



HAL
open science

Stability of discrete systems involving shocks and friction

Patrick Ballard, Alain Léger, Elaine Pratt

► **To cite this version:**

Patrick Ballard, Alain Léger, Elaine Pratt. Stability of discrete systems involving shocks and friction. Conference CMIS on Contact Mechanics, Jul 2005, Hannover, Germany. pp.343-351, 10.1007/3-540-31761-9_38 . hal-00106845

HAL Id: hal-00106845

<https://hal.science/hal-00106845>

Submitted on 24 Oct 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Stability of discrete systems involving shocks and friction

P. Ballard, A. Léger, and E. Pratt

Laboratoire de Mécanique et d'Acoustique - CNRS, 31, Chemin Joseph Aiguier,
13402 Marseille Cedex20, France,
ballard@lma.cnrs-mrs.fr, leger@lma.cnrs-mrs.fr, elaine@lma.cnrs-mrs.fr

Abstract. The stability of the equilibrium states of a simple mechanical system with unilateral contact and Coulomb friction is explored. When the external force is constant, the equilibrium states are completely determined by the mechanical properties of the system and the stability or instability of each of these states is proved. When the external force varies in time two stability results are given.

1 Introduction

The aim of this work is to determine the stability of equilibrium states for a mechanical system involving unilateral contact with Coulomb friction. Because of the nonsmoothness of such systems due to the presence of frictional contact, the usual methods used to determine the stability of equilibrium states are practically inoperative. The analysis is performed here by direct integration of the dynamics with initial data close to equilibrium. The stability or instability results are proved by considering the discrete dynamical system induced by the numerical integration method. The convergence of this method has been proved and used to establish the existence result.

We begin by setting the equations and by giving the existence and uniqueness results. In the case of a constant external force the set of equilibria is determined explicitly by the mechanical parameters of the system, namely its stiffness, friction coefficient and external forces. The stability results concerning all these equilibria are stated and the proof is briefly sketched.

We then consider the case where the external force depends on time. In fact we operate a small time dependant perturbation of the constant external force previously studied. Interesting results concerning the equilibria of the perturbed problem and the evolution in time of solutions to this problem with initial data out of equilibrium are given.

2 The basic equations and the dynamics

We restrict our attention to the mass-spring system, now quite classical since the work of Klarbring [4], represented on Figure 1.

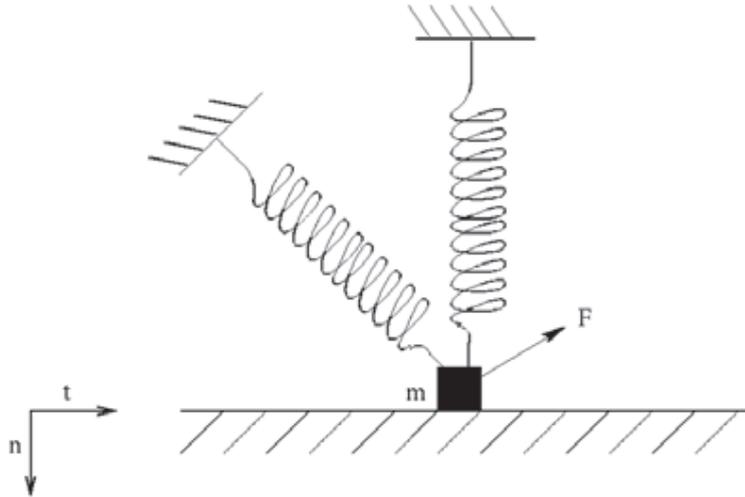


Fig. 1. The mass-spring model.

n and t denote respectively the normal and tangential components of the displacement U and of the reaction R of the mass m . We recall the nonregularized unilateral contact and Coulomb friction laws in which μ denotes the friction coefficient:

$$\begin{cases} U_n \leq 0, R_n \leq 0, U_n \cdot R_n = 0, \\ |R_t| \leq \mu |R_n|, \\ |R_t| < \mu |R_n| \implies \dot{U}_t = 0, \\ |R_t| = \mu |R_n| \implies \dot{U}_t = -\lambda R_t, \lambda \geq 0, \end{cases} \quad (1)$$

We denote by $MMA([0, T]; \mathbb{R}^2)$ (motions with measure acceleration) the vector space of those integrable functions of $[0, T]$ into \mathbb{R}^2 whose second derivative in the sense of distributions is a measure. It is nothing but the space of integrals of functions of bounded variation over $[0, T]$. Functions U in MMA are continuous and admit left and right derivatives (in the classical sense) \dot{U}^- , \dot{U}^+ , at any point, both being functions of bounded variation. We recall that a function of bounded variation, being a uniform limit of a sequence of step functions, is universally integrable (integrable with respect to any measure).

