}(\nabla, q_i)$. The kernel is exactly the middle de Rham complex for $End(E)$. Thus, the deformation theory is controlled by an L_∞-algebra $\mathbb{H}^\cdot MDR(Y, End(E))$. $\square$

Corollary 2.10. *Suppose $(E, \nabla) \in \mathcal{M}_{DR}(\vec{g})$ is a point corresponding to an irreducible representation. Then it is a smooth point of the moduli stack, and has only scalar automorphisms so it is also a smooth point of the moduli space where the tangent space is $\mathbb{H}^1 MDR(Y, End(E))$.*

Proof: Decompose into the trace-free part and the scalars: $End(E) = End'(E) \oplus \mathcal{O}$. The trace of the obstruction map is zero, and $\mathbb{H}^1 MDR(Y, \mathcal{O}) = 0$ since $Y \cong \mathbb{P}^1$. Thus, the deformations and obstructions are given by $\mathbb{H}^1 MDR(Y, End'(E))$ and $\mathbb{H}^2 MDR(Y, End'(E))$ respectively. Poincaré duality for the middle cohomology, plus the fact that $End'(E)$ is self-dual, gives $\mathbb{H}^2 MDR(Y, End'(E)) \cong \mathbb{H}^0 MDR(Y, End'(E)) = 0$ since E has no trace-free endomorphisms because it is irreducible. Thus the space of obstructions vanishes, and the tangent space is given by $\mathbb{H}^1 MDR(Y, End(E)) = \mathbb{H}^1 MDR(Y, End'(E))$. $\square$

2.7. **Dimension counting.** From the previous discussion of deformations and obstructions, we find the following boiled-down statement.

Proposition 2.11. *Suppose (E, ∇) is a stable point in $M_{DR}^s(\vec{g})$. Then the dimension of the moduli space at the given point (or any other stable point) is obtained by a naive dimension count:*

$$\dim(M_{DR}^s(\vec{g})) = \sum_{i=1}^{n} \dim(\mathbf{c}(g_i)) - 2r^2 + 2.$$

The factor $2(r^2 - 1)$ corresponds to the fact that the conjugation action factors through $PGL(r)$ and the product identity lies in $SL(r)$.

The same dimension count holds for the open subset of irreducible representations $M_B^{\mathrm{irr}}(\vec{g})$ if it is nonempty.

Proof: Apply Corollary 2.10. The tangent space is $\mathbb{H}^1 MDR(Y, End(E))$ and

$$\mathbb{H}^0 MDR(Y, End(E)) = \mathbb{H}^2 MDR(Y, End(E)) = \mathbb{C}$$

since E is irreducible. Obtain the dimension count by using the fact that $MDR(Y, End(E))$ is the kernel of the map (2.3), noting

$$\dim(\mathbf{c}(g_i)) = r^2 - \dim End(E)_{q_i}^0,$$

and calculating the Euler characteristic. The Riemann-Hilbert correspondence gives the corresponding statement for $M_B^{\mathrm{irr}}(\vec{g})$. $\square$

We introduce the *defect*. It may be seen as playing a role in the dimension count, but is also foremost related to Katz's algorithm as we shall explain later. For each g_i, let $\nu(g_i)$ be

the maximal multiplicity of an eigenvalue. The centralizer of a matrix A in the conjugacy class $\mathbf{c}(g_i)$ is the set of block-diagonal matrices, and the dimension of the conjugacy class is the number of positions which are not in the block-diagonal pieces.

Think visually of shifting all of the diagonal blocks to the left of the matrix. In other words, transpose each square diagonal block with the rectangle consisting of all places to the left of it in the same rows. In the resulting picture, the square blocks are now arrayed from top to bottom flush with the left edge of the matrix. The dimension of the conjugacy class is still the number of positions which are not in these blocks. This leftover part can be divided into two regions: a big rectangle of size $r \times (r - \nu(g_i))$ which is everything to the right of the biggest block, plus a union of other rectangular regions corresponding in each row to the positions to the right of the edge of the corresponding block for that row, but to the left of the size of the biggest block. The second piece might be empty, indeed it is empty exactly in the case when the blocks all have the same size. We obtain the crude estimate

$$\dim(\mathbf{c}(g_i)) \geq r(r - \nu(g_i)),$$

leading to the crude estimate for the dimension of the moduli space as

$$\dim(M_{DR}^s(\vec{g})) \geq nr^2 - r\sum_{i=1}^{n} \nu(g_i) - 2r^2 + 2 = 2 + r\left((n-2)r - \sum_{i=1}^{n}\nu(g_i)\right).$$

In view of this formula already, it seems reasonable to consider the quantity

$$\delta(\vec{g}) := (n-2)r - \sum_{i=1}^{n}\nu(g_i).$$

We call this the *defect* because it enters into Katz's algorithm in a remarkably elegant way: if β is a convolution object corresponding to a choice of maximal-multiplicity eigenvalue for each g_i, then the new rank of the Katz-transformed local monodromy data is

$$r' = r + \delta(\vec{g}).$$

We will want to run Katz's algorithm when $\delta(\vec{g})$ is negative. We can do so until we get to a vector whose defect is positive.

In terms of the defect, the crude dimension count says $\dim(M_{DR}^s(\vec{g})) \geq 2 + r\delta(\vec{g})$. In order to refine the dimension count, introduce the *superdefect* denoted locally by

$$\sigma(g_i) := \dim(\mathbf{c}(g_i)) - r(r - \eta(g_i)),$$

and globally by

$$\sigma(\vec{g}) := \sum_{i=1}^{n}\sigma(g_i).$$

These quantities, which are always ≥ 0, are just the differences between the crude dimension counts and the actual dimensions. Thus we have, when the stable open set is nonempty,

$$\dim(M_{DR}^s(\vec{g})) = 2 + r\delta(\vec{g}) + \sigma(\vec{g}).$$

In view of the possibility of applying Katz's algorithm to decrease the rank whenever $\delta < 0$, the remaining case to investigate is when $\delta(\vec{g}) \geq 0$. Under this hypothesis, the dimension of

the moduli space is always ≥ 2. The cases of dimension 0 were the subject of Katz's original book: they arise when one ends up with a rank one local system, at which point it will no longer be possible to find a convoluter satisfying Convention 3.1. The case of dimension 2 is particularly interesting, although unfortunately our construction of §6 (Corollary 6.10) will not apply.

Lemma 2.12. *The superdefect $\sigma(g_i)$ vanishes if and only if all of the eigenvalues of g_i have the same multiplicity. In the domain $\delta(\vec{g}) \geq 0$, the dimension of the moduli space will be exactly 2, if and only if $\delta = 0$ and the superdefects $\sigma(g_i)$ all vanish.*

Proof: Note that $\sigma(g_i)$ is the number of places left over in the complement of the pushed-left diagonal blocks, after taking out the big rectangle. This vanishes only if all of the blocks have the same size. The last statement follows from $\sigma(\vec{g}) \geq 0$.   □

If $\vec{g}$ is an example of the case $\delta = \sigma = 0$ then any multiple (meaning to multiply all of the divisors g_i by the same amount) is also an example. Thus the examples of this case come in families which are indexed by an integer $d \geq 0$. Following Kostov, write the type of g_i as a partition of r, for example (d, d) indicates a divisor of the form $d[a] + d[b]$ supported at two eigenvalues a and b both with multiplicity d. Then $\vec{g}$ has type given by a "polymultiplicity vector" $PMV(\vec{g})$ which is a vector of partitions. Applying the definition of the defect we immediately see the following, due to Kostov in the paper [90] where he investigates explicitly the resulting list of cases:

Lemma 2.13. *In the domain $\delta(\vec{g}) \geq 0$, the only cases where $\dim(M_{DR}^s(\vec{g})) = 2$ are the following four:*
—*for $n = 4$, $d = r/2$ and $PMV(\vec{g}) = ((d, d), (d, d), (d, d), (d, d))$;*
—*for $n = 3$, $d = r/3$ and $PMV(\vec{g}) = ((d, d, d), (d, d, d), (d, d, d))$;*
—*for $n = 3$, $d = r/4$ and $PMV(\vec{g}) = ((2d, 2d), (d, d, d, d), (d, d, d, d))$; and*
—*for $n = 3$, $d = r/6$ and $PMV(\vec{g}) = ((3d, 3d), (2d, 2d, 2d), (d, d, d, d, d, d))$.*

Proof: Apply Lemma 2.12. Vanishing of the superdefect means that g_i is of type $(d_i, \ldots, d_i)$ for some $d_i = r/b_i$. Vanishing of the defect says $\sum_i (1/b_i) = n - 2$, and the only solutions with b_i integers ≥ 2 are

$$2 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}, \ 1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}, \ 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}, \ 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

These give the cases of the lemma.   □

Exercise 2. *Classify the possible polymultiplicity vectors for $\overline{g}$ in the cases when the moduli space has dimension 4 and 6.*

3. The diagonal configuration and its blowing up

The convolution operation comes from the diagonal configuration consisting of vertical and horizontal lines plus the diagonal. This kind of configuration is a recurring theme in Hirzebruch's work [69].

In what follows, put $Y := \mathbb{P}^1$ and $Z := \mathbb{P}^1$, and look at the product $Z \times Y$. We have the projections $\xi : Z \times Y \to Z$ and $\eta : Z \times Y \to Y$.

Fix a subset of distinct points $Q := \{q_1, \ldots, q_n\} \subset \mathbb{P}^1$ and let Q_Y or Q_Z denote this subset considered as a divisor in Y or Z respectively. Let

$$D := Z \times Q_Y \cup Z \times Q_Z \times Y \cup \Delta \subset Z \times Y.$$

be the divisor obtained by using the divisors Q_Z and Q_Y in the vertical and horizontal directions, and adding the diagonal. Denote also by η and ξ the projections

$$\xi : (Z \times Y) - D \longrightarrow Z - Q_Z, \quad \eta : (Z \times Y) - D \longrightarrow Y - Q_Y.$$

The pair $(Z \times Y, D)$ is the *diagonal configuration.*

The divisor D does not have normal crossings, indeed the diagonal meets the other components in a series of triple points. In order to obtain a variety with normal crossings compactification compatible with the projection maps, we have to blow up the diagonal configuration at these triple crossing points . Let X be the resulting variety, thus we have a birational map

$$X \to Y \times Z$$

obtained by blowing up the points $(q_1, q_1), \ldots, (q_n, q_n)$. Let $J \subset X$ denote the reduced inverse image of the divisor D. We have a decomposition

$$J = T + U_1 + \cdots + U_n + H_1 + \cdots + H_n + V_1 + \cdots + V_n,$$

where:
—T is the strict transform of the diagonal Δ;
—H_i is the horizontal strict transform of $Z \times \{q_i\}$;
—V_i is the vertical strict transform of $\{q_i\} \times Y$; and
—U_i is the exceptional divisor lying over (q_i, q_i).

These intersect as follows: each U_i meets T, H_i and V_i in three distinct points. Also H_i meets V_j for $i \neq j$. These intersections are transverse, and there are no other intersections.

Let $\xi : X \to Z$ denote the first projection. It is seen as going in the vertical direction, so it contracts $U_i + V_i$ to the point q_i and indeed $\xi^{-1}(q_i) = U_i + V_i$. Let $\eta : X \to Y$ denote the second projection going in the horizontal direction, so $\eta^{-1}(q_i) = U_i + H_i$.

The other divisor components are mapped isomorphically onto the bases of these projections:

$$\xi : T \xrightarrow{\cong} Z, \quad \xi : H_i \xrightarrow{\cong} Z$$

and

$$\eta : T \xrightarrow{\cong} Y, \quad \eta : V_i \xrightarrow{\cong} Y.$$

These divisor components intersect transversally all fibers of ξ or η respectively.

3.1. Convoluters—the Betti version.

The basic setup of Katz's convolution operation is to take a local system on Y, pull it back to $Y \times Z$ or the blow-up X, tensor with a rank one local system, and push forward to Z using $\mathbb{R}^1 \xi_*$. Obviously, the first step in understanding and calculating this, is to understand the rank one objects [25] [56] over (X, J). We look at the "Betti" case of local systems or representations of the fundamental group.

The birational blowing-up morphism is an isomorphism outside J and D, that is

$$X - J \xrightarrow{\cong} (Z \times Y) - D.$$

In particular, local systems on one or the other are the same thing. We denote generically by β our convoluter, in this case a local system. It is given by a representation of the fundamental group into $\mathbb{C}^*$, a representation which factors through the abelianization as

$$\beta : H_1(X - J, \mathbb{Z}) \to \mathbb{C}^*.$$

The loops γ_{H_i}, γ_{V_i} and γ_T going around the respectively denoted components of J, generate the first homology of $H_1(X - J)$. For our calculations it is convenient to include γ_{U_i} also as generators. These are subject to the following relations:

$$\gamma_T + \sum_{i=1}^n \gamma_{H_i} = 0, \quad \gamma_T + \sum_{i=1}^n \gamma_{V_i} = 0, \quad \gamma_{U_i} = \gamma_T + \gamma_{H_i} + \gamma_{V_i} \quad (i = 1, \ldots, n),$$

coming respectively from a vertical $\mathbb{P}^1$ intersecting T and the H_i; from a horizontal $\mathbb{P}^1$ intersecting T and the V_i; and for $i = 1, \ldots, n$ from a small $\mathcal{C}^\infty$ deformation of the exceptional U_i, which meets T, H_i and V_i and intersects the undeformed U_i negatively. That these generate the module of relations, can be seen from a Leray spectral sequence argument.

Denote by β^{H_i} the monodromy of β on the loop γ_{H_i} and similarly for the other generators. Thus, specifying a local system of rank one on $X - J$ comes down to specifying

$$\beta^{H_i}, \quad \beta^{V_i}, \quad \beta^{U_i}, \quad \beta^T \in \mathbb{C}^*,$$

subject to the relations

$$\beta^T \cdot \prod_{i=1}^n \beta^{H_i} = 1, \quad \beta^T \cdot \prod_{i=1}^n \beta^{V_i} = 1, \quad \beta^{U_i} = \beta^T \cdot \beta^{H_i} \cdot \beta^{V_i}.$$

Of course the last relations mean that β^{U_i} are redundant.

We will use our convoluters to define a convolution operation, in which the diagonal Δ plays a primordial role. It will be important to have nontrivial monodromy around the diagonal. To simplify notation set

$$\chi := \beta^T$$

and make the following convention.

Convention 3.1. *The monodromy around the diagonal of our convoluter is nontrivial, that is $\chi \neq 1$.*

3.2. **Convoluters—the de Rham version.** We will find it most convenient to restrict to convolution with rank one logarithmic connections on the trivial bundle, that is de Rham objects of the form $(\mathcal{O}_X, d + \beta)$ where β is a one form on X with logarithmic poles along J. The more general case can be viewed as being subsumed by the theory of parabolic logarithmic λ-connections [112], see §7.

