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ON THE MULTIPLICITIES OF A MOTIVE

BRUNO KAHN

Preliminary version

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INTRODUCTION

The aim of this article, in the vein of [1], is to study abstractly the properties of categories of pure motives and to make clear(er) which of them are formal and which are of a more arithmetic-geometric nature.

We work with a rigid tensor category \mathcal{A} such that $K = \text{End}(\mathbf{1})$ is a field of characteristic 0. Our main tool, which was the motivation for this work, is the *multiplicities* of an object $M \in \mathcal{A}$ when \mathcal{A} is semi-simple: they are a collection of central scalars which relates the categorical trace with the ring-theoretic trace (Proposition 1.2). It turns out that the property for these multiplicities to be *integers* or, better, to be so after extending scalars from K to its algebraic closure, is very well-behaved and is satisfied in many important cases. Namely:

- The full subcategory \mathcal{A}_{int} of \mathcal{A} formed by such objects is thick, tensor, rigid, contains the finite-dimensional objects in the sense of Kimura-O’Sullivan and is preserved under tensor functors to another semi-simple rigid category (Corollary 2.5).
- $\mathcal{A}_{\text{int}} = \mathcal{A}$ if \mathcal{A} is of “homological origin” (Theorem 4.5). The category of pure motives over a field modulo numerical equivalence is semi-simple thanks to Jannsen’s theorem [4], and of homological origin.

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When the multiplicities are integers, we prove that the zeta function of an endomorphism f of M is rational (with an explicit formula) and satisfies a functional equation when f is invertible (Theorem 3.2): in the case of motives over a finite field, this shows that these depend on less than the existence of a Weil cohomology theory. We also get some elementary cases where homological equivalence equals numerical equivalence for formal reasons in Proposition 4.6: of course, this remains far from leading to a proof of this famous standard conjecture!

In Section 5, we set out to formulate a version of the Tate conjecture for motives over a finite field in an abstract set-up. Surprisingly, most of the known equivalent versions of this conjecture carry out in this abstract context: see Theorem 5.4 and Corollary 5.10.

1. MULTIPLICITIES IN SEMI-SIMPLE RIGID TENSOR CATEGORIES

Let \mathcal{A} be a rigid K -linear tensor category, where K is a field of characteristic 0; we also assume that $\text{End}(\mathbf{1}) = K$. In the sequel of this article, we shall abbreviate this by saying that \mathcal{A} is a *rigid K -category*.

Let $M \in \mathcal{A}$. The *trace* of an endomorphism $f \in \text{End}(M)$ is the element $\text{tr}(f) \in \text{End}(\mathbf{1}) = K$ defined by the composition

$$\mathbf{1} \xrightarrow{\eta} M^* \otimes M \xrightarrow{1 \otimes f} M^* \otimes M \xrightarrow{R} M \otimes M^* \xrightarrow{\varepsilon} \mathbf{1}$$

where R is the switch and η, ε are the duality structures of M . The trace is K -linear and has the following properties:

$$(1.1) \quad \text{tr}(fg) = \text{tr}(gf), \quad \text{tr}(f \otimes g) = \text{tr}(f) \text{tr}(g), \quad \text{tr}({}^t f) = \text{tr}(f).$$

Suppose that \mathcal{A} is semi-simple. Then $\text{End}_{\mathcal{A}}(M)$ is a semi-simple K -algebra, hence has its own trace, and we want to compare the categorical trace with the ring-theoretic trace. We normalise conventions as follows:

1.1. Definition. a) Let A be a finite-dimensional simple K -algebra. We write:

- $Z(A)$ for the centre of A ;
- $\delta(A) = [Z(A) : K]$;
- $d(A) = [A : Z(A)]^{1/2}$.

We define the *reduced trace* of A as

$$\text{Trd}_A = \text{Tr}_{Z(A)/K} \circ \text{Tr}_{A/Z(A)}.$$

If $A = \prod A_i$ is semi-simple, with simple components A_i , we define $\text{Trd}_A := \sum_i \text{Trd}_{A_i}$.

b) If $A = \text{End}_{\mathcal{A}}(M)$, we set

- $Z_i(M) = Z(A_i)$;

- $\delta_i(M) = \delta(A_i)$;
- $d_i(M) = d(A_i)$;
- $\text{Trd}_M = \text{Trd}_A$.

1.2. Proposition. *There exists a unique element $\mu(M) \in \text{End}(M)$ such that*

$$\text{tr}(f) = \text{Trd}_M(\mu(M)f)$$

for any $f \in \text{End}(M)$. Moreover, $\mu(M)$ is central and invertible. Hence, if (e_i) denotes the set of central idempotents of $A = \text{End}(M)$ corresponding to its simple factors A_i , we may write

$$\mu(M) = \sum_i \mu_i(M)e_i$$

with $\mu_i(M) \in Z_i(M)$.

Proof. Since $\text{End}(M)$ is semi-simple, $(f, g) \mapsto \text{Trd}_M(fg)$ is nondegenerate, which proves the existence and uniqueness of $\mu(M)$. Moreover,

$$\text{Trd}_M(\mu(M)fg) = \text{tr}(fg) = \text{tr}(gf) = \text{Trd}_M(\mu(M)gf) = \text{Trd}_M(f\mu(M)g)$$

and the non-degeneracy also yields the centrality of $\mu(M)$. This element is invertible because the ideal \mathcal{N} is 0 for \mathcal{A} [1, 7.1.7]. The last assertion is obvious. \square

1.3. Lemma. a) *We have $\mu(M^*) = {}^t\mu(M)$.*

b) *Suppose K algebraically closed and M simple. Then $\mu(M) = \dim(M)$.*

Proof. a) follows easily from (1.1) and the fact that the transposition induces an anti-isomorphism from $\text{End}(M)$ onto $\text{End}(M^*)$. b) is obvious, since then $\text{End}(M) = K$ (recall that, by definition, $\dim(M) = \text{tr}(1_M)$). \square

1.4. Remark. If \mathcal{A} is pseudo-abelian (hence abelian), the idempotents e_i of Proposition 1.2 yield the decomposition $M = \bigoplus M_i$ of M into its *isotypical components*. In particular, if S is simple, then $\mu(S^n) = \mu(S)$ for any $n \geq 1$.

On the other hand, it is difficult to relate $\mu(M_1), \mu(M_2)$ and $\mu(M_1 \otimes M_2)$ in general because it is difficult to say something of the map $\text{End}(M_1) \otimes_K \text{End}(M_2) \rightarrow \text{End}(M_1 \otimes M_2)$: it is not even true in general that such a homomorphism sends the centre into the centre. For the same reason, it is difficult to state general facts on the behaviour of the invariant μ under tensor functors. We shall see that this situation improves considerably in the case of *geometrically integral type*, treated in the next section.

