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Semi-classical theory of quiet lasers. I: Principles

Jacques ARNAUD ^{*}; Laurent CHUSSEAU [†]; Fabrice PHILIPPE [‡]

22nd January 2007

Abstract

When light originating from a laser diode driven by non-fluctuating electrical currents is incident on a photo-detector, the photo-current does not fluctuate much. Precisely, this means that the variance of the number of photo-electrons counted over a large time interval is much smaller than the average number of photo-electrons. At non-zero Fourier frequency Ω the photo-current power spectrum is of the form $\Omega^2/(1 + \Omega^2)$ and thus vanishes as $\Omega \rightarrow 0$, a conclusion equivalent to the one given above. The purpose of this paper is to show that results such as the one just cited may be derived from a (semi-classical) theory in which neither the optical field nor the electron wave-function are quantized. We first observe that almost any medium may be described by a circuit and distinguish (possibly non-linear) conservative elements such as pure capacitances, and conductances that represent the atom-field coupling. The theory rests on the non-relativistic approximation. Nyquist noise sources (in which the Planck term $\hbar\omega/2$ is being restored) are associated with positive or negative conductances, and the law of average-energy conservation is enforced. We consider mainly second-order correlations in stationary linearized regimes.

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1 Introduction

”Comprendre”, c’est comprendre *autrement* (“Comprehend” means comprehend *differently*).

In the present introduction we outline our objective, main concepts and approximations employed, key results, and describe how the paper is organized.

Scope of the paper. Laser noise impairs the operation of optical communication systems and the measurement of small displacements or small rotation rates with the help of optical interferometry. Even though laser light is far superior to thermal light, minute fluctuations restrict the ultimate performances. Signal-to-noise ratios, displacement sensitivities, and so on, depend mainly of the spectral densities, or correlations, of the photo-currents. It is therefore important to have at our disposal formulas enabling us to evaluate these quantities for configurations of practical interest, in a form as simple as possible.

We are mostly concerned with basic concepts leaving out detailed practical calculations. Non-essential noise sources such as mechanical vibrations are ignored. Real lasers involve many secondary effects that are presently neglected for the sake of clarity. For example, because of the large size of the cavity in comparison with wavelength, lasers tend to oscillate on more than one mode. Even if the side-mode powers are much reduced with the help of distributed feed-backs or secondary cavities, small-power side modes may significantly influence laser-noise properties, particularly near the shot-noise level. Side-mode powers should probably be less than 40 dB below the main mode power to be insignificant. In the case of gas lasers, multiple levels, atomic collisions, thermal motions, and so on, may strongly influence noise properties, but these effects are neglected here.

The main purpose of this paper is to show that, contrary to what most previous works imply, the properties of quiet lasers may be understood on the basis of a simple semi-classical theory, that is, a theory in which neither the optical field nor the electron wave-function are quantized. The electrons may be uncoupled to one another (dilute atom gases) or strongly coupled as is the case in semiconductors. This theory (proposed by one of us in papers from 1986 on, and in book form in 1989 [1]) is accurate and easy to apply, yet little known. The physical concepts are hopefully better explained in the present paper than in previous ones. Once the necessary assumptions have been agreed upon, laser noise formulas for various configurations follow from elementary mathematics. In particular, operator algebra is not needed.

In this first part, the principles are presented and elementary results are derived. Advanced results relating to the noise of lasers incorporating multilevel atoms or having spatially varying phase-amplitude coupling factors, the linewidth of inhomogeneously broadened lasers, and the role of electrical feedbacks, are listed but the derivations are postponed.

The reader should be aware that our interpretations of the basic mechanisms behind

quiet-laser operation¹, the rôle of the Petermann K-factor² and non-linear gain, on some respects differ from those found elsewhere in the literature. For simple laser models we observe however exact agreement between our results and those derived from Quantum Optics, whenever a comparison can be made.

On the other hand, theories found in most of the Optical Engineering literature are devised to fit closely with observations, but often involve so many arbitrary parameters that agreement between theory and experiments is not conclusive as far as fundamental concepts are concerned. Detailed comparison between some of our theoretical results and experiments could require specially constructed lasers.

Main concepts of the theory and approximations. Lasers are akin to any oscillator, or clock, whether mechanical or electrical. The only difference that exists between a microwave oscillator such as a reflex klystron, for example, and a laser relates to the different electronic responses to alternating fields. In a microwave tube the electron motion is usually not harmonic and its coupling to a single-frequency electromagnetic field may be understood accurately only through numerical calculations. In contradistinction, masers and lasers employ basically two-level molecules or atoms, and this results in simplified treatments³. But the phenomena of stimulated emission and absorption are essentially the

¹Most early semi-classical studies of laser amplitude and phase noise consider as essential the field spontaneously emitted by upper-state atoms. From our view-point, spontaneous emission is irrelevant at the fundamental level. From our view-point a constant pump current entails a constant photo-current under ideal conditions. In a recent book [2] the basic mechanism behind quiet-laser operation is described instead as follows: "Although the noise generated in the external resistor is far below the shot-noise level, this does not mean that the carrier injection into the active region is regulated[...]. The carriers supplied by the external circuit are injected stochastically across the depletion layer before they reach the active region [i.e., they are Poisson-distributed.]" The authors then introduce potential fluctuations to explain the observed quiet radiation. It may be, however, that these authors description is just another way of describing the same Physics as in the present paper.

²It was observed early by E.I. Gordon [3] that, in the linear regime, laser line-widths are enhanced above those given by the well-known Schawlow-Townes formula for various circuits involving lumped elements or transmission lines, see [4, p. 120]. The linewidth-enhancement is ascribed here to the fact that in such circuits gain and loss regions occur at different locations. The line-width enhancement factor is observed in strictly single-mode laser or maser oscillators. The K-factor effect discovered by Petermann [5] is of great practical importance for some laser diodes. It is not, in our opinion, fundamentally different from the effect which has just been described, and, contrary to a wide-spread belief, it cannot be applied directly to above-threshold lasers. The law of average-energy conservation then tells us that quiet light should be observed with a quiet pump, leaving aside non-unity quantum efficiency, current leakage, and so on, irrespectively of the value of the K-factor value. An entirely different view point has been recently expressed [6]: "Because of the non-orthogonality of the laser eigenmodes, the noise from other modes is homodyned into the lasing mode, leading to an excess noise in the lasing mode, *which is the Petermann excess noise* (our emphasis).

³For two-level atoms, upward electron jumps (stimulated absorption) and downward jumps (stimulated emission) may be treated symmetrically according to the time-dependent Schrödinger equation. Strictly speaking, the two-level approximation holds rigorously only for electrons immersed in a magnetic field, the lower energy state corresponding to the case where the electron magnetic moment points in the direction of the field and the higher energy state corresponding to the electron magnetic moment pointing in the

same for every oscillator. The noise properties are also similar. Let us quote the Nobel-prize winner W. E. Lamb, Jr. [7, p. 208]: "Whether a charge q moving with velocity v in an electrical field \mathcal{E} will gain or loose energy depends on the algebraic sign of the product $ev\mathcal{E}$ [...]. If the charge is loosing energy, this is equivalent to stimulated emission. [...] In the domain of electronics, a triode vacuum-tube radio-frequency oscillator was developed by L. de Forest in 1912. This was in fact the first maser oscillator made by man".

The lasers considered oscillate in a single electromagnetic mode in the steady state. Only stationary⁴ fluctuations of the currents driving the active elements are allowed. The system elements are supposed to be time-independent.

- Basic set-up.

An optical set up involves three basic components. First a light source driven by an electrical current (called the pump). Second, an optical circuit involving slits, lenses, beam-splitters, resonators, and so on, which we view as being conservative, that is, free of loss or gain. Third, light detectors delivering photo-currents. Light sources deliver optical power while light detectors absorb optical power. Ideally, the detector photo-currents could be employed to pump the light sources so that the complete system could operate in an autonomous manner. Such equilibrium configurations will be discussed near the end of the paper to clarify the concepts. But in general we are dealing with a stationary non-equilibrium system.

In many experiments, we only need to know time-averaged photo-currents. This information suffices for example to verify that light passing through an opaque plate pierced with two holes exhibits interference patterns. The experiment is performed by measuring the time-averaged photo-currents issued from an array of detectors located behind the plate. Other experiments involving the transmission of information through an optical fiber require that the fluctuations of the photo-current about its mean be known⁵. We restrict ourselves to stationary non-relativistic configurations⁶.

opposite direction. In the case of atoms the electron energy is bounded from below but may extend to arbitrarily large values. The symmetry between stimulated emission and stimulated absorption therefore rests on the approximation that two levels only are important. In particular, the scattering states are ignored. The two-level approximation may cause apparent violation of oscillator-strength sum rules and of the equivalence between oscillator-strength values when different optical-field gauges are employed. These difficulties are un-consequential in the present theory.

⁴A fluctuation is called "stationary" when correlations of all orders are independent of the initial time. This adjective is employed differently in the expression "stationary states" where "stationary" means that the electron wave-function modulus is time independent.

⁵A light beam carries information if it is modulated in amplitude or phase. Small modulations may be obtained from the present theory by ignoring the noise sources, but they are not discussed explicitly for the sake of brevity. The information to be transmitted is corrupted by natural fluctuations.

⁶That is, the free-space permeability μ_o is set equal to zero, or, equivalently, the speed of light in free space, c , is set at being infinite. These quantities therefore nowhere enter into the theory, and questions having to do with special relativity are irrelevant. More precisely, we consider electron velocities v much

In previous semi-classical theories the spontaneously emitted field is considered to be the fundamental source of noise. The classical optical field is supposed to be incremented by the field spontaneously emitted by upper-state atoms with a phase uniformly distributed between 0 and 2π (hence the randomness). Instead, we view noise as basically originating from *stimulated* electron jumps from one state to another. Spontaneous electron decay is neglected for simplicity in the major part of the paper. Nonetheless a large number of results relevant to the field of Stationary Quantum Optics may be obtained.

- Non-fluctuating driving currents.

We will almost exclusively consider laser diodes driven by constant (i.e., non-fluctuating) electrical currents. Such currents may be obtained for example from a battery or a large charged capacitance and a cold series resistance. This conclusion follows from the Nyquist formula derived from Classical Statistical Mechanics that says that at material absolute temperature $T_m = 0\text{K}$ (precisely $\hbar\Omega \ll k_B T_m \ll \hbar\omega$, where Ω denotes the Fourier frequency and ω the optical frequency) no fluctuations are involved in a state of thermal equilibrium. Detailed analysis shows that the Nyquist formula holds also when a steady current flows through the resistance as long as the Ohm law remains applicable. Alternatively, we may generate non-fluctuating currents from space-charge-limited cathodic emission. It is now-a-day possible to inject in a device one electron at a time. A discrete realistic picture of a non-fluctuating current is accordingly the regular injection of electrons, say one every nano-second, if only small Fourier frequencies are considered. This discrete picture will be employed in numerical simulations. Let us emphasize that the present theory is unable to describe the system response to individual electrons.

- Non-fluctuating radiation.

It occurred as a surprised to the physics community when Golubev and other [8] proved theoretically in 1984 on the basis of the Quantum Optics laws that lasers driven by a quiet pump (e.g., a non-fluctuating current) deliver sub-Poissonian (or "quiet") photon streams. From our viewpoint, this observation would be better

smaller than c and transition frequencies ω much smaller than mc^2/\hbar , where m denotes the electron mass. We acknowledge that under these conditions some atomic properties are being overlooked. Relativistic effects that do not require electromagnetic-field quantization are: 1) the apparent increase of the electron mass, 2) the value of the electron magnetic moment $\mu = \mu_B$ derived from the Dirac equation, and 3) the spin-orbit energy splitting. This splitting, which results from the fact that, crudely speaking, atomic electrons perform circular motions at velocity v in nuclei electrical fields and thus "see" magnetic fields, is in fact small in hydrogen atoms, but becomes important for heavier atoms because v/c is not negligible. Electromagnetic field quantization is apparently required to evaluate: 1) the (Lamb) energy splitting between $2P_{1/2}$ and $2S_{1/2}$ hydrogenic states, 2) the correction $1 + \alpha/2\pi\dots$ to the electron magnetic moment, where the fine-structure constant $\alpha \approx 1/137$ is set equal to zero in the non-relativistic approximation, and 3) the radiative decay of excited-state atoms. In that case however indirect approximate methods based on Statistical Mechanics or the Classical Maxwell Equations with retarded potentials may be employed.

expressed by saying that when a laser is driven by a non-fluctuating current and the output light is incident on a photo-detector, the photo-current does not fluctuate much. In the latter formulation the notion of laser light statistics is being by-passed. The above prediction then may be viewed as a strictly classical result, resulting from the law of conservation of the average energy, as we discuss below. What is *non-classical* (i.e., quantum in nature) from our view-point are the shot-noise fluctuations. This so-called "Schottky effect" has been observed long ago in vacuum tubes. This is perhaps for this historical reason that the Schottky effect is often referred to as being a "classical effect". But because it originates from the discreteness of the electric charge, it should be viewed instead as an intrinsically quantum effect. If one considers integration times large compared with the duration between successive photo-electrons, the discrete character of the electrical charge flow tends to be washed out, the theory becomes classical in nature, and accordingly a non-fluctuating photo-current is obtained⁷.

- Law of average-energy conservation.

Let us explain in some detail how the law of average energy conservation is being employed. The electrical pump raises atoms initially in the absorbing state at rate $R_e(t)$ and thus supplies a power $\hbar\omega_e R_e(t)$, where $\hbar\omega_e$ denotes the atoms transition energy. When these atoms decay back, the energy in the optical resonator is incremented⁸. Conversely, the energy in the optical resonator is employed to raise detecting atoms initially in the absorbing state, with transition energy $\hbar\omega_a$, to the upper state at a rate $R_a(t)$. This power is delivered to the external load, perhaps followed by an electronic amplifier. Ideally, we have $\hbar\omega_a \approx \hbar\omega_e \equiv \hbar\omega_o$, in which case the source-detector configuration may be viewed as reversible⁹. The law of con-

⁷It is interesting to note that similar concepts (relating this time to the conservation of the average angular momentum rather than to the average energy) were recently advanced by C.S. Unnikrishnan [9]. The author shows that, if a pair of electrons in the singlet state is emitted, and the magnetic moments are detected at separate locations at angles differing by θ , the only correlation consistent with conservation of the average angular momentum is the quantum result $-\cos(\theta)/4$, if the readings are normalized to $\pm 1/2$. What is strictly "quantum" is the discreteness of the electron spin. In the large-spin S limit the correlation evaluated from classical considerations is obtained. B. d'Espagnat, though challenging that author's interpretation of Bell's results, seems to agree with his factual conclusions [10].

⁸The concept of "light energy" is understood only in a restricted sense. In order to determine the energy contained in a laser resonator at some time, say $t = 0$, one may cut-off the pump and measure the number of subsequent photo-detection events. It should be noted, however, that semiconductors (incorporated in particular in laser diodes) contain some energy of their own that cannot easily be separated out from the field energy. Even though the optical field is not quantized, the word "photon" is employed occasionally as an other name for the energy of loss-less resonators. Precisely, the resonator energy is written as $(m + 1/2)\hbar\omega_o$, where m is called the number of photons in the resonator, ω_o the resonator (angular) frequency, and \hbar the Planck constant (divided by 2π). Likewise, the word "photon rate" is another name for electromagnetic power divided by $\hbar\omega_o$.

⁹This situation is analogous to that of the reversible heat engines discovered by Carnot in 1824. Reversibility occurs when bodies are contacted only when their temperatures are nearly the same. As Carnot

ervation of energy then says that the integral from $t = 0$ to $t = T$ of the power difference $(R_e(t) - R_a(t))\hbar\omega_o$ is equal to the system energy increment from $t = 0$ to $t = T$, consisting of atomic and field energy increments. These, however, are finite. It follows that in the limit $T \rightarrow \infty$, we must have $R_e(t) \approx R_a(t)$. More precisely, $\lim_{T \rightarrow \infty} \langle R_e(t) \rangle_T - \langle R_a(t) \rangle_T = 0$, where the subscripts T refer to average values taken over a time duration T . In other words, input and output rate spectral densities must be the same when they are averaged over a sufficiently large time duration. In the Fourier domain Ω , this means that the input and output rate spectral densities must be the same in the limit $\Omega \rightarrow 0$.

In the case of laser diodes, the number n of electrons in the conduction band fluctuates as a consequence of the lase-diode dynamics. The Fermi-Dirac law then tells us that the potential applied to the diode fluctuates, and should be written as $U(t)$. If J denotes the non-fluctuating pump current the input power $JU(t)$ is no longer a constant. However, detailed calculations show that the fluctuations of U have a negligible effect on the energy balance, so that the previous argument still holds.

- Circuit representation.

The configurations investigated in this paper are described in terms of conservative elements such as capacitances and inductances, whose values may be obtained from separate classical measurements, as is done in conventional electronics¹⁰, and positive and negative conductances.

- Average conductances.

To define the latter, consider atoms treated according to the (non-relativistic) Schrödinger equation. The one-electron wave-function $\psi(x)$ describes an ensemble of identically-prepared systems. According to Born, $|\psi(x, t)|^2$ denotes the probability density of finding the electron at x if a position measurement is performed at time t , and $|\psi(p, t)|^2$ denotes the probability density of finding the electron momentum as p if a momentum measurement is performed at time t , where $\psi(p)$ is essentially the Fourier transform of $\psi(x)$. These atoms are supposed to be located between parallel perfectly-conducting plates with a spatially-uniform optical field in-between, and thus the electrons are submitted, besides static electrical fields, to electrical fields at some optical frequency ω . The usual electric-dipole and rotating-wave approximations may

acknowledged, a non-zero power (non-zero heat flow) occurs only when there is some temperature difference between the contacted bodies. However, it is legitimate to consider the limit in which this temperature difference tends to zero. If this is the case, the mechanical energy delivered *per cycle* tends to a well defined limiting value. Cycles are then very slow and the power generated is very small.

¹⁰The evaluation of a capacitance from its geometric dimensions is straightforward. If one insists in evaluating inductances from their geometric dimensions one needs suppose that they contain electrons. The latter have magnetic moments and the magnetic permeability μ may be much larger than μ_o just above the Curie temperature T_C , being of the form $(T - T_C)^{-1.6}$. In that case a non-zero inductance is obtained with μ_o set equal to zero.

be employed. The Quantum Mechanical (QM)-averaged induced current is proportional to the QM electron average momentum p . Next, we consider electrons which interact with the field only for a finite duration τ , either because they are given some transverse motion, or because they tunnel out from the interaction volume into conductors. We end up with an expression of the average conductance G "seen" by the optical potential applied to the plates, which may be positive or negative, depending on the initial conditions. Detailed calculations according to the above lines are in fact unnecessary because we suppose that the (positive or negative) conductances are known from separate measurements. We initially neglect the frequency-dependence of the conductance G that characterizes the field-atom coupling, assuming a large homogeneous broadening due, e.g., to frequent atomic collisions. As matter of fact, $G(\omega)$ peaks at some transition frequency ω_a , and an improved field-atom model is in terms of a circuit consisting of a capacitance in parallel with an inductance and small resistance in series. To conclude, the system is described by a *circuit*, consisting of interconnected (conservative) capacitances, inductances, and positive and negative (linear) conductances. We may allow, however, the conductances to depend on parameters such as the number n of atoms or of electrons in the conduction band, and on the emitted photon rate R .

- Nyquist noise at optical frequencies.

As hinted above, optical set-ups are viewed as black boxes characterized by in-going and out-going photo-currents, whose statistical properties are either prescribed or sought for. Once a medium has been described by a circuit we are concerned with potentials and currents varying at, or near, some optical frequency ω_o . These will be called "optical potentials", $V(t)$, and "optical currents", $I(t)$, respectively, to distinguish them from static potentials, U , and slowly-varying currents, $j(t)$. We introduce optical potentials (or electric fields) and optical currents (or magnetic fields) for the sole purpose of ensuring that the photo-currents conserve the average energy in the sense explained above.

One needs the quantum form given by Nyquist in his celebrated paper, supplemented by the term $\hbar\omega_o/2$ previously suggested by Planck. The complete formula will be referred to as the "Nyquist-like" formula. An experimental verification of that formula at a temperature of 1.6K is illustrated for example at the beginning of Gardiner's book on Quantum Noise [11]. As a matter of fact, only the Planck term is important in the major part of this paper because we suppose that absorber atoms are all in the lower state (T=0K) while all the emitter atoms are in the higher state (complete population inversion). Various methods will be presented showing that, in that case, the induced-current-fluctuations spectral density is equal to $\hbar\omega_o G$, where G denotes the absolute value of the conductance. The current noise sources are independent of one another. The Nyquist current noises may be supposed to be normal, i.e., jointly

gaussian distributed¹¹.