In the present case, then, a convoluter is just a section

$$\beta \in H^0(X, \Omega^1_X(\log J)).$$

Given β we denote by β^{H_i}, β^{V_i}, β^{U_i} and β^T its residues along H_i, V_i, U_i and T respectively. Note that $H^1(X, \mathcal{O}_X) = 0$, so by Deligne's mixed Hodge theory

$$\int : H^0(X, \Omega_X^1(\log J)) \xrightarrow{\cong} H^1((X - J)^{\mathrm{top}}, \mathbb{C})$$

and the integrals over γ_{H_i} etc. are $2\pi\sqrt{-1}$ times the residues β^{H_i} etc. In particular, the structure of $H_1(X - J, \mathbb{Z})$ recalled in the previous subsection implies that β is determined by its residues, and these are subject to the equations

$$\beta^T + \sum_{i=1}^n \beta^{H_i} = 0, \quad \beta^T + \sum_{i=1}^n \beta^{V_i} = 0, \quad \beta^{U_i} = \beta^T + \beta^{H_i} + \beta^{V_i}.$$

The de Rham convolution object β gives rise to a local system, or Betti convoluter φ with

$$\varphi^{H_i} = e^{2\pi\sqrt{-1}\beta^{H_i}}, \dots.$$

The analogue of Convention 3.1 is:

Convention 3.2. *The residue β^T of β along the diagonal is not an integer.*

4. MIDDLE CONVOLUTION—BETTI VERSION

In this section we will work with the divisor $D \subset (Z \times Y)$, and do our computations in braid-group style [113] [132] [45]. One could alternatively use the blowing up (Z, J) and give a treatment similar to the one we will give in the de Rham case later.

The discussion of this section is the complex geometric version of Katz's construction. Katz gave a geometric definition of middle convolution in [86, 2.7-2.9]. His formulae there, stated in the context of perverse ℓ-adic sheaves, work universally in any geometric context. The complex geometric version was defined more explicitly, and first exploited by Dettweiler and Reiter [44] [45] [46], and Völklein, Strambach, [139] [132]. They write down explicit matrices but the motivation comes from braid-style computations. Kostov proposes an ingenious version of the construction which doesn't refer to the geometric picture, but instead is based on the possibility of multiplying the connection matrix by a scalar to get to the case of integer eigenvalues [91]. And, Crawley-Boevey views the construction, again in algebraic terms, as something about root systems. Boalch considered a particular example of middle convolution in a non-rigid case [15], and the link with Katz's construction was made in [47]. In [45] following [86, Chap. 5.1] it is shown that the explicit matrix definition of MC has a geometric or cohomological interpretation as a higher direct image—this is the point of view we adopt here. The braid-style calculations of group cohomology necessary to get the local form of monodromy out of this geometric definition were done in [45] but using the Pochammer basis for the group cohomology classes, rather than a standard basis as we shall use here. In spite of the numerous references on this subject, we go through the details, where possible keeping simplifying assumptions for our expository purpose.

4.1. **Definitions.** Recall that

$$\eta : Z \times Y \to Y, \quad \xi : Z \times Y \to Z$$

are the projections, and use the same name for the projections on the open subset $(Z \times Y) - D$. A *convoluter* β is a rank one local system on $(Z \times Y) - D$.

Define the *raw convolution* RC_β as follows. If L is a local system on $Y - Q_Y$ then put

$$RC_\beta(L) := R^1\xi_*(\beta \otimes (\eta^*L)).$$

It is a local system on $Z - A_Z$. The *middle convolution* $MC_\beta(L)$ will be a subsystem of $RC_\beta(L)$ the kernel of the map to some natural essentially local systems coming from triviality of certain pieces of the local monodromy transformations. It corresponds to replacing the cohomology of the fibers in the $R^1\xi_*$ construction, by the middle cohomology discussed in §2.1 above.

In order to make explicit calculations, we will adopt the viewpoint of homology rather than cohomology. Let $RC_\beta^*(L)$ denote the local system obtained by taking the homology of the fibers with coefficients in $\beta \otimes \eta^*L$. Let $MC_\beta^*(L)$ denote the quotient corresponding to "middle homology" defined by duality with middle cohomology (the precise definition will appear in our discussion below). If we let β^* and L^* denote the dual local systems, then by the duality between homology and cohomology we have

$$RC_\beta^*(L) = (RC_{\beta^*}(L^*))^*$$

and similarly for MC^*. Thus it is equivalent if we look at homology, and it is easier to visualize geometrically classes in homology with local coefficients.

4.2. Computations in group homology.

We would like to calculate the local monodromy transformations of the raw and then middle convolutions. In order to do this, we transform the question into a computation of the action of the fundamental group of the base, on the group homology of the fiber. See [139] [45] [42] for example.

In order to speak of fundamental groups, we need to choose basepoints. Choose a basepoint $b \in Y - Q$. For $z \in Z - Q_Z$ the fiber of $X - J$ over z is $Y - Q - \{z\}$. This has (z, b) as basepoint whenever $z \neq b$. In particular, in order to get a fibration of based spaces we should additionally take the point b out of the base. For this reason, put $Q_Z^b := Q_z \cup \{b\}$.

On the other hand, we would like to consider the fundamental group of $Z - Q_Z^b$. Choose another basepoint $c \in Z - Q_Z^b \subset \mathbb{P}^1 \cong Y$. In the fiber over $c \in Z$ we have the complement of Q_Y and should also take out the diagonal point (c, c). Thus, let $Q_Y^c := Q_Y \cup \{c\}$.

The fiber $Y_c = \{c\} \times (Y - Q_Y^c)$ of the projection $\xi : X - J \to Z - Q_Z^b$ over c, is an open Riemann surface pointed by the basepoint (c, b).

Let $\Gamma := \pi_1(Y_c, (c, b))) \cong \pi_1(Y - Q_Y^c, b)$. It is a free group. The fundamental group of the base $\Upsilon := \pi_1(Z - Q_Z^b, c)$ acts on Γ. We will describe the action in greater detail below. Denote the action by $u \mapsto (\gamma \mapsto \mu(u, \gamma))$ for $u \in \Upsilon$, $\gamma \in \Gamma$.

Make the convention for group composition that ab means b followed by a. That way, a monodromy representation indicates transport of sections along the path and satisfies $\rho(ab) = \rho(a)\rho(b)$.

The local system $RC_\beta^*(L)$ restricted to $Z - Q_Z^b$ can be described as follows. The local system L corresponds to a representation ρ of Γ on the vector space L_b, invariant with respect to the action of Υ in the sense that

$$\rho(\mu(u, \gamma)) = \rho(\gamma)$$

for any $\gamma \in \Gamma$ and $u \in \Upsilon$. Similarly, the local system β corresponds to a pair of characters

$$\beta^{c,\cdot} : \Gamma \to \mathbb{C}^*, \quad \beta^{\cdot,b} : \Upsilon \to \mathbb{C}^*$$

and the first of these is again invariant, $\beta^{c,\cdot}(\mu(u,\gamma)) = \beta^{c,\cdot}(\gamma)$. Tensoring together we obtain a representation denoted

$$\rho^\beta : \gamma \mapsto \beta^{c,\cdot}(\gamma)\rho(\gamma),$$

again invariant with respect to the action of Υ. The local system $RC_\beta(L)$ (resp. $RC_\beta^*(L)$) corresponds to the vector space $H^1(\Gamma, \rho^\beta)$ (resp. $H_1(\Gamma, \rho^\beta)$). The action of Υ is obtained by the natural action, tensored with the character $\beta^{\cdot,b}$. This tensorization is due to the fact that the local system β is not trivial on the basepoint section $z \mapsto (z, b)$ over $Z - Q_Z^b$.

Denote by V the vector space $\mathbb{C}^r$ on which the representation ρ^β is defined. Denote by $H^1(\Gamma, V)$ the cohomology and $H_1(\Gamma, V)$ the homology.

Fix generators for Γ as follows: we have loops $\alpha_1, \ldots, \alpha_n$ going clockwise around the points $q_1, \ldots, q_n$ in the standard way, and δ going clockwise around the point $(c, c) \in \Delta$ in the fiber $\{c\} \times (Y - Q_Y^c)$. The group Γ has generators α_i, δ subject to the single relation

$$\delta \alpha_1 \cdots \alpha_n = 1.$$

Using this relation any one of the generators could be ignored but it will be more convenient to keep all of them.

The character $\beta^{c,\cdot}$ acts on these generators as follows: $\beta^{c,\cdot}(\alpha_i) = \beta^{V_i}$, $\beta^{c,\cdot}(\delta) = \beta^T$. Thus we have

$$\rho^\beta(\delta) = \beta^T \cdot 1^r, \quad \rho^\beta(\alpha_i) = \beta^{V_i}\rho(\alpha_i), \quad i = 1, \ldots, n.$$

The homology $H_1(\Gamma, V)$ is the homology at degree one of the sequence

$$C_2(\Gamma, V) \to C_1(\Gamma, V) \to C_0(\Gamma, V) = V.$$

Furthermore, $C_1(\Gamma, V)$ is the $\mathbb{C}$-vector space formally generated by the symbols $G(\gamma, v)$ where $\gamma \in \Gamma$ and $v \in V$, subject only to the relation of $\mathbb{C}$-linearity in the variable v. And $C_2(\Gamma, V)$ is generated by symbols $Q(\gamma, \xi, v)$ where $\gamma, \xi \in \Gamma$ and $v \in V$. Geometrically, $G(\gamma, v)$ represents a cycle which starts with value v and continues along the path γ. And $Q(\gamma, \xi, v)$ represents a simplex whose sides are γ, ξ and $\xi\gamma$ with coefficient v at the starting point.

The boundary operators for the complex are

$$\partial G(\gamma, v) = \gamma(v) - v, \quad \partial Q(\gamma, \xi, v) = G(\gamma, v) + G(\xi, \gamma(v)) - G(\xi\gamma, v), \quad \partial \circ \partial = 0.$$

We work with the vector space $C_1(\Gamma, V)/\partial C_2(\Gamma, V)$ denoted just $C_1/\partial C_2$ for short. It is finite dimensional, and its elements are $\mathbb{C}$-linear combinations of classes denoted $G[\gamma, v]$ which are now subject to the relations that this symbol is $\mathbb{C}$-linear in v, and that

$$G[\xi\gamma, v] = G[\gamma, v] + G[\xi, \gamma(v)].$$

If we fix a basis $\{v_j\}$ for V then from the set of generators of Γ we obtain a basis for $C_1/\partial C_2$ consisting of the $G[\alpha_i, v_j]$ and $G[\delta, v_j]$. It will be useful in what follows to have a formula for multiple products. For example

$$G[\eta\xi\gamma, v] = G[\gamma, v] + G[\xi, \gamma(v)] + G[\eta, \xi\gamma(v)]$$

and more generally $G[\gamma_1 \cdots \gamma_m, v] = \sum_{i=1}^m G[\gamma_i, \gamma_{i+1} \cdots \gamma_m(v)]$. Similarly for the inverse, the equation

$$0 = G[\gamma\gamma^{-1}, v] = G[\gamma^{-1}, v] + G[\gamma, \gamma^{-1}(v)]$$

gives $G[\gamma^{-1}, v] = -G[\gamma, \gamma^{-1}(v)]$.

Now consider the action of Υ. What we call the "natural action" is the one coming from the action on the explicit generators written above. This corresponds to tensoring to trivialize the restriction of the local system on the basepoint section. In the end, since the local system is not trivial on the basepoint section, we will have to take the natural action tensored with the character $\beta^{\cdot,b}$.

The generator u_i of Υ corresponds to a path where the point c goes around the point q_i. There is some choice about how to arrange this picture, with respect to the picture of the standard generators of Γ. Think of the points $q_1, \ldots, q_n$ as lined up in a row, with the basepoint b off to one side so that the points are arrayed from left to right when viewed from b. The paths α_i go straight from b to q_i, once around clockwise, then back to b. On the other hand, let c be on the other side of the row of points q_i. We obtain a number of paths $\delta = \delta_1, \ldots, \delta_n$ going from b to c, around clockwise, and back to b. These are defined by saying that the starting and ending path for δ_i goes just to the left of the point q_i, for $1 < i \leq n$ the path goes between q_{i-1} and q_i. We have the relation

$$\delta_{i+1} = \alpha_i \delta_i \alpha_i^{-1}.$$

In particular the δ_i are all conjugate to $\delta = \delta_1$, which implies that $\rho^\beta(\delta_i)$ are always multiplication by β^T.

Now, define u_k as the path which sends c straight to q_k, around clockwise, and back to its starting point. This happens on the other side of our picture from the paths starting at b.

With this picture, the action of u_k doesn't change the α_j for $j \neq k$. On the other hand, we have a Dehn twist between δ_k and α_k. Notice that the introduction of the different conjugates δ_k allows us to represent these Dehn twists uniformly for each k; if we try to write down the formula with $\delta = \delta_1$ it becomes more complicated.

A geometric look at the picture of c going clockwise around q_k yields:

Proposition 4.1. *The action of u_k on Γ is given by*

$$\mu(u_k, \alpha_k) = \delta_k^{-1} \alpha_k \delta_k,$$

and

$$\mu(u_k, \delta_k) = \delta_k^{-1} \alpha_k^{-1} \delta_k \alpha_k \delta_k.$$

$\square$

Exercise 3. *Define conjugates ζ_i where the point c goes in between q_{i-1} and q_i, around b, and back. Describe the action of ζ_i. We have the relations $u_1 \cdots u_{i-1} \zeta_i u_i \cdots u_n = 1$ in Υ. Check that the action defined by the above formulae for the u_i plus the formulae for ζ_i, satisfies these relations.*

4.3. **The local monodromy transformations.** Now we would like to compute the eigenvalues of the monodromy transformations. This computation is local around one of the points q_k. For our present purposes we only ask for the Jordan normal form of the monodromy transformation. More precise information, in fact the explicit monodromy matrices with respect to the Pochammer basis, are obtained in [45], [42, Lemma 3.3.5, Proposition 3.3.6]. Our computation is along these lines but we don't need to consider the Pochammer elements.