2. INTEGRAL MULTIPLICITIES

In all this section, \mathcal{A} is a semi-simple rigid K -category.

2.1. Definition. a) An object $M \in \mathcal{A}$ is of *integral type* if the scalars $\mu_i(M)$ of Proposition 1.2 belong to \mathbf{Z} .

b) M is *geometrically of integral type* if $M_{\bar{K}} \in \mathcal{A}_{\bar{K}}$ is of integral type, where \bar{K} is an algebraic closure of K .

c) \mathcal{A} is of *integral type* (resp. *geometrically of integral type*) if every $M \in \mathcal{A}$ is of integral type (resp. geometrically of integral type).

2.2. Proposition. a) If M is of integral type, we have

$$(2.1) \quad \mu_i(M) = \frac{\mathrm{tr}(e_i)}{\delta_i(M)d_i(M)}$$

for any i .

b) Direct sums and direct summands of objects of integral type are of integral type. Similarly for geometrically of integral type. In particular, \mathcal{A} is of integral type (resp. geometrically of integral type) if and only if its pseudo-abelian envelope is.

c) If M is geometrically of integral type, then it is of integral type. Moreover, if this is the case, the invariants $\mu_i(M)$ are “geometric” in the sense that if L/K is any extension, then $\mu_i(M) = \mu_{i,\alpha}(M_L)$ for any simple factor $A_{i,\alpha}$ of $A_i \otimes_K L$.

d) $M \in \mathcal{A}$ is geometrically of integral type if and only if, in $\mathcal{A}_{\bar{K}}^{\natural}$, the dimension of every simple summand of $M_{\bar{K}}$ is an integer.

e) If M is finite-dimensional in the sense of Kimura-O’Sullivan, then M is geometrically of integral type.

f) If M is geometrically of integral type, so is M^* .

Proof. a) and b) are obvious. For c), we have the decomposition

$$Z_i(M) \otimes_K \bar{K} \xrightarrow{\sim} \prod_{\alpha} \bar{K}$$

where α runs through the distinct K -embeddings of $Z_i(M)$ into \bar{K} . Correspondingly, $A_i \otimes_K \bar{K}$ decomposes as a direct product

$$A_i \otimes_K \bar{K} \simeq \prod_{\alpha} A_i^{\alpha}$$

with A_i^{α} simple over \bar{K} . This gives a decomposition

$$e_i \otimes_K 1 = \sum_{\alpha} e_i^{\alpha}$$

into central idempotents. But clearly, $\mu(M_{\bar{K}}) = \mu(M) \otimes_K 1$. By hypothesis, the images of $\mu_i(M)$ in \bar{K} under the embeddings α are rational integers, which implies that $\mu_i(M)$ is itself a rational integer. The additional claim of c) immediately follows from this proof.

d) follows immediately from Lemma 1.3 a). For e), we may assume that K is algebraically closed, \mathcal{A} pseudo-abelian and M simple; then $\dim(M)$ is an integer since M is finite dimensional [1, Th. 9.1.7]. Finally, f) follows from Lemma 1.3 b). \square

2.3. Theorem. *Let $M, N \in \mathcal{A}$ be geometrically of integral type, (e_i) the central idempotents of $\text{End}(M)$ and (f_j) the central idempotents of $\text{End}(N)$. For a pair (i, j) , let A_{ij} be the semi-simple algebra $(e_i \otimes f_j) \text{End}(M \otimes N)(e_i \otimes f_j)$. Then one has formulas of the type*

$$\mu_i(M)\mu_j(N) = \sum_k m_k \mu_k(M \otimes N)$$

where k indexes the simple factors of A_{ij} and the m_k are integers ≥ 0 . Moreover, for any k , there is such a formula with $m_k > 0$. In particular, $M \otimes N$ is geometrically of integral type.

Proof. We proceed in 2 steps:

1) $\text{End}(M)$ and $\text{End}(N)$ are split. By Proposition 2.2 b), we may assume that \mathcal{A} is pseudo-abelian. This allows us to assume M and N simple, hence $\text{End}(M) = \text{End}(N) = K$ and $A_{ij} = \text{End}(M \otimes N)$. Using Formula (2.1) to compute $\text{tr}(1_M \otimes 1_N)$ in two different ways, we get the formula

$$(2.2) \quad \mu(M)\mu(N) = \sum m_k \mu_k(M \otimes N)$$

with $m_k = \delta_k(M \otimes N)d_k(M \otimes N)$.

Coming back to the case where $\bar{\mathcal{A}}$ is not necessarily pseudo-abelian, this gives the formula

$$m_k = \delta_k(A_{ij}) \frac{d_k(A_{ij})}{d_i(M)d_j(N)}$$

and the previous argument shows ungrudgingly that this is an integer.

2) The general case. Extending scalars to \bar{K} and using Proposition 2.2 c), we are reduced to 1) as follows: for any $\alpha : Z_i(M) \rightarrow \bar{K}$ and any $\beta : Z_j(N) \rightarrow \bar{K}$, we have a formula with obvious notation:

$$\mu_i^\alpha(M_{\bar{K}})\mu_j^\beta(N_{\bar{K}}) = \sum_k \sum_\gamma m_k^\gamma \mu_k^\gamma((M \otimes N)_{\bar{K}})$$

where, for each k , γ runs through the embeddings of $Z_k(M \otimes N)$ into \bar{K} . By Remark 1.4, this gives a formula as wanted.

It remains to prove the last assertion: for this, it suffices to show that, given a simple factor A_k of A_{ij} , there is a pair (α, β) such that

$$\mathrm{Hom}_{\bar{K}}(A_k \otimes_K \bar{K}, (e_i^\alpha \otimes f_j^\beta)(A_{ij} \otimes_K \bar{K})(e_i^\alpha \otimes f_j^\beta)) \neq 0.$$

This is obvious, since $\mathrm{Hom}_K(A_k, A_{ij}) \neq 0$ and $A_{ij} \otimes_K \bar{K} = \prod_{\alpha, \beta} (e_i^\alpha \otimes f_j^\beta)(A_{ij} \otimes_K \bar{K})(e_i^\alpha \otimes f_j^\beta)$. \square

2.4. Corollary. *Assume that M and N are simple and that, in Theorem 2.3, all terms $\mu_k(M \otimes N)$ have the same sign. Then we have $|\mu_k(M \otimes N)| \leq |\mu(M)\mu(N)|$ for all k . If $|\mu(M)| = |\mu(N)| = 1$, then $A = \mathrm{End}(M \otimes N)$ is “geometrically simple” in the sense that $A \otimes_K \bar{K}$ is a matrix algebra over $Z(M \otimes N) \otimes_K \bar{K}$ (otherwise said, A is an Azumaya algebra over its centre). Moreover, $\mu(M \otimes N) = \mu(M)\mu(N)$.*

Proof. This follows from the last statement of Theorem 2.3. In the case where $|\mu(M)| = |\mu(N)| = 1$, Formula (2.2) gives the conclusion. \square

2.5. Corollary. *a) The full subcategory $\mathcal{A}_{\mathrm{int}}$ of \mathcal{A} consisting of geometrically integral objects is a thick rigid tensor subcategory of \mathcal{A} , containing the finite dimensional objects.*

b) Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a \otimes -functor to another rigid semi-simple K -category. Then $F(\mathcal{A}_{\mathrm{int}}) \subseteq \mathcal{B}_{\mathrm{int}}$.