- Dependence of G on frequency ω .

In general, the conductance G depends on frequency. A well-known theorem says that in the linear regime the Nyquist formula is applicable to frequency-dependent conductances $G(\omega)$, as long as the temperature is uniform.

- Dependence of G on the number n of electrons.

Both the average conductance G and the spectral density of the induced-current fluctuations are proportional to the number n of electrons as long as these electrons are not coupled directly to one another. In semiconductors the electrons are directly coupled to one another and the conductance G depends non-linearly on the number n of electrons. At $T_m = 0$ a gain proportional to $n^{1/3}$ would indeed be appropriate. However, because the fluctuations of n are small, a conductance of the form $G(n) = G_o + (dG/dn)(n - n_o)$ is often employed.

- Dependence of G on the emitted (or absorbed) rate R .

In semiconductors the conductance G may depend significantly not only on the number n of electrons in the conduction band, but also, explicitly, on the emitted power. This effect, which may be due to spectral-hole burning, is called here "gain compression" (another name is "non-linear gain"). It plays a significant role in laser-diodes operation, increasing in particular the laser-diode relaxation-oscillation damping. We show in a section under what conditions the usual Nyquist-like formula remains applicable under such circumstances.

- Linear and linearized regimes.

Only two limiting cases are considered, namely the linear regime and the linearized regime. In the linear regime optical potentials and currents are proportional to the fundamental noise sources. The response of linear systems to specified sources is straightforward, but dispersion effects need investigation. This regime is applicable to lasers below the so-called "threshold" driving current and, usually, to attenuators and amplifiers.

In the linearized regime, on the other hand, one first needs evaluate average optical potentials and currents ignoring the noise sources. This is the so-called "steady state". Next, one supposes that the *deviations* of the optical potentials and currents from their average values, denoted by Δ , are proportional to the fundamental noise sources. The latter enter again when flowing powers are being evaluated, that is,

¹¹Within our linear or linearized approximations the noise currents are therefore also normal and cross-correlations of any order may be obtained from second-order cross-correlations. It follows that measurable noise currents are time reversible, and thus do not reflect the fact that the optical circuit elements are causal.

current noise sources are not given for free, so to speak, but do enter in the power balance. This is the main reason why our theory differs from previous semi-classical theories.

The linearized regime is applicable to well-above-threshold lasers. The intermediate situation in which the system is neither linear nor can be linearized that may occur for closed-to-threshold lasers is not considered. As said above, we treat only the stationary regime found when a laser is driven by a constant current, possibly supplemented by stationary fluctuations, and no element is explicitly time-dependent, in which case photo-detection events are stationary as well.

We assume that the atomic polarization may be adiabatically eliminated, so that our equations involve only the optical field, proportional to the optical potential V , and the numbers of electrons in various levels. The latter derive from rate equations that may sometimes be simplified by neglecting time derivatives ("slaving principle"). Because spontaneous decay plays only a secondary role in our theory it is ignored for the sake of simplicity in the major part of this introductory paper, but is briefly discussed in Section 4.4.

- Potential fluctuations and correlations.

In laser diodes employing semi-conducting materials, a constant-current drive J entails a static potential U across the diode that slightly exceeds E_g/e , where E_g denotes the semiconductor energy gap, because the bottom of the conduction band is filled up with n electrons, according to the Fermi-Dirac distribution. Likewise, there are n holes at the top of the valence bands. The rate equations that we shall introduce later on involve random fluctuations of n , and thus fluctuations ΔU of the potential U . This fluctuation is very small, yet measurable. One may also measure the correlation between ΔU and the detected current fluctuation ΔJ . One may define this correlation in such a way that it is independent of any linear optical loss that may occur between the laser and the detector. From our view-point, ΔU is a small secondary effect that may initially be neglected.

- Light spectrum.

The light spectrum is a well defined quantity. To observe it, it suffices to introduce before the photo-detector a very narrow-band, cold and linear, filter whose response is centered at some frequency ω_o . The average photo-current $\langle J(t) \rangle$ is proportional to the light spectral density $\mathcal{S}(\omega_o)$.

In the linear regime, the light spectrum may be evaluated from the modulus square of the system response to the Nyquist-like noise sources. Instead of using a narrow linewidth filter as said above, the light spectrum may be derived from the photo-current spectrum, because the latter can be shown to be an auto-convolution of the light spectrum.

In the *linearized regime*, the light spectrum may be evaluated by first neglecting amplitude fluctuations and considering frequency fluctuation $\Delta\omega(t)$, the latter being defined from the time derivative of the phase fluctuations of the optical wave incident on the photo-detector, which may be evaluated from the linearized-system response to the Nyquist-like noise sources. Experimentally, frequency noise may be converted to photo-current noise if the detector is preceded by frequency-selective optical circuits. A dual-detector arrangement is then advisable.

Key results. When light originating from a laser diode driven by non-fluctuating electrical currents is incident on a photo-detector, the photo-current does not fluctuate much. Precisely, this means that the variance of the number of photo-electrons counted over a large time interval is much smaller than the average number of photo-electrons. As we shall see, this is a consequence of the law of average energy conservation. Lasers having that property are called "quiet lasers". Viewed in another way, at high power, the photo-current reduced power spectrum (the adjective "reduced" meaning that the spectrum singularity at $\Omega = 0$ has been removed) is of the form $\Omega^2/(1 + \Omega^2)$, where Ω denotes the Fourier frequency, and thus the spectral density vanishes at $\Omega = 0$. This conclusion is equivalent to the one given earlier concerning the photo-count variance. We will say that light is sub-Poissonian when the spectral density of the photo-current is less than the average rate at small Fourier frequencies¹².

The conclusion that for quiet lasers photo-current spectral densities vanish at zero Fourier frequency holds as long as the elements involved are *conservative*. Accordingly, the conclusion holds irrespectively of dispersion (related to the so-called "Petermann K-factor"), of the value of the phase-amplitude coupling factor (introduced independently in 1967 by Haken and Lax and usually denoted by α), and of the amount of gain compression (introduced by Chanin and alternatively called "non-linear gain"). These effects do affect however the photo-current spectral density at non-zero Fourier frequencies, the laser linewidth, and other laser properties. Note that conventional vacuum tubes with space-charge-limited cathode emission such as reflex klystrons should also emit quiet electromagnetic radiation. We do not know whether this has actually been observed, nor whether it can be observed in consideration of the klystron modest efficiency, and of thermal or flicker noises.

Organization of the paper. Besides the introduction and the conclusion, the paper consists of six sections. The first one gives an account of the most relevant results in Physics. The second one lists mathematical results relating to deterministic or random functions. The third one is a discussion of the Circuit Theory and the Classical and Quantum equations of motion. The fourth one offers methods of establishing that the

¹²Some authors call instead light "sub-Poissonian" when its normalized correlation (to be later defined) at zero time delay is less than unity. The two definitions are in general non-equivalent.

spectral density of Nyquist-like noise sources associated with a conductance is proportional to the absolute value of the conductance. Next, a number of results of practical interest are derived in the linear and linearized regimes to illustrate how the general theory is employed.

Aside from historical works, many citations relate to our own work (1986-2006). It is our intention to provide in a later version of this paper a more comprehensive list. Some important references, not cited here, may be traced back, however, from the recent papers cited.

2 Physics

According to the latin poet Lucretius, a follower of Democritus, there are no forbidden territories to knowledge: "...we must not only give a correct account of celestial matter, explaining in what way the wandering of the sun and moon occur and by what power things happen on earth. We must also take special care and employ keen reasoning to see where the soul and the nature of mind come from,..." . And indeed, the three most fundamental questions: what is the origin of the world? what is life? what is mind? remain subjects of scientific examination. Needless to say, the present paper addresses much more restricted questions.

We will first recall how Physics evolved from the early times to present, no attempt being made to follow strictly the course of history. The theory of light or particle motion and the theory of heat followed independent paths for a long time. The Einstein contributions proved crucial to re-unite these two fields early in the 20th century. We may distinguish "pictures" based on our in-born or acquired concepts of space and time that may not answer all legitimate questions nor be accurate in every circumstances, and complete theories. Quantum theory is considered by most physicist as being accurate and complete, although some questions of interpretation remain hotly debated. We will consider in some detail the theory of waves and trajectories that are essential to understand the mechanisms behind vacuum-tube and laser operation. We also offer view-points concerning the Quantum Theory of Light.

2.1 Early times

From the time of emergence of the amphibians, earth, a highly heterogeneous stuff, is our living place. On it, we experience a variety of feelings. We feel the pull of gravity, breath air, get heat from the fire and the sun, and feed on plants growing on earth and water. Our experience, both as human beings and as physicists, is based on these living conditions. One may presume that natural selection led human beings to an intuitive understanding of geometrical-physical-chemical quantities such as space, time, weight, warmth, flavor, and so on. At some point in the evolutionary process a degree of abstraction, made possible by an enlarged brain, facilitated our fight for survival. An example of abstract thinking is the association with space of the number 3, corresponding to the number of perceived

dimensions. People "in the street" may however wish to distinguish the two horizontal-plane dimensions and the vertical dimension, considering that, for the latter, up and down are non-equivalent directions. It may not be obvious to them that the distinction between "up" and "down" is caused by the earth gravitational field, and that people living on the other-side of the earth have the same feelings as we do in their every-day life, even though, with respect to our own reference frame, they are "up-side-down". As we shall see, analogous considerations may apply to time, according to Boltzmann.

Another naturally evolving concept is indeed the distinction between past and future and physical causality: matter acts on matter only at a later time. The so-called "arrow of time" is a much debated subject. According to Boltzmann, in an infinite universe, there may be large-scale spontaneous fluctuations of the entropy (that one may crudely describe as expressing disorder). Past \rightarrow future would correspond to the direction of increasing entropy. There may be times where the entropy decreases, instead of increasing. But the distinction is purely a matter of convention (in analogy with the "up and down" distinction mentioned above). This view point is consistent with the fact that the fundamental equations of Physics are (with the exception of the rarely occurring neutral-kaon decay) invariant under a change from t to $-t$. There has been objection to the Boltzmann view-point, however, and most recent authors would rather ascribe the time arrow to cosmic evolution, with the universe starting at the "big-bang" time in a state of very low entropy.

In contrast with the rational view concerning causality, the magic way of thinking presupposes the existence of causal relationships between our desires, fears, or incantations, and facts. Now-a-days, magic thinking co-exists with rational thinking probably because it gives people sharing similar beliefs a sense of togetherness and helps a few individuals acquire authority and power. The consequences of irrationality are perhaps too remote to be of concern to most.

The control of fire by man some 500 000 years ago and drastic climatic changes that occurred, mainly in Europe, some 23 000 years ago, triggered evolutionary events. Likewise, the practice of growing crops made possible a population explosion some 10 000 years ago, particularly in Egypt, and gave an incentive for measuring geometrical figures, precisely accounting for elapsed times, and measuring weights. Let us now consider more precisely what is meant by space and heat.

Empedocle (\sim 500 BC) viewed the world as being made up of four elements, namely earth, water, air and fire. These elements remain a source of inspiration for poets and scientists alike, but they are not considered anymore as having a fundamental nature. Democritus (\sim 400 BC) pictured reality as a collection of interacting identical particles that cannot be split ("a-toms"). Aristotle wrote in his *Metaphysics VIII*: "Democritus apparently assumes three differences in substances; for he says that the underlying body is one and the same in material, but differ in shape, position, and inter-contact". This picture may still be viewed as being basically accurate.

The present work is not concerned with the cosmos per se. Yet, one cannot ignore that observations of the sky have been a source of inspiration in the past and remain

very much so at present. Early observers distinguished stars from planets, the latter moving apparently with respect to the former. The ancient Greeks (Ptolemy) conceived a complicated system of rotating spheres aimed at explaining the apparent motion of these celestial objects. Aristarchus (310-230 BC), however, realized that the earth was rotating about itself and about the sun, the latter being considered to be located at the center of the universe. This *heliocentric* system was rediscovered by Copernicus (1473-1543) and popularized by Galileo Galilei (burned at stake in Rome in 1600 for heresy). Next came the establishment of the three laws of planetary motion by Kepler, the dynamical explanation of these laws by Newton, and the deeper theory proposed by Einstein in 1917, which so far appears to be in good agreement with observations.

When two bodies are in thermal contact they tend to reach the same temperature. Thus, two differently constructed thermometers may be calibrated one against the other by placing them in the same bath and comparing their readings. In the cases of a thermal contact, the hotter body loses an amount of heat gained by the colder one but the converse never occurs. It may well be that the condition of heat-engine reversibility, discovered by Carnot in 1824, could have been made at a much earlier time and could have served as a basis for subsequent developments in Physics. The present attitude is rather that one should derive the laws of Thermodynamics from Classical or Quantum theories. It may be however that, to the contrary, the latter theories cannot be formulated unambiguously without the former.

2.2 How physicists see the world now-a-day

Beyond a qualitative understanding of the nature of heat, early observers were able to perform measurements of temperature and gas pressure with fair accuracy. Temperatures were measured through the expansion of gases at atmospheric pressure, linear interpolation being made between the freezing (0°C) and boiling (100°C) water temperatures. The concept of absolute zero of temperature emerged through the observation that extrapolated gas volumes would vanish at a negative temperature, now known to be $-273.15^{\circ}\text{C}=0$ kelvin. The Classical Theory of Heat was established in the 18th and 19th centuries mainly by Black, Carnot and Boltzmann. The major contribution is due to Carnot (1824) who introduced the concept of heat-engine *reversibility*. The fact that hot bodies radiate power was known very early (some reptiles possess highly-sensitive thermal-radiation detectors). It is however only in the 19th century that the proportionality of the total radiated power to the fourth power of the absolute temperature was established. Difficulties relating to the theory of blackbody radiation led Einstein around 1905 to the conclusion that Classical Physics ought to be replaced by a more fundamental theory, namely the Quantum Theory, even though important conclusions may be reached without it. Another motivation for studying in some detail the theory of heat is that lasers are in some sense heat engines. They may be “pumped” by radiations originating from a hot body such as the sun. But, just as is the case for heat engines, a cold body is also required to absorb the radiation

resulting from the de-excitation of the lower atomic levels. Lasers are able to convert heat into work in the form of radiation, but their efficiency is limited by the second law of thermodynamics. Output-power average values and fluctuations may be similar for lasers and heat engines.

Approximately 500 km above ground level, space is nearly empty so that laser light gets attenuated only as a result of diffraction. Microwave and light beams may therefore carry large amounts of data in space over great distances. The grand picture we now have is that of a world 13 billions years old and 13 billions light-years across containing about 10^{11} galaxies. Our own galaxy (milky way) apparently possesses at its center a spinning black hole with a mass of 3.5 millions solar masses. Penzias and Wilson discovered in 1965 the cosmic background microwave radiation, which accurately follows the Planck law for a temperature of 2.73 kelvins. This cosmic black-body radiation is almost isotropic. Yet, minute changes of intensity according to the direction of observation have been measured, which provide precious information concerning the state of the universe some 300 000 years after the "big-bang". Numerous observations relating to ordinary stars such as the sun, neutron stars, quasars, black holes are particularly relevant to high-energy physics. It is expected that gravitational waves emitted for example by binary stars or collapsing stars will be discovered within the next ten years or so. Their detection may require sophisticated laser interferometers operating in space. In such interferometers, laser noise plays a crucial role. Reactors aim at creating on earth conditions similar to those occurring in the sun interior, i.e., temperatures of millions of kelvins, and to deliver energy, perhaps by the year 2050. An alternative technique employs powerful lasers shooting at a deuterium-tritium target. A reduction of the laser-beam wave-front fluctuations are essential in that application. For a review of the present views concerning the universe, see for example [12].

2.3 Epistemology

Epistemology is the study of the origin, nature, methods and limits of knowledge. Undoubtedly, Physics is an experimental science. Its purpose is to predict the outcome of observations, or at least average values of such observations, over a large number of similar systems, from a few principles using Mathematics as a language. Observations are required to set aside as much as possible human subjectivity. This is done by performing a large number of "blind" experiments, the same procedure being repeated again and again in independent laboratories. A physical theory should be "falsifiable", that is, one should be able to realize, or at least conceive, an experiment capable of disproving it. The average value $\langle a \rangle$ of a quantity a is calculated by summing $ap(a)$, where $p(a)$ is the probability density of a . It is apparently difficult to provide an unambiguous definition of the word "probability". Let us quote Dose [13] "There is a fundamental mistrust in probability theory among physicists. The need to extract as comprehensive information as possible from a given set of data is in many cases not as pressing as in other fields since active

experiments can be repeated in principle until the obtained results satisfy preset precision requirements. [In other fields] the available data should be exploited with every conceivable care and effort". As data comes in our estimate of the probability $p(a)$ improves, and eventually approaches an objective value, defined according to the frequentists view-point.

In practice, most scientific progresses were accomplished with the help of intuitively appealing *pictures*, describing how things happen in our familiar three-dimensional space and evolve in the course of time. These pictures are supposed to tell us how things are behind the scene, or to suggest calculations whose outcome may be compared to experimental results. Let us quote Kelvin: "I am never content until I have constructed a mechanical model of the subject I am studying. If I succeed in making one, I understand; otherwise I do not". But many models, helpful at a time, are often discarded later on in favor of more abstract view-points. The Democritus picture of reality has been worked out in modern time by Bernouilli, Laplace and a few others. Given perfectly accurate observations made at some time, called "initial conditions", the theory is supposed to predict the outcome of future observations if the system observed is not perturbed meanwhile. (Poincaré, however, pointed out that for some systems, e.g., three or more interacting bodies in Celestial Mechanics, the error grows quickly in the course of time when the initial conditions are not known with perfect accuracy). The equations that describe ideal motions are time reversible, so that when the system is known with perfect accuracy at a time its state in the past as well as in the future is predictable. Predictions for earlier times (perhaps a misnomer) make sense if measurements were then made but not revealed to the physicist. What we have just described is sometimes referred to as the "Classical Paradigm".

Reality is surely a concept of practical value. Anyone wishes to distinguish reality, as something having a degree of permanency, from illusions or dreams that are transitory in nature. On some matters, the opinions of a large number of people are sought, supposing that their agreement would prevent individual failures. In that sense, reality may exist independently of observers and be *revealed* by observations. But according to Bohr the purpose of Physics is not to discover what nature *is*, but to discover what we can say consistently about it. One may stick to the view-point that observations may relate only to a complete set up, including the preparation and measurement devices, the latter being considered classical [14]. A specific measurement device is described in [15]. In effect, the object to be measured should be able to switch another object involving a large number of degrees of freedom from one metastable state to another. In Quantum Mechanics, a system whose preparation is somewhat uncertain is described by a statistical operator ρ . Some authors go one step further and advocate the use of a probability density $p(\rho)$ [16]. For an ensemble of systems described by a pure state $\psi(x)$, measurement of the electron position suffers from an uncertainty Δx and measurement of the electron momentum suffers from an uncertainty Δp . Quantum mechanics tells us that $\Delta x \Delta p \geq \hbar/2$. However, one can measure the electron energy accurately in a time interval as short as one pleases [17]. The advanced notions mentioned in the present paragraph are not needed in this paper. It seems to us nonetheless that some understanding of the Physics conceptual difficulties is

useful.

Seemingly reasonable pictures may fail to agree with observations in special circumstances. Indeed, consider a source and two measuring apparatuses, one located on the left of the source and the other on the right. Using Stapp [18] terminology, apparatuses may be set up to measure either size (large or small) or color (black or white) but not both at the same time. It is observed¹³ that (l,b), (w,w) and (b,l) never occur, where large, small, white and black have been abbreviated by their first letters. The first term in these expressions correspond to the left-apparatus outcome and the second term to the right-apparatus outcome. In writing "large", for example, we of course imply that the apparatus has been set up to measure size, while in writing "black", for example, we imply that the apparatus has been set up to measure color. Let us now attempt to explain the above observations on the basis of the following picture: Assume that there exist four kind of particles, namely (lw), (lb), (sw) and (sb). The source is supposed to shoot out one of these particles to the left and one to the right according to some probability law (there are all-together 16 probabilities summing up to unity, but only 4 of them will be considered). However, the fact that (l,b) never occurs implies that $\text{prob}(lw,lb)=0$. Here, " $\text{prob}(lw,lb)=0$ " means that the source is not allowed to shoot out a particle of the kind lw on the left and a particle of the kind lb on the right. Indeed, if it were allowed to do so, the left apparatus, set up to measure size, would sometimes give "l", while the right apparatus, set up to measure color, would sometimes give "b", contrary to observation. For the same reason, the source is not allowed to shoot out "lb" on the left and "lb" on the right, a condition that we write as $\text{prob}(lb,lb)=0$. Next we note that the observation that (w,w) never occurs implies that $\text{prob}(lw,lw)=0$. Finally, the observation that (b,l) never occurs implies that $\text{prob}(lb,lw)=0$. Accordingly, the probability that "l" be found on both sides is, considering the four possible combinations, $\text{prob}(l,l) = \text{prob}(lb,lb) + \text{prob}(lw,lb) + \text{prob}(lb,lw) + \text{prob}(lw,lw) = 0$, where the probabilities obtained above have been employed. Observations reveal, however, that if "l" is found on the left side, the probability that "l" be found also on the right side is equal to 0.065, i.e., is non-zero. It follows that the picture of a source shooting out two particles disagrees with observations. Of course, the non-zero probability quoted above (6.25 per cent) applies to elementary particles having only two attributes, each of them exhibiting any-one of two possible values, not to macroscopic objects that may have other, measurable, attributes. It is frequently the case that an effect deemed impossible according to Classical Mechanics, for example the transmission of a particle through (or above) a barrier of greater energy, is in fact observed (tunneling). This is because it is considered impossible, even in principle, to measure particle energies on top of the barrier.