To reduce notation put
$$\chi := \beta^T, \quad \beta_i := \beta^{H_i}.$$
The values of β^{V_i} don't matter, as we are initially calculating the natural action trivialized over the basepoint section. Our previous formulae become
$$\rho^\beta(\delta_k) = \chi, \quad \rho^\beta(\alpha_i) = \beta_i \rho(\alpha_i).$$
For the action of u_k, we have $u_k \cdot G[\alpha_i, v_j] = G[\alpha_i, v_j], \ i \neq 1$. On the other hand,
$$u_k \cdot G[\alpha_k, v_j] = G[\delta_k^{-1}\alpha_k\delta_k, v_j] = G[\delta_k^{-1}, \alpha_k\delta_k(v_j)] + G[\alpha_k, \delta_k(v_j)] + G[\delta_k, v_j]$$
$$= G[\alpha_k, \delta_k(v_j)] + G[\delta_k, v_j] - G[\delta_k, \delta_k^{-1}\alpha_k\delta_k(v_j)]$$
and using the formula for the action of δ_k which is by multiplication by χ (in particular it commutes with the α_k),
$$u_k \cdot G[\alpha_k, v_j] = \chi G[\alpha_k, v_j] + G[\delta_k, (1 - \alpha_k)(v_j)].$$
Finally,
$$u_k \cdot G[\delta_k, v_j] = G[\delta_k^{-1}\alpha_k^{-1}\delta_k\alpha_k\delta_k, v_j]$$
$$= -G[\delta_k, \delta_k^{-1}\alpha_k^{-1}\delta_k\alpha_k\delta_k(v_j)] - G[\alpha_k, \alpha_k^{-1}\delta_k\alpha_k\delta(v_j)] + G[\delta_k, \alpha_k\delta_k(v_j)] + G[\alpha_k, \delta_k(v_j)] + G[\delta_k, v_j]$$
$$= -\chi G[\delta_k, v_j] - \chi^2 G[\alpha_k, v_j] + \chi G[\delta_k, \alpha_k(v_j)] + \chi G[\alpha_k, v_j] + G[\delta_k, v_j]$$
$$= (\chi - \chi^2)G[\alpha_k, v_j] + G[\delta_k, (\chi(\alpha_k - 1) + 1)v_j].$$

Lemma 4.2. *Suppose v_j is an eigenvector of $\rho(\alpha_k)$ with eigenvalue $r_{k,j}$. Denote $\chi := \beta^T$. Then the two-dimensional subspace of $C_1/\partial C_2$ generated by $G[\alpha_k, v_j]$ and $G[\delta_k, v_j]$ is invariant under the transformation u_k, and on this subspace (with the two generators taken as basis vectors) the transformation u_k has matrix*
$$u_k|_{\langle G[\alpha_k, v_j], G[\delta_k, v_j]\rangle} = \begin{pmatrix} \chi & (\chi - \chi^2) \\ (1 - \beta_k r_{k,j}) & 1 + \chi(\beta_k r_{k,j} - 1) \end{pmatrix}.$$

Proof: In the computations above, the action of α_k is by the representation ρ^β, and v_j is an eigenvector of $\rho^\beta(\alpha_k)$ but this time with eigenvalue $\beta_k r_{k,j}$. Thus we should take the previous formulae and replace $\rho^\beta(\alpha_k)v_j$ by $\beta_k r_{k,j}v_j$, which gives the stated matrix. $\square$

Corollary 4.3. *In the situation of the previous lemma, the eigenvalues of u_k acting on the two dimensional subspace $\langle G[\alpha_k, v_j], G[\delta_k, v_j]\rangle$ are 1 and $\chi\beta_k r_{k,j}$.*

Proof: The determinant of the matrix in the lemma is

$$\chi + \chi^2(\beta_k r_{k,j} - 1) - (\chi - \chi^2)(1 - \beta_k r_{k,j}) = \chi + \chi^2\beta_k r_{k,j} - \chi^2 - \chi + \chi\beta_k r_{k,j} + \chi^2 - \chi^2\beta_k r_{k,j}$$
$$= \chi\beta_k r_{k,j}.$$

The trace is $1 + \chi\beta_k r_{k,j}$. The eigenvalues satisfy two equations which clearly hold for 1 and $\chi\beta_k r_{k,j}$. $\qquad\square$

Suppose $\chi\beta_k r_{k,j} = 1$, then the matrix in the above lemma is

$$\begin{pmatrix} \chi & (\chi - \chi^2) \\ (1 - \chi^{-1}) & 2 - \chi \end{pmatrix} = 1 + (\chi - 1)\begin{pmatrix} 1 & -\chi \\ \chi^{-1} & -1 \end{pmatrix},$$

that is 1 plus a rank one matrix whose square is zero. In this case the 2×2 matrix of Lemma 4.2 is not semisimple. Therefore, keep the following restriction on our eigenvalues.

Convention 4.4. *We assume that β has the property that $\chi\beta_k r_{k,j} \neq 1$ for all eigenvalues $r_{k,j}$ of $\rho(\alpha_k)$. In other words, the matrix $\chi\rho^\beta(\alpha_k)$ has only nontrivial eigenvalues.*

Lemma 4.5. *Suppose Convention 2.1 so $\rho(\alpha_k)$ is a semisimple matrix, and Convention 4.4 so that $\chi\beta_k r_{k,j} \neq 1$. Then the monodromy transformation of $C_1/\partial C_2$ around u_k looks up to conjugacy like*

$$\chi\rho^\beta(\alpha_k) \oplus 1^m$$

where m is given by a dimension count.

Proof: The map

$$V \oplus V \to C_1/\partial C_2, \quad (u,v) \mapsto G[\alpha_k, u] + G[\delta_k, v]$$

is injective. The action of $u_k \in \Upsilon$ preserves the image and there it acts as $\chi\rho^\beta(\alpha_k) \oplus 1^r$. Thus the multiplicity of an eigenvalue in the action of u_k is at least as big as its multiplicity in $\chi\rho^\beta(\alpha_k)$.

On the other hand, the images of the maps $V \to H_1(\Gamma, V)$ given by $v \mapsto G[\alpha_i, v]$ for $i \neq k$, span a subspace on which $u_k \in \Upsilon$ acts trivially, and with the subspace of the previous paragraph, these two subspaces generate $C_1/\partial C_2$. We obtain a surjective u_k-equivariant map from a representation of the form $\chi\rho^\beta(\alpha_k) \oplus 1^{m'}$ to $C_1/\partial C_2$. Thus, the multiplicity of a nontrivial eigenvalue in u_k is at most its multiplicity in $\chi\rho^\beta(\alpha_k)$. This surjection also shows that the action of u_k on $C_1/\partial C_2$ is semisimple.

The condition that the eigenvalues of $\chi\rho^\beta(\alpha_k)$ be all nontrivial means that the multiplicities are the same as their multiplicities in u_k. This gives the direct sum decomposition of the lemma. $\qquad\square$

Exercise 4. *Calculate the action of ζ_k. After going to the action on the homology which is the kernel of the boundary map*

$$H_1 = Z_1/\partial C_2 = \ker(C_1/\partial C_2 \to C_0 \cong V),$$

the operator ζ_k should act by multiplication by a scalar. After tensoring with $\beta^{\cdot,b}$ it should give the identity since the local system $RC_\beta^*(L)$ doesn't depend on the choice of basepoint and hence extends across $\{b\}$.

We now consider the action of Υ on $H_1 := \ker(C_1/\partial C_2 \to C_0)$.

Proposition 4.6. *Suppose that ρ is an irreducible representation of rank $r > 2$. Suppose that the monodromy transformations $\rho(\alpha_i)$ are semisimple (Convention 2.1), suppose that $\chi \neq 1$ (Convention 3.1), and suppose that the eigenvalues $r_{k,j}$ of $\rho(\alpha_k)$ are different from $\chi^{-1}\beta_k^{-1}$ (Convention 4.4). The dimension of $H_1(\Gamma, \rho^\beta)$ is equal to $(n-1)r$. The group Υ acting on $H_1(\Gamma, \rho^\beta)$ by the raw convolution representation has the following effect on the generators:*

$$u_k \mapsto \beta^{V_k} \otimes (\chi \cdot \rho^\beta(\alpha_k) \oplus 1^{(n-2)r}).$$

Proof: The conditions of Proposition 2.4 apply even to the dual local system, therefore $H_0 = 0$. In particular, the boundary map is surjective onto C_0. Recall that $H_2 = 0$ because we are looking at an open curve. The Euler characteristic of the complement of $n + 1$ points is $(n - 1)$ which gives $\dim H_1 = (n - 1)r$.

On the other hand, the natural action of Υ on $C_0 \cong V$ is trivial. Thus, for the monodromy transformations of the u_k, the kernel H_1 of the boundary map contains all of the nontrivial part. For this action, using the dimension count and Lemma 4.5, the matrix of the action of u_k is $\chi \cdot \rho^\beta(\alpha_k) \oplus 1^{(n-2)r}$. As pointed out at the start of the computation, we then have to tensor with the character $\beta^{\cdot,b}$ to get the representation corresponding to the raw convolution. $\square$

4.4. **Middling.** Suppose that ρ^β has some eigenvalues equal to 1 around a point q_i. The loop around that point, with the eigenvector as coefficient, gives a cycle in $H_1(\Gamma, \rho^\beta)$ which will be covariant under Υ. Going to the middle convolution, replacing H_1 by the middle version, gets rid of these invariant cycles.

We are assuming that $\chi \neq 1$ so this behavior doesn't occur at the point c, and in particular the point c serves as a point where there are no fixed vector so we can apply Lemma 2.6 above.

Let $F_i \subset L_b$ denote the subspace of vectors fixed by the monodromy transformation $\rho^\beta(\alpha_i)$. Since we are assuming that the local monodromy transformations are unipotent, the dimension of F_i is equal to the multiplicity of 1 as eigenvalue of ρ^β. We get a map

$$\phi_i : F_i \to H_1(\Gamma, \rho^\beta), \quad \phi_i(f) := G[\alpha_i, f].$$

These put together to give $\phi : \bigoplus_{i=1}^n F_i \to H_1(\Gamma, \rho^\beta)$. Recall that

$$MH_1(\Gamma, \rho^\beta) := \frac{H_1(\Gamma, \rho^\beta)}{\phi \bigoplus_{i=1}^k F_i}.$$

The group Υ acts on $MH_1(\Gamma, \rho^\beta)$. As before, we can calculate with the natural action trivialized on the basepoint section, which should then be tensored with the character $\beta^{\cdot,b}$ to obtain the middle coconvolution $MC_\beta^*(L)$.

Lemma 4.7. *The map ϕ is equivariant for the natural action on the target, and with u_k acting trivially on F_i for $i \neq k$, and by multiplication by χ on F_k. Also, ϕ is injective.*

Proof: From the previous subsection, the action of u_k preserves $G[\alpha_i, f]$ for $i \neq k$. If f is a fixed vector for $\rho^\beta(\alpha_k)$ then it is an eigenvector with eigenvalue $\beta_k r_{k,j} = 1$. In the matrix of Lemma 4.2, we get that the image of $G[\alpha_k, f]$ is $\chi G[\alpha_k, f]$. Injectivity of ϕ follows from Lemma 2.6. $\square$

This lemma leads to the computation of the monodromy action of u_k on the middle homology. Write $V = V' \oplus V''$ where $\rho(\alpha_k)$ acts with eigenvalue β_k^{-1} on V', and with eigenvalues distinct from β_k^{-1} on V''. Thus, for $\rho^\beta(\alpha_k)$ the fixed subspace is V' with its complement V''. Recall 4.6 that before tensoring with $\beta^{\cdot,b}$, the natural action of u_k on $H_1(\Gamma, \rho^\beta)$ decomposes as $\chi\rho^\beta(\alpha_k) \oplus 1^{(n-2)r}$. The underlying vector space decomposes as $V' \oplus V'' \oplus \mathbb{C}^{(n-2)r}$, and u_k acts by χ on V', by eigenvalues different from χ on V'', and trivially on $\mathbb{C}^{(n-2)r}$.

Convention 4.4 says that if $r_{k,j}$ is an eigenvalue of $\rho(\alpha_k)$ then $\chi\beta_k r_{k,j} \neq 1$. The $\beta_k r_{k,j}$ are the eigenvalues of $\rho^\beta(\alpha_k)$. This condition therefore says that the eigenvalues of u_k on V'' are different from 1. Convention 3.1 says that $\chi \neq 1$. Therefore the three subspaces in the above decomposition of $H_1(\Gamma, \rho^\beta)$ are distinguished by the eigenvalues of u_k.

Lemma 4.7 now implies that ϕ sends F_i into the part $\mathbb{C}^{(n-2)r}$, and sends F_k into the part V'. On the other hand, V' is the space of fixed vectors of u_k, isomorphic (and indeed, equal) to F_k. Therefore, in the middle homology there is no remaining eigenspace for χ, the term V'' remains intact, and the trivial eigenspace is reduced by an appropriate amount, to a size given by the dimension count. We can state this as follows.

Proposition 4.8. *Assume Conventions 2.1, 3.1, and 4.4. The action of $u_k \in \Upsilon$ on the middle homology is semisimple, and we have a decomposition*

$$MH_1(\Gamma, \rho^\beta) = V'' \oplus \mathbb{C}^m,$$

where V'' is the direct sum of all eigenspaces of $\rho(\alpha_k)$ for eigenvalues different from β_k^{-1}, and m is given by a dimension count. The natural action of u_k on V'' is by $\chi\beta_k\rho(\alpha_k)$, and the natural action on $\mathbb{C}^m$ is trivial. The middle convolution action is obtained by multiplying everything by β^{V_k}.

$\square$

To put this another way, suppose a is an eigenvalue of $\rho(\alpha_k)$ of multiplicity $m_k(a)$. Then the corresponding eigenvalue of the action of u_k on the middle convolution is:
—$\beta^{V_k}\chi\beta^{H_k}a = \beta^{U_k}a$ with the same multiplicity $m_k(a)$ when $\beta^{H_k}a \neq 1$; or
—β^{V_k} with multiplicity $m_k(a) + \dim MH_1 - r$ when $a = (\beta^{H_k})^{-1}$.
This is seen by recalling that $\beta_k := \beta^{H_k}$, $\chi := \beta^T$ and $\beta^{U_k} = \beta^{V_k}\beta^T\beta^{H_k}$.

Exercise 5. *The product of all the eigenvalues for all singular points must be 1. As a reality-check, see that this remains true for the middle convolution with the above formulae.*

4.5. **The Katz transformation on the level of local monodromy.** We create some notation for describing the effect of the middle convolution operation on local monodromy. Let $\mathbb{L}$ denote an abelian group with the group law written multiplicatively. Define $Div(\mathbb{L})$ to be the free abelian group generated by points of $\mathbb{L}$. An element of $Div(\mathbb{L})$ is thus a finite linear combination $g = \sum_{\alpha \in \mathbb{L}} m(\alpha) \cdot [\alpha]$ with $m(\alpha) \in \mathbb{Z}$ and $m(\alpha) = 0$ for almost all $\alpha \in \mathbb{L}$. The divisor is *effective* if all the coefficients are positive $m(\alpha) \geq 0$.

The elements of $\mathbb{L}$ are thought of as representing possible eigenvalues, and elements of $Div(\mathbb{L})$ represent conjugacy classes of semisimple matrices with these eigenvalues. The cases of interest are $\mathbb{L} = \mathbb{G}_m$, which applies to the Betti case of the present chapter, and $\mathbb{L} = \mathbb{A}^1$ which will apply for the de Rham case in the next chapter.

Since we are restricting in this paper to the case of semisimple local monodromy, we can use the simpler $Div(\mathbb{L})$ rather than the set of all Jordan normal forms [86] [91] [34] [35] [123] etc. The rank of the matrix is the degree of the divisor, that is the sum of the coefficients $m(\alpha)$. Denote this by $|g|$. Define the *determinant* to be $det(g) := \prod_{\alpha \in \mathbb{L}} \alpha^{m(\alpha)} \in \mathbb{L}$, well defined since almost all factors are the identity element $1_{\mathbb{L}}$. For obvious reasons when the operation of $\mathbb{L}$ is conventionally denoted additively we write $Tr(g)$ rather than $det(g)$.

Fix n. A *local monodromy vector* is an n-tuple of elements of $Div(\mathbb{L})$, denoted

$$\vec{g} = (g_1, \ldots, g_n) \in Div(\mathbb{L})^n,$$

such that the degrees are the same, $|g_1| = \ldots = |g_n|$. Denote this common degree by $r(\vec{g})$ and call it the *rank* of $\vec{g}$ because it will correspond to the rank of the local system. Define the *total determinant* to be the product

$$Det(\vec{g}) := det(g_1) \cdots det(g_n).$$

In order to be a candidate for the local monodromy vector of a local system, we must have $Det(\vec{g}) = 1$.