Proof. a) follows from Proposition 2.2 and Theorem 2.3; b) follows from Proposition 2.2 d). \square

3. APPLICATION: THE ZETA FUNCTION OF AN ENDOMORPHISM

3.1. Definition. Let \mathcal{A} be a rigid K -category, $M \in \mathcal{A}$ and $f \in \mathrm{End}(M)$. The *zeta function* of f is

$$Z(f, t) = \exp \left(\sum_{k \geq 1} \mathrm{tr}(f^k) \frac{t^k}{k} \right) \in K[[t]].$$

3.2. Theorem. *Suppose that \mathcal{A} is semi-simple and that $M \in \mathcal{A}$ is of integral type. Then,*

a) For any $f \in \mathrm{End}(M)$, $Z(f, t) \in K(t)$. More precisely, one has with the notation of Definition 1.1

$$Z(f, t) = \prod_i \mathrm{Nrd}_{A_i}(e_i - e_i f t)^{-\mu_i(M)}$$

where, for all i , $\mathrm{Nrd}_{A_i}(e_i - e_i f t) := N_{Z_i(M)/F} \mathrm{Nrd}_{A_i/Z_i(M)}(e_i - e_i f t)$ denotes the inverse reduced characteristic polynomial of the element

$e_i f$ if A_i .

b) If f is invertible, one has the functional equation

$$Z(f^{-1}, t^{-1}) = (-t)^{\chi(M)} \det(f) Z(f, t)$$

where $\chi(M) = \text{tr}(1_M)$ and $\det(f) = \prod_i \text{Nrd}_{A_i}(e_i f)^{\mu_i(M)}$.

Proof. a) Applying the formula of Proposition 1.2, we get

$$\begin{aligned} Z(f, t) &= \exp \left(\sum_{k \geq 1} \text{Trd}_M(\mu(M) f^k) \frac{t^k}{k} \right) \\ &= \exp \left(\sum_{k \geq 1} \sum_i \text{Trd}_M(\mu_i(M) e_i f^k) \frac{t^k}{k} \right) \\ &= \prod_i \exp \left(\sum_{k \geq 1} \text{Trd}_{A_i}((e_i f)^k) \frac{t^k}{k} \right)^{\mu_i(M)} \end{aligned}$$

and the conclusion follows from the well-known linear algebra identity

$$\exp \left(\sum_{k \geq 1} \text{Trd}_{A_i}((e_i f)^k) \frac{t^k}{k} \right) = \text{Nrd}_{A_i}(e_i - e_i f t)^{-1}.$$

For b), we write

$$\text{Nrd}_{A_i}(e_i - e_i f^{-1} t^{-1}) = \text{Nrd}_{A_i}(-e_i f^{-1} t^{-1}) \text{Nrd}_{A_i}(e_i - e_i f t)$$

hence

$$\begin{aligned} Z(f^{-1}, t^{-1}) &= \prod_i \text{Nrd}_{A_i}(e_i - e_i f^{-1} t^{-1})^{-\mu_i(M)} \\ &= \prod_i \text{Nrd}_{A_i}(-e_i f^{-1} t^{-1})^{-\mu_i(M)} \text{Nrd}_{A_i}(e_i - e_i f t)^{-\mu_i(M)} \\ &= \prod_i \text{Nrd}_{A_i}(-e_i f^{-1} t^{-1})^{-\mu_i(M)} Z(f, t) \end{aligned}$$

and

$$\begin{aligned} \prod_i \text{Nrd}_{A_i}(-e_i f^{-1} t^{-1})^{-\mu_i(M)} &= \\ &= (-t)^{\sum_i \mu_i(M) d_i(M) \delta_i(M)} \prod_i \text{Nrd}_{A_i}(e_i f)^{\mu_i(M)} = (-t)^{\chi(M)} \det(f). \end{aligned}$$

□

3.3. Remark. The definition of \det shows that

$$\det(1 - ft) = Z(f, t)^{-1}$$

if the left hand side is computed in $\mathcal{A}_{K(t)}$.

4. MULTIPLICITIES IN RIGID TENSOR CATEGORIES OF HOMOLOGICAL TYPE

4.1. Definition. a) A rigid K -category \mathcal{A} is of *homological type* if there exists a tensor functor

$$H : \mathcal{A} \rightarrow \text{Vec}_L^\pm$$

where L is an extension of K and Vec_L^\pm is the tensor category of $\mathbf{Z}/2$ -graded finite-dimensional L -vector spaces, provided with the Koszul rule for the commutativity constraint. We say that H is a *realisation* of \mathcal{A} .

We say that \mathcal{A} is *neutrally of homological type* if one may choose $L = K$.

b) A semi-simple rigid K -category $\bar{\mathcal{A}}$ is of *homological origin* (*resp. neutrally of homological origin*) if it is \otimes -equivalent to \mathcal{A}/\mathcal{N} , where \mathcal{A} is a rigid K -category of homological type (*resp. neutrally of homological type*) and $\mathcal{N} = \mathcal{N}(\mathcal{A})$ is the ideal of morphisms universally of trace 0.

