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ON THE MULTIPLICITIES OF A MOTIVE

BRUNO KAHN

Abstract. We introduce the notion of multiplicities for an object M in a semi-simple rigid tensor category $\mathcal{A}$, as a collection of central scalars which relate the categorical trace with the ring-theoretic trace. Multiplicities turn out to be rational integers in important cases, most notably when $\mathcal{A}$ is of "homological origin". We show that this integrality condition has simple consequences, like the rationality and a functional equation for the zeta function of an (invertible) endomorphism. An example is the category of pure motives modulo numerical equivalence with rational coefficients over a field k; if k is finite and M is of abelian type, its multiplicities are all equal to ± 1.

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Introduction

The aim of this article, in the spirit of [1], is to study abstractly the properties of categories of pure motives and to make clear(er) which of them are formal and which are of a more arithmetic-geometric nature.

We work with a rigid additive tensor category $\mathcal{A}$ such that $K = \mathrm{End}(\mathbf{1})$ is a field of characteristic 0. We shall be interested in the *multiplicities* of an object $M \in \mathcal{A}$: when $\mathcal{A}$ is semi-simple, they are a collection of central scalars which relates the categorical trace with the

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ring-theoretic trace (Proposition 1.2). It turns out that the condition for these multiplicities to be *integers* or, better, to be so after extending scalars from K to its algebraic closure, is very well-behaved and is satisfied in many important cases. Namely:

- The full subcategory $\mathcal{A}_{\mathrm{int}}$ of $\mathcal{A}$ formed by such objects is thick, tensor, rigid, contains the "Schur-finite" objects (those which are killed by a nonzero Schur functor), and is preserved under tensor functors to another semi-simple rigid category (Corollary 2.6).
- $\mathcal{A}_{\mathrm{int}} = \mathcal{A}$ if $\mathcal{A}$ is of "homological origin" (Theorem 4.6). The category of pure motives over a field modulo numerical equivalence is semi-simple thanks to Jannsen's theorem [7], and of homological origin.

When the multiplicities are integers, we prove that the zeta function of an endomorphism f of M is rational (with an explicit formula) and satisfies a functional equation if f is invertible (Theorem 3.2): in the case of motives over a finite field, this shows that these depend on less than the existence of a Weil cohomology theory. We also get some elementary cases where homological equivalence equals numerical equivalence for formal reasons in Proposition 4.10 c): of course, this remains far from leading to a proof of this famous standard conjecture!

In Section 6, we formulate a version of the Tate conjecture for motives over a finite field in an abstract set-up. My initial motivation was to see what multiplicities had to say on this conjecture; this turns out to be disappointing (see Theorem 6.4 and Remark 6.5) but I find it amusing and perhaps enlightening that most of its known equivalent versions carry out in this abstract context: see Theorem 6.4 and Corollary 6.10. The proof of [9] that under the Tate conjecture and Kimura-O'Sullivan finite dimensionality, rational and numerical equivalences agree over a finite field also carries out abstractly: see Theorem 6.12.

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Terminology and notation. Let $\mathcal{A}$ be a rigid K-linear tensor category, where K is a field of characteristic 0; we also assume that $\mathrm{End}(\mathbf{1}) = K$. In the sequel of this article, we shall abbreviate this by saying that $\mathcal{A}$ is a *rigid K-category*. Since we shall refer to Deligne's article [4] several times, it is worth stressing that we do not assume

$\mathcal{A}$ abelian, unlike in *loc. cit.* We write $\mathcal{A}^\natural$ for the pseudo-abelian hull (idempotent completion) of $\mathcal{A}$.

If $M \in \mathcal{A}$, we shall say (as has become common practice) that M is *Schur-finite* if there exists a nonzero Schur functor $\mathbb{S}$ such that $\mathbb{S}(M) = 0$ and *finite-dimensional* (in the sense of Kimura-O'Sullivan) if $M \simeq M_+ \oplus M_-$ where M_+ (*resp.* M_-) is killed by some nonzero exterior (*resp.* symmetric) power functor. We say that M_+ is *positive* and M_- is *negative*. It is known that finite-dimensional implies Schur-finite (*cf.* [4, 1.7]). For properties of finite-dimensional objects (*resp.* of Schur functors) we refer to [12] and [1, §9] (*resp.* to [4]).

1. Multiplicities in semi-simple rigid tensor categories

Let $M \in \mathcal{A}$. The *trace* of an endomorphism $f \in \mathrm{End}(M)$ is the element $\mathrm{tr}(f) \in \mathrm{End}(\mathbf{1}) = K$ defined by the composition

$$\mathbf{1} \xrightarrow{\eta} M^* \otimes M \xrightarrow{1\otimes f} M^* \otimes M \xrightarrow{R} M \otimes M^* \xrightarrow{\varepsilon} \mathbf{1}$$

where R is the switch and η, ε are the duality structures of M.

Special case. We shall denote the trace of 1_M by $\chi(M)$ and call it the *Euler characteristic of* M.

The trace is K-linear and has the following properties:

$$\mathrm{tr}(fg) = \mathrm{tr}(gf), \quad \mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g), \quad \mathrm{tr}({}^t f) = \mathrm{tr}(f). \tag{1.1}$$

Suppose that $\mathcal{A}$ is semi-simple. Then $\mathrm{End}_{\mathcal{A}}(M)$ is a semi-simple K-algebra, hence has its own trace, and we want to compare the categorical trace with the ring-theoretic trace. We normalise conventions as follows:

1.1. **Definition.** a) Let A be a finite-dimensional simple K-algebra. We write:

- $Z(A)$ for the centre of A;
- $\delta(A) = [Z(A) : K]$;
- $d(A) = [A : Z(A)]^{1/2}$.

We define the *reduced trace* of A as

$$\mathrm{Trd}_A = \mathrm{Tr}_{Z(A)/K} \circ \mathrm{Trd}_{A/Z(A)}\,.$$

If $A = \prod A_i$ is semi-simple, with simple components A_i, we define $\mathrm{Trd}_A := \sum_i \mathrm{Trd}_{A_i}$.

b) If $A = \mathrm{End}_{\mathcal{A}}(M)$, we set

- $Z_i(M) = Z(A_i)$;
- $\delta_i(M) = \delta(A_i)$;
- $d_i(M) = d(A_i)$;
- $\mathrm{Trd}_M = \mathrm{Trd}_A$.

1.2. **Proposition.** *There exists a unique element* $\mu(M) \in \mathrm{End}(M)$ *such that*

$$\mathrm{tr}(f) = \mathrm{Trd}_M(\mu(M)f)$$

for any $f \in \mathrm{End}(M)$*. Moreover,* $\mu(M)$ *is central and invertible. Hence, if* (e_i) *denotes the set of central idempotents of* $A = \mathrm{End}(M)$ *corresponding to its simple factors* A_i*, we may write*

$$\mu(M) = \sum_i \mu_i(M)e_i$$

with $\mu_i(M) \in Z_i(M)$*.*

Proof. Since $\mathrm{End}(M)$ is semi-simple, $(f, g) \mapsto \mathrm{Trd}_M(fg)$ is nondegenerate, which proves the existence and uniqueness of $\mu(M)$. Moreover,

$$\mathrm{Trd}_M(\mu(M)fg) = \mathrm{tr}(fg) = \mathrm{tr}(gf) = \mathrm{Trd}_M(\mu(M)gf) = \mathrm{Trd}_M(f\mu(M)g)$$

and the non-degeneracy also yields the centrality of $\mu(M)$. This element is invertible because the ideal $\mathcal{N}$ is 0 for $\mathcal{A}$ [1, 7.1.7]. The last assertion is obvious. □

1.3. **Lemma.** *a) We have* $\mu(M^*) = {}^t\mu(M)$*.*
b) Suppose that K *is algebraically closed and* M *is simple. Then* $\mu(M) = \chi(M)$*.*

Proof. a) follows easily from (1.1) and the fact that the transposition induces an anti-isomorphism from $\mathrm{End}(M)$ onto $\mathrm{End}(M^*)$. b) is obvious, since then $\mathrm{End}(M) = K$ (recall that, by definition, $\chi(M) = \mathrm{tr}(1_M)$). □

1.4. **Remark.** If $\mathcal{A}$ is pseudo-abelian (hence abelian, [7, Lemma 2]), the idempotents e_i of Proposition 1.2 yield the decomposition $M = \bigoplus M_i$ of M into its *isotypical components*. In particular, if S is simple, then $\mu(S^n) = \mu(S)$ for any $n \geq 1$.
On the other hand, it is difficult to relate $\mu(M_1), \mu(M_2)$ and $\mu(M_1 \otimes M_2)$ in general because it is difficult to say something of the map $\mathrm{End}(M_1) \otimes_K \mathrm{End}(M_2) \to \mathrm{End}(M_1 \otimes M_2)$: it is not even true in general that such a homomorphism sends the centre into the centre. For the same reason, it is difficult to state general facts on the behaviour of the invariant μ under tensor functors. We shall see that the situation improves considerably in the case of *geometrically integral type*, discussed in the next section.