A *convoluter* is a function

$$\beta : H_1(X - J, \mathbb{Z}) \to \mathbb{L}$$

which, in view of the generators and relations for $H_1(X - J, \mathbb{Z})$, can be thought of as a vector

$$\beta = (\beta^{H_1}, \ldots, \beta^{V_1}, \ldots, \beta^{U_1}, \ldots, \beta^T) \in \mathbb{L}^{3n+1}$$

subject to the relations

$$\beta^{H_1} \cdots \beta^{H_n} \cdot \beta^T = 1, \quad \beta^{V_1} \cdots \beta^{V_n} \cdot \beta^T = 1, \quad \beta^{U_i} = \beta^{H_i} \beta^{V_i} \beta^T \ (1 \le i \le n).$$

As pointed out above in the Betti (§3.1) and de Rham (§3.2) cases, a convoluter contains the data necessary for defining a rank one object on X, which will also be denoted by β. The coefficients correspond to the local monodromy around the divisors V_i, H_i, U_i and the diagonal T respectively. In this picture the group $\mathbb{L}$ is the group of possible local monodromy for rank one objects, which depends on what kind of object we are considering.

The *Katz operation on semisimple local monodromy* assigns to a local monodromy vector $\vec{g}$ and a convoluter β for the same number of points n, a new local monodromy vector $\kappa(\beta, \vec{g})$. This is defined concretely as follows.

Define the *defect* $\delta(\beta, \vec{g})$, which is going to be the difference between the rank of the original local system, and the rank of the new local system obtained by middle convolution. Write out the coefficients

$$g_i = \sum_\alpha m_i(\alpha) \cdot [\alpha],$$

where for clarity we denote by $[a]$ the point $a \in \mathbb{L}$ considered as a divisor. The defect is defined as

$$\delta(\beta, \vec{g}) := (n-2)r - \sum_{i=1}^n m(\beta^{H_i, -1}).$$

Corollary 2.5, applied to the divisor $K := Q \cup \{c\}$ with $n + 1$ points, says that

$$(4.1) \qquad\qquad \dim MH^1(\Gamma, \rho^\beta) = r + \delta(\beta, \vec{g}).$$

If no term β is specified, it means to choose any β such that $\beta^{H_i, -1} := (\beta^{H_i})^{-1} \in \mathbb{L}$ is a point of maximal multiplicity for g_i, the resulting $\delta(\vec{g})$ obviously doesn't depend on which choice is made. This is the same formula as considered in §2.7.

Define the *local Katz transformation* at the point q_i by

$$\kappa_i(\beta, \vec{g}) := (m_i(\beta^{H_i, -1}) + \delta(\beta, \vec{g})) \cdot [\beta^{V_i}] + \sum_{\alpha\beta^{H_i} \neq 1} m_i(\alpha) \cdot [\alpha\beta^{U_i}].$$

The *global Katz transformation* is defined by

$$\vec{\kappa}(\beta, \vec{g}) := \left(\kappa_1(\beta, \vec{g}), \ldots, \kappa_n(\beta, \vec{g}) \right).$$

Scholium 4.9. *Suppose ρ is a representation of rank r on $Y - Q_Y$ satisfying Convention 2.1 that the local monodromy transformations are semisimple. Suppose β is a convoluter, a rank one local system on $X - J$. Assume that β satisfies Convention 3.1 that $\chi \neq 1$, and that Convention 4.4 holds: $\chi\rho^\beta(\alpha_k)$ have no trivial eigenvectors.*

Let $\vec{g} \in Div(\mathbb{G}_m)^n$ denote the vector of local monodromy data for ρ, and define the defect $\delta(\beta, \vec{g})$ as above.

Under these conditions, the middle coconvolution $MC^*_\beta(\rho)$ and the middle convolution $MC_\beta(\rho)$ are local systems on $Z - Q_Z \cong Y - Q_Y$ of rank

$$r' = r + \delta(\beta, \vec{g}),$$

whose local monodromy transformations are semisimple and have local monodromy types given by the Katz transformation

$$\vec{\kappa}(\beta, \vec{g}).$$

Proof: We have done the computations for the middle coconvolution in the previous subsection. The same is true for the middle convolution by Poincaré-Verdier duality. The change in ranks is formula (4.1) above, which makes the defect appear in the multiplicity of the new eigenvalue as described after Proposition 4.8. $\qquad\qquad\square$

4.6. The Katz morphism on Betti moduli spaces.
This construction extends to giving a morphism on the level of moduli spaces:

Theorem 4.10. *Let $M_B(\mathbb{P}^1, Q; \vec{g})$ denote the Betti moduli space of local systems on $\mathbb{P}^1 - Q$ having semisimple local monodromy transformations corresponding to $\vec{g}$. Suppose β is a rank one local system on $(Z \times Y) - D$. Suppose that $(\beta, \vec{g})$ satisfy Conventions 3.1 and 4.4. Then the middle convolution construction $L \mapsto MC_\beta(L)$ gives a morphism of moduli spaces*

$$MC_\beta : M_B(\mathbb{P}^1, Q; \vec{g}) \rightarrow M_B(\mathbb{P}^1, Q; \vec{\kappa}(\beta, \vec{g})).$$

This is sort of obvious, although technically speaking it requires some work: we should carry out the middle convolution construction in the context of local systems of modules over a ring. The fact that the H^0 and H^2 terms vanish, so the dimension of H^1 never jumps, is the basic thing which makes it work. Notice that our Conventions 3.1 and 4.4 are only conditions on $\beta, \vec{g}$, in particular they don't require defining open subsets of the moduli spaces.

This type of morphism between moduli spaces was considered in [35] and other places. It is clearly related to the theory of representations of the braid group such as the Burau representation, see [98] [105].

4.7. Involutivity.
One of the main properties of Katz's construction is its involutivity. This implies that it gives an isomorphism of moduli spaces. The involutivity is basic to the constitution of an algorithm: one can go forward to see if a local system with transformed local monodromy data should exist, and if one is found then one can go backward to give back a local system with the original local monodromy data.

Katz shows associativity of the convolution operator which allows him to deduce involutivity [86, 2.9.7]. Later proofs were also given in the algebraic setting by Völklein, Dettweiler-Reiter, and Crawley-Boevey and Shaw.

Katz's proof didn't rely on the Fourier transform interpretation, which nevertheless furnishes a conceptual reason for involutivity: convolution can be interpreted as a composition of two Fourier transform operators using also tensor products with rank one systems. The Fourier transform is involutive by analogy with classical real analysis, so its composition two times and also with tensoring by an invertible rank one system, is involutive with an appropriate change of convoluter as described below.

It would be interesting to use connections with irregular singularities, and "wild" harmonic theory, to make this argument precise in the complex geometric setting. This would involve Bloch-Esnault [13], Sabbah [124] and Szabo [133]. See also [15] and the recent preprint [67]. Very recently Aker and Szabo have contructed an involutive Nahm transform for parabolic Higgs bundles [3] which should lead to a complex analytic version of the Fourier transform construction.

For the middle convolution operation, involutivity can already be seen on the level of local monodromy data.

Proposition 4.11. *Let $c : X \to X$ be the automorphism which flips the factors and let β^* be the dual local system whose monodromy transformations are the inverses. Then*

$$\vec{\kappa}(c^*\beta^*, \vec{\kappa}(\beta, \vec{g})) = \vec{g}.$$

Proof: We will be making changes of variables in the sums, so it is convenient to have the following formula for the Katz transformation in terms of $g_i = \sum_\alpha m_i(\alpha)[\alpha]$ and the defect $d := \delta(\beta, \vec{g})$:

$$\kappa_i(\beta, \vec{g}) = (m_i(\beta^{H_i, -1}) + d)[\beta^{V_i}] - m_i(\beta^{H_i, -1})[\beta^{V_i}\beta^T] + \sum_\alpha m_i(\alpha)[\alpha\beta^{U_i}].$$

Put $\gamma := c^*\beta^*$. In particular we have $\gamma^{H_i} = \beta^{V_i,-1}$, $\gamma^{V_i} = \beta^{H_i,-1}$, $\gamma^{U_i} = \beta^{U_i,-1}$, and $\gamma^T = \beta^{T,-1}$. Write $\vec{g}' = (g'_1, \ldots, g'_n) := \vec{\kappa}(\beta, \vec{g})$ and let r' be the rank, $m'(\alpha)$ the multiplicities and d' the defect with respect to γ. One calculates that $d' = -d$ so the defects cancel out and at least on the level of ranks we have $\mathrm{rk}(\vec{\kappa}(\gamma, \vec{g}')) = r$. We can write

$$\kappa_i(\gamma, \vec{g}') = (m'_i(\gamma^{H_i,-1}) + d')[\gamma^{V_i}] - (m'_i(\gamma^{H_i,-1}))[\gamma^{V_i}\gamma^T] + \sum_{\alpha'} m'_i(\alpha')[\alpha'\gamma^{U_i}]$$

$$= m_i(\beta^{H_i,-1})[\beta^{H_i,-1}] - (m_i(\beta^{H_i,-1}) + d)[\beta^{V_i}\beta^{U_i,-1}] + \sum_{\alpha} m'_i(\alpha')[\alpha\beta^{U_i,-1}].$$

The sum in the last term amounts to looking at g'_i but translated by $\beta^{U_i,-1}$, in other words

$$\sum_{\alpha} m'_i(\alpha')[\alpha\beta^{U_i,-1}] =$$

$$(m_i(\beta^{H_i,-1}) + d)[\beta^{V_i}\beta^{U_i,-1}] - m_i(\beta^{H_i,-1})[\beta^{V_i}\beta^T\beta^{U_i,-1}] + \sum_{\alpha} m_i(\alpha)[\alpha\beta^{U_i}\beta^{U_i,-1}].$$

After some textual cancellation, our full expression becomes

$$\kappa_i(\gamma, \vec{g}') = m_i(\beta^{H_i,-1})[\beta^{H_i,-1}] - m_i(\beta^{H_i,-1})[\beta^{V_i}\beta^T\beta^{U_i,-1}] + \sum_{\alpha} m_i(\alpha)[\alpha] = g_i.$$

This completes the proof.   □

Katz has also shown by direct calculation that the virtual dimensions of the moduli spaces for $\vec{g}$ and $\vec{\kappa}(\beta, \vec{g})$ are the same.

Finally, we state the involutivity of the middle convolution morphism itself. We have seen the involutivity on the level of local monodromy data, so it makes sense to look at the composition of the middle convolution morphisms.

Theorem 4.12. *The composition*

$$M_{DR}(P, Q; \vec{g}) \overset{MC(\beta)}{\longrightarrow} M_{DR}(P, Q; \kappa(\beta, \vec{g})) \overset{MC(c^*\beta^*)}{\longrightarrow} M_{DR}(P, Q; \vec{g})$$

is the identity, if we are in the situation of Theorem 4.10 for both of the morphisms.

We don't describe the proof here but refer to Katz [86], Völklein [139], Dettweiler-Reiter [44], and more recently Crawley-Boevey and Shaw [35].

4.8. **Detecting emptiness of the moduli space.** One of the main features of Katz's construction is that it permits us to detect whether a given moduli space is empty or not in terms of the next moduli space in the algorithm. In other words,

$$M_B(\vec{g}) = \emptyset \Leftrightarrow M_B(\kappa(\beta, \vec{g})) = \emptyset$$

assuming Conventions 3.1 and 4.4.

This is specially the case when $M_B(\kappa(\beta, \vec{g}))$ is not even defined because one of the divisors in the vector $\kappa(\beta, \vec{g})$ is no longer effective. It is comforting to work this case out explicitly. Let $d = \delta(\beta, \vec{g})$ denote the defect. The multiplicities in the local divisors $\kappa_i(\beta, d, g_i)$ are either the same as in g_i, or else they are changed by adding d. In particular, if $d \geq 0$ then

we will never get to a noneffective divisor. Thus we may assume that $d < 0$. Suppose that $\kappa_i(\beta, d, g_i)$ becomes noneffective. The only multiplicity which changes is $m_i(\beta^{H_i, -1})$, which becomes

$$m_i(\beta^{H_i, -1}) + d.$$

In particular, we are in the current situation, only if

$$m_i(\beta^{H_i, -1}) + d < 0.$$

Plugging in the formula for the defect, we have

$$m_i(\beta_{H_i}^{-1}) + r(n - 2) - \sum_{j=1}^{n} m_j(\beta^{H_i, -1}) < 0,$$

and adding r to both sides and simplifying we get

$$\sum_{j \neq i} (r - m_j(\beta^{H_i, -1})) < r.$$

This says that the sum for $j \neq i$ of the ranks of the matrices $\rho^\beta(\alpha_j) - 1$ is $< r$. Since these matrices generate the action of the group algebra on the vector space V, under this condition the action cannot be irreducible. So, ρ^β and hence ρ is not irreducible. Thus, we have the following lemma.

Lemma 4.13. *Suppose that $\vec{g}$ consists of effective divisors, and at least one of the divisors in $\kappa(\beta, \vec{g})$ is not effective. In this case, the representation ρ cannot be irreducible. In the case where $\vec{g}$ is automatically irreducible, this means that the moduli space $M_B(\vec{g})$ is empty.*

$\square$

4.9. **Running Katz's algorithm (Kostov's program).** Kostov invented the protocol of applying Katz's algorithm to the nonrigid case. Suppose $\vec{g}$ is a local monodromy vector. Choose a convoluter β so that $(\beta^{H_i})^{-1}$ is an eigenvalue of maximal multiplicity for g_i. Thus $\delta(\beta, \vec{g}) = \delta(\vec{g})$. If $\delta(\vec{g}) < 0$ and if the pair $(\beta, \vec{g})$ satisfies Conventions 3.1 and 4.4, then we obtain an isomorphism of moduli spaces for $\vec{g}$ and the Katz-transformed vector $\vec{\kappa}(\beta, \vec{g})$. The rank strictly decreases, so we can keep going on in the same way, until we get to $r = 1$ or more generally to a case where all of the local monodromy matrices are diagonal; to an impossibility result; to the problem discussed in the subsequent paragraph below; or until we get into the range $\delta \geq 0$. If we hit an impossibility result anywhere along the way, then the original moduli space was empty. If we hit $r = 1$ then the original moduli space was a point. If we get into the range $\delta \geq 0$ then according to Kostov we expect that the moduli space should be nonempty, with a direct construction of some points [91] [92]. Crawley-Boevey and Shaw [35] gave a different construction covering cases not treated in [91] [92], and prove in some cases that the moduli space is a complete intersection. We will discuss a Higgs-bundle version of the direct construction in §6.3 below.

The problem with the previous paragraph is that somewhere along the way, we might hit a vector $\vec{g}$ for which every choice of β corresponding to maximal multiplicities, dissatisfies either Convention 3.1 or Convention 4.4. In this case the algorithm no longer makes sense as

we have described it. Apparently it can be made to work anyway, but this goes beyond the scope of the present discussion and we refer to the papers of Kostov and Crawley-Boevey. Instead, we will just point out that it doesn't happen if the original eigenvalues are sufficiently general.