4.2. Lemma. *If \mathcal{A} is of homological type, \mathcal{A}/\mathcal{N} is semi-simple. If moreover it is neutrally of homological type and the corresponding realization H is faithful, the functor $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}$ has the idempotent lifting property.*

Proof. The first statement follows from [2, Th. 1 a)]. For the second, let $M \in \mathcal{A}$ and \bar{M} its image in $\bar{\mathcal{A}}$. The hypothesis implies that $\text{End}_{\mathcal{A}}(M)$ is a finite-dimensional K -algebra. Let \mathcal{R} be its radical: it is nilpotent and contained in $\mathcal{N}(M, M)$ by [2, Th. 1 a)]. Thus $\text{End}_{\bar{\mathcal{A}}}(\bar{M})$ is a quotient of the semi-simple algebra $\text{End}_{\mathcal{A}}(M)/\mathcal{R}$. Therefore we may lift orthogonal idempotents of $\text{End}_{\bar{\mathcal{A}}}(\bar{M})$ to orthogonal idempotents of $\text{End}_{\mathcal{A}}(M)$, first in $\text{End}_{\mathcal{A}}(M)/\mathcal{R}$ and then in $\text{End}_{\mathcal{A}}(M)$ itself. \square

4.3. Lemma. *Let E be an extension of K . If $\bar{\mathcal{A}}$ is of homological origin, then $\bar{\mathcal{A}}_E := \bar{\mathcal{A}} \otimes_K E$ is also of homological origin.*

Proof. Let \mathcal{A} of homological type be such that $\mathcal{A}/\mathcal{N} \simeq \bar{\mathcal{A}}$, and let $H : \mathcal{A} \rightarrow \text{Vec}_L^\pm$ be a realisation of \mathcal{A} . Consider the tensor functor

$$H_E : \mathcal{A}_E \rightarrow \text{Vec}_{L \otimes_K E}^\pm$$

given by $H_E(M) = H(M) \otimes_K E$. Here $L \otimes_K E$ is not a field in general, but we can map it to one of its residue fields L' . Then the composite functor

$$H' : \mathcal{A}_E \rightarrow \text{Vec}_{L'}^\pm$$

is a tensor functor. To conclude, it suffices to observe that $\bar{\mathcal{A}}_E \simeq \mathcal{A}_E/\mathcal{N}(\mathcal{A}_E)$ by [2, Lemme 1]. \square

4.4. Lemma. *Suppose that $\bar{\mathcal{A}}$ is neutrally of homological origin. Then the pseudo-abelian envelope of $\bar{\mathcal{A}}$ is also neutrally of homological origin.*

Proof. A realization H with coefficients K extends to the pseudo-abelian envelope \mathcal{A}^\natural of \mathcal{A} , since Vec_K^\pm is pseudo-abelian. On the other hand, Lemma 4.2 implies that $\mathcal{A}^\natural/\mathcal{N}^\natural$ is pseudo-abelian, where \mathcal{N}^\natural is the ideal \mathcal{N} of \mathcal{A}^\natural ; but the obvious functor $\mathcal{A}/\mathcal{N} \rightarrow \mathcal{A}^\natural/\mathcal{N}^\natural$ is clearly a pseudo-abelian envelope. \square

4.5. Theorem. *If $\bar{\mathcal{A}}$ is of homological origin, it is geometrically of integral type.*

Proof. By Lemma 4.3, we may assume K algebraically closed. Choose $(\mathcal{A}, H : \mathcal{A} \rightarrow \text{Vec}_L^\pm)$ such that $\bar{\mathcal{A}} \simeq \mathcal{A}/\mathcal{N}$. Without loss of generality, we may also assume that L is algebraically closed. The functor H canonically extends to a realisation $H_L : \mathcal{A}_L \rightarrow \text{Vec}_L^\pm$. By [2, Lemme 1], $\mathcal{N}(\mathcal{A}_L) = \mathcal{N} \otimes_K L$. Denote the functor of extension of scalars $\bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}_L$ by $M \mapsto M_L$. Clearly, $\mu(M_L) = \mu(M) \otimes_K 1$, and the simple factors of $\text{End}_{\bar{\mathcal{A}}_L}(M_L) = \text{End}_{\bar{\mathcal{A}}}(M) \otimes_K L$ are the same as those of $\text{End}_{\bar{\mathcal{A}}}(M)$. All this reduces us to the case where $K = L$ is algebraically closed. Without loss of generality we may further assume H to be faithful. Finally, Lemma 4.4 reduces us to the case where $\bar{\mathcal{A}}$ is pseudo-abelian. We then have to prove that $\mu(S) \in \mathbf{Z}$ for any simple $S \in \bar{\mathcal{A}}$.

Let \tilde{S} be an object of \mathcal{A} which maps to S . By hypothesis, $\text{End}_{\bar{\mathcal{A}}}(S) = K$. Thus $\mu(S) \in K$, hence $\text{tr}(1_S) = \mu(S)$. On the other hand,

$$(4.1) \quad \text{tr}(1_S) = \text{tr}(1_{\tilde{S}}) = \dim_{\text{gr}} H(\tilde{S}) \in \mathbf{Z}.$$

\square

4.6. Proposition. *Let $\bar{\mathcal{A}}$ be of homological origin; let \mathcal{A} be of homological type with $\mathcal{A}/\mathcal{N} = \bar{\mathcal{A}}$, and let $H : \mathcal{A} \rightarrow \text{Vec}_L^\pm$ be a realization functor. Then*

- a) *For any simple object $S \in (\bar{\mathcal{A}}_L)^\natural$, $d(S) \mid \mu(S)$.*
- b) *Suppose $L = K$, H faithful and let \tilde{S} be a lift of S in \mathcal{A}^\natural . If $H(\tilde{S})$ is purely even or odd, then the nilpotence level r of $\mathcal{N}(\tilde{S}, \tilde{S})$ is bounded by*

$$r \leq |\mu(S)|/d(S).$$

In particular, if $|\mu(S)| = 1$, then $\text{End}_{\mathcal{A}}(\tilde{S})$ is a field.

Proof. a) Since $\mathcal{A}_L/\mathcal{N}(\mathcal{A}_L) = \bar{\mathcal{A}}_L$, $\bar{\mathcal{A}}_L$ is neutrally of homological origin; up to quotienting \mathcal{A}_L and replacing K by L , we may assume $L = K$ and H faithful. Then \mathcal{A} is semi-primary and, as in the proof of Theorem 4.5, we may further assume that \mathcal{A} and $\bar{\mathcal{A}}$ are pseudo-abelian.

