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An extension to the Wiener space of the arbitrary functions principle

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Abstract

The arbitrary functions principle says that the fractional part of $nX$ converges stably to an independent random variable uniformly distributed on the unit interval, as soon as the random variable $X$ possesses a density or a characteristic function vanishing at infinity. We prove a similar property for random variables defined on the Wiener space when the stochastic measure $dB_s$ is crumpled on itself. To cite this article: N. Bouleau, C. R. Acad. Sci. Paris, Ser. I 343, (2006), 329-332.

Résumé


1 Introduction

Let us denote $\{x\}$ the fractional part of the real number $x$ and $\stackrel{d}{\Rightarrow}$ the weak convergence of random variables. Let $(X,Y)$ be a pair of random variables
with values in $\mathbb{R} \times \mathbb{R}^r$, we refer to the following property or its extensions as the arbitrary functions principle:

$$\left(\{nX\}, Y\right) \overset{d}{\Rightarrow} (U, Y) \quad (1)$$

where $U$ is uniformly distributed on $[0, 1]$ independent of $Y$.

This property is satisfied when $X$ has a density or more generally a characteristic function vanishing at infinity. (cf [5] Chap. VIII §92 and §93, [2], [4]). It yields an approximation property of $X$ by the random variable $X_n = X - \frac{1}{n}\{nX\} = \frac{\lfloor nX \rfloor}{n}$ where $\lfloor x \rfloor$ denotes the entire part of $x$.

**Proposition 1.** Let $X$ be a real random variable with density and $Y$ a random variable with values in $\mathbb{R}^r$. Let $X_n = \frac{\lfloor nX \rfloor}{n}$

a) For all $\varphi \in C^1 \cap \text{Lip}(\mathbb{R})$ and for all integrable random variable $Z$,

$$(n(\varphi(X_n) - \varphi(X)), Y) \overset{d}{\Rightarrow} (-U\varphi'(X), Y)$$

$$n^2\mathbb{E}[(\varphi(X_n) - \varphi(X))^2] \rightarrow \frac{1}{3}\mathbb{E}[\varphi^2(X)Z]$$

where $U$ is uniformly distributed on $[0, 1]$ independent of $(X, Y)$.

b) $\forall \psi \in L^1([0, 1])$

$$(\psi(n(X_n - X)), Y) \overset{d}{\Rightarrow} (\psi(-U), Y)$$

under any probability measure $\tilde{P} \ll P$.

We extend such results to random variables defined on the Wiener space.

## 2 Periodic isometries.

Let $(B_t)$ be a standard $d$-dimensional Brownian motion and let $m$ be the Wiener measure, law of $B$. Let $t \mapsto M_t$ be a bounded deterministic measurable map, periodic with unit period, into the space of orthogonal $d \times d$-matrices such that $\int_0^1 M_t ds = 0$ (e.g. a rotation in $\mathbb{R}^d$ of angle $2\pi t$). The transform $B_t \mapsto \int_0^t M_s dB_s$ defines an isometric endomorphism in $L^p(m)$, $1 \leq p \leq \infty$. Let be $M_n(s) = M(ns)$ and $T_n = T_{M_n}$. The transposed of the matrix $N$ is denoted $N^*$. 

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Proposition 2. Let be \(X \in L^1(m)\). Let \(\bar{m}\) be a probability measure absolutely continuous w.r. to \(m\). Under \(\bar{m}\) we have
\[
(T_n(X), B) \xrightarrow{d} (X(w), B).
\]

The weak convergence acts on \(\mathbb{R} \times C([0, 1])\) and \(X(w)\) denotes a random variable with the same law as \(X\) had under \(m\) function of a Brownian motion \(W\) independent of \(B\).

Proof. a) If \(X = \exp\{i \int_0^1 \xi_dB + \frac{1}{2} \int_0^1 |\xi|^2 ds\}\) for some element \(\xi \in L^2([0, 1], \mathbb{R}^d)\), we have \(T_n(X) = \exp\{i \int_0^1 \xi^*_n M_n(s)dB_s + \frac{1}{2} \int_0^1 |\xi|^2 ds\}\).