In Kostov's notation, a "nongenericity relation" is a subset of the eigenvalues counted with multiplicities, of the same rank $r' \in 1, \ldots, r-1$ at each point q_i, such that the product of them all is 1. Any nontrivial sub-local system has monodromy sub-data which give a nongenericity relation.

Kostov says that a monodromy data vector $\vec{g}$ is *1-generic* if there is no nongenericity relation of rank $r' = 1$. This is the same as saying that there is no equation $a_1 \cdots a_n = 1$ such that a_i is an eigenvalue of g_i.

Lemma 4.14. *Suppose $\vec{g}$ is 1-generic, and suppose β is a convoluter such that each $(\beta^{H_i})^{-1}$ is an eigenvalue of g_i. Then the pair $(\beta, \vec{g})$ satisfies Conventions 3.1 and 4.4 and we get a middle convolution isomorphism between moduli spaces.*

Proof: It is trivial that the pair satisfies the conditions. In order to get an isomorphism we also need to have the same conditions for the inverse pair $(c^*\beta^*, \vec{\kappa}(\beta, \vec{g}))$. Convention 3.1 for $c^*\beta^*$ is equivalent to Convention 3.1 for β. For Convention 4.4 note that the eigenvalues of $\kappa_i(\beta, \vec{g})$ are either $\varphi = \beta^{U_i}\alpha$ for eigenvalues α of g_i with $\beta^{H_i}\alpha \neq 1$, or else $\varphi = \beta^{V_i}$. Convention 4.4 for the inverse pair thus requires for these φ

$$(c^*\beta^*)^T(c^*\beta^*)^{H_i}\varphi \neq 1.$$

Recalling that $(c^*\beta^*)^T(c^*\beta^*)^{H_i} = \beta^{T,-1}\beta^{V_i,-1}$, the condition becomes

$$\beta^{T,-1}\beta^{V_i,-1} \cdot \beta^{U_i}\alpha \neq 1, \quad \text{for } \beta^{H_i}\alpha \neq 1,$$

$$\beta^{T,-1}\beta^{V_i,-1} \cdot \beta^{V_i} \neq 1.$$

The first is verified by tautology and the second is Convention 3.1. □

In Katz's original rigid case, a nongenericity relation among eigenvalues of highest multiplicity automatically causes the local system to become reducible, and meeting such a nongenericity relation anywhere along the way rules out existence of any irreducible rigid local system. I would like to thank the referee for pointing out the following very interesting example, which shows that there can be a nongenericity relation among other eigenvalues, even for an irreducible rigid local system. The example consists of a local system of rank 3 with 3 singular points having local monodromy eigenvalues $(a, b, c), (u, v, w), (g, h, h)$. It is rigid, and exists even with a nongenericity relation of the form $aug = 1$. If there is no other nongenericity relation then the local system cannot be reducible (by looking at the block of size 2). One can construct this system by convolution of a hypergeometric system $(a', b'), (u', v'), (g', h')$ with a convoluter having $\beta^{H_i} = \beta^{V_i} = x, y$, or $z = (h')^{-1}$ ($i = 1, 2,$ or 3). As an exercise in applying the Katz transformation, the convoluted system is

$$\begin{pmatrix} a'xy^{-1}z^{-1} \\ b'xy^{-1}z^{-1} \\ x \end{pmatrix}, \quad \begin{pmatrix} u'yx^{-1}z^{-1} \\ v'yx^{-1}z^{-1} \\ y \end{pmatrix}, \quad \begin{pmatrix} g'zx^{-1}y^{-1} \\ z \\ z \end{pmatrix}.$$

Thus $aug = a'xy^{-1}z^{-1}u'yx^{-1}z^{-1}g'zx^{-1}y^{-1} = a'u'g'x^{-1}y^{-1}z^{-1}$ can be equal to 1 by an appropriate choice of x, y, z.

For nonrigid local systems the situation is even less clear and we will be happy with the following result.

Proposition 4.15. *Fix Kostov's polymultiplicity vector (PMV) [89] etc. containing the multiplicities of eigenvalues in the divisors g_i. The variety of all possible $\vec{g}$ with this polymultiplicity vector, is a disjoint union of d connected open subsets of tori, where d is the pgcd of all the multiplicities in $\vec{g}$. If $\vec{g}$ is a sufficiently general point in any of these connected components, then we can run Katz's algorithm until we hit either an empty moduli space for the reason discussed in §4.8, or the case of all diagonal local monodromy (i.e. rank one system tensored with $\mathbb{C}^r$), or the case $\delta \geq 0$ which will be discussed in §6.3 below. The monodromy vectors encountered along the way are always themselves general points, in particular they are 1-generic.*

Proof: Invertibility of the transformation on local monodromy data (Lemma 4.11) plus its continuity with respect to the eigenvalue parameters if the PMV is fixed, imply that for $\vec{g}$ general in its connected component, and β general in the variety of possible choices given that the β^{H_i} come from $\vec{g}$ (that is, general among the possible choices of β^{V_i}), the resulting $\vec{\kappa}(\beta, \vec{g})$ is again general in its connected component. Thus, formally applying a sequence of Katz transformations as for the algorithm, we encounter only general local monodromy vectors.

If the PMV is not simple, that is if the pgcd of all the multiplicities is $d \geq 2$, then there can exist a nongenericity relation even for general $\vec{g}$. However, the nongenericity relation is always of rank at least r/d, and the case $d = r$ is the degenerate one with only diagonal matrices. Thus, a general $\vec{g}$ in any connected component is always 1-generic, except in the degenerate diagonal case. $\square$

In case of a non-simple PMV, the variety in the previous proposition has some components where there is a nongenericity relation. If the moduli space has dimension 2, when we get to $\delta = 0$ and $\sigma = 0$ Kostov shows in [90] that all local systems are reducible for the nongeneric components. The case of dimension 2 is somewhat special and is not covered by our construction in §6.3.

Roberts studies the geographical implications of Katz's algorithm in the rigid case [123], and it would be good to extend his results to the nonrigid case.

5. Middle convolution—the de Rham version

The de Rham version involves replacing local systems by logarithmic connections [116] [13] [75]. Middle convolution in the "Fuchsian" case of connections on the trivial bundle has been extensively considered [89] etc., [63] [64] [45] [46] [55] [33]. In our treatment we don't distinguish between trivial and nontrivial underlying bundles, so in a certain sense we consider less information than these references, on the other hand our approach places things in an abstract setting.

In order to use the logarithmic de Rham complex, it is essential to have a morphism between smooth projective varieties with normal crossings divisors. Thus we use the blowing-up X with its divisor $J \subset X$ described in §3. The second projection gives a map $\xi : (X, J) \to (Z, Q)$ in good position, meaning that the inverse image of Q is the divisor $U + V \subset J$ which has normal crossings.

For a vector bundle with logarithmic connection (E, ∇) on (Y, Q_Y) and a de Rham convoluter $\beta \in H^0(X, \Omega_X^1(\log J))$, define a vector bundle with logarithmic connection on X:

$$(F, \nabla_F) := \eta^*(E, \nabla) \otimes (\mathcal{O}_X, d + \beta).$$

The divisor $HT := H_1 + \ldots + H_n + T \subset J$ is transverse to the fibers of ξ. In a relative version of the discussion of §, 2.3 we can define the middle relative de Rham complex with respect to HT, by the exact sequence

(5.1) $0 \to MDR(X/Z, F; HT) \to DR(X/Y, F) \to \mathcal{F}_{HT/Z}^0[-1] \to 0.$

For $z \in Z$, denote by X_z the fiber of ξ over z, and let

$$MDR(X_z, F; HT_z) := MDR(X/Z, F; HT)|_{X_z}$$

with similar notation for the full de Rham complex. Over points $z \in Z - Q$ this is the same thing as the middle de Rham complex for $(X_z, HT_z) \cong (Y, Q + \{z\})$ considered in §2.3. In order to have a good base-change theory, we impose the following.

Convention 5.1. *For every $z \in Z$, the degree 0 and 2 hypercohomology groups of the restriction $MDR(X_z, F; HT_z)$ vanish.*

This condition implies that $\mathbb{R}^1\xi_* MDR(X/Z, F; HT)$ is locally free over Z with fiber over a point z equal to $\mathbb{H}^1 MDR(X_z, F; HT_z)$. It has a logarithmic Gauss-Manin connection denoted by $\nabla_{GM,\mathrm{mid}}$, and we define the *de Rham middle convolution* as

$$MC_\beta(E, \nabla) := (\mathbb{R}^1\xi_* MDR(X/Z, F; HT), \nabla_{GM,\mathrm{mid}}),$$

a vector bundle with logarithmic connection on (Z, Q_Z).

The restriction of the quotient term in (5.1) to a point $z \in Z$ is just a skyscraper sheaf placed in cohomological degree 1, so it has no $\mathbb{H}^0$ or $\mathbb{H}^2$. The long exact sequence for the higher derived direct image of the exact sequence (5.1) therefore gives the following.

Lemma 5.2. *Suppose that Convention 5.1 holds. Then the same vanishing holds for the full de Rham complex, the $\mathbb{R}^1\xi_* DR(X/Z, F)$ is again a vector bundle compatible with base change, and we have a short exact sequence*

(5.2) $0 \to \mathbb{R}^1 f_* MDR(X/Z, F; HT) \to \mathbb{R}^1 f_* DR(X/Z, F) \to \mathbb{R}^0 f_*(\mathcal{F}_{HT/Z}^0) \to 0.$

This short exact sequence is compatible with the Gauss-Manin connections $\nabla_{GM,\mathrm{mid}}$ on the left and ∇_{GM} in the middle.

$\square$

The classical definition of the Gauss-Manin connection is as the connecting map for the short exact sequence of complexes

(5.3) $0 \to DR(X/Z, F) \otimes \xi^*\Omega_Z^1(\log Q)[-1] \to DR(X, F) \to DR(X/Z, F) \to 0.$

When $q \in Q$ is a singular point, the de Rham complex $DR(X, F)$, which by convention means the logarithmic de Rham complex with respect to J, can be restricted to a complex $DR(X, F)|_{X_q}$ on the fiber $X_q \subset J$. We obtain a restriction of (5.3) to X_q. Note that $\Omega_Z^1(\log Q)_q \cong \mathbb{C}$ and the residue of ∇_{GM} at q is the endomorphism

$$\mathbb{H}^1 DR(X_q, F) \to \mathbb{H}^2(DR(X_q, F) \otimes_{\mathbb{C}} \Omega_Z^1(\log Q)_q[-1]) = \mathbb{H}^1 DR(X_q, F)$$

induced by the connecting map for the restriction of (5.3).

The expression as a connecting map is not very convenient for calculating the eigenvalues. The calculation was done by Katz in [84] (thanks to H. Esnault for pointing out this reference). Without going through all of the details, here is the conclusion. In our case, $q = q_i$ for some $i = 1, \ldots, n$, and the singular fiber X_q consists of two components $X_q = U_i \cup V_i$ meeting in a point $w_i := U_i \cap V_i$. We have a short exact sequence

(5.4) $$0 \to DR(U_i, F_{U_i}(-w_i)) \to DR(X_q, F_{X_q}) \to DR(V_i, F_{V_i}) \to 0.$$

Note that HT meets X_q in a collection of smooth points distinct from the crossing point w_i. Thus the exact sequence defining the middle de Rham complex is compatible with (5.4), and we have the same short exact sequence for middle de Rham complexes
(5.5)
$$0 \to MDR(U_i, F_{U_i}(-w_i), HT_{U_i}) \to MDR(X_q, F_{X_q}, HT_{X_q}) \to MDR(V_i, F_{V_i}, HT_{V_i}) \to 0.$$

We refine Convention 5.1 to apply to each of the components:

Convention 5.3. *For $j = 0, 2$ we require that*

$$\mathbb{H}^j MDR(U_i, F_{U_i}(-w_i), HT_{U_i}) = 0, \quad \mathbb{H}^j MDR(V_i, F_{V_i}, HT_{V_i}) = 0.$$

Assuming Convention 3.2, this condition for all the q_i implies Convention 5.1. For points $z \in Z - Q$, Lemma 2.8 provides the required vanishing.

Proposition 5.4. *Assuming Convention 5.3, we get a short exact sequence from (5.5) on the level of $\mathbb{H}^1$. The residues of ∇_F along U_i and V_i give endomorphisms of F_{U_i} and F_{V_i}. These fit into a diagram*

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{H}^1 MDR(U_i, F_{U_i}(-w_i)) & \to & \mathbb{H}^1 MDR(X_q, F_{X_q}) & \to & \mathbb{H}^1 MDR(V_i, F_{V_i}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{H}^1 MDR(U_i, F_{U_i}(-w_i)) & \to & \mathbb{H}^1 MDR(X_q, F_{X_q}) & \to & \mathbb{H}^1 MDR(V_i, F_{V_i}) & \to & 0
\end{array}$$

where the endomorphism of $\mathbb{H}^1 DR(X_q, F_{X_q})$ is the residue of the middle Gauss-Manin connection $\nabla_{GM,\mathrm{mid}}$ at $q = q_i$. For brevity the notations HT_{U_i} etc. have been omitted.

In our case, the endomorphisms of the left and right terms will be semisimple. This will imply that the residue of $\nabla_{GM,\mathrm{mid}}$ is semisimple, as long as we know that the endomorphisms on the left and right don't have any common eigenvalues. We can state this as the following lemma.

Lemma 5.5. *Let ψ_{U_i} and ψ_{V_i} be the endomorphisms of $\mathbb{H}^1 MDR(U_i, F_{U_i}(-w_i), HT_{U_i})$ and $\mathbb{H}^1 MDR(V_i, F_{V_i}, HT_{V_i})$ respectively, determined by the endomorphisms of F_{U_i} and F_{V_i} given by the residues of ∇ along U_i and V_i. Suppose that these endomorphisms are semisimple, and don't have any common eigenvalues. Then the residue of $\nabla_{GM,\mathrm{mid}}$ at q_i is semisimple and isomorphic to $\psi_{U_i} \oplus \psi_{V_i}$.*

$\square$

On U_i and V_i we have a logarithmic structure also at the point w_i. However, this point is not included in the "middle" part which is just HT_{U_i} or HT_{V_i}.

We now turn to the fact that F is the pullback of (E, ∇), tensored with $(\mathcal{O}_X, d + \beta)$. From the above discussion, the main problem is to calculate the restrictions

$$\eta^*(E, \nabla)|_{U_i}, \quad \eta^*(E, \nabla)|_{V_i}, \quad \beta|_{U_i}, \quad \beta|_{V_i}.$$

We can define the restriction of a logarithmic one-form β to V_i as follows. It depends on the pullback of the coordinate function t from Z (where $t(q_i) = 0$). Set $b := \mathrm{res}(\beta, V_i)$, then $\beta - b\frac{dt}{t}$ is a logarithmic form having zero residue along V_i, thus it is in the kernel of the residue map on logarithmic forms which maps by restriction to $\Omega^1_{V_i}(\log)$. Define

$$\beta|^t_{V_i} := (\beta - b\frac{dt}{t})|_{V_i}.$$

It is a logarithmic form on V_i whose residues along $H_j \cap V_i$ are just β^{H_j}, for $j \neq i$. This determines the restriction, and it has residue at the intersection point

$$\mathrm{res}(\beta|^t_{V_i}, w_i) = -\sum_{j \neq i} \beta^{H_j}.$$

The similarly-defined restriction $\beta|^t_{U_i}$ is a logarithmic form on U_i whose residues along $H_i \cap U_i$ and $T \cap U_i$ are respectively β^{H_i} and β^T, so

$$\mathrm{res}(\beta|^t_{U_i}, w_i) = -\beta^{H_i} - \beta^T.$$

The relation $\beta^T + \sum_i \beta^{H_i} = 0$ gives $\mathrm{res}(\beta|^t_{V_i}, U_i \cap V_i) + \mathrm{res}(\beta|^t_{U_i}, U_i \cap V_i) = 0$, characteristic of the fact that these restrictions correspond to a single logarithmic form on $X_{q_i} = U_i \cup V_i$.