The evolution problem, formulated along the lines of MOREAU [7], [8], is the following:

Problem \mathcal{P} . Find $U \in MMA([0, T]; \mathbb{R}^2)$ and $R \in \mathcal{M}([0, T]; \mathbb{R}^2)$ such that:

- $U(0) = U_0 \quad ; \quad \dot{U}^+(0) = V_0$
- $\ddot{U} + K \cdot U = F + R, \quad \text{in } [0, T]$
- $U_n \leq 0, \quad R_n \leq 0, \quad U_n R_n = 0$
- $$\int_{[0, T]} \left[R_t \cdot (V - \dot{U}_T^+) - \mu R_n (|V| - |\dot{U}_t^+|) \right] \geq 0, \quad (2)$$

 $\forall V \in C^0([0, T]; \mathbb{R})$
- $U_n(t) = 0 \quad \implies \quad \dot{U}_n^+(t) = -e \dot{U}_n^-(t), \quad \text{in }]0, T].$

F denotes the external force, $e \in [0, 1]$ a real constant (the so-called restitution coefficient) and (U_0, V_0) some initial condition, assumed to be compatible with the unilateral constraint, that is

$$U_{0n} \leq 0 \quad \text{and} \quad U_{0n} = 0 \quad \implies \quad V_{0n} \leq 0. \quad (3)$$

In the following we consider completely inelastic shocks, i.e. $e = 0$. In this case, the unilateral contact conditions and the shock law can be replaced in (2) by

$$U_n \leq 0 \quad \text{with} \quad \begin{cases} U_n < 0 \implies R_n = 0, \\ U_n = 0 \implies \dot{U}_n^+ \leq 0, R_n \leq 0, \dot{U}_n^+ R_n = 0. \end{cases} \quad (4)$$

It has been recently established that problem (\mathcal{P}) has a solution as soon as the data F is an integrable function of $[0, T]$. The proof of this result uses the convergence of a time discretization, and is heavily inspired by the proof given by Monteiro-Marques in [6]. The main qualitative point of this proof is that the time discretization is the same as that of the numerical software *NSCD* [3] and will be used in the following as a discrete dynamical system.

It has also been obtained recently [1] that, for given initial data, the trajectory is not generally unique, and that uniqueness holds only if the data F is an analytical function of $[0, T]$.

3 The set of equilibria

The equilibrium states of problem (\mathcal{P}), which have been explored in [2], are the set of displacements U and reactions R satisfying, in addition to conditions (1), the following system:

$$\begin{cases} K_t \cdot U_t + W \cdot U_n = F_t + R_t \\ W \cdot U_t + K_n \cdot U_n = F_n + R_n, \end{cases} \quad (5)$$

where (F_t, F_n) denotes the external force and $K = \begin{pmatrix} K_t & W \\ W & K_n \end{pmatrix}$ the stiffness matrix of the system of springs.

Looking first for solutions without contact, immediatly leads to:

$$\begin{cases} U_n = \frac{A}{\det K}, & U_t = \frac{K_n \cdot F_t - W \cdot F_n}{\det K} \\ R_n = R_t = 0, \end{cases} \quad (6)$$

where quantity A is defined as $A = K_t \cdot F_n - W \cdot F_t$. Conditions (1) then imply that (6) will be an equilibrium solution only if $A \leq 0$.

Looking for equilibrium solutions in contact with the obstacle, that is such that $U_n = 0$, we find that system (1), (5) reduces to:

$$\begin{cases} W \cdot U_t = F_n + R_n \\ K_t \cdot U_t = F_t + R_t \\ |R_t| \leq \mu |R_n|, \end{cases} \quad (7)$$

which gives

$$\begin{cases} R_t = \frac{K_t}{W} R_n + \frac{A}{W} \\ |R_t| \leq \mu |R_n|. \end{cases} \quad (8)$$

The solutions of (8) determine the equilibria in the $\{R_t, R_n\}$ plane. As shown on figure 2, these equilibria belong to the intersection of an affine line which represents the equilibrium equation and of the Coulomb cone.