Let $\tilde{S} \in \mathcal{A}$ mapping to S . By Wedderburn's theorem, the map $\text{End}_{\mathcal{A}}(\tilde{S}) \rightarrow \text{End}_{\bar{\mathcal{A}}}(S)$ has a ring-theoretic section σ . This makes $H(\tilde{S})$ a module over the division ring $\text{End}_{\bar{\mathcal{A}}}(S)$. Therefore $\dim_K H^\varepsilon(\tilde{S})$ is divisible by $\dim_K \text{End}_{\bar{\mathcal{A}}}(S) = \delta(S)d(S)^2$ for $\varepsilon = \pm 1$. On the other hand,

$$\dim_K H^+(\tilde{S}) - \dim_K H^-(\tilde{S}) = \mu(S)\delta(S)d(S)$$

by Proposition 2.2 a). Therefore, $\delta(S)d(S)^2$ divides $\mu(S)\delta(S)d(S)$, which means that $d(S)$ divides $\mu(S)$, as claimed.

b) Let $\mathcal{N} = \mathcal{N}(\tilde{S}, \tilde{S})$ and consider the filtration $(\mathcal{N}^i H(\tilde{S}))_{0 \leq i \leq r-1}$. Note that $\mathcal{N}^i H(\tilde{S}) = \mathcal{N}^{i+1} H(\tilde{S}) \iff \mathcal{N}^i = 0$ since \mathcal{N} is a nilpotent set of endomorphisms of $H(\tilde{S})$. The associated graded $(\text{gr}^i H(\tilde{S}))_{0 \leq i \leq r-1}$ is a graded $\text{End}_{\bar{\mathcal{A}}}(S)$ -module, and $\text{gr}^i H(\tilde{S}) \neq 0$ for all $i < r$. Hence $r \leq \dim H(S)/\dim \text{End}_{\bar{\mathcal{A}}}(S) \leq |\mu(S)|/d(S)$. \square

4.7. Remark. Coming back to the zeta function of an endomorphism, suppose that \mathcal{A} is of homological type; let $M \in \mathcal{A}$ and $f \in \text{End}_{\mathcal{A}}(M)$. If H is a realization of \mathcal{A} , we have by the usual computation

$$Z(f, t) = \det(1 - H(f)t)^{-1} = \frac{\det(1 - ft \mid H^-(M))}{\det(1 - ft \mid H^+(M))}.$$

Let \bar{M} be the image of M in $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{N}$ and \bar{f} be the image of f in $\text{End}_{\bar{\mathcal{A}}}(\bar{M})$. Since $Z(f, t) = Z(\bar{f}, t)$, we get from Theorem 4.5 and Theorem 3.2 a) the identity

$$\frac{\det(1 - ft \mid H^+(M))}{\det(1 - ft \mid H^-(M))} = \prod_i \text{Nrd}_{A_i}(e_i - e_i \bar{f}t)^{\mu_i(M)}.$$

Suppose for example that \bar{M} is simple; the identity reduces to

$$\frac{\det(1 - ft \mid H^+(M))}{\det(1 - ft \mid H^-(M))} = \text{Nrd}_A(1 - \bar{f}t)^{\mu(M)}$$

where $A = \text{End}_{\bar{\mathcal{A}}}(\bar{M})$.

Supposing further that $\mu(M) > 0$ to fix ideas, we find that the inverse characteristic polynomial of f acting on $H^-(M)$ (with coefficients in L) divides the one for $H^+(M)$, and the quotient has coefficients in K . This does not imply, however, that $H^-(M) = 0$.

5. AN ABSTRACT VERSION OF THE TATE CONJECTURE

5.A. Automorphisms of the identity functor. Let \mathcal{A} be a rigid K -category, and let F be an \otimes -endomorphism of the identity functor of \mathcal{A} . By [6, I.5.2.2], F is then an isomorphism. Concretely, F is given

by an automorphism $F_M \in \text{End}(M)$ for every object $M \in \mathcal{A}$; F_M is natural in M , and further:

$$\begin{aligned} F_{M \oplus N} &= F_M \oplus F_N \\ F_{M \otimes N} &= F_M \otimes F_N \\ F_{M^*} &= {}^t F_M^{-1} \text{ (cf. [6, I, (3.2.3.6)])}. \end{aligned}$$

5.1. Definition. The *zeta function* (relative to F) of an object $M \in \mathcal{A}$; it is:

$$Z_F(M, t) = Z(F_M, t).$$

5.2. Lemma. *The zeta function is additive in M :*

$$Z_F(M \oplus N, t) = Z_F(M, t) Z_F(N, t).$$

It is multiplicative in M in the following sense:

$$Z_F(M \otimes N, t) = Z_F(M, t) * Z_F(N, t)$$

where $*$ is the unique law on $1 + tK[[t]]$ such that, identically, $f * (gh) = (f * g)(f * h)$ and

$$(1 - at)^{-1} * (1 - bt)^{-1} = (1 - abt)^{-1}.$$

(Explicitly: if $f(t) = \exp(\sum_{n \geq 1} a_n \frac{t^n}{n})$ and $g(t) = \exp(\sum_{n \geq 1} b_n \frac{t^n}{n})$, then $f * g(t) = \exp(\sum_{n \geq 1} a_n b_n \frac{t^n}{n})$.)

If moreover \mathcal{A} is semi-simple of integral type, then

- (1) $Z_F(M, t) \in K(t)$ for any $M \in \mathcal{A}$;
- (2) $Z_F(M^*, t^{-1}) = (-t)^{\chi(M)} \det(F_M) Z_F(M, t)$;
- (3) for S simple,

$$Z_F(S, t) = P_S(t)^{-\chi(S)/\deg(F_S)}$$

where $P_S(t)$ is the inverse minimum polynomial of F_S over K and $\deg(F_S) = \deg(P_S) = [K[F_S] : K]$.

Proof. The additivity is obvious; the multiplicativity follows from the identities

$$\text{tr}(F_{M \otimes N}^n) = \text{tr}(F_M^n \otimes F_N^n) = \text{tr}(F_M^n) \text{tr}(F_N^n).$$

(1), (2) and (3) follow from Theorem 3.2: (1) from part a), (2) from part b) by noting that $Z({}^t F_S^{-1}, t^{-1}) = Z(F_S^{-1}, t^{-1})$, and (3) from part a) again by noting that F_S is in the centre of $\text{End}_{\mathcal{A}}(S)$ (use Proposition 2.2 a)). \square

5.B. The semi-simple case.

5.3. Definition. In the above, suppose \mathcal{A} semi-simple of integral type. We say that (\mathcal{A}, F) verifies the Tate conjecture if, for any $M \in \mathcal{A}$, $K[F_M]$ is the centre of $\text{End}_{\mathcal{A}}(M)$.

5.4. Theorem (cf. [3, Th. 2.7]). *Let \mathcal{A} be a semi-simple rigid pseudo-abelian K -category of integral type, and let $F \in \text{Aut}^{\otimes}(Id_{\mathcal{A}})$. Then the following conditions are equivalent:*

- (i) Given a simple object $S \in \mathcal{A}$, $F_S = 1_S$ implies $S = \mathbf{1}$.
- (ii) For any $M \in \mathcal{A}$, $\text{ord}_{t=1} Z_F(M, t) = -\dim_K \mathcal{A}(\mathbf{1}, M)$.
- (iii) For $S, T \in \mathcal{A}$ simple, $P_S = P_T \Rightarrow S \simeq T$.
- (iv) For $M, N \in \mathcal{A}$, $Z_F(M, t) = Z_F(N, t) \Rightarrow M \simeq N$.
- (v) (\mathcal{A}, F) verifies the Tate conjecture.