We are not concerned in the present paper with Physics in general but only with stationary configurations, so that the epistemology of that part of Physics could perhaps be made more precise. The system is allowed to run in an autonomous manner, that

¹³In reality, we are referring here to Quantum Mechanical (QM) predictions rather than to real observations. There has been, however, so many experimental observations that agree with QM, that one may overlook the fact that observations have perhaps not been made for the system presently considered.

is without any external action impressed upon it, and there is a continuous record of some quantities, particularly the times at which photo-electrons are emitted or absorbed. Systems on which we may act from the outside are not considered. Photo-electrons may be accelerated to such high energies by static fields that no ambiguity occurs concerning their occurrence times. The question asked to the physicist then resembles the one asked to people attempting to recover missing letters from impaired manuscripts: can you determine the missing letters from the known part of the text? In the present situation one would like to be able to tell whether an event occurred during some small time interval, given the rest of the record. Or at least give the *probability* that such an event occurs in the specified time interval. In other words, given a large collection of similar systems, on what fraction of them does an event occur? Instead of being given impaired records, we may be given information concerning the various components that constitute the system, such as lenses, semi-conductors, and so on, characterized by earlier, independent measurements. These measurements are deterministic in nature because they are performed in the classical high-field (yet, usually, linear) regime. In view of the observed uncertainty, spontaneous noise sources must obviously be introduced somewhere in the theory. We consider that the noise sources are located solely at emitters and absorbers, viewed as being similar in nature. Pound [19] described earlier lasers in terms of a Nyquist theorem extended to negative temperatures.

2.4 Waves and trajectories

Physics courses usually first describe how the motion of masses may be obtained from the Newtonian equations. But it might be preferable to let students get first familiarity with classical waves, for example by observing capillary waves on the surface of a mercury bath. Such waves are described by a real function of space and time that one may denote $\psi(x, t)$ in one space dimension. One reason (to be explained in more detail subsequently) to consider waves as being of primary interest is that the law of refraction follows in a logical manner from the wave concept, but does not from the ray concept. Once wave concepts have been sufficiently clarified, the many-fold connections existing between waves on the one hand, and particles or light rays on the other hand, may be pointed out. Note that, historically, the motions of macroscopic bodies and light rays were established first (around 1600) and the properties of waves later on (around 1800 for light and 1900 for particles). Few precise results concerning waves seem to have been reported at the time of the ancient Greece. Yet, casual observation of the sea under gently blowing winds suffices to reveal important features. Had such observations been made, the course of discoveries in Science would perhaps have been quite different from what actually occurred.

Waves at the surface of constant-depth seas propagate at constant speed u . In realistic conditions there is some dissipation and the wave amplitude may decrease but the wave speed remains essentially unchanged. This is a striking example of a physical object whose speed does not vary, no force being impressed upon it. The only condition required is

that the medium parameters (the sea depth in the present situation) do not vary from one location to another.

In the 1630s Galileo observed that macroscopic objects move at a constant speed when no force is exerted on them, in contradiction with the then-prevailing Aristotle teaching. A related finding by Galileo is the principle of special relativity: The laws of Physics established in some inertial laboratory are the same in another laboratory moving at a constant speed with respect to the first. In the year 1637 Descartes proposed the following interpretation for the refraction of light rays at the interface between two transparent media such as air and water. Descartes associates with a light ray a momentum that he calls "determination" having the direction of the ray and a modulus depending on the medium considered but not on direction. He observes that the x -component of the momentum should not vary at the interface as a consequence of the uniformity of the system in that direction, justifying this assertion by a mechanical analogy, namely a ball traversing a thin sheet. The law of refraction asserting that $\cos(\theta_1)/\cos(\theta_2)$, where the angles are defined with respect to the x -axis and the subscripts 1,2 refer to the two media, does not depend on the ray direction, follows from the above concepts¹⁴. Note that Descartes was only concerned with trajectories in space, i.e., he was not interested in the motion of light pulses in time, so that questions sometimes raised as to whether light pulses propagate faster or slower in air or in water are not relevant to his discussion.

No one at the time suggested that there may be a connection between particles or light rays on the one hand, and waves on the other hand. The wave properties of light were discovered by Grimaldi, reported in 1665, and explained by Huygens in 1678. The wave properties of particles were discovered much later by Davisson and Germer in 1927. In modern terms the Galileo, Descartes (and later Newton) concepts imply that particles and light rays obey ordinary differential equations. But without the wave concept the law of refraction for light or for particles relies on observation and intuition rather than logic.

A wave packet has finite duration but includes many wave crests. A key concept is that of group velocity defined as the velocity of the peak of a wave packet, or short pulse. In particular, what is usually called the "velocity" of a (non-relativistic) body is the group velocity of its associated wave. But usually wave packets spread out in the course of time. In the non-linear regime though, wave packets, called *solitons*, may exhibit particle-like behavior in the sense they do not disperse. Bore-like solitary waves created by horse-drawn barges were first reported by Russell in 1844.

Let us be more precise about waves. As said above, waves are very familiar to us, particularly gravity waves (not to be confused with the Einstein gravitational waves) on the sea generated by wind, or capillary waves generated on the surface of a lake by a falling stone. Simple reasoning and observations lead among other results to the law of refraction. Waves are defined by a real function $\psi(x, t)$ for one space coordinate x , and time

¹⁴The law of refraction is most commonly written as $n_1 \sin(i_1) = n_2 \sin(i_2)$, where the n are refractive indices and the i angles are defined with respect to the normal to the interface.

t , obeying a partial differential equation. If the wave equation is unaffected by space and time translations we may set $\psi(x, t) = f(x - ut)$ for arbitrary speeds u . This results into an ordinary differential equation for the function $f(x)$ which in general admits solutions. Let us begin our discussion with monochromatic (single-frequency) waves propagating in the x direction in a conservative linear and space-time invariant medium. The wavelength λ is the distance between adjacent crests at a given time. We define the wave number $k = 2\pi/\lambda$. The wave-period T is the time it takes a crest to come back, at a given location. We define the frequency $\omega = 2\pi/T$. It follows from the above definitions that the velocity of a crest, called the phase velocity, is $u = \omega/k$. Such waves propagate at constant speed without any action being exerted on them. For linear waves there is a definite relationship between ω and k independent of the wave amplitude, called the dispersion equation. For gravity waves in deep inviscid (non-viscous) waters we have, for example, $\omega = \sqrt{gk}$, where $g \approx 9.81\text{m/s}^2$ is the earth acceleration. When the water depth h is not large compared with wavelength (shallow water), the dispersion relation involves h as a parameter [20, see ref. 8].

The above considerations may be related to mechanical effects. Indeed, if a wave carrying a power P is fully absorbed, the absorber is submitted to a force F satisfying the relation $P/\omega = F/k$. This ratio, called "wave action", depends on the nature of the wave but does not vary if some parameter is changed smoothly, either in space or in time. For a wave of finite duration τ , the energy collected by the absorber is $E = P\tau$ and the momentum received (product of its mass and velocity) is $p = F\tau$.

If the water depth h is changed at some time $t = 0$ from, say, 1m to 2m, it is observed that k is unchanged as a consequence of the wave continuity. But invariance of k implies a frequency change since the dispersion equation depends on h . In that case, the wave speed changes at time t . Likewise, If the water depth h changes at some location $x = 0$ from, say, 1m to 2m, it is observed that ω is unchanged as a consequence of the wave continuity. But invariance of ω implies a wave number change since the dispersion equation depends on h . In that case the wave speed changes at $x = 0$.

Consider now a monochromatic wave (fixed frequency ω) propagating in two dimensions with coordinates x, y . The direction of propagation is defined as being perpendicular to the crests and the wavelength $\lambda = 2\pi/k$ is defined as the distance between adjacent crests at a given time. But one may also define a wavelength λ_x in the direction x as the distance between adjacent crests *in the x -direction* at a given time. Let the wave be incident obliquely on the interface between two media, the x -axis. For gravity waves the two media may correspond for example to $h(y) = 1\text{m}, y > 0$ and $h(y) = 2\text{m}, y < 0$. Because of the continuity of the wave, λ_x is the same in the two media. If we further assume that the propagation is *isotropic*, that is, that k does not depend on the direction of propagation of the wave in the x, y plane, the law of refraction follows, namely that $k_x = k_1 \cos(\theta_1) = k_2 \cos(\theta_2)$, where the subscripts 1,2 refer to $y > 0$ and $y < 0$ respectively, and the angles θ are defined with respect to the interface, that is, to the x -axis. The law of refraction therefore follows from wave continuity and isotropy alone.

Questions relating to the velocity of light pulses are important for the transmission of information. A wave-packet containing many wave crests moves at the so-called "group velocity" $v = d\omega/dk$, which often differs much from the phase velocity u defined above. Considering only two waves at frequency ω and $\omega + d\omega$, the relation $v = d\omega/dk$ may be visualized as a kind of Moiré effect. Wave crests move inside the packet, being generated at one end of the packet and dying off at the other end. For waveguides we have $uv = c^2$, $v < c$, $u > c$. For matter waves associated with a particle the group velocity v coincides with the particle velocity. Since the energy $E = p^2/(2m)$ and $p = mv$, a previous relation reads $p^2/(2m\omega) = p/k$. It follows that $u = \omega/k = p/2m = v/2$. For gravity waves the dispersion relation gives instead $u = 2v$. A general result applicable to loss-less waves is that the group velocity v is the ratio of the transmitted power P and the energy stored per unit length. It never exceeds the speed of light c in free space.

Wave solutions of the form $\psi(x, t) = \psi(x - ut)$, where $\psi(x)$ is some given function and u a constant, exist also for non-linear wave equations. When the $\psi(x)$ function is localized in x , the invariant wave-form is called a solitary wave. In some cases, solitary waves exhibit transformations akin to those of particles when two waves collide and are called "solitons" in the sense that the soliton integrity is preserved.

As said before, most continuous media may be modeled by discrete circuits. For example, a transmission line may be modeled by series inductances and parallel capacitances. Free space may be modeled by electrical rings in which electrical charges move freely and magnetic rings in which (hypothetical) magnetic charges would move freely. If each electrical ring is interlaced with four magnetic rings and conversely, the Maxwell equations in free space obtain in the small-period limit.

Under confinement along the x -direction, waves at some fixed frequency ω may be viewed as superpositions of "transverse modes". For a transverse mode the wave-function factorizes into the product of a transverse function $\psi(x; \omega)$ and a function of the form $\exp(ik(\omega)z - i\omega t)$. Another connection between waves and rays rests on the representation of transverse modes by ray *manifolds*. These are not however independent rays. A phase condition is imposed on them that leads to approximate expressions of $\psi(x; \omega)$ and $k(\omega)$. Note the analogy with Quantum-Mechanics stationary states, z and t being interchanged.

Thus the wave-particle connection is many fold. First the medium in which the wave propagates may be approximated by a discrete sequence of elements, for example a periodic sequence of springs and masses for acoustical waves and electrical inductance-capacitance circuits for electromagnetic waves, with a period allowed to tend to zero at the end of the calculations. One motivation for introducing this discreteness is that computer simulations require it anyway. A more subtle one is that some divergences may be removed in that way. We have mentioned above capillary waves on a mercury bath. They may be treated by considering the forces binding together the mercury molecules and their inertia, ending up with equations of Fluid Mechanics. Like-wise, acoustical waves in air may be described through the collision of molecules in some limit (isothermal or adiabatic). Second, wave modes may be described approximately (WKB approximation) by ray *manifolds*. Third,

one may consider the behavior of wave packets in the high-frequency limit and liken these wave packets average trajectories to those of macroscopic bodies.

We have described above the motion of light and particles in terms of waves. Semi-classical theories such as the one employed in the present paper rest indeed on such wave concepts, namely Quantum Mechanics for describing electrons, and Circuit Theory for describing the relationship between potentials and currents. The speed of light in free-space is irrelevant in that theory. Quantization then only means that electrons are identical point particles. When particles such as electrons are electrically charged they may be accelerated to arbitrarily large energies by static electrical potentials. Being then in the classical domain there is no ambiguity concerning their arrival time. Uncharged point particles such as neutrons could conceivably be accelerated similarly by gravitational fields, even though this may turn out to be difficult in practice.

2.5 Atoms and elements

Around 1927 it was discovered theoretically by de Broglie and subsequently verified experimentally that a wave of wave-number $k = mv/\hbar$ should be associated with electrons of velocity v . An approximate solution for the motion of an electron following a closed classical path in the neighborhood of a positively charged nucleus thus amounts to prescribe that an integral number n of wavelengths $2\pi/k$ fits along the closed classical path. These discrete solutions are called "stationary states" and n is essentially the principal quantum number. According to the Pauli principle, at most two electrons (with spin $\pm\hbar/2$) may be ascribed to each of these states. At $T = 0\text{K}$ and without excitation by other particles, only the lowest-energy states are filled with electrons. Different elements (H, He, Li...) differ by the number Z of protons in their nucleus.

In the next paragraph we recall how the chemical and electronic properties of the various elements found in nature follow from the above principle, and describe what happens when atoms get closer and closer to one another to form crystals. Then we recall the basic properties of semi-conductors.

2.6 Electron states

We summarize below the most basic concepts concerning elements found in nature and their electron states. The simplest element is the hydrogen atom consisting of a proton with an electrical charge e and a mass much larger than the electron mass m , so that for most purposes the proton may be considered as being fixed in space. This proton attracts one electron of charge $-e$ so that the assembly is neutral. According to Classical Mechanics the electron may circle around the proton at a distance r with a velocity v such that the centrifugal force be balanced by the attraction from the proton, namely $mv^2/r = e^2/(4\pi\epsilon_0 r^2)$. From this view-point any distance r may occur, the velocity v being appropriately chosen. According to Quantum Theory a wave-length $2\pi\hbar/mv$ is associated

with electrons moving at velocity v . The resonance condition is that an integral number n of wavelengths fits within the electron path perimeter $2\pi r$. According to this model, due to Bohr, there is only a discrete sequence of allowed electron energies, corresponding to $n = 1, 2, \dots$. The more exact theory due to Schrödinger leads to symmetrical ground states, called s-states, and anti-symmetrical 3-time degenerate first-excited states, called p-states.

Let us now consider the different elements found in nature. These elements were classified by Mendeleïev in 1869 on empirical grounds. Helium nuclei consist of two protons, lithium nuclei of three protons, and so on, with an equal number of electrons, so that atoms are electrically neutral. (There may be various numbers of neutrons bound to the protons, which depart from the number of protons by a few units, corresponding to different isotopes. Neutrons are not considered in the present discussion). Most elements have an outer layer consisting of a number of electrons going from 1 (e.g., sodium) to 8 (e.g., neon). Particularly important are 3-5 crystals, such as gallium-arsenide.

2.7 Semi-conductors

Our purpose here is to give readers unfamiliar with solid-state physics an overview of the most important phenomena. For silicon, the number of outer electrons is 4. Two silicon atoms (or more) may bind to one another by exchanging electrons of opposite spins (covalent binding). When two atoms are approaching one another, their electronic states get perturbed. As it happens, the isolated-atom electron s-state acquires an energy greater than the isolated-atom electron p-state. For a large number N of atoms, the atomic separation a sets up at a value that minimizes the total energy. The original s-states then split into N states that are so-closely spaced in energy that they form an almost continuous band of states called the conduction band. The original 3-fold degenerate p-states split into N states that are so-closely spaced in energy that they form three almost continuous band of states called the valence bands. Because the degeneracy is lifted these three bands should be distinguished. They are called respectively the heavy-hole band, the light-hole band and the split-off band. For our purposes, only the heavy-hole band needs be considered.

The separation in energy between the bottom of the conduction band and the top of the valence band is called the band gap E_g , often expressed in electron-volts. At $T=0K$, the lower-energy valence band is filled with electrons while the higher-energy conduction band is empty. At that temperature the electrons are unable to respond to an external field because no state is available to them (except perhaps at extremely-high fields). If an electron is introduced in the conduction band by some means it moves in response to an electrical field with an apparent mass m_c smaller than the free-space mass m . If, on the other hand, an electron is removed from the valence band one says that a "hole" has been introduced. This hole is ascribed a positive charge e and a mass usually larger than m .

When two materials having different band gaps are contacted the band gap centers align approximately, and potential steps occur both in the conduction and valence bands. In the case of a double-hetero-junction the lower-band-gap material is sandwiched between

two higher-band-gap materials. The potential steps tend to confine both free electrons and free holes in the central low-band-gap material (e.g., GaAs). Being confined in the same volume electrons and holes may easily interact.

As the band-gap decreases electrons may undergo virtual transfers from one band to the other more easily. As a consequence the material is more easily polarized by external (static or optical) electrical fields. In other words, the material permittivity $\epsilon(\omega)$ increases as the band gap decreases. This is why the permittivity (or refractive index) of the low-band-gap gallium-arsenide is significantly larger than the permittivity (or refractive index) of the large-band-gap aluminum arsenide. When a small-band-gap semi-conductor (GaAs) is sandwiched between two higher-band-gap semiconductors (AlAs), the higher-index material may guide optical waves. This fact is important for the guidance of optical waves in laser diodes employing double-hetero-junctions. An happy circumstance is therefore that electrons, holes, and light, may all get confined in the central part of the double-hetero-junctions considered.

Gallium possesses 3 electrons in the outer shell and arsenide possesses 5 electrons. Equal numbers of these atoms may associate to form a crystal of gallium-arsenide (GaAs), a material particularly important in Opto-Electronics. The reason for this importance is that, unlike silicon, this is a "direct band-gap" material. In direct band-gap materials the minimum of the conduction-band energy and the valence-band maximum energy correspond to the same electron momentum. Accordingly, electrons lying at the bottom of the conduction band may get easily transferred to the top of the valence band, and conversely, the law of momentum conservation being then fulfilled. In such a process, an energy E_g is absorbed by light through stimulated or spontaneous emission processes. Unfortunately, this energy may also be absorbed by another electron (Auger effect) that subsequently cascades down, its energy being converted into heat.

Finally, one should say a word about doping, considering as an example a silicon crystal. When a small number of silicon atoms are replaced by arsenic atoms, these atoms, referred to as "impurities", easily deliver an electron (n-doping). Conversely, when a small number of silicon atoms are replaced by gallium atoms these atoms easily capture electrons (p-doping). A p-n diode consists of two contacting semi-conductors, one with p-doping and one with n-doping. Electron currents may be injected into p-n diodes, and in particular into double-hetero-junctions. This is the current referred to in this paper as the laser-diode driving current or the photo-current of a quantum photo-detector. It is denoted by J .

The above discussion hopefully provides the essential concepts that one needs to get some understanding of the electrical behavior of laser diodes. Note that we denote by z the coordinate along which the optical wave propagates (junction plane) and by x the direction perpendicular to the semiconductor layers. Guidance along the transverse y direction is also considered.

2.8 Classical detectors and generators

The first high-frequency oscillator was probably a triode, with a feed-back mechanism from the anode to the grid controlling the current flow. But we are particularly interested in the reflex klystron discovered by the Varian brothers in 1937, which may deliver electromagnetic radiation up to a frequency of about 10 GHz, and employs space-charge-limited cathodic emission (see e.g. [21]). It was discovered in 1940 by Thompson, North and Harris [22] that the current emitted by space-charge-limited cathodes is strongly sub-Poissonian¹⁵.

Classical detectors are usually $p - n$ junctions that exhibit non-linear current-potential characteristics. If a sinusoidal potential is applied to the diode the current exhibits a non-zero average value, which is a measure of the applied potential amplitude.