2. Integral multiplicities

In all this section, $\mathcal{A}$ is a semi-simple rigid K-category.

2.1. **Definition.** a) An object $M \in \mathcal{A}$ is *of integral type* if the scalars $\mu_i(M)$ of Proposition 1.2 belong to $\mathbf{Z}$.
b) M is *geometrically of integral type* if $M_{\bar{K}} \in \mathcal{A}_{\bar{K}}$ is of integral type, where $\bar{K}$ is an algebraic closure of K.
c) $\mathcal{A}$ is *of integral type* (*resp. geometrically of integral type*) if every $M \in \mathcal{A}$ is of integral type (*resp.* geometrically of integral type).

2.2. **Proposition.** *a) If M is of integral type, we have*

$$\mu_i(M) = \frac{\mathrm{tr}(e_i)}{\delta_i(M) d_i(M)} \tag{2.1}$$

for any i.
b) Direct sums and direct summands of objects of integral type are of integral type. Similarly for geometrically of integral type. In particular, $\mathcal{A}$ is of integral type (resp. geometrically of integral type) if and only if its pseudo-abelian envelope is.
c) If M is geometrically of integral type, then it is of integral type. Moreover, if this is the case, the invariants $\mu_i(M)$ are "geometric" in the sense that if L/K is any extension, then $\mu_i(M) = \mu_{i,\alpha}(M_L)$ for any simple factor $A_{i,\alpha}$ of $A_i \otimes_K L$.
d) $M \in \mathcal{A}$ is geometrically of integral type if and only if, in $\mathcal{A}^\natural_{\bar{K}}$, the Euler characteristic of every simple summand of $M_{\bar{K}}$ is an integer.
e) If M is Schur-finite, it is geometrically of integral type.
f) If M is (geometrically) of integral type, so is M^*.

Proof. a) and b) are obvious. For c), we have the decomposition

$$Z_i(M) \otimes_K \bar{K} \xrightarrow{\sim} \prod_\alpha \bar{K}$$

where α runs through the distinct K-embeddings of $Z_i(M)$ into $\bar{K}$. Correspondingly, $A_i \otimes_K \bar{K}$ decomposes as a direct product

$$A_i \otimes_K \bar{K} \simeq \prod_\alpha A_i^\alpha$$

with A_i^α simple over $\bar{K}$. This gives a decomposition

$$e_i \otimes_K 1 = \sum_\alpha e_i^\alpha$$

into central idempotents. But clearly, $\mu(M_{\bar{K}}) = \mu(M) \otimes_K 1$. By hypothesis, the images of $\mu_i(M)$ in $\bar{K}$ under the embeddings α are rational integers, which implies that $\mu_i(M)$ is itself a rational integer. The additional claim of c) immediately follows from this proof.

d) follows immediately from Lemma 1.3 a).

For e), if M is Schur-finite, so is $M_{\bar{K}} \in \mathcal{A}^{\natural}_{\bar{K}}$; all simple direct summands of $M_{\bar{K}}$ are Schur-finite as well, hence their Euler characteristics are rational integers. This immediately follows from the main result of [4], but one can more elementarily use Proposition 2.2.2 of A. del Padrone's thesis [5], which generalises the case of finite-dimensional objects [1, 7.2.4 and 9.1.7]. The conclusion now follows from d).

Finally, f) follows from Lemma 1.3 b). □

2.3. **Remark.** C. Weibel raised the question whether the converse of e) is true. I don't know any counterexample; it holds at least if $\mathcal{A}_{\mathrm{int}}$ is of homological origin in the sense of Definition 4.1 b) (see Theorem 4.6).

2.4. **Theorem.** *Let $M, N \in \mathcal{A}$ be geometrically of integral type, (e_i) the central idempotents of* $\mathrm{End}(M)$ *and (f_j) the central idempotents of* $\mathrm{End}(N)$*. For a pair (i,j), let A_{ij} be the semi-simple algebra $(e_i \otimes f_j)\,\mathrm{End}(M \otimes N)(e_i \otimes f_j)$. Then one has formulas of the type*

$$\mu_i(M)\mu_j(N) = \sum_k m_k \mu_k(M \otimes N)$$

where k indexes the simple factors of A_{ij} and the m_k are integers ≥ 0. Moreover, for any k, there is such a formula with $m_k > 0$.
In particular, $M \otimes N$ is geometrically of integral type.

Proof. We proceed in 2 steps:

1) $\mathrm{End}(M)$ and $\mathrm{End}(N)$ are split. By Proposition 2.2 b), we may assume that $\mathcal{A}$ is pseudo-abelian. This allows us to assume M and N *simple*, hence $\mathrm{End}(M) = \mathrm{End}(N) = K$ and $A_{ij} = \mathrm{End}(M \otimes N)$. Using Formula (2.1) to compute $\mathrm{tr}(1_M \otimes 1_N)$ in two different ways, we get the formula

$$\mu(M)\mu(N) = \sum m_k \mu_k(M \otimes N) \tag{2.2}$$

with $m_k = \delta_k(M \otimes N) d_k(M \otimes N)$.

Coming back to the case where $\mathcal{A}$ is not necessarily pseudo-abelian and M, N not necessarily simple, this gives the formula

$$m_k = \delta_k(A_{ij}) \frac{d_k(A_{ij})}{d_i(M) d_j(N)}$$

(see Remark 1.4), and the previous argument shows ungrievously that this is an integer.

2) The general case. Extending scalars to $\bar{K}$ and using Proposition 2.2 c), we are reduced to 1) as follows: for any $\alpha : Z_i(M) \to \bar{K}$ and

any $\beta : Z_j(M) \to \bar{K}$, we have a formula with obvious notation:

$$\mu_i^\alpha(M_{\bar{K}})\mu_j^\beta(N_{\bar{K}}) = \sum_k \sum_\gamma m_k^\gamma \mu_k^\gamma((M \otimes N)_{\bar{K}})$$

where, for each k, γ runs through the embeddings of $Z_k(M \otimes N)$ into $\bar{K}$. By Remark 1.4, this gives a formula as wanted.

It remains to prove that, given a simple factor A_k of A_{ij}, one may find a formula with $m_k > 0$. For this, it suffices to show that there is a pair (α, β) such that

$$\mathrm{Hom}_{\bar{K}}(A_k \otimes_K \bar{K}, (e_i^\alpha \otimes f_j^\beta)(A_{ij} \otimes_K \bar{K})(e_i^\alpha \otimes f_j^\beta)) \neq 0.$$

This is obvious, since $\mathrm{Hom}_K(A_k, A_{ij}) \neq 0$ and $A_{ij} \otimes_K \bar{K} = \prod_{\alpha,\beta}(e_i^\alpha \otimes f_j^\beta)(A_{ij} \otimes_K \bar{K})(e_i^\alpha \otimes f_j^\beta))$. □

2.5. **Corollary.** *Assume that M and N are simple and that, in Theorem 2.4, all terms $\mu_k(M \otimes N)$ have the same sign. Then we have $|\mu_k(M \otimes N)| \leq |\mu(M)\mu(N)|$ for all k. If $|\mu(M)| = |\mu(N)| = 1$, then $A = \mathrm{End}(M \otimes N)$ is "geometrically simple" in the sense that $A \otimes_K \bar{K}$ is a matrix algebra over $Z(M \otimes N) \otimes_K \bar{K}$ (otherwise said, A is an Azumaya algebra over its centre). Moreover, $\mu(M \otimes N) = \mu(M)\mu(N)$.*

Proof. This follows from the last statement of Theorem 2.4. In the case where $|\mu(M)| = |\mu(N)| = 1$, Formula (2.2) gives the conclusion. □

2.6. **Corollary.** *a) The full subcategory $\mathcal{A}_{\mathrm{int}}$ of $\mathcal{A}$ consisting of geometrically integral objects is a thick rigid tensor subcategory of $\mathcal{A}$ containing the Schur-finite objects.*
b) Let $F : \mathcal{A} \to \mathcal{B}$ be a $\otimes$-functor to another rigid semi-simple K-category. Then $F(\mathcal{A}_{\mathrm{int}}) \subseteq \mathcal{B}_{\mathrm{int}}$.