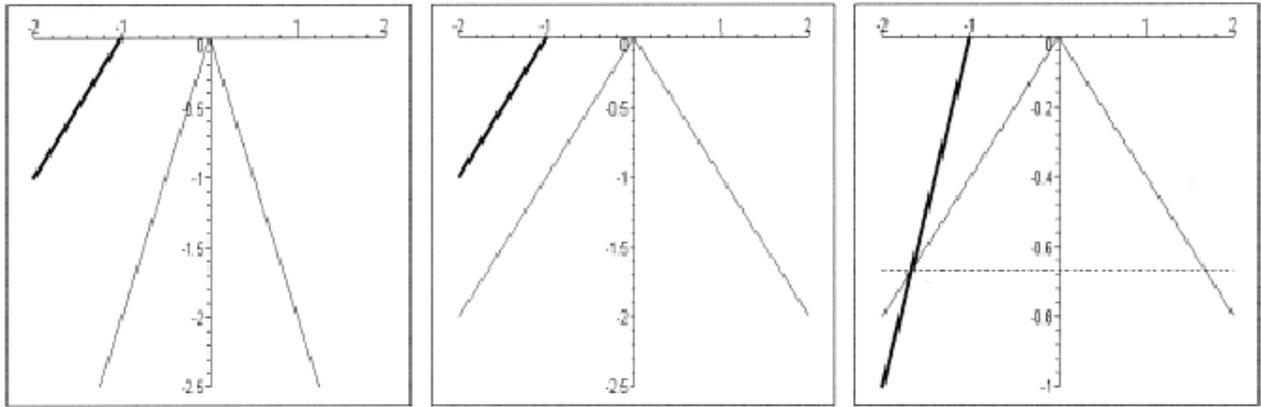
The dependance of the set of equilibria on the stiffness parameters, friction coefficient and external load is summarized in table 1.

4 Stability analysis

4.1 Main steps of the analysis

Having investigated the equilibrium states under constant data, we analyze the stability of these equilibria. As announced in the introduction, this stability analysis consists in choosing an initial data for problem (\mathcal{P}) in a neighborhood of any of the equilibria in a classical phase space, and studying the evolution in time of the distance between the corresponding trajectory and the equilibrium. This analysis uses estimates on the iterates of the time discretization, considered as a dynamical system, that was used to establish the existence result.

Proposition 1 summarizes the stability results in the case of a constant force. We conclude this section by studying the case where the external force varies in time.

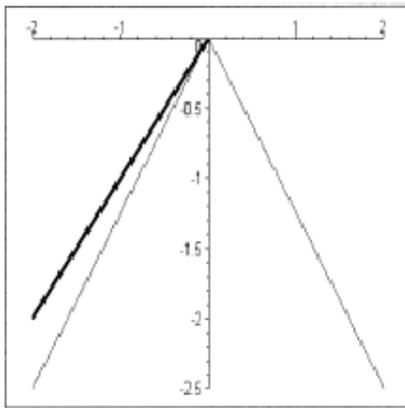


$$\mu < K_t/W$$

$$\mu = K_t/W$$

$$\mu > K_t/W$$

2.a: $A \equiv K_t F_n - W F_t < 0$

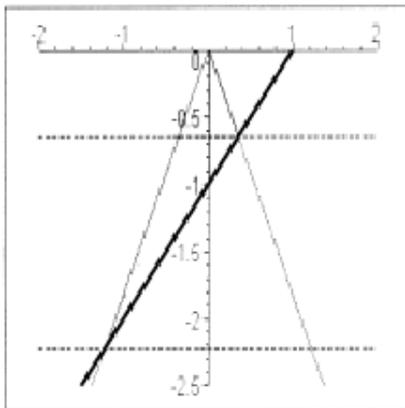


$$\mu < K_t/W$$

$$\mu = K_t/W$$

$$\mu > K_t/W$$

2.b: $A \equiv K_n F_n - W F_t = 0$



$$\mu < K_t/W$$

$$\mu = K_t/W$$

$$\mu > K_t/W$$

2.c: $A \equiv K_t F_n - W F_t > 0$

Fig. 2. The sets of equilibria in the $\{R_t, R_n\}$ plane.

Table 1. The equilibrium states with respect to parameters A and μ .

	$A < 0$	$A = 0$	$A > 0$
$\mu < \frac{K_T}{W}$	1 solution without contact	1 solution in grazing contact	2 solutions in impending positive and negative slip + infinitely many solutions in strict sticked contact
$\mu = \frac{K_T}{W}$	1 solution without contact	1 solution in grazing contact + infinitely many solutions in impending positive slip	1 solution in impending negative slip + infinitely many solutions in strict sticked contact
$\mu > \frac{K_T}{W}$	1 solution without contact + 1 solution in impending positive slip	1 solution in grazing contact + infinitely many solutions in strict sticked contact	1 solution in impending negative slip + infinitely many solutions in strict sticked contact

Proposition 1. *Under a constant external force*

- *if there are only two equilibria, the one which is in contact is unstable;*
- *if there are infinitely many equilibrium states in impending sliding, they are all unstable, except the vertex of the cone which is Lyapunov stable;*
- *as soon as there are infinitely many equilibrium states in strictly sticked contact, all the equilibria are Lyapunov stable;*
- *if the parameters are such that the vertex of the cone is the unique equilibrium state, then this equilibrium is asymptotically stable.*

These results are obtained thanks to several technical lemmas, of which we give hereafter three significative examples.