Putting \(Z^n_t = \int_0^t \xi^*_s M_n(s)dB_s\) gives \((Z^n, Z^n)_t = \int_0^t \xi^*_s M_n(s)\xi_s ds = \int_0^t |\xi|^2(s)ds\) which is a continuous function. Now by proposition 1, \(\int_0^t \xi^*_s M_n(s)ds \rightarrow \int_0^t \xi^*_s M(s)ds = 0\) which implies by Ascoli theorem \(\sup_t |\int_0^t \xi^*_s M_n(s)ds| \rightarrow 0\). The argument of H. Rootzén [6] applies and yields \((\int_0^t \xi^*_s dW, B) \xrightarrow{d} (\int_0^t \xi_dW, B)\) giving the result in this case by continuity of the exponential function.

b) When \(X \in L^1(m)\), we approximate \(X\) by linear combination of exponentials of the preceding type and consider the characteristic functions. The inequality
\[
|\mathbb{E}[e^{iuT_n(X)}] - \mathbb{E}[e^{iuT(X_k)}]| \leq |u| |\mathbb{E}|T_n(X) - T(X_k)| = |u| \|X - X_k\|_{L^1}
\]
gives the result.

c) This extends to the case \(\bar{m} \ll m\) by the properties of stable convergence.

3  Approximation of the Ornstein-Uhlenbeck structure.

From now on, we assume for simplicity that \((B)\) is one-dimensional. Let \(\theta\) be a periodic real function with unit period such that \(\int_0^1 \theta(s)ds = 0\) and \(\int_0^1 \theta^2(s)ds = 1\). We consider the transform \(R_n\) of the space \(L^2_{\mathbb{C}}(m)\) defined by its action on the Wiener chaos:

If \(X = \int_{s_1<...<s_k} \hat{f}(s_1, \ldots, s_k)dB_{s_1} \ldots dB_{s_k}\) for \(\hat{f} \in L^2_{\text{sym}}([0, 1]^k, \mathbb{C})\),
\[
R_n(X) = \int_{s_1<...<s_k} \hat{f}(s_1, \ldots, s_k)e^{i\frac{n}{\pi}\theta(s_1)}dB_{s_1} \ldots e^{i\frac{n}{\pi}\theta(s_k)}dB_{s_k}.
\]
$R_n$ is an isometry from $L^2_2(m)$ into itself. From $n(e^{\frac{1}{n}\sum_{p=1}^k \theta(ns_p)} - 1) = i \sum_{p=1}^k \theta(ns_p) \int_0^1 e^{\alpha \frac{1}{n} \sum_p \theta(ns_p)} d\alpha$ it follows that if $X$ belongs to the $k$-th chaos
\[ \|n(R_n(X) - X)\|_{L^2}^2 \leq k^2 \|X\|_{L^2}^2 \|\theta\|_\infty^2. \]
In other words, denoting $A$ the Ornstein-Uhlenbeck operator, $X \in \mathcal{D}(A)$ implies
\[ \|n(R_n(X) - X)\|_{L^2} \leq 2 \|AX\|_{L^2} \|\theta\|_\infty \]
and this leads to

**Proposition 3.** If $X \in \mathcal{D}(A)$
\[ (-in(R_n(X) - X), B) \overset{d}{=} (X^\#(\omega, w), B) \]
where $W$ is an Brownian motion independent of $B$ and $X^\# = \int_0^1 D_s X \, dW_s$.

**Proof.** If $X$ belongs to the $k$-th chaos, expanding the exponential by its Taylor series gives
\[ n(R_n(X) - X) = i \int_{s_1 < \cdots < s_k} \hat{f}(s_1, \ldots, s_k) \sum_{p=1}^k \theta(ns_p) dB_{s_1} \ldots dB_{s_k} + Q_n \]
with $\|Q_n\|^2 \leq \frac{1}{k^2} \|\theta\|^2_\infty \|X\|^2$.