Proof. a) follows from Proposition 2.2 and Theorem 2.4. b) follows from Proposition 2.2 d). □

3. Application: the zeta function of an endomorphism

3.1. **Definition.** Let $\mathcal{A}$ be a rigid K-category, $M \in \mathcal{A}$ and $f \in \mathrm{End}(M)$. The *zeta function* of f is

$$Z(f,t) = \exp\left(\sum_{k \geq 1} \mathrm{tr}(f^n)\frac{t^n}{n}\right) \in K[[t]].$$

3.2. **Theorem.** *Suppose that $\mathcal{A}$ is semi-simple and that $M \in \mathcal{A}$ is of integral type. Then,*

a) For any $f \in \mathrm{End}(M)$, $Z(f,t) \in K(t)$. More precisely, one has with the notation of Definition 1.1

$$Z(f,t) = \prod_i \mathrm{Nrd}_{A_i}(e_i - e_i f t)^{-\mu_i(M)}$$

where, for all i, $\mathrm{Nrd}_{A_i}(e_i - e_i f t) := N_{Z_i(M)/F}\,\mathrm{Nrd}_{A_i/Z_i(M)}(e_i - e_i f t)$ denotes the inverse reduced characteristic polynomial of the element $e_i f$ if A_i.
b) If f is invertible, one has the functional equation

$$Z(f^{-1}, t^{-1}) = (-t)^{\chi(M)} \det(f) Z(f,t)$$

where $\chi(M) = \mathrm{tr}(1_M)$ and $\det(f) = \prod_i \mathrm{Nrd}_{A_i}(e_i f)^{\mu_i(M)}$.

Proof. a) Applying the formula of Proposition 1.2, we get

$$\begin{aligned} Z(f,t) &= \exp\left(\sum_{k\geq 1} \mathrm{Trd}_M(\mu(M) f^n)\frac{t^n}{n}\right) \\ &= \exp\left(\sum_{k\geq 1}\sum_i \mathrm{Trd}_M(\mu_i(M) e_i f^n)\frac{t^n}{n}\right) \\ &= \prod_i \exp\left(\sum_{k\geq 1} \mathrm{Trd}_{A_i}((e_i f)^n)\frac{t^n}{n}\right)^{\mu_i(M)} \end{aligned}$$

and the conclusion follows from the well-known linear algebra identity

$$\exp\left(\sum_{k\geq 1} \mathrm{Trd}_{A_i}((e_i f)^n)\frac{t^n}{n}\right) = \mathrm{Nrd}_{A_i}(e_i - e_i f t)^{-1}.$$

For b), we write

$$\mathrm{Nrd}_{A_i}(e_i - e_i f^{-1} t^{-1}) = \mathrm{Nrd}_{A_i}(-e_i f^{-1} t^{-1})\,\mathrm{Nrd}_{A_1}(e_i - e_i f t)$$

hence

$$\begin{aligned} Z(f^{-1}, t^{-1}) &= \prod_i \mathrm{Nrd}_{A_i}(e_i - e_i f^{-1} t^{-1})^{-\mu_i(M)} \\ &= \prod_i \mathrm{Nrd}_{A_i}(-e_i f^{-1} t^{-1})^{-\mu_i(M)}\,\mathrm{Nrd}_{A_1}(e_i - e_i f t)^{-\mu_i(M)} \\ &= \prod_i \mathrm{Nrd}_{A_i}(-e_i f^{-1} t^{-1})^{-\mu_i(M)} Z(f,t) \end{aligned}$$

and

$$\prod_i \mathrm{Nrd}_{A_i}(-e_i f^{-1} t^{-1})^{-\mu_i(M)} =$$
$$(-t)^{\sum_i \mu_i(M) d_i(M) \delta_i(M)} \prod_i \mathrm{Nrd}_{A_i}(e_i f)^{\mu_i(M)} = (-t)^{\chi(M)} \det(f).$$

□

3.3. **Remark.** The definition of det shows that

$$\det(1 - ft) = Z(f,t)^{-1}$$

if the left hand side is computed in $\mathcal{A}_{K(t)}$.

4. Multiplicities in rigid tensor categories of homological type

4.1. **Definition.** a) A rigid K-category $\mathcal{A}$ is *of homological type* if there exists a tensor functor

$$H : \mathcal{A} \to \mathrm{Vec}_L^{\pm}$$

where L is an extension of K and $\mathrm{Vec}_L^{\pm}$ is the tensor category of $\mathbf{Z}/2$-graded finite-dimensional L-vector spaces, provided with the Koszul rule for the commutativity constraint. We say that H is a *realisation* of $\mathcal{A}$[1].

We say that $\mathcal{A}$ is *neutrally of homological type* if one may choose $L = K$.

b) A semi-simple rigid K-category $\bar{\mathcal{A}}$ is *of homological origin* (*resp. neutrally of homological origin*) if it is $\otimes$-equivalent to $\mathcal{A}/\mathcal{N}$, where $\mathcal{A}$ is a rigid K-category of homological type (*resp.* neutrally of homological type) and $\mathcal{N} = \mathcal{N}(\mathcal{A})$ is the ideal of morphisms universally of trace 0.

4.2. **Lemma.** *If $\mathcal{A}$ is of homological type, $\mathcal{A}/\mathcal{N}$ is semi-simple. If moreover it is neutrally of homological type and the corresponding realisation H is faithful, the functor $\mathcal{A} \to \mathcal{A}/\mathcal{N}$ has the idempotent lifting property.*

Proof. The first statement follows from [3, Th. 1 a)]. For the second, let $M \in \mathcal{A}$ and $\bar{M}$ its image in $\bar{\mathcal{A}}$. The hypothesis implies that $\mathrm{End}_{\mathcal{A}}(M)$ is a finite-dimensional K-algebra. Let $\mathcal{R}$ be its radical: it is nilpotent and contained in $\mathcal{N}(M,M)$ by [3, Th. 1 a)]. Thus $\mathrm{End}_{\bar{\mathcal{A}}}(\bar{M})$ is a quotient of the semi-simple algebra $\mathrm{End}_{\mathcal{A}}(M)/\mathcal{R}$. Therefore we may lift orthogonal idempotents of $\mathrm{End}_{\bar{\mathcal{A}}}(\bar{M})$ to orthogonal idempotents of $\mathrm{End}_{\mathcal{A}}(M)$, first in $\mathrm{End}_{\mathcal{A}}(M)/\mathcal{R}$ and then in $\mathrm{End}_{\mathcal{A}}(M)$ itself. □

[1] When $\mathcal{A}$ is abelian, this is what Deligne calls a super-fibre functor in [4], except that we do not require any exactness or faithfulness property here.

4.3. **Notation.** Let $\mathcal{A}$ be of homological type. For $M \in \mathcal{A}$, we write $\delta_i(M), d_i(M), \mu_i(M)$ for $\delta_i(\bar{M}), d_i(\bar{M}), \mu_i(\bar{M})$, where $\bar{M}$ is the image of M in $\bar{\mathcal{A}}$.