Moreover, these conditions imply:

- (vi) For any simple S , $|\mu(S)| = 1$ and $K[F_S]$ is the centre of $\text{End}_{\mathcal{A}}(S)$.

Proof. We shall prove the following implications:

- (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)
- (ii) \Rightarrow (vi)
- (iii) + (vi) \Rightarrow (v) \Rightarrow (iii).

(i) \Rightarrow (ii): both sides are additive in M so we may assume M simple. If $M = \mathbf{1}$, $Z_F(M, t) = 1/(1-t)$ and the formula is true. If $M \neq \mathbf{1}$, Lemma 5.2 (3) and the hypothesis show that $\text{ord}_{t=1} Z_F(M, t) = 0$ and the formula is also true.

(ii) \Rightarrow (iii): Consider $f(t) = Z_T(S^* \otimes T, t)$. By Lemma 5.2, Formulas (2), (3) and the multiplicativity rule, we see that

$$f(t) = \prod_{i,j} (1 - \alpha_i \alpha_j^{-1} t)^m$$

where $m = -\frac{\chi(S)}{\deg(F_S)} \frac{\chi(T)}{\deg(F_T)}$ and the α_i are the roots of the irreducible polynomial $P_S = P_T$ in a suitable extension of K . Note that (ii) implies that $\text{ord}_{t=1} Z_F(M, t) \leq 0$; the above formula shows that this integer is < 0 . Hence $0 \neq \mathcal{A}(\mathbf{1}, S^* \otimes T) \simeq \mathcal{A}(S, T)$ and $S \simeq T$ by Schur's lemma.

(iii) \Rightarrow (iv): write $M = \bigoplus_{i \in I} S_i^{m_i}$ and $N = \bigoplus_{i \in I} S_i^{n_i}$, where S_i runs through a set of representatives of the isomorphism classes of simple objects of \mathcal{A} . We then have, by Lemma 5.2 (3):

$$Z_F(M, t) = \prod_{i \in I} P_{S_i}(t)^{-m_i \chi(S_i) / \deg(F_{S_i})}$$

$$Z_F(N, t) = \prod_{i \in I} P_{S_i}(t)^{-n_i \chi(S_i) / \deg(F_{S_i})}.$$

By hypothesis, the $P_{S_i}(t)$ are pairwise distinct irreducible polynomials with constant term 1; then $Z_F(M, t) = Z_F(N, t)$ implies $m_i = n_i$ for all i , hence $M \simeq N$.

(iv) \Rightarrow (i): by hypothesis and Lemma 5.2 (3), $Z_F(S, t) = (1-t)^{-\chi(S)}$. Thus $Z_F(S, t) = Z_F(\mathbf{1}, t)^{\chi(S)}$. If $\chi(S) < 0$, this gives $S^{-\chi(S)} \simeq \mathbf{1}$, which implies $\chi(S) = -1$ and $S \simeq \mathbf{1}$, which is absurd since $\chi(\mathbf{1}) = 1$. Thus $\chi(S) \geq 0$, hence $S \simeq \mathbf{1}^{\chi(S)}$, hence $S \simeq \mathbf{1}$ since S is simple.

(ii) \Rightarrow (vi): the same computation as in the proof of (i) \Rightarrow (iii) gives

$$\begin{aligned} \delta(S)d(S)^2 &= \dim \operatorname{End}_{\mathcal{A}}(S) = -\operatorname{ord}_{t=1} Z(S^* \otimes S, t) \\ &= \left(\frac{\chi(S)}{\deg(F_S)} \right)^2 \operatorname{ord}_{t=1} \prod_{i,j} (1 - \alpha_i \alpha_j^{-1} t) = \frac{\chi(S)^2}{\deg(F_S)}. \end{aligned}$$

Using the identity $\chi(S) = \mu(S)d(S)\delta(S)$ (cf. Proposition 2.2 a)), we get

$$\deg(F_S) = \delta(S)\mu(S)^2.$$

But $\deg(F_S) \mid \delta(S)$, hence $\delta(S) = \deg(F_S)$ and $\mu(S)^2 = 1$.

(iii) + (vi) \Rightarrow (v): Let $M = \bigoplus_i S_i^{m_i}$ with $m_i > 0$ and the S_i simple and pairwise nonisomorphic. Then

$$\operatorname{End}_{\mathcal{A}}(M) = \prod_i M_{m_i}(\operatorname{End}_{\mathcal{A}}(S_i))$$

hence the centre of $\operatorname{End}_{\mathcal{A}}(M)$ is the product of the centres of the $\operatorname{End}_{\mathcal{A}}(S_i)$. By (vi), each of these centres is generated by F_{S_i} ; by (iii), the P_{S_i} are pairwise distinct irreducible polynomials, hence the minimum polynomial of F_M must be divisible by their product.

(v) \Rightarrow (iii) (compare [3]): if $P_S = P_T$ but $S \not\cong T$, then $\operatorname{End}_{\mathcal{A}}(S \oplus T) = \operatorname{End}_{\mathcal{A}}(S) \times \operatorname{End}_{\mathcal{A}}(T)$, with centre containing $L \times L$ for $L = K[F_S] = K[F_T]$. But $F_{S \oplus T}$ is killed by $P_S = P_T$, a contradiction. \square

5.5. Remark. Condition (vi) is really weaker than the others: take $F = 1$ in \mathcal{A} the category of linear representations of a finite abelian group over K algebraically closed.