Now restrict (F, ∇_F) to V_i. Since $\eta|_{V_i}$ is the identity, $\eta^*(E, \nabla)|_{V_i} \cong (E, \nabla)$. The restriction of F is therefore

$$(F, \nabla_F)|_{V_i} = (\eta^* E, \eta^* \nabla + \beta)|_{V_i} = (E, \nabla + \beta|^t_{V_i}).$$

The residual endomorphism induced by ∇_F is just scalar multiplication by β^{V_i}.

Next look at the restriction of (F, ∇_F) to U_i. It clearly depends only on the local form of (E, ∇) near the point q_i. We may even localize in an analytic neighborhood, and so assume that E has the form of a trivial bundle $\mathcal{O}^r$ and the connection is given by $\nabla = d + A\frac{dy}{y}$. We use the notation y for our coordinate on Y at the point q_i (which should be the same as t under $Y \cong Z$), also considered as a function on $Z \times Y$ or X.

Now t and y give coordinates on $Z \times Y$. The ratio $u = y/t$ is a coordinate on X, in a neighborhood of the point $H_i \cap U_i$ along U_i. On U_i it corresponds to the linear coordinate which takes the values $u(U_i \cap H_i) = 0$, $u(U_i \cap T) = 1$, $u(U_i \cap V_i) = \infty$.

The relation $\frac{dy}{y} = \frac{du}{u} + \frac{dt}{t}$ allows us to calculate the restriction

$$(\frac{dy}{y})|^t_{U_i} = \frac{du}{u}.$$

The residue of the pullback of $\frac{dy}{y}$ along U_i is equal to 1. Using $(E, \nabla) \cong (\mathcal{O}^r, d + A\frac{dy}{y})$ we get that the restriction of the pullback to U_i is

$$(\eta^* E, \eta^* \nabla + \beta)|_{U_i} = (\mathcal{O}^r_{U_i}, d + A\frac{du}{u} + \beta|^t_{U_i}),$$

and the residue of $\eta^* \nabla + \beta$ along U_i is $A + \beta^{U_i}$ which is an endomorphism of this bundle preserving the logarithmic connection. Here, in canonical terms $\mathcal{O}^r$ corresponds to the fiber E_{q_i} and A corresponds to the residue of ∇ at q_i.

Apply Lemma 5.5 to calculate the residue of $\nabla_{GM,\text{mid}}$. By Convention 2.2, A is semisimple with eigenvalues never differing by a nonzero integer. Invoking either Corollary 2.8 using the fact that the residue of $\beta|^t_{U_i}$ at T_{U_i} is a nonzero scalar (Convention 3.2), or just by direct computation, we have

$$\mathbb{H}^i DR(\mathcal{O}_{U_i}(-U_i \cap V_i)^r, d + A\frac{du}{u} + \beta|^t_{U_i}) = 0, \quad i = 0, 2.$$

A direct computation using the fact that $\Omega^1_{U_i}(\log)(-w_i) \cong \mathcal{O}_{U_i}$ gives

$$\mathbb{H}^1 DR(\mathcal{O}_{U_i}(-U_i \cap V_i)^r, d + A\frac{du}{u} + \beta|^t_{U_i}) \cong H^0(\Omega^1_{U_i}(\log)(-w_i) \cong \mathcal{O}_{U_i})^r = \mathbb{C}^r,$$

and the action of the residue of $\eta^* \nabla + \beta$ is given by the matrix $A + \beta^{U_i}$.

The residue is nontrivial on the diagonal (Convention 3.2) so the middle condition at $U_i \cap T$ has no effect, and the middle condition at the point $U_i \cap H_i$ removes the zero eigenspace of the residue there, that is to say the zero eigenspace of the matrix $A + \beta^{H_i}$. Introduce the following notation: if M is a semisimple matrix then $M^\dagger$ is the same endomorphism but only of the sum of eigenspaces different from zero. Thus the contribution from U to the residue of $\nabla_{GM,\text{mid}}$ on the middle direct image is

$$\text{res}(\nabla_{GM,\text{mid}})_U = (A + \beta^{H_i})^\dagger + (\beta^{U_i} - \beta^{H_i}).$$

Recall that $\beta^{U_i} = \beta^{H_i} + \beta^T + \beta^{V_i}$, giving

$$\text{res}(\nabla_{GM,\text{mid}})_U = (A + \beta^{H_i})^\dagger + \beta^{V_i} + \beta^T.$$

The contribution from V is the cohomology of $\nabla + \beta|^t_{V_i}$, with middle condition at the points $H_j \cap V_i$ for $j \neq i$ and no middle condition at $w_i = U_i \cap V_i$. This contribution occurs with a single eigenvalue which is the residue of β, in our notation β^{V_i}. Let d_i denote the dimension of this cohomology group and $\mathbf{1}^{d_i}$ is the identity matrix of rank d_i. If Lemma 5.5 can be applied then we conclude that the full residue of the Gauss-Manin connection on the middle convolution is given by

(5.6) $$\text{res}(\nabla_{GM,\text{mid}}) = \left[(A + \beta^{H_i})^\dagger + \beta^{V_i} + \beta^T\right] \oplus \left[\beta^{V_i}\mathbf{1}^{d_i}\right].$$

In order to be able to apply Lemma 5.5 we need to know that the eigenvalues of the two pieces don't coincide. We also need something for the first sentence in Corollary 2.8. The following condition is analogous to Convention 4.4 from the Betti case.

Convention 5.6. *For any eigenvalue α of the residue $A = \text{res}(\nabla, q_i)$, we have*

$$\alpha + \beta^{H_i} + \beta^T \notin \mathbb{Z}, \quad \alpha + \beta^{H_i} \notin \mathbb{Z} - \{0\}.$$

The first condition is equivalent to saying $\alpha - \sum_{j\neq i}\beta^{H_j} \notin \mathbb{Z}$, and if each $-\beta^{H_j}$ is an eigenvalue of the residue at q_j then this condition would be a consequence of 1-genericity as in 4.14. The second condition will hold whenever we need to choose $-\beta^{H_i}$ from among the eigenvalues of $\mathrm{res}(\nabla, q_i)$, by Convention 2.2.

Lemma 5.7. *Suppose Conventions 2.2, 3.2 and 5.6 hold. Then the eigenvalues of $(A + \beta^{H_i})^\dagger + \beta^{V_i} + \beta^T$ are distinct from β^{V_i}, so Lemma 5.5 can be applied as above (5.6) with*

$$d_i = (n-2)r - \sum_{j\neq i} m_j(-\beta^{H_j}).$$

Proof: The eigenvalues of $(A + \beta^{H_i})^\dagger + \beta^{V_i} + \beta^T$ are of the form $\alpha + \beta^{H_i} + \beta^T + \beta^{V_i}$ for α eigenvalues of A. Under Convention 5.6 these are different from β^{V_i}. For the dimension of the piece coming from V_i, note that

$$\mathrm{res}(\nabla + \beta|_{V_i}^t, w_i) = A + \mathrm{res}(\beta|_{V_i}^t, w_i) = A - \sum_{j\neq i}\beta^{H_j} = A + \beta^{H_i} + \beta^T.$$

Convention 5.6 says that the eigenvalues here are never integers, also the residues at points of H_{V_i} are never nonzero integers. Thus Corollary 2.8 applies and we can calculate the dimension d_i by using the Euler characteristic which gives the formula as stated. The terms in the sum over $i \neq j$ come from the middle conditions at the points $H_j \cap V_i$; there is no middle condition at the remaining point w_i. $\square$

As in §2.5 and §4.5 above, let $\vec{g} = (g_1, \ldots, g_n)$ denote the residual data for (E, ∇) with $g_i \in Div(\mathbb{A}^1)$ effective divisors representing the multiplicity vectors of the eigenvalues. Asking that the residues lie in conjugacy classes $\mathbf{c}(g_i)$ insures Convention 2.2 automatically, and Conventions 3.2 and 5.6 are conditions only on the pair $(\beta, \vec{g})$. The result of Lemma 5.5 applied as in (5.6) says exactly that the vector of residual data for $\nabla_{GM,\mathrm{mid}}$ is given by $\vec{\kappa}(\beta, \vec{g})$. We can sum up as follows.

Scholium 5.8. *Suppose (E, ∇) is a logarithmic connection on (Y, Q_Y) with semisimple residues (Convention 2.2) corresponding to a vector $\vec{g} \in Div(\mathbb{A}^1)^n$, and suppose β is a de Rham convoluter (§3.2). Suppose that Convention 3.2) that $\beta^T \notin \mathbb{Z}$, and Convention 5.6 that $\alpha + \beta^{H_i} + \beta^T \notin \mathbb{Z}$ and $\alpha + \beta^{H_i} \notin \mathbb{Z} - \{0\}$ for any eigenvalue α of $\mathrm{res}(\nabla, q_i)$. These are conditions on $(\beta, \vec{g})$ only. Then the de Rham middle convolution $MC_\beta(E, \nabla)$ is a logarithmic connection on (Z, Q_Z) with semisimple residues whose vector of residual data is given by the Katz transformation $\vec{\kappa}(\beta, \vec{g})$.*

Theorem 5.9. *Suppose $Q \subset \mathbb{P}^1$ is a set of n points, $\vec{g} \in Div(\mathbb{A}^1)^n$ is a vector of semisimple residual data, and β is a de Rham convoluter. Suppose Conventions 3.2 and 5.6 hold. Then middle convolution induces an isomorphism*

$$MC_\beta : M_{DR}(\mathbb{P}^1, Q; \vec{g}) \stackrel{\cong}{\to} M_{DR}(\mathbb{P}^1, Q; \vec{\kappa}(\beta, \vec{g})).$$

This isomorphism is involutive like in §4.7 (but with $-c^*\beta$ instead of $c^*\beta^*$) and is compatible with the isomorphism of Theorem 4.10 via the Riemann-Hilbert correspondence [45, Theorem 1.2].

6. Harmonic bundles and parabolic structures

There is a notion of *parabolic bundle on Z with parabolic structures at the q_i*. We don't repeat the definition here. These will be called "parabolic bundles" for short. If E is a parabolic bundle then for each q_i and each $\alpha \in \mathbb{R}$ we have an associated graded vector space $Gr_{\alpha,q_i}(E)$ which is finite-dimensional. Multiplication by a local coordinate at q_i gives an isomorphism

$$Gr_{\alpha,q_i}(E) \cong Gr_{\alpha+1,q_i}(E).$$

Define the *residue* of E at p_i to be the associated-graded direct sum

$$\mathrm{res}(E; q_i) := \bigoplus_{0 \le \alpha < 1} Gr_{\alpha,q_i}(E).$$

For a fixed $\lambda \in \mathbb{C}$ there is a notion of *logarithmic λ-connection* ∇ on a parabolic bundle E. The logarithmic structure is with respect to the divisor $Q = q_1 + \ldots + q_n$. For any $\alpha_1, \ldots, \alpha_n$ it induces a λ-connection

$$\nabla : E_{\alpha_1, \ldots, \alpha_n} \to E_{\alpha_1, \ldots, \alpha_n} \otimes \Omega^1_Z(\log Q).$$

With the canonical isomorphism $\Omega^1_Z(\log Q)_{q_i} \cong \mathbb{C}$, a λ-connection on the parabolic bundle E induces an endomorphism called the *residue*

$$\mathrm{res}(\nabla; q_i) : \mathrm{res}(E; q_i) \to \mathrm{res}(E; q_i).$$

The residue of the pair (E, ∇) at a point q_i is defined as the pair of an S^1-graded vector space with endomorphism

$$\mathrm{res}(E, \nabla; q_i) := (\mathrm{res}(E; q_i), \mathrm{res}(\nabla; q_i)).$$

When necessary, we introduce an index to denote the piece $\mathrm{res}(\nabla; q_i)_\alpha$ acting on $Gr_{\alpha,q_i}(E) \subset \mathrm{res}(E; q_i)$.

If $F \subset E$ is a parabolic subbundle compatible with ∇ then we can consider its *parabolic degree*, and the *parabolic slope* is the parabolic degree divided by the rank. We say that (E, ∇) is *stable* (resp. *semistable*) if for any strict parabolic subbundle compatible with ∇, the parabolic slope of F is strictly less than (resp. less than or equal to) that of E.

Given a parabolic bundle E, define its *parabolic type at q_i* to be the divisor on S^1

$$\mathrm{type}(E, q_i) := \sum_{0 \le \alpha < 1} (\dim Gr_{\alpha,q_i}(E)) \cdot [\alpha].$$

This follows the discussion in §4.5 with $\mathbb{L} = S^1$.

Up to isomorphism, the residue at q_i of a parabolic logarithmic λ-connection $\mathrm{res}(E, \nabla; q_i)$ is classified by specifying its type $\sum g_\alpha[\alpha]$ and for each α, specifying the Jordan normal form of an endomorphism of a vector space of dimension g_α.

We say that the residues of (E, ∇) are *semisimple* if the Jordan normal forms are diagonalizable. This means that on each Gr_{α,q_i} the action of $\mathrm{res}(\nabla; q_i)_\alpha$ is semisimple or diagonalizable, so it corresponds to a divisor on $\mathbb{A}^1$. Altogether, when the residues are semisimple, the isomorphism class of the residue of (E, ∇) at q_i is determined by a divisor of total degree r on $S^1 \times \mathbb{A}^1$.

Recall that we have a notion of *tame harmonic bundle* on $Z - Q$. A harmonic bundle consists of a flat connection, and an equivariant harmonic map. The tameness condition means that locally near the singularities, the the flat sections of the connection on sectors have polynomial growth with respect to the harmonic metric. Measuring the growth rate of sections leads to parabolic structures. The flat connection decomposes as

$$D = D' + D'' = (\partial + \overline{\theta}) + (\overline{\partial} + \theta)$$

where $\partial + \overline{\partial}$ is a unitary connection, $\theta + \overline{\theta}$ antipreserves the metric, and θ is holomorphic. Fix $\lambda \in \mathbb{C}$ which allows us to define a differential λ-connection

$$D_\lambda := \lambda D' + D''.$$

The $(0, 1)$ piece, which has contributions from both terms $\lambda D'$ and D'', is a usual holomorphic structure giving rise to a holomorphic vector bundle E. The $(1, 0)$ piece is a holomorphic λ-connection on E.