4.2 Some technical lemmas

The first two lemmas are simple and intuitive. Considering an equilibrium state in contact perturbed by a normal or a tangential velocity, then, under some conditions, the trajectory will come into contact again. More precisely:

Lemma 1. *Let (U^{eq}, R^{eq}) with R_n^{eq} strictly negative be an equilibrium solution, and V_{n0} a perturbation of this equilibrium at time t_0 , then there exists \bar{t} , $t_0 < \bar{t} < +\infty$, such that $U_n(\bar{t}) = 0$.*

Lemma 2. *Let (U^{eq}, R^{eq}) be a grazing equilibrium state ($R_n^{eq} = 0$ and $U_n^{eq} = 0$) and V_{t0} a perturbation of this equilibrium at time t_0 , then there exists \tilde{t} , $t_0 < \tilde{t} < +\infty$, such that $U_n(\tilde{t}) = 0$.*

The last lemma is more technical, and is already very close to a stability result:

Lemma 3. *Let (U^{eq}, R^{eq}) be an equilibrium state with R_n^{eq} strictly negative. Let the dynamics after any perturbation be such that there exists a time $t^* < +\infty$ with $V_t(t^*) = 0$ and $R_n(t^*) < 0$.*

Then, if there exists an equilibrium state $(\hat{U}^{eq}, \hat{R}^{eq})$ such that $R_n^{eq}(t^) = \hat{R}_n^{eq}$, $V_t(t) = 0 \forall t > t^*$.*

4.3 Stability under nonconstant forces

The two theorems given below prove a conjecture recently suggested to the authors by Michel Jean.

Let us choose the external force F under the form $F(t) = F_0 + \epsilon\xi(t)$, where $\xi(t)$ is an analytical function and ϵ a positive parameter. (\mathcal{P}_0) (resp. (\mathcal{P}_ϵ)) denotes problem (\mathcal{P}) where $\epsilon = 0$ (resp. $\epsilon > 0$). In fact (\mathcal{P}_0) represents the problem previously studied, i.e. with a constant external force and (\mathcal{P}_ϵ) a perturbation of (\mathcal{P}_0) .

A solution (U_ϵ, R_ϵ) of problem (\mathcal{P}_ϵ) where U_ϵ is constant will be referred to as a *space-equilibrium* solution.

The two following results are established:

Theorem 1. *The following statements i) and ii) are shown to be equivalent:*
i) (U_0, R_0) is an equilibrium solution of problem (\mathcal{P}_0) such that the reaction (R_{n0}, R_{t0}) is strictly inside the Coulomb cone.

ii) $\exists \epsilon_0 > 0$ such that $\forall \epsilon < \epsilon_0$, the solution (U_ϵ, R_ϵ) of problem (\mathcal{P}_ϵ) , obtained with the equilibrium solution of (\mathcal{P}_0) as initial data, is a space-equilibrium solution with $U_\epsilon = U_0$.

Theorem 2. *Assume ϵ is such that there exists a space-equilibrium solution to problem (\mathcal{P}_ϵ) . Then any solution of (\mathcal{P}_ϵ) , with a nongrazing equilibrium solution of problem (\mathcal{P}_0) as initial data, leads in finite time to a space-equilibrium solution of (\mathcal{P}_ϵ) .*

References

1. Ballard P. and Basseville S. (2005) Existence and uniqueness for dynamical unilateral contact with Coulomb friction: a model problem. *Math. Modelling and Num. Analysis*
2. Basseville S., Leger A. and Pratt E. (2003) Investigation of the equilibrium states and their stability for a simple model with unilateral contact and Coulomb friction. *Arch. Appl. Mechanics* 73: 409-420
3. Jean M. (1999) The nonsmooth contact dynamics method. *Cmp. Meth. Appl. Mech. Engn.* 177: 235-257
4. Klarbring A. (1990) Examples of nonexistence and nonuniqueness of solutions to quasistatic contact problems with friction. *Ing. Arch.* 60: 529-541
5. Martins J.A.C., Monteiro Marques M.D.P. and Gastaldi F. (1994) On an example of nonexistence of solution to a quasistatic frictional contact problem, *Eur. J. Mech. A/Solids* 13: 113-133
6. Monteiro Marques M.D.P. (1993) *Differential Inclusions in Nonsmooth Mechanical Problems*. Birkhäuser, Basel Boston Berlin
7. Moreau J.J. (1988) Standard inelastic shocks and the dynamics of unilateral constraints. In: Del Piero G, Marceri F. (eds) *Unilateral Problems in Structural Analysis*. Springer Verlag, Berlin Heidelberg New York
8. Moreau J.J. (1993) Unilateral contact and dry friction in finite freedom dynamics. In: Moreau J.J., Panagiotopoulos D.D. (eds) *Nonsmooth Mechanics and Applications*. CISM Courses and Lectures 302, Springer-Verlag, Berlin Heidelberg New York.