5.6. Proposition. *Let \mathcal{A} be semi-simple of integral type and let $F \in \operatorname{Aut}^{\otimes}(Id_{\mathcal{A}})$.*

a) *The Tate conjecture is true for (\mathcal{A}, F) if and only if it is true for $(\mathcal{A}^{\natural}, F)$, where \mathcal{A}^{\natural} is the pseudo-abelian envelope of \mathcal{A} and F is extended to \mathcal{A}^{\natural} naturally.*

b) *If \mathcal{A} is geometrically of integral type, the Tate conjecture is invariant under extension of scalars: if L is an extension of K , then (\mathcal{A}, F) verifies the Tate conjecture if and only if (\mathcal{A}_L, F) verifies the Tate conjecture.*

Proof. a) “If” is obvious. For “only if”, let $M = (N, e) \in \mathcal{A}^\natural$ where $N \in \mathcal{A}$ and e is an idempotent of N . Write $M = \bigoplus_{i \in I} S_i^{m_i}$ and $N = \bigoplus_{i \in I} S_i^{n_i}$ as in the proof of Theorem 5.4, (iii) \Rightarrow (iv). We have

$$\text{End}(M) = \prod_i M_{m_i}(\text{End}(S_i)), \quad \text{End}(N) = \prod_i M_{n_i}(\text{End}(S_i)).$$

Letting Z_i denote the centre of $\text{End}(S_i)$, we get

$$Z(\text{End}(M)) = \prod_{m_i > 0} Z_i, \quad Z(\text{End}(N)) = \prod_{n_i > 0} Z_i.$$

By hypothesis, $Z(\text{End}(N))$ is generated by F_N as a K -algebra; this implies that Z_i is generated by F_{S_i} for all i and that the P_{S_i} are pairwise distinct. Hence F_M generates $Z(\text{End}(M))$ as well.

b) This is obvious since the centre of a semi-simple algebra behaves well under extension of scalars. \square

5.7. Corollary. *If (\mathcal{A}, F) verifies the Tate conjecture, then the conditions of Theorem 5.4 hold in \mathcal{A} even if \mathcal{A} is not pseudo-abelian.*

Proof. This is obvious except for (ii) and (iv); but by Proposition 5.6 a), $(\mathcal{A}^\natural, F)$ verifies the Tate conjecture; by Theorem 5.4, \mathcal{A}^\natural also verifies conditions (ii) and (iv), which a fortiori hold in its full subcategory \mathcal{A} . \square

5.8. Proposition. *Suppose that (\mathcal{A}, F) verifies the Tate conjecture. Let $S \in \mathcal{A}$ be a simple object.*

a) *If $\chi(S) \geq 0$, then $\Lambda^{\chi(M)+1}(M) = 0$; if $\chi(S) < 0$, then $S^{-\chi(M)+1}(M) = 0$.*

b) *\mathcal{A} is finite-dimensional in the sense of Kimura-O’Sullivan; more precisely, there exists a unique \otimes - $\mathbf{Z}/2$ -grading of \mathcal{A} such that S simple is positive (resp. negative) if and only if $\chi(S) > 0$ (resp. < 0).*

Proof. a) By Theorem 5.4 (iv), it suffices to see that $Z_F(N, t) = 1$ for $N = \Lambda^{\chi(M)+1}(M)$ (resp. $N = S^{-\chi(M)+1}(M)$): this follows (somehow...) from the computations of [1, 7.2.4]. b) is an immediate consequence (see also [1, 9.2.1]). \square

5.C. The homological case.

5.9. Theorem. *Let \mathcal{A} be of homological type, provided with a faithful realization functor $H : \mathcal{A} \rightarrow \text{Vec}_L^\pm$. Let $F \in \text{Aut}^\otimes(\text{Id}_{\mathcal{A}})$, and let us still denote by F its image in $\text{Aut}^\otimes(\text{Id}_{\bar{\mathcal{A}}})$, where $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{N}$. Note that F acts on H by functoriality. Consider the following conditions on an object $M \in \mathcal{A}$:*

- (i) $\bar{M} \in \bar{\mathcal{A}}$ verifies Condition (ii) of Theorem 5.4.

- (ii) The map $\mathcal{A}(\mathbf{1}, M) \otimes_K L \rightarrow H(M)^F$ is surjective and the composition $H(M)^F \rightarrow H(M) \rightarrow H(M)_F$ is an isomorphism (simplicity at 1).
- (iii) The map $\mathcal{A}(\mathbf{1}, M) \otimes_K L \rightarrow H(M)^F$ is surjective and $\mathcal{N}(\mathbf{1}, M) = 0$.
- (iv) The sign conjecture holds for M .
- (v) $H^-(M)^F = 0$.

Then

- (1) (i) + (v) \iff (ii) + (iii).
- (2) (i) + (iv) \Rightarrow (v) \Rightarrow (iv).
- (3) (ii) for M and M^* \iff (iii) for M and M^* .

Proof. These are classical arguments that only need to be put straight in this abstract context.

Note that $H^-(\mathbf{1}) = 0$, so that $\mathcal{A}(\mathbf{1}, M) \otimes_K L \rightarrow H(M)^F$ actually lands into $H^+(M)^F$; denote its image by $\mathcal{A}(\mathbf{1}, M)L$. By definition of \mathcal{N} , the projection $\mathcal{A}(\mathbf{1}, M) \otimes_K L \rightarrow \bar{\mathcal{A}}(\mathbf{1}, M) \otimes_K L$ factors through $\mathcal{A}(\mathbf{1}, M)L$. The diagram

$$\begin{array}{ccc} \mathcal{A}(\mathbf{1}, M)L & \hookrightarrow & H^+(M)^F \\ \text{surj} \downarrow & & \\ \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) \otimes_K L & & \end{array}$$

gives the inequalities

$$\dim_L H^+(M)^F \geq \dim_L \mathcal{A}(\mathbf{1}, M)L \geq \dim_K \bar{\mathcal{A}}(\mathbf{1}, \bar{M}).$$

On the other hand,

$$\text{ord}_{t=1} Z_F(M, t) =$$

$$\begin{aligned} & \text{ord}_{t=1} \det(1 - F_M t \mid H^-(M)) - \text{ord}_{t=1} \det(1 - F_M t \mid H^+(M)) \\ & = \dim_L H^-(M)^{F^\infty} - \dim_L H^+(M)^{F^\infty} \end{aligned}$$

where $H^\pm(M)^{F^\infty}$ denotes the characteristic subspace of $H^\pm(M)$ for the eigenvalue 1 under the action of F .

(1) Suppose that $H^-(M)^F = 0$. Then $H^-(M)^{F^\infty} = 0$ and, under (i), we have

$$\begin{aligned} \dim_L H^+(M)^F & \geq \dim_L \mathcal{A}(\mathbf{1}, M)L \geq \dim_K \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) \\ & = \dim_L H^+(M)^{F^\infty} \geq \dim_L H^+(M)^F \end{aligned}$$

hence we have equality everywhere, and (ii) and (iii) are true. Conversely, (ii) + (iii) gives isomorphisms $\bar{\mathcal{A}}(\mathbf{1}, M)_L \xrightarrow{\sim} H(M)^F \xrightarrow{\sim}$

$H(M)^{F^\infty}$. In particular, $H^-(M)^F = 0$ and we have $\dim_K \bar{\mathcal{A}}(\mathbf{1}, M) = \dim_L H^+(M)^{F^\infty}$, hence (i) and (v). Thus, (i) + (v) \iff (ii) + (iii).