4.4. **Lemma.** *Let E be an extension of K. If $\bar{\mathcal{A}}$ is of homological origin, then $\bar{\mathcal{A}}_E := \bar{\mathcal{A}} \otimes_K E$ is also of homological origin.*

Proof. Let $\mathcal{A}$ of homological type be such that $\mathcal{A}/\mathcal{N} \simeq \bar{\mathcal{A}}$, and let $H : \mathcal{A} \to \mathrm{Vec}_L^{\pm}$ be a realisation of $\mathcal{A}$. Consider the tensor functor

$$H_E : \mathcal{A}_E \to \mathrm{Vec}^{\pm}_{L \otimes_K E}$$

given by $H_E(M) = H(M) \otimes_K E$. Here $L \otimes_K E$ is not a field in general, but we can map it to one of its residue fields L'. Then the composite functor

$$H' : \mathcal{A}_E \to \mathrm{Vec}^{\pm}_{L'}$$

is a tensor functor. To conclude, it suffices to observe that $\bar{\mathcal{A}}_E \simeq \mathcal{A}_E/\mathcal{N}(\mathcal{A}_E)$ by [3, Lemme 1]. □

4.5. **Lemma.** *Suppose that $\bar{\mathcal{A}}$ is neutrally of homological origin. Then the pseudo-abelian envelope of $\bar{\mathcal{A}}$ is also neutrally of homological origin.*

Proof. A realisation H with coefficients K extends to the pseudo-abelian envelope $\mathcal{A}^\natural$ of $\mathcal{A}$, since $\mathrm{Vec}_K^{\pm}$ is pseudo-abelian. On the other hand, Lemma 4.2 implies that $\mathcal{A}^\natural/\mathcal{N}^\natural$ is pseudo-abelian, where $\mathcal{N}^\natural$ is the ideal $\mathcal{N}$ of $\mathcal{A}^\natural$; but the obvious functor $\mathcal{A}/\mathcal{N} \to \mathcal{A}^\natural/\mathcal{N}^\natural$ is clearly a pseudo-abelian envelope. □

4.6. **Theorem.** *If $\bar{\mathcal{A}}$ is of homological origin, any $\bar{M} \in \bar{\mathcal{A}}$ is Schur-finite; in particular, $\bar{\mathcal{A}}$ is geometrically of integral type.*

Proof. Choose $(\mathcal{A}, H : \mathcal{A} \to \mathrm{Vec}_L^{\pm})$ as in the proof of Lemma 4.4. Without loss of generality, we may assume that H is faithful. Lift $\bar{M}$ to $M \in \mathcal{A}$. Then $H(M)$ is finite-dimensional, hence Schur-finite, which implies that M and therefore $\bar{M}$ is Schur-finite. The conclusion now follows from Proposition 2.2 e). □

4.7. **Remark.** The converse of Theorem 4.6 holds: namely, if every object of $\bar{\mathcal{A}}$ is Schur-finite, then the same is true in $(\bar{\mathcal{A}}_{\bar{K}})^\natural$. By [4, 0.6 and following remark], $(\bar{\mathcal{A}}_{\bar{K}})^\natural$ is $\otimes$-equivalent to $Rep(G, \varepsilon)$, where (G, ε) is a super-affine group scheme over $\bar{K}$; in particular, $(\bar{\mathcal{A}}_{\bar{K}})^\natural$ and hence $\bar{\mathcal{A}}$ admits a realisation into $\mathrm{Vec}_{\bar{K}}$. This shows that *if $\bar{\mathcal{A}}$ is of homological origin, then it is actually of homological type.* Another approach to this idea is the one in [2], using $\otimes$-sections.

However, in the case of pure motives, one wants of course to study the general situation of Definition 4.1 b), which is the one that arises naturally! This is what we do in the remainder of this section.

4.8. **Definition.** Let $\mathcal{A}$ be of homological type, and let $H : \mathcal{A} \to \mathrm{Vec}_L^{\pm}$ be a realisation functor. Given $M \in \mathcal{A}$, we say that *the sign conjecture holds for M* (with respect to H) if there exists $p \in \mathrm{End}_{\mathcal{A}}(M)$ such that $H(p)$ is the identity on $H^+(M)$ and is 0 on $H^-(M)$.

4.9. **Lemma** (cf. [1, 9.2.1]). *With the notation of Definition 4.8:*
a) If M is finite-dimensional, it verifies the sign conjecture.
b) The converse is true if H is faithful and $\mathcal{N}(M, M)$ is a nilideal (this is always the case if $L = K$). □

4.10. **Proposition.** *Let $\mathcal{A}$ be of homological type, $\bar{\mathcal{A}} := \mathcal{A}/\mathcal{N}$ and let $H : \mathcal{A} \to \mathrm{Vec}_L^{\pm}$ be a realisation functor. Then*
a) For any simple object $S \in (\bar{\mathcal{A}}_L)^\natural$, $d(S) \mid \mu(S)$.
b) Suppose H faithful. Let $M \in \mathcal{A}$ verify the sign conjecture. Then the nilpotence level r of $\mathcal{N}(M, M)$ verifies

$$r < \prod_i \left(\frac{|\mu_i(M)|}{e_i(M)} + 1\right)$$

where $e_i(M) = d(S_i)$, with S_i a simple summand of $\bar{M} \in \bar{\mathcal{A}}^\natural$ corresponding to its i-th isotypical component (see Remark 1.4).
c) If $\bar{M}$ is isotypical and $\mu(M) = \pm 1$, then $\mathcal{N}(M, M) = 0$.

Proof. a) Since $\mathcal{A}_L/\mathcal{N}(\mathcal{A}_L) = \bar{\mathcal{A}}_L$, $\bar{\mathcal{A}}_L$ is neutrally of homological origin; up to quotienting $\mathcal{A}_L$ and replacing K by L, we may assume $L = K$ and H faithful. Then $\mathcal{A}$ is semi-primary and, as in the proof of Theorem 4.6, we may further assume that $\mathcal{A}$ and $\bar{\mathcal{A}}$ are pseudo-abelian.

Let $\tilde{S} \in \mathcal{A}$ mapping to S. By Wedderburn's theorem, the map $\mathrm{End}_{\mathcal{A}}(\tilde{S}) \to \mathrm{End}_{\bar{\mathcal{A}}}(S)$ has a ring-theoretic section σ. This makes $H(\tilde{S})$ a module over the division ring $\mathrm{End}_{\bar{\mathcal{A}}}(S)$. Therefore $\dim_K H^\varepsilon(\tilde{S})$ is divisible by $\dim_K \mathrm{End}_{\bar{\mathcal{A}}}(S) = \delta(S)d(S)^2$ for $\varepsilon = \pm 1$. On the other hand,

$$\dim_K H^+(\tilde{S}) - \dim_K H^-(\tilde{S}) = \mu(S)\delta(S)d(S)$$

by Proposition 2.2 a). Therefore, $\delta(S)d(S)^2$ divides $\mu(S)\delta(S)d(S)$, which means that $d(S)$ divides $\mu(S)$, as claimed.

b) Assume first $L = K$. Without loss of generality, we may also assume $\mathcal{A}$ pseudo-abelian. Let $\mathcal{N} = \mathcal{N}(M, M)$ and consider the filtration $(\mathcal{N}^i H(M))_{0 \le i \le r-1}$. Note that $\mathcal{N}^i H(M) = \mathcal{N}^{i+1} H(M) \iff \mathcal{N}^i = 0$ since $\mathcal{N}$ is a nilpotent set of endomorphisms of $H(M)$. The associated graded $(\mathrm{gr}^i H(M))_{0 \le i \le r-1}$ is a graded $\mathrm{End}_{\bar{\mathcal{A}}}(\bar{M})$-module, and $\mathrm{gr}^i H(M) \neq 0$ for all $i < r$.

Since M verifies the sign conjecture, for each primary summand $\bar{M}_i$ of $\bar{M}$, with lift M_i in $\mathcal{A}$, one has either $H^+(M_i) = 0$ or $H^-(M_i) = 0$. The proof of a) then shows that $\mathrm{gr} H(M_i)$ is an $\mathrm{End}_{\bar{\mathcal{A}}}(\bar{M}_i)$-module

of length $\frac{|\mu(S_i)|}{d(S_i)}$ where S_i is the associated simple object. Note that $\mathrm{End}_{\bar{\mathcal{A}}}(\bar{M}) = \prod_i \mathrm{End}_{\bar{\mathcal{A}}}(\bar{M}_i)$: it follows that $\mathrm{gr}H(M)$ is an $\mathrm{End}_{\bar{\mathcal{A}}}(\bar{M})$-module of length $\prod(\frac{|\mu(S_i)|}{d(S_i)}+1)-1$. Hence the inequality.