2.9 Quantum detectors and generators

The first man-made quantum oscillators involving discrete matter levels were masers, operating at microwave frequencies. Subsequently maser action was discovered to occur naturally near some stars. The first laser, generating visible light, was discovered by Maiman in 1959. The fact that space-charge-limited cathodes generate light with sub-Poisson statistics was first demonstrated by Teich and Saleh in 1983 [23].

Quantum detectors involve transitions between atomic or molecular states. Of particular interest is the visible-light-photon-counter, see [2, p. 181], which has a quantum efficiency of 0.88, a gain of 30 000, a time response of 2ns, but unfortunately a rather large dark count of 20 000 counts per second.

2.10 Quantum Theory of Light

The Maxwell theory of electromagnetic waves suggests that radiated heat consists of electromagnetic waves of some sort. In 1862 Maxwell wrote "[electromagnetic waves travel] at a speed so nearly that of light that it seems we have strong reason to conclude that light itself (including radiant heat and other radiations) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws." Measurements on black-body spectra were performed around 1900 with the help of gratings of appropriate periods, and described by a formula that involves the universal constant \hbar . On the other hand the wave properties of electrons were discovered only decades later because the concept that electrons might possess wave-like behavior ought to wait for the observation that atoms emit light at well-defined frequencies, and because of the technical difficulty of sending electrons emitted from a small-area source on crystals in a very good vacuum. The interpretation of the observed diffraction patterns involves the

¹⁵One of us (J.A.) set up in 1954 an experiment demonstrating that below threshold reflex klytrons behave as negative conductances, and that, in confirmation of the Thompson and others discovery just cited, the current is strongly sub-Poissonian (unpublished), but the expected sub-Poissonian nature of the emitted radiation was not measured.

constant \hbar . It is perhaps not preposterous to suggest that these two key discoveries could have occurred in the reversed order. Had this be the case, the Planck constant would have been considered as being fundamentally related to atomic behavior, and the subsequent appearance of the same constant in black-body radiation would have been viewed as a consequence of the atomic theory.

Most physicists opinion is that the photon concept is essential to understand "non-classical" (e.g., sub-shot-noise) states of light. For a Quantum Theory of Light, see e.g., [24]. Let us recall some of the arguments given in favor of the photon concept, which evolved into the modern second-quantization procedures. The most obvious one is that γ -rays have been observed with an energy $\hbar\omega \approx 1$ micro-joule that appear as point particles because they are detected at precise times and locations, with precise energies and momenta¹⁶ independently of the nature of the absorbing material, whether it be steel or lead, say. γ -ray photons and ultra-relativistic charged particles indeed look alike. But in the non-relativistic regime, light and electrons behaviors are quite distinct.

Light-quanta (later on called "photons") were introduced by Einstein on the basis of the following argument. Consider a collection of two-level atoms in a state of thermal equilibrium with the black-body radiation field. When an atom in the upper state decays to the lower state by emitting light spontaneously it recoils if the light emission is *directed* but would not if light were radiated (almost) isotropically. Einstein calculations indicate that a directed emission is required if the Maxwellian atomic velocity distribution is to be recovered. What is actually observed in free-space is that when a short light pulse brings an atom to the 2s-state it may take a long time, on the order of 1 second, before the atom decays spontaneously to the 1s-state because the 2s \rightarrow 1s transition is forbidden to first order. When this event in fact occurs, say at time $t = 0$, the atom losses an energy E and recoils with a momentum of magnitude E/c in some direction. A time $t = R/c$ later, an atom located at a distance R from the first one gets excited, gaining an energy E and a momentum of magnitude E/c . After completion of these processes energy and momentum are conserved quantities. However, from time 0 to t , these conservation laws seem to be violated. This is why one usually postulates that some agent, called "light", carries energy and momentum at speed c from one atom to the other¹⁷. For an arbitrary distance R between the two atoms see [25]. These authors comment that "In a sense, every photon is virtual, being emitted and then, sooner or later, absorbed", and "virtual photons are messenger particles that cannot be directly detected". This conclusion fits well with our paper philosophy.

Another argument in favor of the concept that light consists of lumps of energy $\hbar\omega$

¹⁶The mathematical uncertainty relations: $\Delta t \Delta \omega \approx \Delta x \Delta k \approx 1$ relating a function to its Fourier transform, may be converted into the following physical uncertainty relations: $\Delta t \Delta E/E \approx (\Delta x/c) \Delta p/p \approx \hbar/E < 10^{-21}$ second if $E > mc^2$, setting $E = \hbar\omega, p = \hbar k = E/c$.

¹⁷Note that ancient philosophers thought that light was moving from the eye to the candle (say), rather than the opposite. This view-point does not fit well with the observation made by Römer that light propagates at a finite speed, and the law of causality as it is presently understood.

is the observation that when a light beam of constant small intensity is incident on an ideal photo-detector (i.e., free of dark current and thermal noise) photo-current events sometimes occur long before the required optical energy $\hbar\omega$ has been collected, in apparent violation of the law of conservation of energy. Our interpretation of this observed fact would be as follows. For the sake of clarity let us suppose that the light is applied to the photo-detector (consisting of atoms initially in the ground state) through a potential source $u(t)$ delivering an (unknowable) current $i(t)$. According to the Schrödinger equation that describes atoms submitted to a prescribed time-varying potential there is, at any positive time t , a non-zero probability that the electron be in the upper state. If this is the case the electron may jump back to the ground state through the potential source, in which case a detection event is recorded. Formally, the current trajectories $i(t)$ may be such that the law of conservation of energy is enforced for a single event. At that point, the reader may object that the light power, given essentially by $u(t)i(t)$, is not a constant in this model, contrary to the constant-light-intensity assumption made earlier. But it must be realized that the concept of "constant light intensity" makes sense only if one refers to stochastic or Quantum Mechanical averages over a large number of similar systems. For a single system, there exists no independent way of measuring the "light intensity" as a function of time. The only information one may obtain concerning the intensity of a light beam is through the output of photo-detectors, and this brings us back to the above discussion. A number of authors have shown that many effects that were at a time supposed to prove the reality of the photon concept (photo-electric effect, Compton effect...) may in fact be interpreted in a semi-classical manner. The question remains apparently open.

Note that most Quantum Theories of laser action begin with a discussion of the statistical operator ρ of the optical field in empty loss-less resonators, which are treated in analogy with mechanical oscillators. Pumping and losses are subsequently introduced in an approximate manner by enforcing the preservation of the commutation relations. For the case presently considered (stationary linearized laser) the results of such calculations exactly coincide with ours. One must thus presume that the approximations are the same. This point, however, has apparently not been discussed in detail. Other Quantum Optics treatments consider atoms in either their upper (pumping atoms) or lower (detecting atoms) states introduced at specific times into the optical cavity and spending there a fixed time τ . Whether the atoms leaving the cavity are in their lower or upper state may in principle be measured, and the corresponding probabilities may be evaluated. In that way some properties of stationary lasers may be predicted. However, the flying-atoms configuration is quite different from the one discussed in this paper.

A relevant Quantum-Optics theory considers instead continuous resonant photo-detection processes. That is, atoms in the absorbing state are coupled to the optical cavity at all times. If the cavity is, at some time, described by a statistical operator ρ , the probability density that a counting event occurs is given by $Trace\{E_- \rho E_+\}$, to within a constant factor proportional to the field detecting-atoms coupling. Here, E_- , E_+ are exponential phase operators with the property that $E_{\pm} |m\rangle = |m\pm 1\rangle$, where $|m\rangle$ denotes a state with exactly

m photons (London, 1926). If a detection event does occur at that time, ρ is changed instantaneously to $\rho' = E_- \rho E_+ / \text{Trace}\{E_- \rho E_+\}$ [26]. To our knowledge, the theory has not been applied yet to stationary quiet lasers.

As far as electrons are concerned, some physicists "second-quantize" the Schrödinger wave-function. Some of them present electron second-quantization as a "book keeping" method, convenient when the optical field itself is quantized, rather than as a fundamentally new theory leading up to different conclusions. For example, the statistical properties of non-interacting electron collections that may exchange heat with a reservoir (canonical ensemble) have been obtained in that manner. However, a simpler direct solution may also be found, based on the partition of integers [27].

3 Basic Mathematics

We clarify the notation employed, recall elementary mathematical formulas, and give the main properties of random point-processes. We then offer a simple picture of quiet light generation, and show that random deletion of photo-electrons leaves unaffected the reduced spectrum. This section is entitled "Basic Mathematics" because most results are mathematical in nature, although rigor is overlooked and the physical motivation is pointed out occasionally.

3.1 Units, conventions and notations

Our notations and conventions may differ from the ones employed by engineers, physicists, or experimentalists, which are not always fully consistent. To simplify formulas we sometimes set as unity quantities such as the characteristic conductance of transmission lines. Otherwise, SI units are employed throughout.

Numerical values. The numerical values needed are

$$\begin{aligned}
 e \text{ (absolute electron charge)} &\approx 1.60 \cdot 10^{-19} \text{ coulombs} \\
 m \text{ (electron mass)} &\approx 9.10 \cdot 10^{-31} \text{ kilograms} \\
 \hbar \text{ (Planck constant divided by } 2\pi) &\approx 1.05 \cdot 10^{-34} \text{ joules} \times \text{second} \\
 k_B \text{ (Boltzmann constant)} &\approx 1.38 \cdot 10^{-23} \text{ joules/kelvin}
 \end{aligned} \tag{1}$$

and

$$\frac{1}{4\pi\epsilon_o} = 10^{-7} (2.99792458 \cdot 10^8)^2 \text{ meters/farad} \tag{2}$$

which is *exact*, i.e., not subjected to revision as a consequence of later measurements, and involves a finite number of digits. The constants $e, m, \hbar, 4\pi\epsilon_o, k_B$ are the only ones that

enter into the present theory because, in agreement with the non-relativistic approximation, we set the free-space permeability $\mu_o = 0$ or equivalently $c = \infty$. The quantities shown above may be set as unity without introducing inconsistency or loss of generality¹⁸. The atomic-unit convention would, however, obscure the physical meaning of the formulas and prevent us from checking dimensional accuracy. We sometimes employ as energy unit the electron-volt $\approx 1.60 \cdot 10^{-19}$ joules. Note that capacitances may be evaluated from ϵ_o and their geometric dimensions, namely the electrodes areas and their spacings. As far as inductances are concerned we suppose that they are known from measurement.

Decibels. Usually an amplifier is loaded with a nominal conductance such as 20 millisiemens (resistance of 50 Ω) and the input impedance is equal to that of the load. If the input power of an amplifier is P_{in} and the output power is P_{out} , the amplifier gain in decibel (abbreviation "dB") is defined as $10 \log_{10}(P_{out}/P_{in})$. If the amplifier is linear the gain does not depend on the input power. In terms of the potentials V_{in}, V_{out} at the input and output ports, the gain reads $20 \log_{10}(|V_{out}/V_{in}|)$ dB, because powers are proportional to the modulus-squares of the potentials in the situation considered. Likewise, an attenuation is defined as $10 \log_{10}(P_{in}/P_{out}) = -10 \log_{10}(P_{out}/P_{in})$. A gain of 3dB means that the input power is multiplied by a factor close to 2. Note that dBm means decibels above a power of 1 mW. For example, 30dBm represents approximately a power of 1 watt.

Spectral densities. We generally employ double-side spectral densities, so that the usual shot-noise formula $2e |J|$ for example is written here as $e |J|$, i.e., without a factor of 2 ($-e$ denotes the electron charge and J the average current).

Functions. Different functions are distinguished by explicitly writing out their arguments. For example the Fourier transform of a function $\psi(x)$ is denoted by $\psi(k)$, i.e., with the same symbol, even though they are different functions. One should not confuse a constant U (no argument) with a function $U(x)$, for example. To avoid a confusion between the arguments of a function and products we employ parentheses of different size, e.g., $f(x/d)$ is a function f of x/d , while $f(x/d)$ represents the product of f and x/d . As usual $\cos^2(x)$ means $(\cos(x))^2$, and likewise for other trigonometric functions.

Complex numbers. A complex number is denoted either as $z = \Re\{z\} + i\Im\{z\}$ or as $z = z' + iz''$, and $z^* = z' - iz''$ denotes the complex conjugate of z . We denote $|z|^2 \equiv z^*z = z'^2 + z''^2$. A complex notation is often employed for describing quantities that vary sinusoidally

¹⁸These units amount to taking as length unit the Bohr radius $a_o = 4\pi\epsilon_o\hbar^2/me^2 \approx 0.53 \cdot 10^{-10}$ meters, and as speed unit $v_o = e^2/4\pi\epsilon_o\hbar = \hbar/a_o m \approx 2.19 \cdot 10^6$ meters/second. Note that v_o may be written as αc , where $\alpha \approx 1/137$ denotes the fine-structure constant. Here α is set equal to zero. The Bohr magneton $e\hbar/2m$ equals 1/2 in these units. The electron magnetic moment in these units would be more exactly -0.5005, and the electron spin along some quantization axis $\pm\hbar/2 = \pm 1/2$. Note that the letter μ may denote two entirely different quantities, namely magnetic moments and magnetic permeabilities.

in time. According to the complex notation, the function $i(t) = \sqrt{2}|I| \cos(\omega t + \phi)$, where the frequency ω and the phase ϕ are real constants, is written as $i(t) = \sqrt{2}\Re\{I \exp(-i\omega t)\}$, where the complex number I is defined as $I = |I| \exp(-i\phi)$. Similar definitions apply to potentials $v(t)$ varying sinusoidally in time, that is $v(t) = \sqrt{2}\Re\{V \exp(-i\omega t)\}$. We have chosen to introduce the factor $\sqrt{2}$ so that the average optical power, defined as the time average of the current-potential product $v(t)i(t)$ be simply equal to the real part of the product V^*I , i.e., without the factor 1/2 that would otherwise occur. V and I are called rms (root-mean-square) complex potentials and currents, respectively, or more briefly, optical potentials and currents. The minus sign in $\exp(-i\omega t)$ is employed in optics because waves propagating forward in space then involve a term of the form $\exp(ikx)$, where k denotes the wave-number, that is, with a plus sign. But for slow variations the function $j(t) = \sqrt{2}|J| \cos(\Omega t + \phi)$ is denoted $j(t) = \sqrt{2}\Re\{J \exp(j\Omega t)\}$, where the complex number J is defined as $J = |J| \exp(j\phi)$, as is usually done in electrical engineering. The complex notation considerably simplifies calculations for real, causal, linear and time-invariant circuits submitted to sinusoidal potentials or currents. Even though the squares of i and j are both equal to -1 these two numbers should be distinguished. In the book mentioned earlier a "bi-complex" notation was introduced to describe in an exact manner sinusoidally-modulated sinusoidal signals. This bi-complex notation is recalled in Appendix C.

In schematics, current sources are represented by circles with an arrow in them, while potential sources are represented by a circle with + and - signs, to define what is meant by positive current or positive potential, as shown later in Fig. 2.

Relative noise, normalized correlation and variance. We are mostly concerned with photo-currents $j(t) \equiv -e\mathcal{D}(t)$. Here, $\mathcal{D}(t)$ is the sum over k of $\delta(t - t_k)$ -distributions where the t_k form a stationary point (or "event") process of average rate D . We call "reduced spectrum" $\mathcal{S}_{\Delta D}(\Omega)$ the spectral density $\mathcal{S}_{\mathcal{D}}(\Omega)$ of that process with the singularity at $\Omega = 0$ removed. The "relative spectrum" $\mathcal{S}_{\Delta D/D}(\Omega)$ is obtained by dividing $\mathcal{S}_{\Delta D}(\Omega)$ by D^2 . Finally, the "relative noise" $\mathcal{N}(\Omega) \equiv \mathcal{S}_{\Delta D/D}(\Omega) - 1/D$ vanishes for shot-noise, also referred to as the "standard quantum limit". We are mostly interested in circumstances where the relative noise is negative. In the engineering literature most authors call "relative-intensity noise" the quantity $2\mathcal{S}_{\Delta D/D}(\Omega)$, and express it in decibels/hertz. Our main objection to using it is that, unlike the relative noise employed in this paper, the relative-intensity noise does not enjoy the property of being independent of (cold, linear) attenuations. Further, the expression "decibel per hertz" is difficult to comprehend.

We call "normalized correlation" $g(\tau)$ the auto-correlation of the $\mathcal{D}(t)$ process with the singularity at $\tau = 0$ being removed, divided by the square of the average rate D . $g(\tau)$ may take any non-negative value. The reason why we do not use the Quantum Optics notation $g^{(2)}(\tau)$ is two-fold. One is that correlations of order other than the second are not employed, so that no confusion may arise. More importantly, $g^{(2)}(\tau)$ is usually defined in terms of the so-called "light intensity" $I(t)$ according to $g^{(2)}(\tau) \equiv \langle I(0)I(\tau) \rangle / I^2$. For mathematical

reasons a quantity so-defined cannot be less than unity. Since $g^{(2)}(\tau)$ -values less than unity are apparently measured, the Quantum-Optics view point is that $I(t)$ should be considered as an operator instead of an ordinary function of time. In this paper the concept of "light intensity" does not enter and $g(\tau)$ refers to photo-currents exclusively.

We denote by $\mathcal{V}(T)$ the variance of the number of photo-detection events occurring within the time-interval T , divided by D , minus 1. In Quantum Optics, the Mandel Q -parameter is defined by a similarly looking expression, namely $Q \equiv \text{variance}(m)/\langle m \rangle - 1$, but the operator m sometimes refers to the number of photons in the cavity, rather than to the number of photo-electrons. Thus our $\mathcal{V}(T)$ and the Q -factor may have different physical meanings. Some authors employ the Fano factor $\mathcal{F} \equiv Q + 1$. Sometimes in the literature, however, the Fano factor refers, not to the number of photons in a cavity, but to the normalized spectrum of electrical-current fluctuations. Again, because of this ambiguity we set the (double-sided) spectral density of the current driving a laser diode as $\xi \langle J \rangle$, where $\xi = 0$ for a quiet pump and $\xi = 1$ for a Poissonian pump, and consider that the Fano factor \mathcal{F} relates instead to the number of photons in an optical resonator.

If $P(t)dt$ denotes the probability that an event occurs between t and $t + dt$, $P(t)$ is called a probability *density*. The word "density", however, being clear from the context, may be omitted. These notions will be explained in more details in Section 3.5.

3.2 Fourier transforms

Mathematical relations that we found most useful are listed below. The Fourier transform $\psi(k)$ of the function $\psi(x)$ and the reciprocal relation are

$$\psi(k) = \int_{-\infty}^{\infty} dx \exp(-ikx)\psi(x) \quad (3)$$

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp(ikx)\psi(k), \quad (4)$$

where $i^2 = -1$, and k is called the wave-number. Note that the position of the 2π factor varies from one author and another, without of course affecting the end results. Obviously, the Fourier transform of $(ik)^n \psi(k)$ is equal to the n th derivative of $\psi(x)$ with respect to x . If $\psi(x)$ is real, we have $\psi(k)^* = \psi(-k)$.

Relations similar to (3) and (4) hold with x changed to t , k to ω and (to be consistent with our conventions for optical signals) i changed to $-i$. Then the element of integration in (4) is $d\nu \equiv d\omega/2\pi$, where ν denotes as usual the optical frequency.

Note the following physical application: For particles moving in time-independent potentials $V(x)$, stationary states $\psi(x)$ are real functions of x (to within an arbitrary over-all phase factor that we set equal to 1). If furthermore $V(x)$ is an even function of x , $\psi(x)$ is either an even or odd function of x . It follows from the above considerations that the $\psi(k)$ -functions are, respectively, real even or imaginary odd. In the present mathematical section we set $\hbar = 1$ and do not distinguish the electron momentum p from the wave-number k .

We will need the following expression of the Dirac δ -distribution¹⁹

$$\delta(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \exp(ikx), \quad (5)$$

implying that its Fourier transform is unity. Using this expression one may prove that

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} |\psi(k)|^2 = \int_{-\infty}^{+\infty} dx |\psi(x)|^2. \quad (6)$$

Let $|\psi(x)|^2 dx$ be interpreted as the probability of finding the position of a particle between x and $x + dx$ if a measurement is performed, and $P(p)dp \equiv |\psi(p)|^2 dp$ be the probability of finding the electron momentum between p and $p + dp$ if a measurement is performed (this latter measurement may be accomplished by letting the particle free at some time t and observing its position on some far-away screen). We are led to define the wave function in momentum space as

$$\psi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \exp(-ipx) \psi(x), \quad (7)$$

according to (6). In general the wave-function depends on time, and the average value of p , evaluated at some time t , depends on time and is denoted $\langle p(t) \rangle$.