Measuring the growth rate of sections in a holomorphic frame, leads to an extension of E as a parabolic bundle over Z with parabolic structure along Q, again denoted E. The connection $\nabla = (D_\lambda)^{1,0}$ is a logarithmic λ-connection on the parabolic bundle E. The parabolic logarithmic λ-connection (E, ∇) is *polystable*, in other words a direct sum of stable objects of the same slope.

Conversely, given a parabolic logarithmic λ-connection (E, ∇), if it is polystable then there exists an essentially unique structure of harmonic bundle given by a harmonic metric on E over $X - Q$ with the appropriate growth rates determined by the parabolic structure. The metric connection is unique and the metric is unique up to a positive real scalar on each stable piece. The Higgs case is $\lambda = 0$.

In keeping with Convention 2.1, we would like to insure that the monodromy transformations are semisimple. We furthermore assume that the residues of θ vanish. This amounts to restricting to representations where the local monodromy eigenvalues are in $S^1 \subset \mathbb{G}_m$ together with trivial filtered local system structures at the singularities.

Convention 6.1. *The residue of the Higgs field θ on the associated-graded of the parabolic bundle $Gr_{\alpha,q_i}(E)$ at any point $q_i \in Q$ is equal to zero.*

Lemma 6.2. *If (E, θ) is a polystable parabolic Higgs bundle of degree zero satisfying Convention 6.1 then the monodromy transformations of the corresponding local system are semisimple, with eigenvalues in S^1 corresponding to $e^{2\pi i \alpha}$ for α the parabolic weights.*

Proof: This follows from the local considerations shown in [129]. □

The moduli space of parabolic logarithmic λ-connections can be constructed, see many references on parabolic bundles included in the bibliography below. This moduli space becomes isomorphic (as a real analytic space possibly with singularities) to a space of harmonic bundles which can be constructed as in Hitchin's original case [70], see also Fujiki [54]. As λ varies we get a family over the affine line, which is the *nonabelian Hodge filtration* on the moduli space. Glueing with the complex conjugate we get the twistor space for Hitchin's hyperkähler structure [70] [54]. In our case of an open curve, some further work is needed, see [114] for example. We should deal with the transformation of residual types which occurs when we change λ [129], a situation which appears to reflect some kind of weight-two

phenomenon corresponding to the punctures. In the case of quasi-unipotent monodromy we should be able to deal with the problem by looking at local systems with unipotent monodromy on a DM-curve. This general moduli problem will not be considered any more here, but constitutes a good source of further questions: first and foremost we would like to have Katz isomorphisms between these moduli spaces coming from a parabolic middle convolution.

6.1. **Cyclotomic harmonic bundles.** Our construction of Higgs bundles will be based on a trick to insure stability. Recall that $\mathbb{C}^*$ acts on the space of parabolic Higgs bundles by $t : (E, \theta) \mapsto (E, t\theta)$. If we assume that the residue of θ is unipotent (or even equal to zero if we want to keep with Convention 2.1) then this action preserves the residue of the parabolic structure and θ, so by [129] it preserves the local type of monodromy transformations of the corresponding local system.

Recall that a *complex variation of Hodge structure* is a harmonic bundle which is a fixed point for the full action of $\mathbb{C}^*$ or equivalently for the action of S^1. A variant is to look at the action of a finite cyclic subgroup of roots of unity $\mu_m \subset \mathbb{C}^*$. Recall [70] [30] that the action of S^1 preserves the harmonic metric structure. A harmonic bundle which is a fixed point of the action of μ_m will be called a *cyclotomic harmonic bundle*, where $m \geq 2$ is considered as fixed for now—later we can say "m-cyclotomic" if we need to specify m.

When we say that (E, θ) is a fixed point this means that it is provided with an additional structure of an action of μ_m on E such that for any $u \in \mu_m$ and $e \in E$ we have $(u\theta)(ue) = u(\theta e)$.

The structure of a cyclotomic harmonic bundle is very similar to the structure of a complex variation of Hodge structure. The group of characters of μ_m is canonically isomorphic to $\mathbb{Z}/m\mathbb{Z}$ because we have defined μ_m as coming from a privileged embedding $\mu_m \subset \mathbb{C}^*$. The action of μ_m on E provides a decomposition according to characters

$$E = \bigoplus_{p \in \mathbb{Z}/m\mathbb{Z}} E^p,$$

and the formula $(u\theta)(ue) = u(\theta e)$ then says

$$\theta : E^p \to E^{p+1} \otimes \Omega_Z^1(\log Q).$$

The only difference with the case of variations of Hodge structure is that $p + 1$ is taken in the quotient group $\mathbb{Z}/m\mathbb{Z}$ so θ includes a piece of the form

$$\theta : E^m \to E^1 \otimes \Omega_Z^1(\log Q).$$

Thus, θ is no longer required to be a nilpotent transformation. In the parabolic case, the decomposition is compatible with the parabolic structure.

The action of $\mu_m \subset S^1$ preserves the differential operators of the harmonic bundle, so by averaging we can always choose a μ_m-invariant harmonic metric. The decomposition of E is then orthogonal and the pieces E^p are preserved by the metric connection $\partial + \overline{\partial}$. The complex conjugate $\overline{\theta}$ goes from E^p to $A^{0,1}(E^{p-1})$. Thus, in all respects this looks like a complex variation of Hodge structure except that the Hodge decomposition is viewed circularly and the Kodaira-Spencer components can go all the way around the circle.

Hitchin in [72] gave a construction of a subspace of Higgs bundles which corresponded to an analogue of Teichmüller space. In Hitchin's construction, a basic variation of Hodge structure is modified by adding a new term in the Higgs field. In this sense, the notion of cyclotomic Higgs field is a variant on [72]. Hitchin's Teichmüller Higgs bundles can sometimes be cyclotomic: in the notation of [72] if $\alpha_m \neq 0$ but $\alpha_{m-1} = \ldots = \alpha_2 = 0$ then the Higgs field written down there is cyclotomic.

The correspondence between Higgs bundles and local systems is compatible with the action of μ_m, and this helps with the stability condition. Suppose (E, θ) is an m-cyclotomic Higgs bundle, that is a bundle with action of μ_m (or equivalently a decomposition as above) compatible with the action on θ. We say that it is *cyclotomically semistable* (resp. *cyclotomically stable*) if for any μ_m-invariant sub-Higgs bundle, the slope is smaller (resp. strictly smaller) than the slope of E.

Proposition 6.3. *Suppose (E, θ) is a cyclotomically stable parabolic cyclotomic Higgs bundle of parabolic degree zero. Then (E, θ) is polystable as a regular parabolic Higgs bundle, and it has a μ_m-equivariant harmonic metric with growth rates corresponding to the parabolic structure.*

Proof: The β-subbundle is μ_m-invariant, so cyclotomic semistability implies semistability. The socle is μ_m-invariant, so cyclotomic polystability implies polystability. Then [128], Theorem 1 which took into account the possibility of having the action of a group such as μ_m, provides an invariant harmonic metric. The growth rates are governed by the choice of initial metric, as discussed in [128] §10 and [129]. □

It is interesting to note that the cyclotomic Higgs bundles play a special role in the compactification of the de Rham moduli space. Recall that the compactification puts at infinity a divisor obtained by dividing $M_{Higgs} - \{\theta \text{ nilpotent}\}$ by the action of $\mathbb{C}^*$. If we take the quotient in the sense of stacks, then the compactification becomes a DM stack and the stacky points with automorphism group μ_m are exactly the cyclotomic Higgs bundles.

6.2. **The maximal case.** Traditionally one of the easiest cases is when the E^p are line bundles. For this, take $m = r$ equal to the rank.

Lemma 6.4. *Suppose $E = \bigoplus E^p$ is an r-cyclotomic parabolic Higgs bundle of rank r with θ not nilpotent. This means that the E^p are parabolic line bundles and every component $\theta^p : E^p \to E^{p+1} \otimes \Omega^1_Z$ is nonzero. Then (E, θ) is cyclotomically stable, hence polystable as an ordinary parabolic Higgs bundle.*

Proof: Non-nilpotence of θ requires that all the components be nonzero, in particular all of the bundles E^p are nonzero. Since their number is equal to the rank, they must be line bundles. Suppose $F \subset E$ is a μ_r-invariant saturated subbundle. It decomposes as a direct sum of $F^p \subset E^p$. If any one of the F^p is nonzero then all of them are nonzero because every component θ^p is nonzero and they go around in a circle. If F is saturated we get $F^p = E^p$. Thus, there are no μ_r-invariant subbundles of rank strictly between 0 and r, so E is vacuously cyclotomically stable. By Proposition 6.3, (E, θ) is polystable in the ordinary sense. □

The structure of a non-nilpotent r-cyclotomic parabolic Higgs bundle is particularly easy to understand. It consists of a collection of parabolic line bundles $E^1, \ldots, E^r$, together with

morphisms of parabolic bundles

$$\theta^p : E^p \to E^{p+1} \otimes \Omega^1_Z(\log Q).$$

Convention 6.1 says that we want $\mathrm{res}(\theta^p, q_i)$ to induce the zero map on the associated-graded Gr_{α,q_i} for any α and any singular point $q_i \in Q$. In view of the fact that we are dealing with parabolic line bundles, there is only one weight on each side, and the residue map is automatically zero unless the two weights are the same. This will appear in our criterion below.

6.3. **Explicit construction.** A parabolic line bundle on (Y, Q) is always of the form

$$E = \mathcal{O}_Y(k)(a_1 q_1 + \ldots + a_n q_n)$$

with $k \in \mathbb{Z}$ and $a_i \in [0, 1)$. Written in this way, the vector $[k; a_1, \ldots, a_n]$ is uniquely determined and uniquely determines E, and we may use it as notation. Suppose we are given two parabolic line bundles $E = [k; a_1, \ldots, a_n]$ and $E' = [k'; a'_1, \ldots, a'_n]$. A map from E to E' will consist of a holomorphic map

$$f : \mathcal{O}_Y(k) \to \mathcal{O}_Y(k'),$$

such that if $a_i > a'_i$ then $f(q_i) = 0$. In this description we use the fact that $a_i, a'_i \in [0, 1)$, in particular $|a_i - a'_i| < 1$ always.

In order to have a map which furthermore induces the zero map on associated graded spaces at each q_i,

$$Gr_{\alpha,q_i}(f) = 0 : Gr_{\alpha,q_i}(E) \to Gr_{\alpha,q_i}(E'),$$

we should require that $f(q_i) = 0$ also when $a_i = a'_i$. Thus, the description of these maps (which we call zero-residue maps) is that $f(q_i) = 0$ when $a_i \geq a'_i$.

Finally, we get to a description of a map

$$f : E \to E' \otimes \Omega^1_Y(\log Q)$$

inducing zero on the residues as required by Convention 6.1. Recall that $\Omega^1_Y(\log Q) = \mathcal{O}_Y(n-2)$. Thus, such a map f is the same thing as a holomorphic section

$$f \in \Gamma(Y, \mathcal{O}_Y(k' - k + n - 2))$$

such that $f(q_i) = 0$ whenever $a_i \geq a'_i$, or we can also say

$$f \in \Gamma(Y, \mathcal{O}_Y(k' - k + n - 2 - \#\{i, \ a_i \geq a'_i\})).$$

We get the following lemma.

Lemma 6.5. *Suppose $E = [k; a_1, \ldots, a_n]$ and $E' = [k'; a'_1, \ldots, a'_n]$ are parabolic line bundles on (Y, Q). Then there exists a nontrivial zero-residue map*

$$f : E \to E' \otimes \Omega^1_Y(\log Q)$$

if and only if

$$\#\{i, \ a_i \geq a'_i\} \leq k' - k + n - 2.$$

If equality holds then the map f has no zeroes other than as required for the residues, and is unique up to a scalar constant. If the inequality is strict then f has other zeroes.

$$\Box$$

Suppose we are given a vector of divisors $\vec{g} = (g_1, \ldots, g_n) \in Div(S^1)n$, with $deg(g_i) = r$. Choose the standard section of $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ to identify $S^1 \cong [0,1)$. Write

$$g_i = \sum_{\alpha \in [0,1)} m_i(\alpha)[\alpha].$$

A sequence $a_{i,1}, a_{i,2}, \ldots, a_{i,r} \in [0,1)$ is called an *arrangement of g_i* if each α occurs in the sequence with multiplicity $m_i(\alpha)$. This is to say that the sequence of exponentials of the $a_{i,j}$ is a possible sequence of eigenvalues along the diagonal, for a matrix in the conjugacy class $C(g_i)$.

An arrangement is called *good* if the number of indices t with $a_{i,t} \geq a_{i,t+1}$ is minimal. Let $T(g_i)$ be the minimal number of such indices, thus the arrangement is good if

$$\#\{t,\ a_{i,t} \geq a_{i,t+1}\} = T(g_i).$$

Here, and always below, the indices are taken modulo r, for example if $t = r$ then $a_{i,t+1} = a_{i,1}$. Enumerate the indices t as above, in increasing order $t_1 < \ldots < t_p$. We can thus write our arrangement as a "sawtooth":

$$a_{i,1} < a_{i,2} < \ldots < a_{i,t_1}$$
$$a_{i,t_1} \geq a_{i,t_1+1}$$
$$a_{i,t_1+1} < \ldots < a_{i,t_2}$$
$$a_{i,t_2} \geq a_{i,t_2+1}$$
$$\vdots$$
$$a_{i,t_p} \geq a_{i,t_p+1}$$
$$a_{i,t_p+1} < \ldots < a_{i,r}$$

with $a_{i,r} < a_{i,1}$ unless t_p happens to be r. Now let

$$g_{i,j} := [a_{i,t_j+1}] + [a_{i,t_j+2}] + \ldots + [a_{i,t_{j+1}-1}] + [a_{i,t_{j+1}}],$$

with $t_{j+1} := t_1$ when $j = p$ and the terms in $g_{i,p}$ adapted appropriately. These are reduced effective divisors, that is each eigenvalue occurs with multiplicity at most 1, because the sequences are strictly increasing in between the t_j. And we have a decomposition

$$g_i = g_{i,1} + \ldots + g_{i,p}.$$

Notice that

$$t_{j+1} = \deg(g_{i,j}) + t_j \pmod{r}.$$

Conversely, given a pair of p-uples written $(t_1, \ldots, t_p; g_{i,1}, \ldots, g_{i,p})$ with t_j an increasing sequence in $1, \ldots, r$ and the $g_{i,j}$ giving a decomposition of g_i into reduced effective divisors, we get an arrangement. The arrangement is good if p is minimal and equal to the maximal multiplicity in g_i. There is a one-to-one correspondence between such notations and arrangements for g_i.