In general we extend scalars from K to L. Let $s = \prod_i(\frac{|\mu_i(M)|}{e_i(M)}+1)-1$. Applying the result to the category $\mathcal{A}L$ with the same objects as $\mathcal{A}$ and such that $\mathcal{A}L(M,N) = H(\mathcal{A}(M,N))L \subseteq \mathrm{Hom}_L(H(M),H(N))$, we get $(\mathcal{N}(M,M)L)^s = 0$, hence $\mathcal{N}(M,M)^s \subseteq (\mathcal{N}(M,M)L)^s = 0$.

c) This follows immediately from b). □

4.11. **Remark.** In case M is finite dimensional, we have another bound for the nilpotence level of $\mathcal{N}(M,M)$ (valid without assuming H faithful). For simplicity, suppose that M is either positive or negative, and let $n = |\chi(M)|$. By [12, Corollary 10.2], $f^n = 0$ for all $f \in \mathcal{N}(M,M)$, which implies that $\mathcal{N}(M,M)^{n^2} = 0$ by a theorem of Razmyslov improving the Nagata-Higman bound [15, 4.3]. I learned of this improvement from Alessio del Padrone, who has generalised this bound to any finite-dimensional M with $n = |\chi(M^+)| + |\chi(M^-)|$ [5, 2.4.10].

These two bounds have completely different behaviours: in Proposition 4.10 c) the former is optimal while the latter is not, but on the other hand if M is a direct sum of n invertible objects pairwise non-isomorphic, the bound of Proposition 4.10 b) is $2^n - 1$ while the other one is n^2.

As del Padrone pointed out, a third unrelated nilpotency bound is the one predicted by the Bloch-Beilinson-Murre conjecture for the Chow motive of a smooth projective variety X (namely, $\dim(X)+1$ [8, Strong conj. 2.1]).

4.12. **Remark.** Coming back to the zeta function of an endomorphism, suppose that $\mathcal{A}$ is of homological type; let $M \in \mathcal{A}$ and $f \in \mathrm{End}_{\mathcal{A}}(M)$. If H is a realisation of $\mathcal{A}$, we have by the usual computation

$$Z(f,t) = \det(1-H(f)t)^{-1} = \frac{\det(1-ft \mid H^-(M))}{\det(1-ft \mid H^+(M))}.$$

Let $\bar{M}$ be the image of M in $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{N}$ and $\bar{f}$ be the image of f in $\mathrm{End}_{\bar{\mathcal{A}}}(\bar{M})$. Since $Z(f,t) = Z(\bar{f},t)$, we get from Theorem 4.6 and Theorem 3.2 a) the identity

$$\frac{\det(1-ft \mid H^+(M))}{\det(1-ft \mid H^-(M))} = \prod_i \mathrm{Nrd}_{A_i}(e_i - e_i\bar{f}t)^{\mu_i(M)}.$$

Suppose for example that $\bar{M}$ is simple; the identity reduces to

$$\frac{\det(1-ft \mid H^+(M))}{\det(1-ft \mid H^-(M))} = \mathrm{Nrd}_A(1-\bar{f}t)^{\mu(M)}$$

where $A = \mathrm{End}_{\bar{\mathcal{A}}}(\bar{M})$.

Supposing further that $\mu(M) > 0$ to fix ideas, we find that the inverse characteristic polynomial of f acting on $H^-(M)$ (with coefficients in L) *divides* the one for $H^+(M)$, and the quotient has coefficients in K. This does not imply, however, that $H^-(M) = 0$.

5. Examples

In our first example, let $\mathcal{A}$ be the rigid K-category of vector bundles over $\mathbf{P}^1_K$. It is of homological type, with realisation functor $H : \mathcal{A} \to \mathrm{Vec}_L$ for $L = K(t)$ given by the generic fibre. Its indecomposable objects are the $\mathcal{O}(n)$ for $n \in \mathbf{Z}$: they are all of multiplicity 1 but $\mathcal{A}(\mathcal{O}(p), \mathcal{O}(q)) = \mathcal{N}(\mathcal{O}(p), \mathcal{O}(q)) \neq 0$ whenever $p < q$. This shows that the condition "isotypical" is necessary in Proposition 4.10 c) (I am indebted to Yves André for pointing out this example). If one extends scalars from K to L in the style of the proof of this proposition, one finds $\mathcal{A}L(\mathcal{O}(p), \mathcal{O}(q)) = L$ whenever $p < q$.

Our main source of examples is, of course, the category $\mathcal{M}_{\mathrm{num}}(k)$ of pure motives over a field k modulo numerical equivalence. By Jannsen's theorem [7] and Theorem 4.6, $\mathcal{M}_{\mathrm{num}}(k)$ is semi-simple and geometrically of integral type. We shall compute the multiplicities in certain cases.

We leave it to the reader to check that the multiplicities of Artin (even Artin-Tate) motives are always $+1$. The next case is that of abelian varieties.

Let A be an abelian variety of dimension g over k. Then we have the Chow-Künneth decomposition

$$h(A) \simeq \bigoplus_{i=0}^{2g} h_i(A)$$

with $h_i(A) \simeq S^i(h_1(A))$ [13]. Moreover,

$$\mathrm{End}\, h_1(A) = \mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbf{Q}.$$

Moreover, $\chi(h_1(A)) = -2g$. From this and Proposition 2.2, we get for A simple:

$$\mu(A) = -\frac{2g}{\delta(\mathrm{End}^0(A))d(\mathrm{End}^0(A))}.$$

We recover the fact that the denominator divides the numerator.

Like Milne [14], we shall say that A has *many endomorphisms* if $\sum_i \delta(\mathrm{End}^0(A_i))d(\mathrm{End}^0(A_i)) = 2g$, where A_i runs through the simple factors of A, or equivalently if $\delta(\mathrm{End}^0(A_i))d(\mathrm{End}^0(A_i)) = 2g_i$ for all i,

where $g_i = \dim A_i$. This terminology is less ambiguous than "having complex multiplication".

5.1. **Definition.** Let $M \in \mathcal{M}_{\mathrm{num}}(k)$ be a pure motive modulo numerical equivalence. Then M is *of abelian type* if it is isomorphic to a direct summand of the tensor product of an Artin-Tate motive and the motive of an abelian variety.

Motives of abelian type are stable under direct sums, direct summands, tensor products and duals. We then have:

5.2. **Theorem.** *a) For A a simple abelian variety, $\mu(h_1(A)) = -1$ if and only if A has many endomorphisms.*

b) If $g = 1$, then $\mu(h_1(A)) = \begin{cases} -1 & \text{if } A \text{ has complex multiplication} \\ -2 & \text{otherwise.} \end{cases}$

c) If A has many endomorphisms, all multiplicities of $h_i(A)$ are equal to $(-1)^i$.

d) If k is a finite field, then the multiplicities of any motive of abelian type are equal to ± 1.

Proof. a) and b) are clear; c) follows from a) and Corollary 2.5, and d) follows from c) since any abelian variety over a finite field has many endomorphisms [18]. □

The next interesting case is that of $t_2(S)$ where S is a surface [10]. If $k = \mathbf{C}$, there are many examples where the Hodge realisation of $t_2(S)$ is simple [16, Ex. 5 and Prop. 14]. A fortiori $t_2(S)$ is simple, and Proposition 2.2 shows that its multiplicity equals its Euler characteristic, i.e. $b^2 - \rho$ where b^2 is the second Betti number and ρ is the Picard number.

6. An abstract version of the Tate(-Beilinson) conjecture

6.A. **Automorphisms of the identity functor.** Let $\mathcal{A}$ be a rigid K-category, and let F be an $\otimes$-*endomorphism of the identity functor of* $\mathcal{A}$. By [17, I.5.2.2], F is then an isomorphism. Concretely, F is given by an automorphism $F_M \in \mathrm{End}(M)$ for every object $M \in \mathcal{A}$; F_M is natural in M, and further:

$$\begin{aligned} F_{M\oplus N} &= F_M \oplus F_N \\ F_{M\otimes N} &= F_M \otimes F_N \\ F_{M^*} &= {}^tF_M^{-1} \ (\textit{cf.}\ [17, \mathrm{I}, (3.2.3.6)]). \end{aligned}$$

6.1. **Definition.** The *zeta function* (relative to F) of an object $M \in \mathcal{A}$ is

$$Z_F(M,t) = Z(F_M, t).$$

6.2. **Lemma.** *The zeta function is additive in M:*

$$Z_F(M \oplus N, t) = Z_F(M,t) Z_F(N,t).$$

It is multiplicative in M in the following sense:

$$Z_F(M \otimes N, t) = Z_F(M,t) * Z_F(N,t)$$

where $*$ is the unique law on $1+tK[[t]]$ such that, identically, $f*(gh) = (f*g)(f*h)$ and

$$(1-at)^{-1} * (1-bt)^{-1} = (1-abt)^{-1}.$$

(Explicitly: if $f(t) = \exp\left(\sum_{n\geq 1} a_n \frac{t^n}{n}\right)$ and $g(t) = \exp\left(\sum_{n\geq 1} b_n \frac{t^n}{n}\right)$, then $f*g(t) = \exp\left(\sum_{n\geq 1} a_n b_n \frac{t^n}{n}\right)$.)
If moreover $\mathcal{A}$ is semi-simple of integral type, then

(1) *$Z_F(M,t) \in K(t)$ for any $M \in \mathcal{A}$;*
(2) *$Z_F(M^*, t^{-1}) = (-t)^{\chi(M)} \det(F_M) Z_F(M,t)$;*
(3) *for S simple,*

$$Z_F(S,t) = P_S(t)^{-\chi(S)/\deg(F_S)}$$

where $P_S(t)$ is the inverse minimum polynomial of F_S over K and $\deg(F_S) = \deg(P_S) = [K[F_S] : K]$.