For two functions $\psi(x)$ and $\phi(x)$ and their respective Fourier transforms $\psi(k)$ and $\phi(k)$ we obtain from the expression in (5) of the $\delta(\cdot)$ distribution the identity

$$\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \psi(k)^* \phi(k) = \int_{-\infty}^{+\infty} dx \psi(x)^* \phi(x), \quad (8)$$

If we set in (8) $\psi(k) \equiv \psi_1(k)$ and $\phi(k) = k^n \psi_2(k)$, we have, using the observation following (4),

$$\langle k^n \rangle_{12} \equiv \int_{-\infty}^{+\infty} \frac{dk}{2\pi} k^n \psi_1(k)^* \psi_2(k) = \int_{-\infty}^{+\infty} dx \psi_1(x)^* \left(\frac{d}{dx} \right)^n \psi_2(x). \quad (9)$$

provided the integrals exist. Note that $\langle k^n \rangle_{11}$ is real according to its definition. If $\psi_1(x)$, $\psi_2(x)$ are real functions of x , it follows from (9) that $\langle k \rangle_{12} = 0$.

For later application, let us suppose that $\psi(x) = \phi(x) = 0$ when $|x| \geq d/2$. Through two integrations by parts we obtain that

$$\int_{-d/2}^{+d/2} dx \psi(x) \left(\frac{d}{dx} \right)^2 \phi(x) = \int_{-d/2}^{+d/2} dx \phi(x) \left(\frac{d}{dx} \right)^2 \psi(x). \quad (10)$$

¹⁹The $\delta(t)$ -distribution may be viewed alternatively as a function equal to $1/h$ for $-h/2 < t < h/2$ and 0 otherwise, so that the area under the function is unity, letting h go to zero at the end of the calculations. Many other forms of the δ -function may be used, with less-singular derivatives than for the one just given.

On the other hand, for any derivable function $\psi(x)$,

$$2\frac{d}{dx}\psi(x) = \left[\left(\frac{d}{dx}\right)^2 x - x\left(\frac{d}{dx}\right)^2\right]\psi(x). \quad (11)$$

Setting $n = 1$ in (9) and using (10) and (11) we find that

$$\begin{aligned} \langle k \rangle_{12} &\equiv \int_{-\infty}^{+\infty} \frac{dk}{2\pi} k \psi_1(k)^* \psi_2(k) \\ &= -i \int_{-d/2}^{+d/2} dx \psi_1(x)^* \frac{d\psi_2(x)}{dx} \\ &= -\frac{i}{2} \int_{-d/2}^{+d/2} dx x \left[\psi_2(x) \frac{d^2\psi_1(x)^*}{dx^2} - \psi_1(x)^* \frac{d^2\psi_2(x)}{dx^2} \right] \end{aligned} \quad (12)$$

If $\psi_1(x)$ and $\psi_2(x)$ are real to within an over-all phase factor, $\langle k \rangle_{12}$ is imaginary according to (12), and the average momentum for a stationary state ($\langle k \rangle_{11}$ or $\langle k \rangle_{22}$) vanishes.

If $\psi_1(x)$ and $\psi_2(x)$ are solutions of the eigen-equations $d^2\psi_{1,2}(x)/dx^2 + e_{1,2}\psi_{1,2} = 0$, we obtain from (12) that

$$\langle k \rangle_{12} = \frac{i(e_1^* - e_2)}{2} \int_{-d/2}^{+d/2} dx x \psi_1(x)^* \psi_2(x) \equiv \frac{i(e_1^* - e_2^*)}{2} x_{12}. \quad (13)$$

On the other hand, setting $n = 2$ in (9) we obtain, after an integration by parts, that

$$\begin{aligned} \langle k^2 \rangle_{12} &\equiv \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \psi_1(k)^* k^2 \psi_2(k) \\ &= - \int_{-\infty}^{+\infty} dx \psi_1(x)^* \frac{d^2\psi_2(x)}{dx^2}. \end{aligned} \quad (14)$$

The above expressions will be employed in Section 4.3 in relation with the time-dependent Schrödinger equation.

3.3 Convolution

Let us consider a real, causal, linear and time-invariant system. These conditions imply that for a potential source $v(t)$ the current $i(t)$ (or more generally the response to a source of any kind) is given by

$$i(t) = \int_{-\infty}^{+\infty} du y(u) v(t-u) \equiv y * v = v * y, \quad (15)$$

where the kernel $y(u)$ is real, equal to 0 for $u < 0$, and middle stars denote convolution products. The Fourier transform of $y * v$ is the product of the Fourier transform of y and

the Fourier transform of v . The kernel $y(t)$ needs not be an ordinary function of t . For example, for a conductance G we have $i(t) = Gv(t)$, and thus $y(t) = G\delta(t)$, where $\delta(\cdot)$ denotes the Dirac distribution.

If $v(t) = V(p)\exp(pt)$ where p denotes a complex number (not to be confused with particle momenta), $i(t) = I(p)\exp(pt)$, where $I(p) = Y(p)V(p)$ and

$$Y(p) = \int_0^{+\infty} d\tau y(\tau) \exp(-p\tau), \quad (16)$$

a Laplace transform, defines $Y(p)$ for complex p . In most of this work we set $p = -i\omega$, but $Y(-i\omega)$ is denoted simply as $Y(\omega)$, and $Y(-\omega) = Y(\omega)^*$. Again, the written-out argument should prevent confusion.

3.4 Random processes

Usually one observes the photo-current $j(t)$ from a photo-detector in a single set-up. The mathematical treatment of noise, on the other hand, rests on the consideration of an arbitrarily large number of macroscopically-identical set ups, averaging referring to these many set-ups. The question thus arises as to what practical conclusions may be drawn from formulas derived from the formalism. The answer is that, provided a stationary system is *ergodic*, the statistics may be obtained from a single set-up. If this is the case, statistical averages are equivalent to time averages. A system may be ergodic only if the correlation $\langle j(0)j(\tau) \rangle$ tends to $\langle j(0) \rangle^2$ as τ tends to infinity. More stringent conditions must be fulfilled, however, that we suppose met. In principle, once the statistical calculations have been performed, one should verify that the system considered is ergodic in order to be able to apply the results to a single system. In practice, this step is omitted.

Real random processes $x(t)$ are called "stationary" when the auto-correlation $R_{xx}(\tau) \equiv \langle x(t)x(t+\tau) \rangle$ does not depend on t . We shall omit the subscript "xx" of R and the word "auto" when no confusion may arise. It follows that for real random processes $R(\tau) = R(-\tau)$ ²⁰. Time reversal of the random-function samples leaves $R(\tau)$ unchanged.

The spectrum $\mathcal{S}_t(\Omega)$ of this process is a real non-negative even function of Ω . It may be obtained by considering a finite duration T , evaluating of average of the modulus square of the Fourier transform of $x(t)$, dividing by T , and letting T go to infinity. The motivation for introducing a subscript "t" is that we intend to introduce later on a spectrum denoted $\mathcal{S}(\Omega)$ (without a subscript) obtained from $\mathcal{S}_t(\Omega)$ by removing the singularity at $\Omega = 0$. Alternatively, the spectrum may be expressed as the Fourier transform of $R(\tau)$ (Wiener-

²⁰To prove it, set $t = -\tau$ and remember that $x(t)$ is an ordinary function of time, not an operator. The same property holds for non-commuting operators if the correlation is defined as $R_{xx}(\tau) \equiv \langle x(t)x(t+\tau) + x(t+\tau)x(t) \rangle / 2$.

Khinchin theorem)

$$\begin{aligned}\mathcal{S}_t(\Omega) &= \lim_{T \rightarrow \infty} \frac{1}{T} \left\langle \left| \int_0^T dt x(t) \exp(j\Omega t) \right|^2 \right\rangle \\ &= \int_{-\infty}^{+\infty} d\tau R(\tau) \exp(j\Omega\tau) = \int_{-\infty}^{+\infty} d\tau R(\tau) \cos(\Omega\tau).\end{aligned}\quad (17)$$

The two above expressions agree in the mean if the integral from 0 to ∞ of $\tau R(\tau)$ is finite [28, p. 336]. We have employed above the electrical-engineering $\exp(j\Omega t)$ notation. To compare with the previous notation in (3) change Ω to k , τ to x , and j to $-i$. Evaluating $\langle (x(\tau) \pm x(0))^2 \rangle$ we notice that $-R(0) \leq R(\tau) \leq R(0)$. This condition does not suffice however to make $R(\tau)$ positive definite, that is, its Fourier transform could be negative.

In the special case where $x(t)$ does not depend on time we have $R(\tau) = \langle x^2 \rangle = \text{constant}$. Substituting in (17) we find that $\mathcal{S}_t(\Omega) = 2\pi \langle x^2 \rangle \delta(\Omega)$, where $\delta(\cdot)$ denotes the Dirac δ -distribution. We are thus led to define a reduced spectrum $\mathcal{S}(\Omega) \equiv \mathcal{S}_t(\Omega) - 2\pi \langle x^2 \rangle \delta(\Omega)$.

Conversely, the correlation may be expressed in terms of the spectrum through the inverse Fourier transform according to

$$R(\tau) = \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \mathcal{S}_t(\Omega) \cos(\Omega\tau).\quad (18)$$

If we define $y(t) \equiv x(t) - \langle x(t) \rangle$ we have $\langle y(t) \rangle = 0$. The function $C(\tau) = \langle y(0)y(\tau) \rangle$ is called the (auto) covariance of the process $x(t)$.

We are mostly interested in the case where $x(t) = \sum_k \delta(t - t_k)$ with $0 < t_k < T$, where the t_k are referred to as "event times" or "points" (point processes are discussed in more detail in section 3.5). We have $\langle x(t) \rangle = D$, where D denotes the average event rate, and the spectrum of $x(t)$ exhibits a singularity $2\pi D^2 \delta(\Omega)$ at $\Omega = 0$. The first expression in (17) gives the above expression for the reduced spectrum

$$\mathcal{S}(\Omega) = \lim_{T \rightarrow \infty} \frac{1}{T} \left\langle \left| \sum_{\text{allowed } k} \exp(j\Omega t_k) \right|^2 \right\rangle,\quad (19)$$

where $\Omega = 2\pi n/T$, $n = 1, 2, \dots$. Note that $n = 0$ is not allowed, but Ω can be made as small as one wishes by setting $n = 1$ and letting T go to infinity. This expression is useful to evaluate spectra through numerical calculations that generate runs, each with a different t_k sequence. Because of the assumed ergodicity, a single run may suffice.

Note that total energies are obtained by integrating over frequency from $-\infty$ to $+\infty$ the quantity $(d\Omega/2\pi) \mathcal{S}(\Omega)$. For two independent processes $x(t)$ and $y(t)$ of spectral densities \mathcal{S}_x and \mathcal{S}_y , respectively, the spectral density of $z(t) = ax(t) + by(t)$ is $\mathcal{S}_z = |a|^2 \mathcal{S}_x + |b|^2 \mathcal{S}_y$.

The moments of a quantity such as x , denoted $\langle x^n \rangle$, are defined as the integrals over x from $-\infty$ to $+\infty$ of $x^n P(x)$, where $n = 1, 2, \dots$ and $P(x)$ denotes the probability density of x , or the sum from $k = 1$ to ∞ of $k^n P(k)$, where $P(k)$ denotes the probability of having the outcome k . Note that the above integrals or sums may not exist, even for well-behaved probability laws. It follows from (17) and (3) that

$$\langle x^2 \rangle = C(0) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \mathcal{S}(\Omega) \quad (20)$$

if $\langle x \rangle = 0$. Thus, if the spectrum shape is known, the variance of x determines the spectrum.

The sources of noise in our theory are narrow-band current sources written as $c(t) = \sqrt{2}(C'(t) \cos(\omega_o t) + C''(t) \sin(\omega_o t))$, where ω_o denotes the average laser frequency, and $C'(t)$, $C''(t)$ are jointly-stationary slowly-varying real functions of time. It can be shown that $c(t)$ is wide-sense stationary if and only if $\langle C'(t) \rangle = \langle C''(t) \rangle = 0$ and the auto and cross correlations fulfill the conditions $R_{C'C'}(\tau) = R_{C''C''}(\tau)$, $R_{C'C''}(\tau) = -R_{C''C'}(\tau)$. Furthermore, we assume that the statistics is independent of a phase change, and this entails that $R_{C'C''}(\tau) = 0$. Let us recall the following result. If $C'(t)$, $C''(t)$ are uncorrelated and their spectra $\mathcal{S}_{C'}(\omega) = \mathcal{S}_{C''}(\omega)$ vanish for $|\omega| > \omega_c$, then $\mathcal{S}_C(\omega) = \mathcal{S}_{C'}(\omega - \omega_o) + \mathcal{S}_{C'}(\omega + \omega_o)$ [28, p. 380].

Consider a constant-amplitude frequency-modulated signal $x(t) = \cos(\omega_o t + \phi(t))$. If $\Delta\omega(t) = d\phi(t)/dt$ is a stationary low-frequency gaussian process of (double-sided) spectral density $\mathcal{S}_{\Delta\omega}$, the spectral density $\mathcal{S}_x(\omega)$ of $x(t)$ is Lorentzian with a full-width at half power (FWHP) $\delta\omega = \mathcal{S}_{\Delta\omega}$ [29, p. 140], that is, is of the form $\mathcal{S}_x(\omega) \propto 1/[1 + (2(\omega - \omega_o)/\delta\omega)^2]$. Note that here $\Delta\omega(t)$ denotes a process, while $\delta\omega$ is a real positive number, the spectral width.

3.5 Point processes

Point processes are sequences of increasing positive real numbers $t_k, k = 1, 2, \dots$. Each k value corresponds to an "event" or "point" occurring at time t_k . We consider M such sequences, labeled by $m = 1, 2, \dots, M$, called runs. As said above, averages denoted by the sign $\langle \cdot \rangle$ refer to sums from $m = 1$ to $m = M$ of some quantity defined for each run divided by M , letting M go to infinity. Clearly, averaging is a linear operation, that is $\langle a + b \rangle = \langle a \rangle + \langle b \rangle$. The full specification of a point process requires correlations of all order but we shall limit ourselves to first-order and second-order quantities. Ideally, photo-currents from $t = 0$ to $t = T$ are of the form $j(t) = -e\mathcal{D}(t)$, $\mathcal{D}(t) \equiv D + \Delta D(t) = \sum_k \delta(t - t_k)$ with $0 < t_k < T$, where $-e$ denotes the electron charge and $\delta(\cdot)$ the Dirac distribution. Experimentally, we may measure a number of quantities relating to $j(t)$ with the help of integrators, narrow-band filters or electron counters. We will be particularly interested in self-excited point processes, the probability of an event depending in that case on previous-events occurrences. An interesting special case is a process with independent identically-distributed increments $w_k \equiv t_{k+1} - t_k$.

To first order, stationary point processes are characterized by their average rate D , also called the "intensity" or "density" of the process. Let $d(t)$ denote the number of events occurring up to time t , that is the number of k values such that $t_k < t$. The (rather intuitive) result that, for some measurement time T , $\langle d(T) \rangle = DT$ is demonstrated in Appendix A.

3.6 Event-rate spectrum

The photo-current spectrum $e^2 \mathcal{S}_{\mathcal{D}}(\Omega)$, where Ω denotes the Fourier frequency, may be measured by letting the photo-current $j(t)$ flow through a narrow-band filter with center frequency Ω . The function $\mathcal{S}_{\mathcal{D}}(\Omega)$ exhibits a $2\pi D^2 \delta(\Omega)$ singularity, but the reduced spectrum $\mathcal{S}_{\Delta D}(\Omega)$ is non-singular, where $\Delta D \equiv \mathcal{D} - D$. In the theory to be subsequently presented, $\mathcal{S}_{\Delta D}(\Omega)$ is obtained by setting in rate equations $d/dt \rightarrow j\Omega$, see Section 7.4.

The relative noise $\mathcal{N}(\Omega)$ is then defined as

$$\mathcal{N}(\Omega) \equiv \mathcal{S}_{\Delta D/D}(\Omega) - \frac{1}{D}, \quad (21)$$

where $\mathcal{S}_{\Delta D/D}(\Omega) = \mathcal{S}_{\Delta D}(\Omega)/D^2$.

One may be interested instead in the normalized second-order correlation function $g(\tau)$, a non-negative even function of the delay time τ . Aside from normalization, $g(\tau)$ is the correlation of $\mathcal{D}(t)$ with the singularity at $\tau = 0$ being removed. Alternatively, $Dg(\tau)d\tau$ may be defined as the probability that an event occurs between τ and $\tau + d\tau$, given that an event occurred at $t = \tau$. As shown in Appendix A, $g(\tau)$ is related to the relative noise defined above by the integral relations

$$\mathcal{N}(\Omega) = \int_{-\infty}^{\infty} d\tau (g(\tau) - 1) \exp(j\Omega\tau) \quad (22)$$

$$g(\tau) - 1 = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \mathcal{N}(\Omega) \exp(j\Omega\tau). \quad (23)$$

The motivation for introducing $g(\tau) - 1$ in (22) is that this quantity tends to 0 as τ tends to infinity because widely separated events are in that limit independent for stationary processes. The above relations are closely related to the Wiener-Khinchin relations. They are established in Appendix A directly for point processes. Note that our definition of "sub-Poissonian" photo-currents is that $\mathcal{N}(0) < 0$. This does not necessarily imply that $g(0) < 1$.

3.7 Photo-count variance

As shown in Appendix A, the normalized variance $\mathcal{V}(T)$ of the number of events occurring during some time T and $g(\tau)$ are related

$$\mathcal{V}(T) \equiv \frac{\langle d(T)^2 \rangle - \langle d(T) \rangle^2}{\langle d(T) \rangle} - 1 = D \int_{-T}^T d\tau \left(1 - \frac{|\tau|}{T}\right) (g(\tau) - 1) \quad (24)$$

$$2D(g(T) - 1) = \frac{d^2(T\mathcal{V}(T))}{dT^2} \quad (25)$$

To deduce the latter relation from the former, it is convenient to first show that

$$d(T\mathcal{V}(T))/dT = 2D \int_0^T d\tau (g(\tau) - 1)$$

and derivate once more with respect to T . In the special case of a Poisson process we have $g(\tau) = 1$, $\mathcal{V}(T) = 0$ and $\mathcal{N}(\Omega) = 0$, that is, $\mathcal{S}_{\Delta D} = D$.

As an example let us consider a high-power laser driven by a non-fluctuating current. The relative noise is obtained In Section 7.4 in the form

$$\mathcal{N}(\Omega) = -\frac{1}{D(1 + (\Omega\tau_p)^2)} \quad (26)$$

where τ_p is the so-called "photon life time" of the resonator. From this expression we obtain, setting $D = 1$ for simplicity, that

$$g(\tau) = 1 - \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} \frac{1}{1 + (\Omega\tau_p)^2} \exp(j\Omega\tau) = 1 - \frac{1}{2\tau_p} \exp\left(-\frac{\tau}{\tau_p}\right) \quad (27)$$

$$g(0) = 1 - \frac{1}{2\tau_p} \quad (28)$$

Of course $g(\tau) \rightarrow 1$ if $\tau \rightarrow \infty$. In the present situation $g(0) < 1$.

3.8 Dark-room picture

For the sake of illustration let us present a simple picture of regular point processes. The initial point process considered is periodic and consists of events occurring at $t = 1, 2, \dots$ time units, that is $t_k = k$. To make this process stationary, one should consider instead the process $t_k = k + \theta$, where θ is uniformly distributed between 0 and 1. Under circumstances to be defined later on (delay times much larger than unity) we may, however, set $\theta = 0$.

In our picture, one person (representing an electron) enters into a dark room every time unit and wanders randomly in the room until he finds the exit. This picture may describe regularly-pumped lasers if we assume that each electron entering the room is converted

instantly into a photon. Photons wander in the optical resonator for some time and then get instantly converted into photo-electrons. In general it cannot be said, however, that one electron generates a photon and conversely. Electrons act collectively. Nevertheless, in some limiting situation, the above picture turns out to be accurate.

In general, the stationary point process of interest should be written as $t_k = k + \theta + \xi_k$, where θ is uniformly distributed between 0 and 1 and the ξ_k are independent of θ , independent of one-another, and distributed according to the same probability density $P(\xi_k) \equiv P(\xi)$ ²¹. In that model the average event rate (or intensity) of the point process is unity.