Lemma 6.6. *The minimal number of t's is equal to the maximum multiplicity in the divisor g_i,*

$$T(g_i) = \max_{\alpha} m_i(\alpha).$$

Proof: It is easy to see that for any α we have $T(g_i) \geq m_i(\alpha)$. On the other hand, we can clearly choose a decomposition into reduced effective divisors $g_i = g_{i,1} + \ldots + g_{i,p}$ with $p = \max_\alpha m_i(\alpha)$. Thus the minimal p is equal to the maximum of the $m_i(\alpha)$. $\qquad\square$

Now suppose that for each $i = 1, \ldots, g$ we have chosen a good arrangement $a_{i,j}$ for g_i. Let

$$(t_{i,1}, \ldots, t_{i,p_i}; g_{i,1}, \ldots, g_{i,p_i})$$

be the notation established above with $p_i = T(g_i)$. For any sequence $k_1, \ldots, k_r$ define parabolic line bundles

$$E^j := [k_j; a_{1,j}, \ldots, a_{n,j}].$$

In order to construct a cyclotomic Higgs bundle $E = \bigoplus_{j=1}^r E^j$, we investigate the possible choice of $k_1, \ldots, k_r$ such that there are nontrivial zero-residue maps

$$\theta^j : E^j \to E^{j+1} \otimes \Omega_Y^1(\log Q)$$

including the case $j = r, j + 1 = 1$. Let τ_j denote the cardinality

$$\tau_j := \#\{i, \ \ a_{i,j} \geq a_{i,j+1}\}.$$

For any sequence of k_j put

$$z_j := k_{j+1} - (\tau_j + k_j + 2 - n).$$

Lemma 6.7. *With the above notations, there exist zero-residue maps θ^j if and only if $z_j \geq 0$ for $j = 1, \ldots, r$. In this case, z_j is the number of extra zeros of θ^j beyond what are required by the zero-residue condition. The z_j are subject to the relation*

(6.1) $$z_1 + \ldots + z_r = \delta(\vec{g}),$$

so there exists a possible choice of z_j or equivalently of k_j if and only if $\delta(\vec{g}) \geq 0$.

Proof: The first statements come from Lemma 6.5. From the definition of $T(g_i)$ and τ_j we get

$$\sum_{j=1}^r \tau_j = \sum_{i=1}^n T(g_i).$$

Thus

$$z_1 + \ldots + z_r = r(n-2) - \sum_{i=1}^n T(g_i) = \delta(\vec{g}).$$

$\qquad\square$

Given $z_j \geq 0$ subject to the relation $z_1 + \ldots + z_r = \delta(\vec{g})$, and given k_1, we obtain the remaining $k_2, \ldots k_r$ from the formula for z_j. Construct the parabolic bundles E^j and nontrivial zero-residue maps θ^j. This yields an r-cyclotomic Higgs bundle (E, θ).

6.4. **The degree.** As pointed out in Lemma 6.4, the (E, θ) constructed this way is cyclotomically stable, hence polystable as a Higgs bundle. To finish the construction of a local system we need to insure that its parabolic degree vanishes. The parabolic degree of $E^j = [k_j; a_{1,j}, \ldots, a_{n,j}]$ is

$$\deg^{\mathrm{par}}(E^j) = k + a_{1,j} + \ldots + a_{n,j}.$$

Adding up gives the parabolic degree of E:

$$\deg^{\mathrm{par}}(E) = \sum_{j=1}^{r} k_j + \sum_{i,j} a_{i,j} = \sum_{j=1}^{r} k_j + \sum_{i,\alpha} m_i(\alpha)\alpha.$$

By induction,

$$k_{j+1} = k_1 + z_1 + \ldots + z_j + \tau_1 + \ldots + \tau_j + j(2 - n).$$

We have

$$\deg^{\mathrm{par}}(E) = P + k_1 r + \sum_{j=1}^{r} (r - j)(z_j + \tau_j)$$

where

$$P := \sum_{i,\alpha} m_i(\alpha)\alpha + \sum_{j=1}^{r} j(r - j)(2 - n)$$

represents the piece which doesn't depend on the choice of arrangements or of k_j. The condition $Det(\vec{g}) = 1$ says that $P \in \mathbb{Z}$.

Recall that τ_j is the number of i such that $a_{i,j} \geq a_{i,j+1}$. This is the same as the number of i such that $j \in \{t_{i,1}, \ldots, t_{i,p_i}\}$. The terms involving τ_j can be recast as a sum over the elements $t_{i,j}$. We conclude the following formula for the parabolic degree of the Higgs bundle we have constructed:

$$(6.2) \qquad \deg^{\mathrm{par}}(E) = P + k_1 r + \sum_{i=1}^{n} \sum_{j=1}^{p_i} (r - t_{i,j}) + \sum_{j=1}^{r} (r - j) z_j.$$

Theorem 6.8. *Suppose given a vector of divisors $\vec{g} \in Div(S^1)^n$ such that $Det(\vec{g}) = 1$. Suppose that the defect is strictly positive,*

$$r(n - 2) - \sum T(g_i) = \delta(\vec{g}) > 0.$$

Fix any collection of good arrangements $a_{i,j}$ for $\vec{g}$. Then it is possible to choose the k_j subject to the constraint

$$k_{j+1} \geq \tau_j + k_j + 2 - n,$$

such that $\deg^{\mathrm{par}}(E) = 0$.

Proof: Fixing the collection of good arrangements, the terms in (6.2) involving $t_{i,j}$ are fixed. Because of the strictly positive defect, there is a nontrivial choice of $z_1, \ldots, z_j$. Geometrically this means that we have a choice as to how many zeros θ_j can have.

Put $z_r := \delta(\vec{g}) - 1$ and $z_j = 0$ for all but one value of $j = j'$ in which case $z_{j'} = 1$. Make the convention here that if $j' = r$ then $z_r := \delta(\vec{g})$ instead. With this choice we get

$$\sum_{j=1}^{r}(r - j)z_j = r - j',$$

and by choosing j' appropriately this can take on any value between 0 and $r-1$. In particular, modulo r it can take on all values. By adjusting k_1 appropriately, $\deg^{\mathrm{par}}(E)$ can take on any integer value. $\qquad\square$

Theorem 6.9. *Suppose given a vector of divisors $\vec{g} \in Div(S^1)^n$ such that $Det(\vec{g}) = 1$. Suppose that the defect is zero,*

$$r(n - 2) - \sum T(g_i) = \delta(\vec{g}) = 0,$$

but the superdefect is strictly positive $\sigma(\vec{g}) > 0$. Then it is possible to choose a good arrangement $\{a_{i,j}\}$ for $\vec{g}$ and k_1, which determines the remaining k_j in the zero-defect case by the constraint

$$k_{j+1} = \tau_j + k_j + 2 - n,$$

such that $\deg^{\mathrm{par}}(E) = 0$.

Proof: In the case where the defect is zero, we are constrained to have $z_j = 0$. In particular, once we fix k_1 then the others are determined. We have the simplified formula

$$\deg^{\mathrm{par}}(E) = P + k_1 r + \sum_{i=1}^{n}\sum_{j=1}^{p_i}(r - t_{i,j}).$$

On the other hand, the fact that the superdefect is nonzero means that for some i there is at least one eigenvalue α' which appears with multiplicity $m_i(\alpha') < p_i = \max_\alpha m_i(\alpha)$. In particular, for any arrangement which we denote now generically by A, we have at least one interval not containing the eigenvalue α'. Define the following operation on arrangements: find an interval $t_{i,j} + 1, \ldots, t_{i,j+1}$ containing α' but such that the preceding interval $t_{i,j-1} + 1, \ldots, t_{i,j}$ doesn't contain α'. Move α' from the one to the other. We get a new arrangement ∂A with the property that all $t_{i',j'}(\partial A)$ are the same as for A, except

$$t_{i,j}(\partial A) = t_{i,j}(A) + 1.$$

Note that ∂A will always be a good arrangement whenever A is good. From this and the above formula we find

$$\deg^{\mathrm{par}}(E(\partial A)) = \deg^{\mathrm{par}}(E(A)) - 1.$$

In particular, iterating the operation $A \mapsto \partial A$ and modifying k_1 we find that $\deg^{\mathrm{par}}(E(A))$ can take on all integer values as A runs through all the good arrangements. $\qquad\square$

Putting together these two theorems we get:

Corollary 6.10. *Suppose $\vec{g} \in Div(S^1)^n$ is a vector of local monodromy data with eigenvalues in $S^1 \subset \mathbb{G}_m$. Suppose the defect is positive $\delta(\vec{g}) \geq 0$. If $\delta = 0$ then suppose that the superdefect is strictly positive; this is equivalent to supposing that the virtual dimension of*

the moduli space is at least 4. *Then there exists a parabolic* r-*cyclotomic Higgs bundle* (E, θ) *of parabolic degree* 0, *cyclotomically stable and polystable in the usual sense, corresponding to a local system with local monodromy data* $\vec{g}$.

The only cases left to be treated are when the moduli space has dimension 2. There are four families as listed in Lemma 2.13. These cases are considered by Kostov in [90]. As he notes there, the determinant of the vector $\frac{\vec{g}}{d}$ is a d-th root of unity. For $d > 1$ and primitive root of unity, it looks like there cannot be an r-cyclotomic Higgs bundle; however a solution exists [92] and one might hope to construct an m-cyclotomic Higgs bundle for smaller m and with some component bundles E^j of rank 2. When the root of unity is not primitive, Kostov shows that there are no irreducible solutions.

7. FURTHER QUESTIONS

It would be good to have the full middle-convolution theory for the general setup of parabolic logarithmic λ-connections [110] [111] [112] [125]. This raises some nontrivial questions such as defining the middle higher direct image in the parabolic setting, obtaining a base-change result analogous to Convention 5.3, and showing polystability of the middle convolution. It was my original goal to treat these questions here but that turned out to be very difficult.

Aker and Szabo have communicated to me their recent preprint [3] in which they do the Nahm transform (essentially the same as Fourier transform) for parabolic Higgs bundles with irregular singularities having poles of order ≤ 2 at infinity. This should allow one to obtain the middle convolution for parabolic Higgs bundles by following Katz's original method.

If the weights of a parabolic structure are rational, i.e. for every point q_i the parabolic type is a divisor concentrated over roots of unity in S^1, then as discussed in [19] [27] [108] [77], the parabolic bundle may also be viewed as a bundle on a Deligne-Mumford stack $Z[\frac{Q}{m}]$ obtained by assigning an integer m to the points q_i. Here m should be chosen to be divisible by all the denominators of the rational weights which occur. In this case, a logarithmic λ-connection on the parabolic bundle may also be viewed as a logarithmic connection on the corresponding DM-bundle. An intermediate case between the non-parabolic case we have discussed in §5 and the general case of parabolic logarithmic λ-connections, would be the case of parabolic logarithmic λ-connections with rational weights. Also assuming that the residue of the connection on $Gr_{\alpha, q_i}(E)$ is the scalar α, these objects would be equivalent to local systems on the DM-stacks $Z[\frac{Q}{m}]$. It should be possible to have a theory of Katz's middle convolution for these objects. The blown-up surface (X, J) would be provided with a stack structure and the singular fibers would be twisted curves [1] [2].

The moduli spaces have numerous additional structures.

Conjecture 7.1. *The isomorphisms between different moduli spaces given by the middle convolution map, preserve the Hodge filtration, the* $\mathbb{C}^*$ *action, and the Hitchin hyperkähler structure when this is defined, that is when the eigenvalues of the local monodromy transformations are in* S^1.

This conjecture can probably be proven by Aker and Szabo with their method [3], indeed they show that the Nahm transform preserves the hyperkähler structure of the moduli spaces.

Völklein points out in [139] that the Katz isomorphisms between various Betti moduli spaces are compatible with the action of the braid group of the points $q_1, \ldots, q_n \in \mathbb{P}^1$. Similarly, the cohomological formulation immediately implies that for $n \geq 4$ the Katz isomorphisms between different de Rham moduli spaces are compatible with the nonabelian Gauss-Manin connection, i.e. the isomonodromic deformation equations. This was used by Boalch to get information about finite Painlevé orbits in [14] [15]. It would be interesting to look further at the dynamics of the braid action and the isomonodromy equations.

Question 7.2. *Which Hodge types can occur at variations of Hodge structure in the moduli spaces? How does the Hodge type change under middle convolution?*

To what extent do we get unexpected or exceptional automorphisms of moduli spaces, due to the possibility of running Katz's algorithm in several different ways? In particular, one could start in the range $\delta \geq 0$, do a series of middle convolutions which go out of this range, then another series to go back. In some cases this should change the local monodromy vector, so we should obtain isomorphisms $M_B(\vec{g}_1) \cong M_B(\vec{g}_2)$ for $g_1 \neq g_2$ in the range $\delta \geq 0$. Say that these two local monodromy vectors are *middle-convolution equivalent* in this case.

Question 7.3. *What is the quotient of the set of local monodromy vectors with $\delta \geq 0$, by the relation of middle-convolution equivalence? In each middle convolution equivalence class, does the operation of going out and back again provide any nontrivial automorphisms of M_B?*

Question 7.4. *Is there a Torelli theorem saying that the isomorphism class of $M_B(\vec{g})$ and/or $M_{DR}(\vec{g})$ possibly with additional structures such as the Hodge filtration, the hyperkähler metric, etc., determines the middle convolution equivalence class of $\vec{g}$ (and maybe the collection of points Q depending on how much structure we are considering)?*

Problem 7.5. *Generalize Roberts' observations on the geography of the Katz algorithm [123] to the nonrigid case.*

It would be good to compare explicitly what is happening in our presentation, which basically follows Kostov's notation and setup, with the notation and setup used by Crawley-Boevey. Note that in Crawley-Boevey's point of view, the Katz operations are root reflections, and he uses several reflections in a row to get into a positive Weyl chamber before giving an explicit construction. This is obviously basically the same procedure as what we are doing here. It would be good to compare the numbers, and also to recover Roberts' results and observations [123] in the Crawley-Boevey formulation.

What is the exact relationship between our de Rham version of the middle convolution, and the algebraic operations on Fuchsian systems considered by Kostov, Haraoka-Yokoyama, Crawley-Boevey?

Theorem 2.9 says that the middle cohomology of $End(E)$ gives the deformation and obstruction theory for the moduli space of representations with fixed conjugacy classes on a curve. Remembering that the middle cohomology is really intersection cohomology, this suggests that we should ask for the geometric interpretation of the intersection cohomology of $End(E)$ in the higher dimensional case. More precisely, is there a natural derived moduli stack of local systems generalizing Kapranov [80] based on the intersection cohomology? And, what kind of geometric objects does this derived moduli stack parametrize?

7.1. **Low-dimensional cases.** One of the main reasons for looking at moduli spaces of representations on the punctured Riemann sphere is that these give many more examples with small dimension, than are obtained from Hitchin's original case of compact Riemann surfaces. This was first pointed out by Hausel [65] with his "toy example". We have constructed local systems whenever the virtual dimension is ≥ 4. In some sense the first case to look at is dimension 2, which has to be one of the cases listed in Lemma 2.13. Unfortunately, our technique of construction broke down in this case, but we can hope to have a variant.

The explicit techniques applied by Gleizer in the rigid case [55] should be applicable to low-dimensional cases.

It would be interesting to compute as explicitly as possible all of the various structures and properties for some concrete low-dimensional cases. For example, what does the $\mathbb{C}^*$ action on the moduli space look like? Some things to study in low-dimensional cases would be: compactifications [65] and their the dynamics [37] [38] [39] [12], the Hitchin system [70] [66], the relationship with Painlevé equations [71] [14] [15] [47], jumps and wall-crossing phenomena such as in [134] [114], real structures and Toledo invariants [24] [26] [106] [140].

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