Proof. Additivity is obvious; multiplicativity follows from the identities

$$\operatorname{tr}(F^n_{M\otimes N}) = \operatorname{tr}(F^n_M \otimes F^n_N) = \operatorname{tr}(F^n_M)\operatorname{tr}(F^n_N).$$

(1), (2) and (3) follow from Theorem 3.2: (1) from part a), (2) from part b) by noting that $Z({}^tF_S^{-1}, t^{-1}) = Z(F_S^{-1}, t^{-1})$, and (3) from part a) again by noting that F_S is in the centre of $\operatorname{End}_{\mathcal{A}}(S)$ (use Proposition 2.2 a)). □

6.B. **The semi-simple case.**

6.3. **Definition.** In the above, suppose $\mathcal{A}$ semi-simple of integral type. We say that $(\mathcal{A}, F)$ *verifies the Tate conjecture* if, for any $M \in \mathcal{A}$, $K[F_M]$ is the centre of $\operatorname{End}_{\mathcal{A}}(M)$.

6.4. **Theorem** (*cf.* [6, Th. 2.7])**.** *Let $\mathcal{A}$ be a semi-simple rigid pseudo-abelian K-category of integral type, and let $F \in \operatorname{Aut}^{\otimes}(Id_{\mathcal{A}})$. Then the following conditions are equivalent:*

(i) *Given a simple object $S \in \mathcal{A}$, $F_S = 1_S$ implies $S = \mathbf{1}$.*
(ii) *For any $M \in \mathcal{A}$, $\operatorname{ord}_{t=1} Z_F(M,t) = -\dim_K \mathcal{A}(\mathbf{1}, M)$.*
(iii) *For $S, T \in \mathcal{A}$ simple, $P_S = P_T \Rightarrow S \simeq T$.*
(iv) *For $M, N \in \mathcal{A}$, $Z_F(M,t) = Z_F(N,t) \Rightarrow M \simeq N$.*
(v) *$(\mathcal{A}, F)$ verifies the Tate conjecture.*

Moreover, these conditions imply:

(vi) *For any simple S, $|\mu(S)| = 1$ and $K[F_S]$ is the centre of the algebra* $\mathrm{End}_{\mathcal{A}}(S)$.

Proof. We shall prove the following implications:
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)
(ii) $\Rightarrow$ (vi)
(iii) + (vi) $\Rightarrow$ (v) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii): both sides are additive in M so we may assume M simple. If $M = \mathbf{1}$, $Z_F(M,t) = 1/(1-t)$ and the formula is true. If $M \neq 1$, Lemma 6.2 (3) and the hypothesis show that $\mathrm{ord}_{t=1} Z_F(M,t) = 0$ and the formula is also true.

(ii) $\Rightarrow$ (iii): Consider $f(t) = Z_T(S^* \otimes T, t)$. By Lemma 6.2, Formulas (2), (3) and the multiplicativity rule, we see that

$$f(t) = \prod_{i,j} (1 - \alpha_i \alpha_j^{-1} t)^m$$

where $m = -\frac{\chi(S)}{\deg(F_S)} \frac{\chi(T)}{\deg(F_T)}$ and the α_i are the roots of the irreducible polynomial $P_S = P_T$ in a suitable extension of K. Note that (ii) implies that $\mathrm{ord}_{t=1} Z_F(M,t) \leq 0$; the above formula shows that this integer is < 0. Hence $0 \neq \mathcal{A}(\mathbf{1}, S^* \otimes T) \simeq \mathcal{A}(S,T)$ and $S \simeq T$ by Schur's lemma.

(iii) $\Rightarrow$ (iv): write $M = \bigoplus_{i \in I} S_i^{m_i}$ and $N = \bigoplus_{i \in I} S_i^{n_i}$, where S_i runs through a set of representatives of the isomorphism classes of simple objects of $\mathcal{A}$. We then have, by Lemma 6.2 (3):

$$Z_F(M,t) = \prod_{i \in I} P_{S_i}(t)^{-m_i \chi(S_i)/\deg(F_{S_i})}$$
$$Z_F(N,t) = \prod_{i \in I} P_{S_i}(t)^{-n_i \chi(S_i)/\deg(F_{S_i})}.$$

By hypothesis, the $P_{S_i}(t)$ are pairwise distinct irreducible polynomials with constant term 1; then $Z_F(M,t) = Z_F(N,t)$ implies $m_i = n_i$ for all i, hence $M \simeq N$.

(iv) $\Rightarrow$ (i): by hypothesis and Lemma 6.2 (3), $Z_F(S,t) = (1-t)^{-\chi(S)}$. Thus $Z_F(S,t) = Z_F(\mathbf{1},t)^{\chi(S)}$. If $\chi(S) < 0$, this gives $S^{-\chi(S)} \simeq \mathbf{1}$, which implies $\chi(S) = -1$ and $S \simeq \mathbf{1}$, which is absurd since $\chi(\mathbf{1}) = 1$. Thus $\chi(S) \geq 0$, hence $S \simeq \mathbf{1}^{\chi(S)}$, hence $S \simeq \mathbf{1}$ since S is simple.

(ii) $\Rightarrow$ (vi): the same computation as in the proof of (i) $\Rightarrow$ (iii) gives

$$\begin{aligned}\delta(S) d(S)^2 &= \dim \mathrm{End}_{\mathcal{A}}(S) = -\mathrm{ord}_{t=1} Z(S^* \otimes S, t) \\ &= \left(\frac{\chi(S)}{\deg(F_S)}\right)^2 \mathrm{ord}_{t=1} \prod_{i,j} (1 - \alpha_i \alpha_j^{-1} t) = \frac{\chi(S)^2}{\deg(F_S)}.\end{aligned}$$

Using the identity $\chi(S) = \mu(S)d(S)\delta(S)$ (*cf.* Proposition 2.2 a)), we get

$$\deg(F_S) = \delta(S)\mu(S)^2.$$

But $\deg(F_S) \mid \delta(S)$, hence $\delta(S) = \deg(F_S)$ and $\mu(S)^2 = 1$.

(iii) + (vi) $\Rightarrow$ (v): Let $M = \bigoplus_i S_i^{m_i}$ with $m_i > 0$ and the S_i simple and pairwise nonisomorphic. Then

$$\mathrm{End}_{\mathcal{A}}(M) = \prod_i M_{m_i}(\mathrm{End}_{\mathcal{A}}(S_i))$$

hence the centre of $\mathrm{End}_{\mathcal{A}}(M)$ is the product of the centres of the $\mathrm{End}_{\mathcal{A}}(S_i)$. By (vi), each of these centres is generated by F_{S_i}; by (iii), the P_{S_i} are pairwise distinct irreducible polynomials, hence the minimum polynomial of F_M must be divisible by their product.

(v) $\Rightarrow$ (iii) (compare [6]): if $P_S = P_T$ but $S \not\simeq T$, then $\mathrm{End}_{\mathcal{A}}(S \oplus T) = \mathrm{End}_{\mathcal{A}}(S) \times \mathrm{End}_{\mathcal{A}}(T)$, with centre containing $L \times L$ for $L = K[F_S] = K[F_T]$. But $F_{S\oplus T}$ is killed by $P_S = P_T$, a contradiction. □

6.5. **Remark.** Condition (vi) is really weaker than the others: take $F = 1$ in $\mathcal{A}$ the category of linear representations of a finite abelian group over K algebraically closed.