Let us treat a special case that may be solved almost by inspection, namely the case where $P(\xi) = 1/\tau_r$ if $0 \leq \xi < \tau_r$ and 0 otherwise, and τ, τ_r are large integers. Consider a pair $i, j \neq i$ of k values such that $i + \xi_i$ may be in the first time slot $(0, dt)$ and $j + \xi_j$ may be in the second time slot $(\tau, \tau + dt)$. Inspection shows that this is possible only if $-\tau_r < i \leq 0, \tau - \tau_r < j \leq \tau$. Ignoring first the restriction $j \neq i$, we find that the probability we are looking for is the number of allowed i, j values, that is, the product of the i, j ranges, times $1/\tau_r^2$, namely $\tau_r^2/\tau_r^2 = 1$. This result is accurate if $\tau \geq \tau_r$. But if $\tau < \tau_r$ one must subtract from the numerator of the previous expression the number of i, j -values that are equal, namely $\tau_r - \tau$, so that the normalized correlation reads

$$\begin{aligned} g(\tau) &= 1, & \tau &\geq \tau_r \\ g(\tau) &= \frac{\tau_r^2 - (\tau_r - \tau)}{\tau_r^2} = 1 - \frac{\tau_r - \tau}{\tau_r^2}, & \tau &< \tau_r. \end{aligned} \quad (29)$$

In particular, $g(0) = 1 - 1/\tau_r$, indicating a modest amount of anti-bunching, remembering that $\tau_r \gg 1$. The same result is obtained for the laser model in (28) if we set $\tau_r = 2\tau_p$ to make the average life-times the same in the two models.

The reduced photo-events spectrum is obtained from $g(\tau)$ through a Fourier transform according to (22) as

$$\begin{aligned} \mathcal{N}(\Omega) &\equiv 2 \int_0^1 dx (x - 1) \cos(\Omega\tau_r x) = 2 \frac{\cos(\Omega\tau_r) - 1}{(\Omega\tau_r)^2} \\ \mathcal{S}(\Omega) &= 1 + \frac{\cos(\Omega\tau_r) - 1}{(\Omega\tau_r)^2/2}, \end{aligned} \quad (30)$$

where we have set $x \equiv \tau/\tau_r$, remembering that the average rate $D = 1$. We note that $\mathcal{S}(0) = 0$, as one expects from the fact that the primary process is regular and that no event has been lost or created. The spectral density of the process considered, given in (30), is illustrated in Fig. 1.

²¹For the special case $P(\xi) = \delta(\xi)$, where $\delta(\cdot)$ denotes the Dirac δ -distribution, each run is periodic. A most appropriate distribution, however, would be the exponential one.

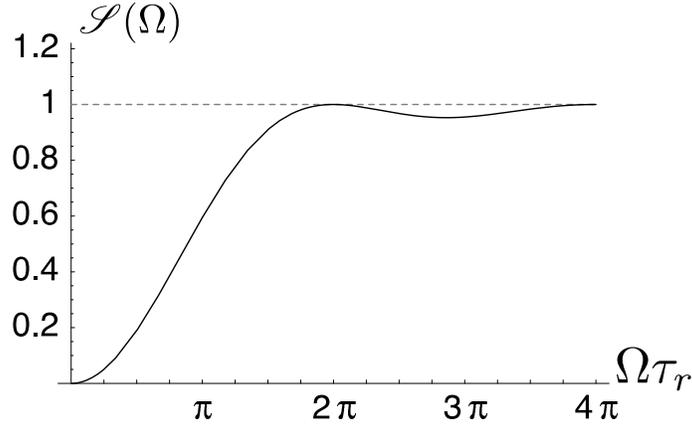


Figure 1: Plain line: spectral density corresponding to the dark room picture, see (30). The dotted line corresponds to the shot-noise level.

Using (24) we obtain in the present model

$$\begin{aligned} \mathcal{V}(T) &= -1 + \frac{\tau_r}{3T}, & T \geq \tau_r \\ \mathcal{V}(T) &= -\frac{T}{\tau_r} + \frac{T^2}{3\tau_r^2}, & T < \tau_r. \end{aligned} \quad (31)$$

It is easy to see that the expression of $g(\tau)$ in (25) is verified in that example.

3.9 Random deletion

Random deletion of events (also called "thinning") means that each event is ascribed a probability $1 - p$ of being deleted. For example, considering the first event of a given run, we flip a coin. If head, that event is preserved (probability 1/2). If tail, it is deleted. The same procedure is applied to the other events of the run and to the events of other runs, each time with a new coin flipping. Obviously the average rate D of the process is multiplied by p . An important result is that the function $g(\tau)$ and thus the other two functions defined above, and in particular the relative noise $\mathcal{N}(\Omega)$, are not affected. Indeed consider the case where there is one event in the time slot $[0, dt]$ and one event in the time slot $[\tau, \tau + d\tau]$, corresponding to a product of 1. In any other circumstances the product is 0. After thinning the probability of having again (1,1) is multiplied by p^2 . But the denominator in the normalized correlation $g(\tau)$ is also multiplied by p^2 , so that the result is unchanged. The average rate may be restored by an appropriate scaling of the time axis. But since in general $g(\tau/p) \neq g(\tau)$, rescaled thinning affects the statistics with the sole exception of Poissonian processes, in which case $g(\tau) = 1$.

4 Classical circuits, Classical Mechanics, Quantum Mechanics

We recall in the present section the basic concepts employed in Circuit Theory, and describe devices useful in radio, microwave, and optical frequency ranges, considering both resonators and transmission lines. Conservative elements such as capacitances, C , and inductances, L , are truly in the realm of the Classical Circuit Theory. The circuit equations provide currents $i(t)$, viewed as (real, linear, causal) responses to specified potential sources $v(t)$. When the field generated by a potential source is applied to electrons, the (Quantum-Mechanical average) current $i(t)$ is, under circumstances to be later discussed, proportional to the potential source $v(t)$, that is, the ratio i/v is a constant called the conductance denoted by G . It is supposed to be known beforehand from separate measurements. This conductance may be positive (and to receive energy) or negative (and to supply energy). The circuit theory is then generalized to non-conservative devices involving, besides C and L , the conductances G . No spontaneous fluctuation is considered at that stage. We will mainly consider sources (and responses) that vary sinusoidally in the course of time at frequency ω . For a closed system, the equations have solutions only for discrete complex values ω_n of ω . We will be particularly interested in circuits that have only one nearly-real frequency, the other ones having large negative imaginary parts, corresponding to strongly damped modes.

We first evaluate the current $i(t)$ induced in a potential source $v(t)$ using the Classical Theory of Motion of electrons. This is a deterministic problem and no fluctuations are involved. (When we are interested in the devices intensity or frequency fluctuations one must take into account the discreteness of the electrical charge and the (thermal) velocity distribution of electrons in conductors). In the realm of Quantum Mechanics we are able to evaluate the average value of the current i and its higher moments at some time t , but not its value as a function of time for a single sample of the ensemble. Quantum jumps (tunneling into a conductor) are mentioned at that point but not treated in full. They are responsible for the fact that the average conductance may be a constant, and for slow fluctuations of the induced current envelope. Spontaneous emission, being not essential in our theory, is only briefly mentioned.

A cursory discussion of detectors and sources of electromagnetic radiation is offered. We consider classical detectors and sources of radiation (e.g., reflex klystrons) whose basic mode of operation may be understood from the Classical Equations of Motion, and quantum detectors and sources of radiation (e.g., masers or lasers) whose basic mode of operation requires Quantum Theory. Finally, we recall the analogy that exists between the propagation of optical beams at some fixed frequency ω , and the behavior of electron wave-functions as a function of time.

4.1 Classical-Circuit Theory

Let us recall basic results. The complex notation often employed for describing quantities that vary sinusoidally in time was recalled in Section 3.1. For strictly sinusoidal potentials and currents represented by the complex numbers V and I , respectively, and linear circuits, we have the generalized Ohm law $I = Y(\omega)V$, where the complex constant of proportionality $Y(\omega)$, called the admittance, and its inverse the impedance $Z(\omega)$, in general depend on the frequency ω , which may vary from minus to plus infinity.

If a potential $v(t)$ is applied to a conductance G (a real number), we have by definition $i(t) = Gv(t)$, or, using the complex notation, $I = GV$, where V , and thus I , are in general complex numbers. We consider in the major part of this paper *ideal* conductances defined as follows: They are supposed to be independent of the driving potential V and to be independent of frequency. Furthermore, they are supposed to have a fixed energy content that may be set equal to zero since only energy differences are relevant. A physical model for ideal conductances is a piece of metal having a large number of inelastic scattering centers. Electrons accelerated by the applied field quickly lose their energy, which is converted into heat. Under such circumstances the electron kinetic energy remains negligible, and thus the total energy is fixed. In contradistinction, the input conductance of a loss-less transmission line of characteristic conductance G_c terminated by an ideal conductance $G = G_c$ (matched load) is equal to G at any frequency. But there is in that case a stored energy equal to $G|V|^2\tau$, where τ denotes the transit time of a pulse along the transmission line (this is power divided by the group velocity times the line length). Thus, a matched transmission line does *not* constitute an ideal conductance in the sense defined above, even though the input conductance G is a real constant.

A capacitance $C = \epsilon_0 A/d_C$ may consist of two parallel perfectly-conducting plates of area A separated by a distance $d_C \ll \sqrt{A}$. The constant ϵ_0 in this formula is called the free-space permittivity. Its exact numerical value is given at the beginning of Section 3. There are two wires connected respectively to the upper and lower plates, so that electrical charges may be introduced or removed. If an electrical charge q is displaced from the (say, lower) plate to the upper plate a potential v appears between the two plates given by $v = q/C$. The energy stored in the capacitance is $E_C = Cv^2/2 = q^2/(2C)$, a result obtained by considering elementary charges dq being displaced from the lower to the upper plate of the initially-uncharged capacitance until a final charge q is reached. If q is a function of time and C is kept constant, we have $v(t) = q(t)/C$. We may set $q = -Ne$, where the number N of electrons is supposed to be so large that q varies almost continuously. As before, e denotes the absolute value of the electron charge.

Let now $v(t)$ be of the sinusoidal form given above. Because the current $i(t)$ represents a flow of electrical charges into one plate or flowing out from the other plate, $i(t)$ is the time-derivative of the electrical charge: $i(t) = dq(t)/dt$. The relation between the complex current I and the complex potential V , as defined above, thus reads $I = -iC\omega V$. The admittance is in the present case $Y(\omega) = -iC\omega$. If we set $Y = G + iB$, we have therefore

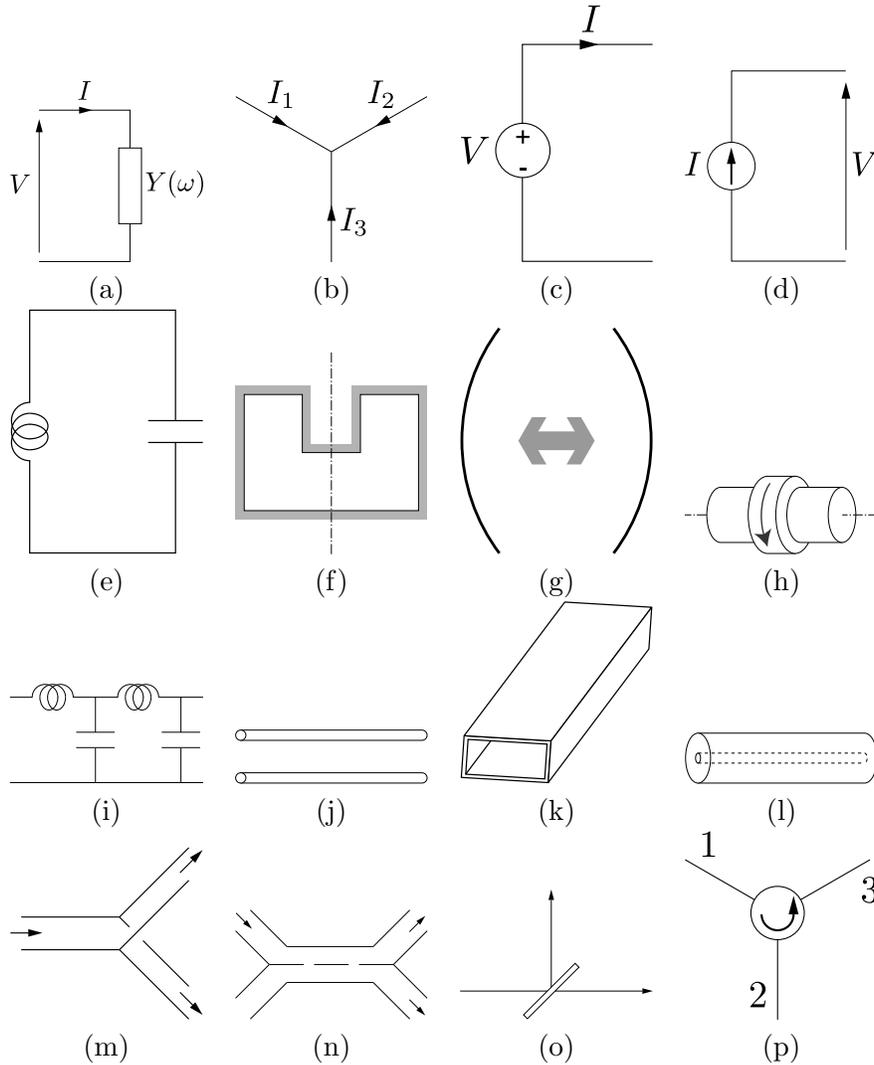


Figure 2: a) There is a linear relation between the potential V and the current I at some frequency ω (generalized Ohm law $I = Y(\omega)V$). b) Illustrates the Kirchhoff law: $I_1 + I_2 + I_3 = 0$. c) Represents a potential source with V independent of I . d) Represents a current source with I independent of V . e) Inductance-capacitance resonating circuit. f) Cavity employed, e.g., in reflex klystrons. g) Fabry-Pérot-type optical resonator with two curved mirrors facing each other. h) Whispering-gallery mode resonator. i) Low-pass filter, j) Parallel conductors, k) Waveguide, l) Optical fiber. Waves may be split in various ways: m) A transmission line is connected to two transmission lines whose characteristic conductances sum up to the original line characteristic conductance. n) Directional coupler. The two holes are spaced a quarter of a wavelength apart. o) The beam splitter is an optical equivalent of the directional coupler. p) The circulator is ideally a loss-less non-reciprocal device.

for an ideal capacitance $G = 0$ and $B = -C\omega$. The stored energy averaged over a period $2\pi/\omega$, reads $\langle E_C \rangle = C |V|^2/2$. In following paragraphs, we will consider a light-emitting device driven by a very large capacitance (instead of, say, a battery) with a very large initial charge q such that the potential $U = q/C$ across the capacitance has the desired value, for example 1 volt. If the light-emitter operation duration is denoted by T , the capacitance supplies a current i during that time, and thus loose a charge $\Delta q = iT$. Because q is very large we have $\Delta q \ll q$ provided the experiment does not last too long. As a consequence the potential U across the capacitance does not vary appreciably. We realize in that manner a constant-potential source, that is a source whose potential does not depend appreciably on the delivered current.

A constant-potential source at optical frequency ω may be realized in a similar manner. Again the capacitance C and the initial charge q are supposed to be arbitrarily large, but we now allow the spacing d_C between the capacitance plates to fluctuate²² at the optical frequency ω . This spacing variation entails a fluctuation of the capacitance, and thus of the potential across the capacitance since the charge is nearly constant as was discussed above. The potential across the capacitance may be written as $U + v(t)$. The important point is that the optical potential $v(t)$ is independent of the current delivered. That is, if atoms are present between the two capacitance plates, processes occurring in the atomic collection have no influence on the field. We have just described an essential component of our circuit-theory schematic. In contradistinction, the potential across a resonating inductance-capacitance circuit modeling a single-mode cavity does depend on atomic processes. For that resonator configuration the assumption that the optical field is nearly constant holds only in the large potential (or large photon number) limit.

An inductance L may be constructed from a cylinder of area A and height $d_L \gg \sqrt{A}$, split along its height, so that an electrical current may flow along the cylinder perimeter. In that case $L = \mu A/d_L$, where μ denotes the permeability. One may assume that the cylinder contains electrons that have magnetic moments. Just above the Curie temperature, μ much exceeds the free-space permeability, so that the latter may be set equal to zero, as was discussed earlier. For an inductance L , the magnetic flux (or magnetic charge) is $\phi(t) = Li(t)$, and the potential across the inductance is $v(t) = d\phi/dt$. It follows that for a constant L , $v(t) = L di(t)/dt$, or, using the complex notation $V = -iL\omega I$. Thus $Y(\omega) \equiv I/V = i/(L\omega)$. The energy stored in an inductance with a current i flowing through it is $E_L = Li^2/2$. For a sinusoidal current represented by the complex number I , the time-averaged energy is $\langle E_L \rangle = L |I|^2/2$. If a large inductance supports a large magnetic flux, the current flowing through the inductance is nearly independent of the potential across the inductance. In that manner, we may realize constant-current sources, either static or oscillating at optical frequencies through a change of L (e.g., by changing a coil length).

²²Practically-minded readers may object that mechanical motion may not be feasible at high frequencies. Let us recall here that the numerical value of ω is arbitrary. What we call "optical" frequency $\omega/2\pi$ may be as low as 1Hz provided that the other frequencies considered be much lower, e.g., 1 mHz.

The linear relationships outlined in previous paragraphs are sometimes referred to as the "generalized Ohm laws". Let us recall that there is a well-known duality between potentials and currents and between electrical charges (expressed in coulombs) and magnetic fluxes (expressed in webers), so that expressions obtained for capacitances may be translated into expressions relating to inductances.

As an application of the above energy formulas, let us consider a circuit consisting of an inductance L and a capacitance C connected in parallel. Since the system is isolated the total admittance must vanish and we obtain the resonance formula $LC\omega_o^2 = 1$, where ω_o denotes the resonant frequency. The sum E of the energy $E_L(t)$ located in the inductance and the energy $E_C(t)$ located in the capacitance does not vary in the course of time. This is twice the time-average energy stored in the capacitance (or inductance). Using above formulas we find that the rms (root-mean-square) field across the capacitance is

$$\mathcal{E} = \sqrt{\frac{E}{\epsilon_o \mathcal{V}}}, \quad (32)$$

where $\mathcal{V} \equiv Ad_C$ denotes the capacitance volume. We later show that when a resonator such as the one presently considered is in a cold environment it eventually reaches a state corresponding to an energy $\hbar\omega_o/2$, where \hbar denotes the Planck constant (divided by 2π). According to the above formula, the so-called "vacuum (rms) field" reads $\mathcal{E}_{vacuum} = \sqrt{\frac{\hbar\omega_o/2}{\epsilon_o \mathcal{V}}}$. The two oppositely-charged capacitance plates attract one another with an average force $F = d(\hbar\omega_o/2)/d(d_c) = \hbar\omega_o/(4d_c)$. For two parallel conducting but neutral plates (involving modes of arbitrarily-high frequencies), there is also an attractive force called "Casimir force". This force however vanishes in the non-relativistic limit.

If two sub-systems are connected to one another by two perfectly conducting wires with a potential $v(t)$ across them and a current $i(t)$ flowing into one of them (the current $-i(t)$ flowing in the other one), the power flowing from one sub-system to the other at some instant t is equal to $v(t)i(t)$. For sinusoidal time-variations, the power averaged over an oscillation period reads $P = \Re\{V^*I\}$.

Finally, let us recall that at a node, that is, at the junction between perfectly conducting wires, the sum of the currents entering into the node vanishes as a consequence of the fact that the electric charge is a conserved quantity. For three wires traversed by currents $i_1(t)$, $i_2(t)$, $i_3(t)$, for example, we have at any instant $i_1(t) + i_2(t) + i_3(t) = 0$. It follows that the complex currents sum up to zero, that is $I_1 + I_2 + I_3 = 0$. Both the real and the imaginary parts of the sum vanish. Such relations are sometimes called "generalized Kirchoff laws". The above discussion suffices to treat circuits consisting of conductances, capacitances and inductances arbitrarily connected to one another. Some circuits require a more complicated description involving for example (non-reciprocal) gyrators. These latter components are useful to separate reflected and incident waves.