(2) Under (iv), we may write $M = M^+ \oplus M^-$, with $H(M^+)$ purely even and $H(M^-)$ purely odd. To prove that $H^-(M)^F = 0$, we may therefore consider separately the cases where M is even and odd.

If M is even, this is obvious. If M is odd, we get, under (i):

$$H^+(M)^F = \mathcal{A}(\mathbf{1}, M) = \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) = 0$$

since $\mathcal{A}(\mathbf{1}, M) \hookrightarrow H^+(M)^F$, and

$$-\dim H^-(M)^{F^\infty} = \dim \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) = 0$$

which shows that (i) + (iv) \Rightarrow (v). For (v) \Rightarrow (iv), we reason as in [5, Proof of Th. 2]: there exists a polynomial $\Pi \in K[t]$ such that Π is divisible by P^- and $\Pi - 1$ is divisible by P^+ , where $P^\varepsilon(t) = \det(t - F \mid H^\varepsilon(M))$; then $\Pi(F) \in \text{End}(M)$ is such that $H(\Pi(M))$ is the identity on $H^+(M)$ and is 0 on $H^-(M)$.

(3) The counit map $M \otimes M^* \rightarrow \mathbf{1}$ gives compatible pairings

$$\mathcal{A}(\mathbf{1}, M)L \times \mathcal{A}(\mathbf{1}, M^*)L \rightarrow L$$

$$\bar{\mathcal{A}}(\mathbf{1}, \bar{M}) \times \mathcal{A}(\mathbf{1}, \bar{M}^*) \rightarrow K$$

$$H(M) \times H(M^*) \rightarrow L.$$

The first and last are perfect pairings: for the first, check it on simple objects thanks to Schur's lemma¹ and for the last, this follows from the structure of the tensor category Vec_K^\pm . Consider now the commutative diagram

$$\begin{array}{ccccc} \bar{\mathcal{A}}(\mathbf{1}, \bar{M})_L & \xleftarrow[\text{surj}]{a} & \mathcal{A}(\mathbf{1}, M)L & \xrightarrow[\text{inj}]{b} & H(M)^F \\ \downarrow \wr & & \downarrow c & & \downarrow d \\ (\bar{\mathcal{A}}(\mathbf{1}, \bar{M}^*)_L)^* & \xrightarrow[\text{inj}]{a^*} & (\mathcal{A}(\mathbf{1}, M^*)L)^* & \xleftarrow[\text{surj}]{b^*} & (H(M^*)^F)^* \simeq H(M)_F. \end{array}$$

Notice that the right vertical map coincides with the one of (ii).

Now assume that b and b^* are isomorphisms. The diagram shows immediately that a, a^* isomorphisms $\Rightarrow d$ isomorphism. Conversely, if d is an isomorphism, so is c ; but then, a and a^* must be isomorphisms. Finally, a is an isomorphism $\Rightarrow \mathcal{A}(\mathbf{1}, M) \rightarrow \bar{\mathcal{A}}(\mathbf{1}, M) \otimes_K L$ is injective $\Rightarrow \mathcal{N}(\mathbf{1}, M) = 0$, as desired. \square

5.10. Corollary. *Let \mathcal{A}, H, F be as in Theorem 5.9, and suppose that \mathcal{A} is pseudo-abelian. Consider the following conditions:*

(i) *The Tate conjecture holds for $(\bar{\mathcal{A}}, F)$.*

¹Or use the definition of the ideal \mathcal{N} .

- (ii) $\mathcal{A} \rightarrow \bar{\mathcal{A}}$ is an equivalence of categories and H induces a fully faithful functor

$$\tilde{H} : \bar{\mathcal{A}}_L \rightarrow \text{Rep}_L(F)_{ss}^{\pm}$$

where the right hand side denotes the \otimes -category of $\mathbf{Z}/2$ -graded L -vector spaces provided with the action of an automorphism F , this action being semi-simple.

- (iii) The sign conjecture holds for any $M \in \mathcal{A}$ (equivalently [1, 9.2.1 c]), \mathcal{A} is a Kimura-O'Sullivan category).
 (iv) For any $M \in \mathcal{A}$, $H^-(M)^F = 0$.

Then (iv) \Rightarrow (iii); moreover (ii) \iff (i) + (iii) \iff (i) + (iv).

If these conditions are verified, then for any simple object $S \in \mathcal{A}_L^{\natural}$, $\text{End}(S)$ is commutative.

Proof. First, (iv) \Rightarrow (iii) by Point 2 of Theorem 5.9. If now (ii) holds, then Conditions (ii) and (iii) of Theorem 5.9 hold for any M , hence so do its conditions (i) and (v) by Point 1 of this theorem. Point 2 also shows that M verifies Condition (iv) of this theorem. This shows that (ii) \Rightarrow (i) + (iii) + (iv) in Corollary 5.10.

Suppose that (i) holds. Then Condition (ii) of Theorem 5.4 holds for any $\bar{M} \in \bar{\mathcal{A}}$. If moreover $H^-(M)^F = 0$ for any $M \in \mathcal{A}$, Conditions (ii) and (iii) of Theorem 5.9 are verified for any $M \in \mathcal{A}$ by Point 1 of this theorem. Applying this to $M = P^* \otimes Q$ for some $P, Q \in \mathcal{A}$, the adjunction isomorphisms

$$\mathcal{A}(P, Q) \simeq \mathcal{A}(\mathbf{1}, P^* \otimes Q)$$

show that $\mathcal{N}(P, Q) = 0$, hence a bijection

$$\bar{\mathcal{A}}(M, N) \otimes_K L \rightarrow \text{Hom}_F(H(M), H(N)).$$

Moreover, since $\bar{\mathcal{A}}$ is semi-simple, $H(F_M)$ is a semi-simple endomorphism of $H(M)$ for any $M \in \mathcal{A}$. This shows that (i) + (iv) \Rightarrow (ii).

Suppose that (i) and (iii) hold. Then Point 2 of Theorem 5.9 shows that $H^-(M)^F = 0$ for any $M \in \mathcal{A}$, thus (i) + (iii) \Rightarrow (iv).

It remains to justify the last claim: it follows from Proposition 4.6 and Condition (vi) of Theorem 5.4. \square

5.11. Remark. In the classical case of motives over a finite field, Conditions (iii) and (iv) hold provided the Weil cohomology H verifies the Weak Lefschetz theorem, by Katz-Messing [5]. It is a little annoying not to be able to dispense of them in this abstract setting, especially in view of Proposition 2.2 b).

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