6.6. **Proposition.** *Let $\mathcal{A}$ be semi-simple of integral type and let $F \in \mathrm{Aut}^{\otimes}(Id_{\mathcal{A}})$.*
a) The Tate conjecture is true for $(\mathcal{A}, F)$ if and only if it is true for $(\mathcal{A}^\natural, F)$, where $\mathcal{A}^\natural$ is the pseudo-abelian envelope of $\mathcal{A}$ and F is extended to $\mathcal{A}^\natural$ naturally.
b) If $\mathcal{A}$ is geometrically of integral type, the Tate conjecture is invariant under extension of scalars: if L is an extension of K, then $(\mathcal{A}, F)$ verifies the Tate conjecture if and only if $(\mathcal{A}_L, F)$ verifies the Tate conjecture.

Proof. a) "If" is obvious. For "only if", let $M = (N, e) \in \mathcal{A}^\natural$ where $N \in \mathcal{A}$ and e is an idempotent of N. Write $M = \bigoplus_{i\in I} S_i^{m_i}$ and $N = \bigoplus_{i\in I} S_i^{n_i}$ as in the proof of Theorem 6.4, (iii) $\Rightarrow$ (iv). We have

$$\mathrm{End}(M) = \prod_i M_{m_i}(\mathrm{End}(S_i)), \quad \mathrm{End}(N) = \prod_i M_{n_i}(\mathrm{End}(S_i)).$$

Letting Z_i denote the centre of $\mathrm{End}(S_i)$, we get

$$Z(\mathrm{End}(M)) = \prod_{m_i>0} Z_i, \quad Z(\mathrm{End}(N)) = \prod_{n_i>0} Z_i.$$

By hypothesis, $Z(\mathrm{End}(N))$ is generated by F_N as a K-algebra; this implies that Z_i is generated by F_{S_i} for all i and that the P_{S_i} are pairwise distinct. Hence F_M generates $Z(\mathrm{End}(M))$ as well.

b) This is obvious since the centre of a semi-simple algebra behaves well under extension of scalars. □

6.7. **Corollary.** *If $(\mathcal{A}, F)$ verifies the Tate conjecture, then the conditions of Theorem 6.4 hold in $\mathcal{A}$ even if $\mathcal{A}$ is not pseudo-abelian.*

Proof. This is obvious except for (ii) and (iv); but by Proposition 6.6 a), $(\mathcal{A}^\natural, F)$ verifies the Tate conjecture; by Theorem 6.4, $\mathcal{A}^\natural$ also verifies conditions (ii) and (iv), which a fortiori hold in its full subcategory $\mathcal{A}$. □

6.8. **Proposition.** *Suppose that $(\mathcal{A}, F)$ verifies the Tate conjecture. Let $S \in \mathcal{A}$ be a simple object.*
a) If $\chi(S) \geq 0$, then $\Lambda^{\chi(S)+1}(S) = 0$; if $\chi(S) < 0$, then $\mathbb{S}^{-\chi(S)+1}(S) = 0$.
b) $\mathcal{A}$ is finite-dimensional in the sense of Kimura-O'Sullivan; more precisely, there exists a unique $\otimes$-$\mathbf{Z}/2$-grading of $\mathcal{A}$ such that S simple is positive (resp. negative) if and only if $\chi(S) > 0$ (resp. < 0).

Proof. a) By Theorem 6.4 (iv), it suffices to see that $Z_F(N, t) = 1$ for $N = \Lambda^{\chi(S)+1}(S)$ (*resp.* $N = \mathbf{S}^{-\chi(S)+1}(S)$): this follows from the computations of [1, 7.2.4]. b) is an immediate consequence (see also [1, 9.2.1]). □

6.C. **The homological case.** Let $\mathcal{A}$ be of homological type, provided with a faithful realisation functor $H : \mathcal{A} \to \mathrm{Vec}_L^{\pm}$. Let $F \in \mathrm{Aut}^{\otimes}(Id_{\mathcal{A}})$, and let us still denote by F its image in $\mathrm{Aut}^{\otimes}(Id_{\bar{\mathcal{A}}})$, where $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{N}$. Note that F acts on H by functoriality.

6.9. **Theorem.** *Consider the following conditions on an object $M \in \mathcal{A}$:*

(i) *$\bar{M} \in \bar{\mathcal{A}}$ verifies Condition* (ii) *of Theorem 6.4.*
(ii) *The map $\mathcal{A}(\mathbf{1}, M) \otimes_K L \to H(M)^F$ is surjective and the composition $H(M)^F \to H(M) \to H(M)_F$ is an isomorphism (semi-simplicity at* 1*).*
(iii) *The map $\mathcal{A}(\mathbf{1}, M) \otimes_K L \to H(M)^F$ is surjective and $\mathcal{N}(\mathbf{1}, M) = 0$.*
(iv) *The sign conjecture holds for M.*
(v) *$H^-(M)^F = 0$.*

Then

(1) (i) + (v) $\iff$ (ii) + (iii).
(2) (i) + (iv) $\Rightarrow$ (v) $\Rightarrow$ (iv).
(3) (ii) *for M and M^** $\iff$ (iii) *for M and M^*.*

Proof. These are classical arguments that only need to be put straight in this abstract context.

Note that $H^-(\mathbf{1}) = 0$, so that $\mathcal{A}(\mathbf{1}, M) \otimes_K L \to H(M)^F$ actually lands into $H^+(M)^F$; denote its image by $\mathcal{A}(\mathbf{1}, M)L$. By definition of $\mathcal{N}$, the projection $\mathcal{A}(\mathbf{1}, M) \otimes_K L \to \bar{\mathcal{A}}(\mathbf{1}, M) \otimes_K L$ factors through $\mathcal{A}(\mathbf{1}, M)L$. The diagram

$$\begin{array}{ccc} \mathcal{A}(\mathbf{1}, M)L & \hookrightarrow & H^+(M)^F \\ {\scriptstyle \text{surj}}\big\downarrow & & \\ \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) \otimes_K L & & \end{array}$$

gives the inequalities

$$\dim_L H^+(M)^F \geq \dim_L \mathcal{A}(\mathbf{1}, M)L \geq \dim_K \bar{\mathcal{A}}(\mathbf{1}, \bar{M}).$$

On the other hand,

$$\begin{aligned} &\operatorname{ord}_{t=1} Z_F(M, t) = \\ &\quad \operatorname{ord}_{t=1} \det(1 - F_M t \mid H^-(M)) - \operatorname{ord}_{t=1} \det(1 - F_M t \mid H^+(M)) \\ &\qquad\qquad = \dim_L H^-(M)^{F^\infty} - \dim_L H^+(M)^{F^\infty} \end{aligned}$$

where $H^\pm(M)^{F^\infty}$ denotes the characteristic subspace of $H^\pm(M)$ for the eigenvalue 1 under the action of F.

(1) Suppose that $H^-(M)^F = 0$. Then $H^-(M)^{F^\infty} = 0$ and, under (i), we have

$$\begin{aligned} \dim_L H^+(M)^F \geq \dim_L \mathcal{A}(\mathbf{1}, M)L &\geq \dim_K \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) \\ &= \dim_L H^+(M)^{F^\infty} \geq \dim_L H^+(M)^F \end{aligned}$$

hence we have equality everywhere, and (ii) and (iii) are true. Conversely, (ii) + (iii) gives isomorphisms $\bar{\mathcal{A}}(\mathbf{1}, M)_L \xrightarrow{\sim} H(M)^F \xrightarrow{\sim} H(M)^{F^\infty}$. In particular, $H^-(M)^F = 0$ and we have $\dim_K \bar{\mathcal{A}}(\mathbf{1}, M) = \dim_L H^+(M)^{F^\infty}$, hence (i) and (v). Thus, (i) + (v) $\iff$ (ii) + (iii).

(2) Under (iv), we may write $M = M^+ \oplus M^-$, with $H(M^+)$ purely even and $H(M^-)$ purely odd. To prove that $H^-(M)^F = 0$, we may therefore consider separately the cases where M is even and odd.