For the sake of illustration and later use, let us generalize the resonator previously considered by introducing in parallel with the capacitance C and the inductance L a con-

ductance G . The relation between a complex current source \mathcal{C} at frequency ω , supposed to be independent of frequency, and the potential V across the circuit reads

$$V(\omega) = \frac{\mathcal{C}}{Y(\omega)} = \frac{\mathcal{C}}{G - i(C\omega - 1/L\omega)}. \quad (33)$$

The power dissipated in the conductance G at frequency ω reads

$$P(\omega) = G |V(\omega)|^2 \approx \frac{G |\mathcal{C}|^2}{G^2 + 4C^2(\omega - \omega_o)^2} \quad (34)$$

in the small-loss approximation. Thus $P(\omega)$ drops by a factor of 2 from its peak value when $2C(\omega_{\pm} - \omega_o) = \pm G$. The full-width at half power (FWHP) $\delta\omega$ of the resonance that is, the difference of (angular) frequencies at which the dissipated power drops by a factor of two, is

$$\delta\omega = \omega_+ - \omega_- = \frac{G}{C} \equiv \frac{1}{\tau_p}, \quad (35)$$

where $\tau_p = C/G$ is sometimes called the "photon lifetime", hence the subscript "p". If the resonator is left alone in a cold environment ($T=0\text{K}$) its classical energy decays according to an $\exp(-t/\tau_p)$ law. For a Fabry-Pérot resonator with mirrors of small power transmissions T_1 , T_2 , respectively, and spacing L , we have

$$\frac{1}{\tau_p} = \frac{T_1 + T_2}{2L/v}, \quad (36)$$

where v denotes the group velocity.

The energy contained in the resonator is twice the average energy contained in the capacitance whose expression was given earlier. We then obtain in the small-loss approximation

$$E(\omega) = C |V(\omega)|^2 = \frac{C |\mathcal{C}|^2}{G^2 + 4C^2(\omega - \omega_o)^2} \approx \frac{\tau_p |\mathcal{C}|^2 / G}{1 + x^2}, \quad (37)$$

where $x \equiv 2\tau_p(\omega - \omega_o)$.

For late use, note the expression of the derivative with respect to ω of the admittance of a linear circuit, submitted to a voltage V

$$iV^2 \frac{dY(\omega)}{d\omega} = -iI^2 \frac{dZ(\omega)}{d\omega} = \sum_k C_k V_k^2 - L_k I_k^2, \quad (38)$$

where the sum is over all the circuit capacitances and inductances. V_k denotes the (complex) voltage across the capacitance C_k and I_k the (complex) current flowing through the

inductance L_k . The circuit resistances or conductances do not enter in the sum. This relation is readily verified for an inductance in series with a resistance and a capacitance in parallel with a conductance. Thus the relation holds for any combination of elements connected in series and in parallel.

For an arbitrary circuit, the task is to "extract", figuratively speaking, the (positive or negative) conductances from the given circuit, each conductance being connected to the conservative circuit that remains after extraction of the conductances. If N (positive or negative) conductances are involved, the circuit becomes an N -port conservative device. For an N -port circuit, we define the vectors $\mathbf{V} \equiv [V_1, V_2, \dots, V_N]^t$ and $\mathbf{I} \equiv [I_1, I_2, \dots, I_N]^t$ where the upper t denotes transposition. The linear relation is written in matrix form $\mathbf{I} = \mathbf{Y}(\omega)\mathbf{V}$, where $\mathbf{Y}(\omega)$ is called the circuit admittance matrix. For a conservative circuit the total entering power $\Re\{\mathbf{V}^{t*}\mathbf{I}\} = 0$. Since this relation must hold for any source this implies that $\mathbf{Y}^{t*} + \mathbf{Y} = \mathbf{0}$.

It is convenient to view the connections between the conservative circuit and the conductances as ideal transmission lines of small length and characteristic conductances G_c . Supposing that $G_c = 1$, the potential V across one of the transmission lines and the current I flowing through the (say, upper) wire, are combined into an ingoing wave whose amplitude is defined as $a = V + I$ and an outgoing wave defined as $b = V - I$. Since under our assumptions the circuit elements are linear, there is a linear relationship between the a -waves and the b -waves. The relation between \mathbf{b} and \mathbf{a} , defined like \mathbf{I} and \mathbf{V} above, may be written in matrix form as $\mathbf{b} = \mathbf{S}\mathbf{a}$, where the \mathbf{S} matrix is called the circuit "scattering matrix". Because the circuit is conservative, the outgoing power equals the ingoing power. It follows that the \mathbf{S} -matrix is unitary, i.e., $\mathbf{S}^{t*}\mathbf{S} = \mathbf{1}$. We need not assume that the circuit is reciprocal, however, that is, the \mathbf{S} -matrix needs not be symmetrical.

We have represented a number of important conservative (loss-less, gain-less) components in either their circuit form, their microwave form, or their optical form in Fig. 2. The origin of the differences is that, as one goes to shorter wavelength (higher frequencies) some circuit elements become too small to be fabricated. It should also be noted that metals, such as copper, that are excellent electrical conductors up to microwave wavelengths, do not behave as electrical conductors any more at optical wavelengths. On the other hand, while it is difficult to find very low-loss dielectrics at microwave frequencies, extremely low-loss glasses exist at optical frequencies. Fig. 2 represents four resonating circuits, that one may call "0-dimensional" devices. Namely, the inductance-capacitance circuit employed up to about 100 MHz, the cavity employed in reflex klystrons and masers for example, the Fabry-Perot resonator consisting of two mirrors facing each other, and the whispering-gallery-mode dielectric resonator, first demonstrated in the microwave range and now-a-days employed in the optical range. Resonators are primarily characterized by their resonant frequency ω_o . Small losses may be characterized by the so-called "photon life-time" τ_p defined earlier. When the resonator size is large compared with wavelength many resonating modes may be present. In most applications it is desirable that only one of them be loss-less, or nearly so (see, e.g. [30]).

Figure 2 represents four one-dimensional devices called "transmission lines". The circuit form is a periodic sequence of series inductances and parallel capacitances. The microwave form consists of two parallel conductors characterized by a characteristic conductance G_c , with waves propagating at the speed of light. Above 1GHz one would rather use waveguides. The optical form is the now-a-day well-known *optical fiber*. A glass fiber (core) in vacuum may guide optical waves by the mechanism of total reflexion. In order to increase the core size without having spurious modes propagating, the core is usually immersed into a lower-refractive-index glass.

Other useful devices are shown in Fig. 2. The power carried by a transmission line of characteristic conductance G_c may be split into two parts simply by connecting it to two transmission lines whose characteristic conductances sum up to G_c . This a three-port reciprocal conservative device. Alternatively, when two transmission lines are put side by side and coupled at two locations separated by a quarter of a wavelength, some of the power incident on a transmission line is transmitted into the other one. This device is called a directional coupler. This 4-port device may be reduced to a 3-port device by putting a matched load at the end of one of the transmission lines. The optical form of a directional coupler is called a beam-splitter, which may simply consist of a flat piece of glass. An important non-reciprocal 3-port device is the *circulator*, which exists in microwave and optical versions. It is intrinsically loss-less: a wave entering into port 1 entirely exits from port 2, a wave entering into port 2 entirely exits from port 3, and a wave entering into port 3 entirely exits from port 1. Such a device is convenient to separate reflected waves from incident waves without introducing losses.

4.2 Classical Equations of Motion

The equations of motion of an electron of charge $-e$ and mass m are first established for the case of a static (time-independent) potential. As an example consider an anode at zero potential and an electron emitted from a cathode at potential $-U$ in vacuum, and look for the electron motion and the induced current. If $i(t)$ denotes the current delivered by the potential source, the power $Ui(t)$ must be equal at any instant to the power delivered to the electron, which is the product of the velocity $p(t)/m$, where p denotes the electron momentum, and the force eU/d exerted upon it, where d denotes the electrode spacing. Since the potential U drops out from this equation, the current is

$$i(t) = \frac{e}{md} p(t). \quad (39)$$

The diode current $i(t) = (e^2U/md^2)t$ increases linearly with time and drops to zero when the electron reaches the anode. Thus, each electron freed from the cathode entails a triangularly-shaped current pulse. If $i(t)$ is integrated over time from $t = 0$ to $t = \tau$ we obtain the absolute value of the electron charge e . We sometimes neglect the pulse duration τ , so that triangularly-shaped current pulses are approximated by $-e\delta(t)$ -functions. The

above theory is applicable only when few electrons are emitted so that space-charge effects may be ignored.

The above expression for the current was established for the case where the potential generates a constant electric field. This is not the case for example if the anode and the cathode are coaxial cylinders of radii r_a and r_c with $r_c < r_a$. The electron is submitted to a force $\mathcal{F}(r)$ inversely proportional to r , assuming radial motion. the induced current $i(t)$ may be obtained from the $p(t)$ trajectory by the same argument as above.

As a second example, consider a one-dimensional square-well, whose potential is equal to 0 for $|x| < d/2$ and infinite (or nearly so) beyond. This potential may be generated by parallel anodes at potential 0 and cathodes at potential $-U$, as shown in Fig. 3. The electron space-time trajectories $x = x(t)$ consist of straight lines with slopes $dx(t)/dt = \pm p/m$, where $p^2/2m = E$ is the electron energy, which may be selected arbitrarily from 0 to eU so that the electron is not captured by the cathodes. The electron is prevented from being captured by the anodes by a strong magnetic field in the x -direction. The quick electron incursions between anodes and cathodes are here neglected. We may consider in particular a lower electron energy E_1 and a higher electron energy E_2 , corresponding to small and large slopes in the $x = x(t)$ diagram, respectively.

The electron motion induces an electrical current $i(t)$ in the potential source, which is proportional to the electron momentum $p(t)$, as said above. In the case of a static potential source the induced current does not correspond to any power delivered or received by the source on the average, so that the electron motion may go on, in principle, for ever.

If now the static potential U is supplemented by a sinusoidal potential $v(t)$ of small amplitude, whose frequency is resonant with the electron motion described above, the *unperturbed* electron momentum $p(t)$ does cause the alternating potential source to receive or generate power, depending of the electron state. However, if we consider a large collection of unperturbed electrons, the power averages out again to zero. It follows that a net energy transfer may be obtained only if we take into account the fact that the alternating potential *perturbs* the electronic motion. In the present classical picture this amounts to bringing all the electrons with the appropriate phase with the alternating potential, an effect called "bunching".

This, however, is not the end of the story. The electron, initially in the lower energy state, may gain enough energy to be captured by a cathode. Once in a cathode, the electron flows through the potential source to the anode, delivering an energy eU to that source, and may be emitted by the anode back into the lower energy state. The net effect of these processes is that some power is being transferred from the alternating potential source to the static potential source, or the converse, the electrons playing an intermediate role.

In more general situations, the Classical Equations of Motion of electrons of charge $-e$, mass m , and potential energy $-eu(x, t)$ are best based on the Hamiltonian formulation. The particle total energy $E(t)$ is expressed as a function of position x , momentum p , and

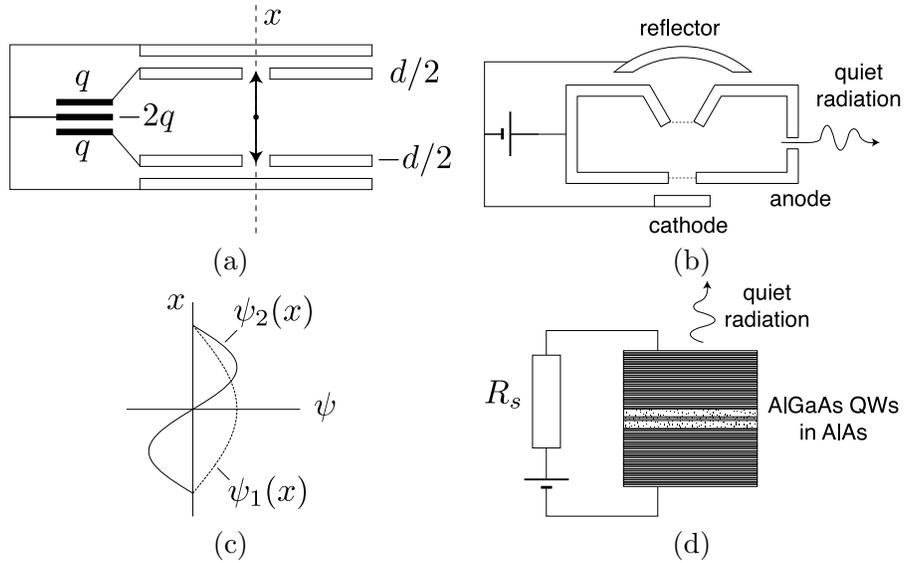


Figure 3: a) Illustrates the potential created by two anodes (inner electrodes) and two cathodes located just outside the anodes. The potential is generated by a large, charged, capacitance, shown on the left. According to the Classical Picture the space-time electron trajectory is almost a zig-zag path, with slight incursions of the electron between the anodes and the cathodes. b) Represents a reflex klystron, which is similar to the previous schematic, but with a resonator added to it. The current is regulated by a space-charge limited cathode. c) Pictures the wave-functions of the ground state and first excited state of a square potential well. d) Represents a "surface-emitting" laser diode. The current is regulated by a large cold resistance R_s .

time t according to the relation

$$H(x, p, t) - E(t) \equiv \frac{p^2}{2m} - eu(x, t) - E(t) = 0, \quad (40)$$

where $p^2/(2m)$ represents the kinetic energy. The Hamiltonian equations read

$$\frac{dx(t)}{dt} = \frac{\partial H(x, p, t)}{\partial p} = \frac{p(t)}{m} \quad (41)$$

$$\frac{dp(t)}{dt} = -\frac{\partial H(x, p, t)}{\partial x} = e \frac{\partial u(x, t)}{\partial x}. \quad (42)$$

The first equation says that the particle momentum $p(t) = mdx(t)/dt$, and the second equation may be written in the Newtonian form $m d^2x(t)/dt^2 = e \partial u(x, t)/\partial x$. Going back to the first example in this section, consider two parallel plates located at $x = 0$ and $x = d$, and at potentials 0 and $-u(t)$, respectively. We have $u(x, t) = -u(t)x/d$, and thus the equation of motion is $md^2x(t)/dt^2 = -eu(t)/d$. The electron decelerates if $u(t) > 0$: The electron is repelled by the negatively-charged cathode.

As far as static conditions are concerned, an electron submitted to a static potential source U is analogous to an electron submitted to the Coulomb potential created by positively-charged nuclei. The potentials, on the order of 1 volt (corresponding to potential energies of $1.6 \cdot 10^{-19}$ joules) are comparable in the two situations. In the case of atoms, however, the distance d is on the order of one nanometer while in the case of two conducting plates the distance can hardly be less than 100 nanometers for practical reasons. As a consequence there exist in the two-plate model many states whose energy is comprised between the lower-state energy $E_1 \approx 0$ and the upper-state energy $E_2 \approx eU$. In both cases the conductance (ratio of the induced current to the applied potential) is initially equal to zero and grows in time linearly until the electron is somehow absorbed.

4.3 Quantum Equations of Motion

The Quantum Equations of Motion of an electron of charge $-e$ and mass m are first established for a static (i.e., time-independent) potential. As an example we consider a one-dimensional square-well, whose potential is equal to 0 for $|x| < d/2$ and infinite (or nearly so) beyond. This potential may be generated by parallel anodes at potential 0 and cathodes at potential $-U$, as shown in Fig. 3. We solve the time-independent Schrödinger equation and obtain in particular a state 1 with lower energy E_1 and a state 2 with higher energy E_2 . As we shall see, these two states correspond to wave-functions $\psi_1(x) = \cos(x)$ and $\psi_2(x) = \sin(2x)$, respectively, leaving aside constants. In the case of a static potential there is no energy exchange between the potential source and the electron when the electron is initially in a stationary state, so that the electron remains in the stationary state, in principle, for ever. There are no energy exchange either, if we perform a time averaging,

when the electron is in a superposition of stationary states. This situation may be compared to the one discussed classically above.

Let now the static potential U be supplemented by a sinusoidal potential $v(t)$ of small amplitude, whose frequency is (in some sense to be defined later) resonant with the electron motion described above. A net energy transfer may be obtained only if we take into account the fact that the alternating potential *perturbs* the electronic motion. In the Classical picture this amounts to bringing all the electrons with the appropriate phase, an effect called "bunching", as said previously. In the Quantum picture (time-dependent Schrödinger equation), one describes the electron wave function $\psi(x, t)$ as the weighted sum of the unperturbed states defined above, with time-dependent coefficients. The theory leads to (Rabi) oscillations between the two states. If averaging is made over a Rabi period, the induced current vanishes. A non-zero conductance is therefore obtained only if the electron is submitted to the optical field only during a small time τ , e.g., because it tunnels out. Thus, initially, the induced current is equal to zero and grows in proportion to time up to time τ . We may evaluate the average conductance "seen" by the potential source. The net effect of these processes is that some power is being transferred from the alternating potential source to the static source, or the converse, the electrons playing an intermediate role. A more realistic model is based on the direct coupling between *bands of states*, assuming that an equilibrium is quickly reached within each band separately.

The quantum treatment is based on the Schrödinger equation

$$[H(x, p, t) - E]\psi(x, t) = 0, \quad E = i\hbar\partial/\partial t, \quad p = -i\hbar\partial/\partial x, \quad (43)$$

where the sign " ∂ " denotes partial derivation. $\psi(x, t)$ is called the wave-function, whose initial value $\psi(x, 0)$ is supposed to be known, and $H(x, p, t) = p^2/(2m) - eu(x, t)$ as in the Classical Equations of Motion, but p and E are now operators of derivation. It is easily shown that, provided $\psi(x, t)$ decreases sufficiently fast as $x \rightarrow \pm\infty$, the integral over all space of $|\psi(x, t)|^2$ does not depend on time. It therefore remains equal to 1 if the initial value is 1, a result consistent with the Born interpretation of the wave function.

Static potentials. Let us suppose that $u(x, t) \equiv u(x)$ does not depend on time. In that case solutions of the above equation of the form $\psi(x, t) = \psi_n(x) \exp(-i\omega_n t)$ may be found, where $n = 1, 2, \dots$. The $\psi_n(x)$ are real functions of x and $E_n \equiv \hbar\omega_n$ that form a complete orthogonal set of functions. For $n = 1, 2$ the wave functions obey the differential equations

$$\begin{aligned} \frac{\hbar^2}{2m} \frac{d^2\psi_1(x)}{dx^2} + eu(x)\psi_1(x) &= -E_1\psi_1(x) \\ \frac{\hbar^2}{2m} \frac{d^2\psi_2(x)}{dx^2} + eu(x)\psi_2(x) &= -E_2\psi_2(x). \end{aligned} \quad (44)$$

with the appropriate boundary conditions. They may be ortho-normalized such that

$$\int_{-\infty}^{+\infty} dx \psi_m(x)\psi_n(x) = \delta_{mn}, \quad (45)$$

where $\delta_{mn} = 1$ if $m = n$ and 0 otherwise.

Potential well. As an example consider an electron of mass m moving along the x axis be reflected by boundaries at $x = -d/2$ and $x = d/2$ where the wave-function is required to vanish, that is, $\psi(\pm d/2) = 0$. The lowest-energy state $n = 1$ and the first excited state $n = 2$ are

$$\psi_1(x, t) = \sqrt{2/d} \cos(\pi x/d) \exp(-i\omega_1 t) \quad (46)$$

$$\psi_2(x, t) = \sqrt{2/d} \sin(2\pi x/d) \exp(-i\omega_2 t) \quad (47)$$

Notice that $\psi_1(x)$ is even in x , while $\psi_2(x)$ is odd in x . Substituting these expressions in the Schrödinger equation (43) with $u(x, t) = 0$, we obtain that

$$\frac{\hbar^2}{2m} \frac{d^2 \psi_n(x)}{dx^2} + \hbar\omega_n \psi_n(x) = 0 \quad (48)$$

provided

$$E_n \equiv \hbar\omega_n = \frac{\pi^2 \hbar^2}{2md^2} n^2 \quad n = 1, 2. \quad (49)$$

We will see later on that optical fields at frequency $\omega_o = \omega_2 - \omega_1 = (3\pi^2 \hbar)/(2md^2)$ may cause the system to evolve from state 1 to state 2 and back. Numerically, $\hbar\omega_o \approx 1.12$ electron-volt if $d = 1$ nano-meter.

For later use let us evaluate

$$\begin{aligned} x_{12} &\equiv \int_{-d/2}^{d/2} dx \, x \, \psi_1(x) \psi_2(x) \\ &= \frac{2}{d} \int_{-d/2}^{d/2} dx \, x \, \cos(\pi x/d) \sin(2\pi x/d) \\ &= \frac{16d}{9\pi^2}, \end{aligned} \quad (50)$$

where we have used the mathematical relation

$$\int_{-\pi/2}^{\pi/2} t \cos(t) \sin(2t) dt = \frac{8}{9}. \quad (51)$$

The parameter x_{12} determines the strength of the atom-field coupling. It is convenient to define a dimensionless *oscillator strength*

$$f \equiv \frac{2m\omega_o}{\hbar} x_{12}^2 = \frac{256}{27\pi^2} \approx 0.96. \quad (52)$$

The maximum possible value of f is 1.