Then using that $\int_{s_1 < \cdots < s_p < \cdots < s_k} h(s_1, \ldots, s_k) \theta(ns_p) dB_{s_1} \ldots dB_{s_p} \ldots dB_{s_k}$ converges stably to $\int_{s_1 < \cdots < s_p < \cdots < s_k} h(s_1, \ldots, s_k) dB_{s_1} \ldots dB_{s_p} \ldots dB_{s_k}$ one gets
\[ -in(R_n(X) - X) \overset{d}{=} \int_{t < s_2 < \cdots < s_k} \hat{f}(t, s_2, \ldots, s_k) dW_t dB_{s_2} \ldots dB_{s_k} + \sum_{t < s_1 < \cdots < s_k} \hat{f}(s_1, t, \ldots, s_k) dB_{s_1} dW_t \ldots dB_{s_k} + \cdots + \int_{s_1 < \cdots < s_{k-1} < t} \hat{f}(s_1, \ldots, s_{k-1}, t) dB_{s_1} \ldots dB_{s_{k-1}} dW_t \]
which equals $\int D_s(X) \, dW_s = X^\#$.

The general case in obtained by approximation of $X$ by $X_k$ for the $\mathbb{D}^{2,2}$ norm and the same argument as in the proof of proposition 2 by the characteristic functions gives the result.

By the properties of stable convergence, the weak convergence of prop. 3 also holds under $\tilde{m} \ll m$. By similar computations we obtain

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Proposition 4. \( \forall X \in \mathcal{D}(A) \)

\[
n^2\mathbb{E}[(R_n(X) - X)^2] \to 2\mathcal{E}[X]
\]

where \( \mathcal{E} \) is the Dirichlet form associated with the Ornstein-Uhlenbeck operator.

Following the same lines, it is possible to show that the theoretical \( \overline{A} \) and practical \( \underline{A} \) bias operators (cf. [1]) defined on the algebra \( \mathcal{L}\{e^{\int \xi dB} ; \xi \in C^1 \} \) by

\[
\begin{align*}
n^2\mathbb{E}[(R_n(X) - X)Y] &= \langle \overline{A}X, Y \rangle_{L^2(m)} \\
n^2\mathbb{E}[(X - R_n(X))R_n(Y)] &= \langle \underline{A}X, Y \rangle_{L^2(m)}
\end{align*}
\]

are defined and equal to \( \underline{A} \).

Comment. The preceding properties are very similar to the results concerning the weak asymptotic error for the resolution of SDEs by the Euler scheme, involving also an “extra”-Brownian motion (cf. [3]).

Nevertheless these results do not use the arbitrary functions principle because a convergence like \( (n \int_0^t (s - \lfloor ns \rfloor) dB_s, B) \xrightarrow{d} (\frac{1}{\sqrt{12}}W + \frac{1}{2}B, B) \) is hidden by a dominating phenomenon \( (\sqrt{n} \int_0^t (B_s - B_{\lfloor ns \rfloor}) dB_s, B) \xrightarrow{d} (\frac{1}{\sqrt{2}}\tilde{W}, B) \) due to the fact that when a sequence of variables in the second (or higher order) chaos converges stably to a Gaussian variable, this one appears to be independent of the first chaos and therefore of \( B \).

The arbitrary functions principle is slightly different, it is a crumpling of the random orthogonal measure \( dB_s \) on itself. This operates even on the first chaos. Concerning the solution of SDEs by the Euler scheme, it is in force for SDEs of the form

\[
\begin{align*}
X^1_t &= x^1_0 + \int_0^t f^{11}(X^1_s)dB_s + \int_0^t f^{12}(X^1_s, X^2_s)ds \\
X^2_t &= x^2_0 + \int_0^t f^{21}(X^1_s)dB_s + \int_0^t f^{22}(X^1_s, X^2_s)ds
\end{align*}
\]

where \( X^1 \) is with values in \( \mathbb{R}^{k_1} \), \( X^2 \) in \( \mathbb{R}^{k_2} \), \( B \) in \( \mathbb{R}^d \) and \( f^{ij} \) are matrices with suitable dimensions which are encountered for the description of mechanical systems under noisy solicitations when the noise depend only on the position of the system and the time. In such equations, integration by parts reduces the stochastic integrals to ordinary integrals and it may be shown that solved by the Euler scheme they present a weak asymptotic error in \( \frac{1}{n} \) instead of \( \frac{1}{\sqrt{n}} \) as usual.
References