If M is even, this is obvious. If M is odd, we get, under (i):

$$H^+(M)^F = \mathcal{A}(\mathbf{1}, M) = \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) = 0$$

since $\mathcal{A}(\mathbf{1}, M) \hookrightarrow H^+(M)^F$, and

$$-\dim H^-(M)^{F^\infty} = \dim \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) = 0$$

which shows that (i) + (iv) $\Rightarrow$ (v). For (v) $\Rightarrow$ (iv), we reason as in [11, Proof of Th. 2]: there exists a polynomial $\Pi \in K[t]$ such that Π is divisible by P^- and $\Pi - 1$ is divisible by P^+, where $P^\varepsilon(t) = \det(t - F \mid$

$H^\varepsilon(M)$); then $\Pi(F) \in \mathrm{End}(M)$ is such that $H(\Pi(M))$ is the identity on $H^+(M)$ and is 0 on $H^-(M)$.

(3) The counit map $M \otimes M^* \to \mathbf{1}$ gives compatible pairings

$$\begin{aligned} \bar{\mathcal{A}}(\mathbf{1}, \bar{M}) \times \mathcal{A}(\mathbf{1}, \bar{M}^*) &\to K \\ \mathcal{A}(\mathbf{1}, M)L \times \mathcal{A}(\mathbf{1}, M^*)L &\to L \\ H(M) \times H(M^*) &\to L. \end{aligned}$$

The first and last are perfect pairings: for the first, check it on simple objects thanks to Schur's lemma[2] and for the last, this follows from the structure of the tensor category $\mathrm{Vec}_L^\pm$. Consider now the commutative diagram

$$\begin{tikzcd} \bar{\mathcal{A}}(\mathbf{1}, \bar{M})_L \arrow[d, "\wr"'] & \mathcal{A}(\mathbf{1}, M)L \arrow[l, "a", "\mathrm{surj}"'] \arrow[r, "b", "\mathrm{inj}"'] \arrow[d, "c"'] & H(M)^F \arrow[d, "d"'] \\ (\bar{\mathcal{A}}(\mathbf{1}, \bar{M}^*)_L)^* \arrow[r, "a^*", "\mathrm{inj}"'] & (\mathcal{A}(\mathbf{1}, M^*)L)^* & (H(M^*)^F)^* \simeq H(M)_F. \arrow[l, "b^*"', "\mathrm{surj}"] \end{tikzcd}$$

Notice that the right vertical map coincides with the one of (ii).

Now assume that b and b^* are isomorphisms. The diagram shows immediately that a, a^* isomorphisms $\Rightarrow d$ isomorphism. Conversely, if d is an isomorphism, so is c; but then, a and a^* must be isomorphisms. Finally, a is an isomorphism $\Rightarrow \mathcal{A}(\mathbf{1}, M) \to \bar{\mathcal{A}}(\mathbf{1}, M) \otimes_K L$ is injective $\Rightarrow \mathcal{N}(\mathbf{1}, M) = 0$, as desired. □

6.10. **Corollary** (*cf.* [19, 2.9]). *Let $\mathcal{A}, H, F$ be as in Theorem 6.9, and suppose that $\mathcal{A}$ is pseudo-abelian. Consider the following conditions:*

(i) *The Tate conjecture holds for $(\bar{\mathcal{A}}, F)$.*

(ii) *$\mathcal{A} \to \bar{\mathcal{A}}$ is an equivalence of categories and H induces a fully faithful functor*

$$\tilde{H} : \bar{\mathcal{A}}_L \to \mathrm{Rep}_L(F)^\pm_{ss}$$

where the right hand side denotes the $\otimes$-category of $\mathbf{Z}/2$-graded L-vector spaces provided with the action of an automorphism F, this action being semi-simple.

(iii) *The sign conjecture holds for any $M \in \mathcal{A}$ (equivalently* [1, 9.2.1 c)], *$\mathcal{A}$ is a Kimura-O'Sullivan category).*

(iv) *For any $M \in \mathcal{A}$, $H^-(M)^F = 0$.*

Then (iv) $\Rightarrow$ (iii)*; moreover* (ii) $\iff$ (i) + (iii) $\iff$ (i) + (iv).
If these conditions are verified, then for any simple object $S \in \mathcal{A}_L^\natural$, $\mathrm{End}(S)$ *is commutative.*

[2]Or use the definition of the ideal $\mathcal{N}$.

Proof. First, (iv) $\Rightarrow$ (iii) by Point 2 of Theorem 6.9. If now (ii) holds, then Conditions (ii) and (iii) of Theorem 6.9 hold for any M, hence so do its conditions (i) and (v) by Point 1 of this theorem. Point 2 also shows that M verifies Condition (iv) of this theorem. This shows that (ii) $\Rightarrow$ (i) + (iii) + (iv) in Corollary 6.10.

Suppose that (i) holds. Then Condition (ii) of Theorem 6.4 holds for any $\bar{M} \in \bar{\mathcal{A}}$. If moreover $H^-(M)^F = 0$ for any $M \in \mathcal{A}$, Conditions (ii) and (iii) of Theorem 6.9 are verified for any $M \in \mathcal{A}$ by Point 1 of this theorem. Applying this to $M = P^* \otimes Q$ for some $P, Q \in \mathcal{A}$, the adjunction isomorphisms

$$\mathcal{A}(P, Q) \simeq \mathcal{A}(\mathbf{1}, P^* \otimes Q)$$

show that $\mathcal{N}(P, Q) = 0$, hence a bijection

$$\bar{\mathcal{A}}(M, N) \otimes_K L \to \mathrm{Hom}_F(H(M), H(N)).$$

Moreover, since $\bar{\mathcal{A}}$ is semi-simple, $H(F_M)$ is a semi-simple endomorphism of $H(M)$ for any $M \in \mathcal{A}$. This shows that (i) + (iv) $\Rightarrow$ (ii).

Suppose that (i) and (iii) hold. Then Point 2 of Theorem 6.9 shows that $H^-(M)^F = 0$ for any $M \in \mathcal{A}$, thus (i) + (iii) $\Rightarrow$ (iv).

It remains to justify the last claim: it follows from Proposition 4.10 and Condition (vi) of Theorem 6.4. □

6.11. **Remark.** In the classical case of motives over a finite field, Conditions (iii) and (iv) of Corollary 6.10 hold provided the Weil cohomology H verifies the Weak Lefschetz theorem, by Katz-Messing [11]. It is a little annoying not to be able to dispense of them in this abstract setting, especially in view of Proposition 2.2 b).

6.D. **The Tate-Beilinson conjecture.** We conclude by transposing the argument of [9] to this abstract context.

6.12. **Theorem** (*cf.* [9, Th. 1]). *Let $\mathcal{A}$ be a rigid K-category provided with a $\otimes$-automorphism F of the identity functor. Suppose that $\mathcal{N}$ is locally nilpotent (e.g. that $\mathcal{A}$ is a Kimura-O'Sullivan category), and that $\bar{\mathcal{A}} = \mathcal{A}/\mathcal{N}$ verifies the Tate conjecture relatively to F. Then $\mathcal{N} = 0$, i.e. $\mathcal{A} = \bar{\mathcal{A}}$.*

Proof. We note that the hypothesis on $\mathcal{N}$ implies that the functor $\mathcal{A} \to \bar{\mathcal{A}}$ is conservative. The argument is the same as in [9]: by rigidity it is sufficient to show that $\mathcal{A}(\mathbf{1}, M) \xrightarrow{\sim} \bar{\mathcal{A}}(\mathbf{1}, \bar{M})$ for any $M \in \mathcal{A}$. By the nilpotence of $\mathcal{N}(M, M)$, we may lift to $\mathrm{End}_{\mathcal{A}}(M)$ an orthogonal system of idempotents of $\bar{M}$ corresponding to a decomposition in simple summands. This reduces us to the case where $\bar{M}$ is simple. There are two cases:

(1) $\bar{M} \simeq \mathbf{1}$. Then $M \simeq 1$ by conservativity, and both Hom groups are isomorphic to K.
(2) $\bar{M} \not\simeq \mathbf{1}$. Then $\bar{\mathcal{A}}(\mathbf{1}, \bar{M}) = 0$ and we have to show that $\mathcal{A}(\mathbf{1}, M) = 0$. By Theorem 6.4 (i), $F_{\bar{M}} \neq 1$, hence by conservativity, $1 - F_M$ is an isomorphism. If now $f \in \mathcal{A}(\mathbf{1}, M)$, we have $f = F_M f$, hence $f = 0$.

□

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