Perturbed motion. We next suppose that a potential source $v(t) = \sqrt{2}V \sin(\omega_o t)$ is applied between the two anodes in Fig. 3. Since the potential varies linearly with x the electron is submitted to a constant optical field. (For most atoms this is an approximation called the "electric dipole approximation"). The equation to solve is

$$\left[\frac{p^2}{2m} - e\sqrt{2}V \sin(\omega_o t)\frac{x}{d} - E\right]\psi(x, t) = 0, \quad E = i\hbar\partial/\partial t, \quad p = -i\hbar\partial/\partial x, \quad (53)$$

We suppose that $\psi(x, 0) = \psi_1(x)$, that is, the electron is initially in the ground state. To obtain an exact expression of $\psi(x, t)$ the wave function should be expressed as an infinite sum of the $\psi_n(x)$ with time-dependent coefficient $a_n(t)$, substituting into the Schrödinger equation, multiplying by $\psi_m(x)$ and integrating over all space. The result is an infinite set of ordinary first-order differential equations for the $a_n(t)$ that can be solved numerically. We consider only the rotating-wave approximation, according to which only terms varying slowly in time are kept, and the $n = 1, 2$ -states [31, p. 39 and followings].

We obtain a solution of the form

$$\psi(x, t) = \cos\left(\frac{\Omega_R}{2}t\right)\psi_1(x) \exp(-i\omega_1 t) - \sin\left(\frac{\Omega_R}{2}t\right)\psi_2(x) \exp(-i\omega_2 t). \quad (54)$$

where $\Omega_R \ll \omega_o$ is the so-called Rabi frequency, to be later evaluated. The pre-factors in (54) ensure that the normalization is preserved, that is, the x -integral of $|\psi(x, t)|^2$ is unity at any t . We obtain an approximate expression of the Rabi frequency

$$\hbar\Omega_R = \frac{e\sqrt{2}V}{d}x_{12}. \quad (55)$$

where we have used the ortho-normality of the wave functions in (45) and introduced x_{12} from its definition in (50). For the potential considered and the value obtained in (52), the above relation reads

$$\hbar\Omega_R = \frac{e\sqrt{2}V}{d}x_{12} = 0.96 \frac{16}{9\pi^2} e\sqrt{2}V \approx 0.17 e\sqrt{2}V. \quad (56)$$

Remember that in our notation V denotes the root-mean-square applied potential. The ratio $\sqrt{2}V/d$, usually denoted by E_o , is the peak applied optical field.

Momentum probability law. Let us now evaluate the momentum probability law $P(p, t)$. The wave functions in momentum space are defined as, see (6) with the Planck constant restored and at $t = 0$,

$$\begin{aligned} \psi_1(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-d/2}^{+d/2} dx \exp\left(-i\frac{px}{\hbar}\right)\psi_1(x) \\ \psi_2(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-d/2}^{+d/2} dx \exp\left(-i\frac{px}{\hbar}\right)\psi_2(x) \end{aligned} \quad (57)$$

where $\psi_1(x), \psi_2(x)$ are given in (46). Setting for brevity $\hbar = 1$, and $\psi_2(p) \equiv i\psi_2''(p)$ the explicit result is

$$\begin{aligned}\psi_1(p) &= \frac{\sqrt{\pi d}}{2} \frac{\cos(pd/2)}{(pd/2)^2 - (\pi/2)^2} \\ \psi_2''(p) &= \sqrt{\pi d} \frac{\sin(pd/2)}{(pd/2)^2 - \pi^2}\end{aligned}\quad (58)$$

Since the Fourier transform (with respect to x) is a linear operation, we obtain from (54) the expression

$$\psi(p, t) = \cos\left(\frac{\Omega_R}{2}t\right)\psi_1(p) \exp(i\omega_o t) - i \sin\left(\frac{\Omega_R}{2}t\right)\psi_2''(p), \quad (59)$$

where the $\psi_{1,2}(p)$ are expressed in terms of the $\psi_{1,2}(x)$ according to (57). Remember from (3) that since $\psi_1(x)$ is real even and $\psi_2(x)$ is real odd, $\psi_1(p)$ is real even and $\psi_2(p) \equiv i\psi_2''(p)$ is imaginary odd. It follows that

$$\begin{aligned}P(p, t) = |\psi(p, t)|^2 &= \cos^2\left(\frac{\Omega_R}{2}t\right)\psi_1(p)^2 + \sin^2\left(\frac{\Omega_R}{2}t\right)\psi_2''(p)^2 \\ &\quad - 2 \sin\left(\frac{\Omega_R}{2}t\right) \cos\left(\frac{\Omega_R}{2}t\right)\psi_1(p)\psi_2''(p) \sin(\omega_o t).\end{aligned}\quad (60)$$

of course, the integral of $P(p, t)$ over all p -values is unity at any time.

Without coupling, that is, when $\Omega_R = 0$, we have $P(p, t) = \psi_1(p)^2$ and thus $\langle p(t) \rangle = 0$ according to the relation below (9). Since the average current induced in the potential source is proportional to $\langle p(t) \rangle$, the average current vanishes. It follows that for stationary states there are no energy exchange between the (static) potential source and the electron on the average.

Average induced current. To evaluate the average momentum when $\Omega_R > 0$, we notice that the first two terms in (60) do not contribute. We thus obtain

$$\langle p(t) \rangle \equiv \int_{-\infty}^{+\infty} dp p P(p, t) = -2 \sin\left(\frac{\Omega_R}{2}t\right) \cos\left(\frac{\Omega_R}{2}t\right) \sin(\omega_o t) p_{12} \quad (61)$$

where

$$\begin{aligned}p_{12} &\equiv -i \int_{-\infty}^{+\infty} dp p \psi_1(p)\psi_2(p) \\ &= \frac{m(E_2 - E_1)}{\hbar} \int_{-d/2}^{+d/2} dx x \psi_1(x)\psi_2(x) \\ &\equiv m\omega_o x_{12}\end{aligned}\quad (62)$$

These relations follow from those given in (13) with $e = 2mE/\hbar^2$. Thus the QM-averaged induced current is

$$\langle i(t) \rangle = \frac{e}{md} \langle p(t) \rangle = \omega_o e \frac{x_{12}}{d} \sin(\Omega_R t) \sin(\omega_o t), \quad (63)$$

where, see (56), $\Omega_R = e\sqrt{2}Vx_{12}/(d\hbar)$. The average conductance "seen" by the (deterministic) alternating potential source $v(t) = \sqrt{2}V \sin(\omega_o t)$ is therefore, initially

$$G(t) \equiv \frac{\langle i(t) \rangle}{v(t)} = \omega_o \frac{e^2}{\hbar} \left(\frac{x_{12}}{d}\right)^2 \frac{\sin(\Omega_R t)}{\Omega_R} \approx \frac{e^2}{\hbar} \left(\frac{x_{12}}{d}\right)^2 \omega_o t = \frac{e^2}{2md^2} t. \quad (64)$$

In the last expression we have assumed an oscillator strength equal to 1 instead of the previously calculated value of 0.96. It is interesting that this expression does not involve \hbar . If some external mechanism interrupts the process after a time τ much smaller than the Rabi period, the time-averaged conductance is given by (64) with t replaced by $\tau/2$. Under such conditions the potential source "sees" a constant positive conductance. A similar discussion applies if the electron is initially in the upper state 2. The conductance then has the same absolute value as before, but is negative.

Electron energy. The electron energy reduces in the present situation to the kinetic energy $E = p^2/(2m)$. For the non-stationary state previously considered we first evaluate $\langle p^2 \rangle$ from (14)

$$\begin{aligned} \langle p^2 \rangle &\equiv \int_{-\infty}^{+\infty} dp p^2 P(p, t) \\ &= \cos^2\left(\frac{\Omega_R t}{2}\right) \int_{-\infty}^{+\infty} dp p^2 \psi_1(p)^2 + \sin^2\left(\frac{\Omega_R t}{2}\right) \int_{-\infty}^{+\infty} dp p^2 \psi_2''(p)^2 \\ &\quad - 2 \sin\left(\frac{\Omega_R t}{2}\right) \cos\left(\frac{\Omega_R t}{2}\right) \sin(\omega_o t) \int_{-\infty}^{+\infty} dp p^2 \psi_1(p) \psi_2''(p) \end{aligned} \quad (65)$$

where $\psi(p, t)$ is given in (59). Thus, using previous expressions

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} = E_1 \cos^2\left(\frac{\Omega_R t}{2}\right) + E_2 \sin^2\left(\frac{\Omega_R t}{2}\right). \quad (66)$$

If we postulate that upon a measurement of the electron energy we may obtain only E_1 or E_2 , the above expression may be interpreted as follows: the probability that the electron be found at time t in state 1 of energy E_1 is $\cos^2(\frac{\Omega_R t}{2})$ and the probability that the electron be found in state 2 of energy E_2 is $\sin^2(\frac{\Omega_R t}{2})$. Note that the integral giving $\langle E^n \rangle$, $n > 1$ diverges in the present model, so that the variance of E cannot be evaluated from the

Schrödinger equation. Under the above postulate that either E_1 or E_2 occurs we obtain: $\text{variance}(E) = (E_2 - E_1)^2 \sin(\Omega_R t)/2$.

Since the electrons are not directly coupled to one another, the induced averaged current is proportional to the total number of electrons. A more appropriate model may be that of nearly-resonant coupling between two narrow bands of states, assuming that an equilibrium is quickly reached within the two bands separately. For example at $T=0\text{K}$, electrons are supposed to decay instantly to the lowest available level of the band considered (only one single-spin state electron being allowed in each level, according to the Pauli principle). This is the situation considered in semiconductors. The lower band is called in that case the valence band and the upper band the conduction band, see the book by Landau and Lifchitz [32, p. 420]. These authors consider from the outset broadened levels and introduce early in their calculation population ratios at thermal equilibrium. The end result is a relation between the variance of the induced current and the conductance (or susceptibility) of the material. In the next section the problem is approached differently, using essentially Statistical Mechanics results.

4.4 Spontaneous decay

Spontaneous decay is the phenomenon by which electrons move from higher-energy levels to lower-energy levels without any field being applied externally. This decay may be non-radiative with the electron energy delivered to other electrons as is the case in the Auger effect, or to nuclei motion (phonons), or radiative with the electron energy being delivered to an optical wave spontaneously emitted from the atom. The optical wave is radiated from the atom almost isotropically on the average, but is directed for individual events, and involves a slight recoil of the atom as a whole. For a collection of many independent atoms the spontaneous-emission rate may be written in the form $S(t) = n(t)/\tau_s + s(t)$, where $n(t)$ denotes the number of atoms in the upper state at time t and τ_s a constant called the radiative life-time. Because in general the spontaneous-emission times of different atoms are independent the spectral density of the fluctuation $s(t)$ is equal to the rate $n(t)/\tau_s$. Thus, spontaneous emission, if not neglected from the outset, enters in our theory in two ways. First it affects the steady-state through the term $\langle n(t) \rangle / \tau_s$, where $\langle \cdot \rangle$ denotes averaging. Second, there is a noise source associated with it, $s(t)$, of known spectral density.

Let us emphasize that neglecting spontaneous emission in no way amounts to neglecting noise altogether. Because spontaneous decay plays only a secondary role in our theory it is ignored for the sake of simplicity in the major part of this introductory paper. In previous semi-classical theories the spontaneously emitted field is considered instead to be the fundamental source of noise. The classical optical field is supposed to be incremented by the field spontaneously emitted by upper-state atoms with a phase uniformly distributed between 0 and 2π (hence the randomness). Such semi-classical theories are in our opinion valid only far above the shot-noise level.

4.5 Sources and detectors

In the next subsections we discuss sources of electromagnetic radiations and ways of detecting such radiations. The most common detectors operate with a low-work function cathode and accelerating potentials. But there exists also quantum detectors whose mode of operation is based on stimulated absorption. As far as sources are concerned, we begin with a vacuum-tube called the "reflex klystron" oscillating in the microwave range of electromagnetic radiation. We then turn to sources of optical waves: thermal radiation and lasers.

Classical photo-detectors Let us recall the basic mode of operation of conventional electronic diodes, photo-detectors and photo-multipliers. Conventional electronic diodes are made up of two parallel plates (labeled in what follows the lower and upper plates) separated by a distance d in vacuum. The lower plate, called "anode" is at zero potential by convention, and the upper plate, called "cathode", is raised at the potential $-U$ with $U > 0$. Suppose that at time $t = 0$ an electron is freed from the upper plate and attracted by the anode²³. Considering only absolute values, the electron momentum increases linearly with time according to the law $p(t) = eUt/d$ until it reaches the anode at time $\tau = d\sqrt{2m/eU}$, where m denotes the electron mass. The electron kinetic energy is then converted into heat. Here τ is neglected. Fig. 4 illustrates in a), the photo-current, represented as a function of time, the output circuit capacitance being taken into account. In b), photo-current spectrum for the case where the output circuit is a resonating circuit tuned at some Fourier frequency Ω_o .

In the case of photo-multipliers, the electron kinetic energy, instead of being dissipated into heat, is employed, at least in part, to free two or more electrons from the anode (secondary-emission effect). The latter are accelerated by a third plate, and so on, so that each electron freed from the cathode by light gets converted into an electron bunch containing n electrons, for example, $n = 10^6$. The situation is the same as if the absolute value e of the electron charge had been multiplied by n . The purpose of photo-multipliers is not to improve the signal-to-noise ratio, which may only degrade. It is to raise the signal to such high levels that the thermal noise of subsequent electronic amplifiers is rendered negligible.

²³To achieve this, the cathode "work function" energy must be overcome by heat (thermo-ionic emission), high electric fields (field emission), electrons (secondary emission), or light (photo-electric emission). Electrons may be freed by thermal motion provided $k_B T$ be of the order of the metal work-function. If nickel is coated with barium oxide, a temperature of 1000 kelvin may suffice. Field emission occurs with kilo-volt potentials if the cathode has the shape of a needle. Electrons may be freed by light provided $\hbar\omega$ exceeds the metal work-function, where \hbar denotes the Planck constant and ω the light frequency. Visible light for example is adequate when the cathode is coated with cesium. The non-zero initial electron velocities are presently neglected, that is, the initial electron momentum $p(0) = 0$. Electrons in a metal are bound to it because they are attracted by their image charge. They may escape, though, because of a tunneling effect whose understanding requires Quantum Mechanics. But once the electron is sufficiently far away from the metal, a Classical Mechanical treatment of its motion is adequate.

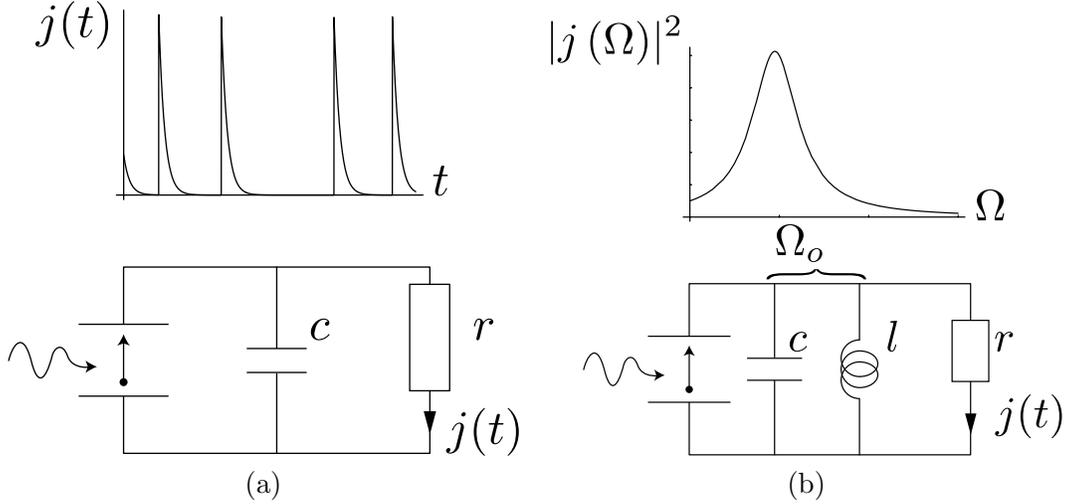


Figure 4: In a), on top the photo-current is represented as a function of time, the output circuit capacitance (lower part) being taken into account. In b) the photo-current spectrum (on top) for the case where the output circuit (lower part) is a resonating circuit tuned at some Fourier frequency Ω_o .

In temperature-limited thermo-ionic diodes electrons are emitted independently of one-another. The electron emission times are therefore Poisson distributed. Thus, the diode current fluctuation $\Delta j(t)$ obeys the so-called shot-noise formula, with a (double-sided) spectral density equal to $e \langle j(t) \rangle$. But when the electron flow is space-charge limited the current is sub-Poissonian, i.e., with a spectral density much smaller than the one just given.

The detectors considered above have been idealized for the sake of simplicity neglecting, e.g., dark currents and thermal noise. As said before, the current flowing out of photo-detectors may be viewed as a sum over the positive integer k of delta-functions of the form $e\delta(t - t_k)$, where the t_k are occurrence times. If this current is transmitted through a low-pass filter individual pulses may overlap, however, and not be distinguishable any more from one another. The current fluctuation then resembles gaussian noise irrespectively of the event-times statistics.

Classical sources By "classical sources" we refer to generators of electromagnetic radiation whose mode of operation may be understood on the basis of the classical law of electronic motion and the classical circuit theory. Let us consider as an example the reflex-klystron oscillator discovered by the Varian brothers. This oscillator generates electromagnetic radiation at microwaves frequencies, and was mostly employed as a low-noise local oscillator in radar receivers until it was superseded by solid-state devices. In our description, this oscillator involves two anodes made up of grids presumed to be transparent to electrons and separated by a distance d . Two cathodes are located just outside the

anodes. The lower one emits electrons, while the upper one plays the role of a reflector, so that the electron motion as a function of time is a zig-zag path (usually the reflector is raised at a potential slightly lower than that of the emitting cathode to prevent electrons from being captured. It also helps finely tune the klystron oscillation frequency). The two anodes are part of a resonator. When the device oscillates there is an alternating potential $v(t)$ between the two anodes.

If the electron emission time is appropriate, the electron loses its energy giving it up to the oscillating potential through an induced current. But since the electron emission times are uniformly distributed along the time axis, the net interaction with the field vanishes. Accordingly, initially, the electrons do not deliver any energy to the oscillating potential. It is as a result of the field action on the electron trajectories that non-zero energy exchanges between the field and the electrons may occur. This effect is called "bunching". Once the electron has lost most of its energy it gets captured by an anode, an event that one may call a jump²⁴. The static-potential energy then gets reduced by eU . In some sense, the electron plays an intermediate role. Indeed, the net effect of a complete electronic cycle is that, for each electronic event, the static potential source delivers an energy eU to the oscillating potential source $v(t)$. Since the cathode emission is space-charge limited the radiated-wave statistics should be sub-Poissonian.

A phenomenon akin to stimulated absorption may be understood similarly. This time, we suppose that the electron is emitted by one of the anodes. Without an alternating field this electron would remain permanently in the neighborhood of the anodes. However, a resonance with an alternating potential may force the electron to oscillate along the x axis with increasing amplitude until its energy reaches the value eU , in which case it gets captured by one of the cathode. The net effect of this electronic process is that the alternating source gives energy to the static source, the opposite of what was discussed in the previous paragraph.

Quantum detectors Quantum photo-detectors (sometimes referred to as "narrow-band" photo-detectors), involve two electron energy levels, coupled to continua, and operate through the process of stimulated absorption. Ideally, the device is reversible in the sense that the electrical energy may be converted back into light energy through the process of stimulated emission. In contradistinction, conventional photo-diodes necessarily dissipate energy in the form of heat.

Quantum sources The best-known light source is thermal radiation. A hot body like the sun radiates energy. The energy inside a closed cavity at absolute temperature T contains an energy given by the law discovered in 1900 by Planck. An important feature of

²⁴In view of the fact that electrons are moving at very low speeds in metals, on the order of $1\mu\text{m/s}$, even in the best electrical conductors, the above statement that jumps are instantaneous may seem surprising. As a matter of fact, the electron appearing on one plate needs not be the same as the electron that hits the other plate. Electrons cannot be distinguished from one another.

this law is that it involves a previously unknown universal constant \hbar with the dimension of action (energy \times time). From the Quantum Mechanical view point, heat excites some electrons to atomic levels higher in energy than the ground state energy. These electrons then may decay spontaneously to the ground state by emitting a quantum of light. A similar mechanism is at work in the so-called "light-emitting diodes" but the spectrum is narrower.

A conventional neon tube generates light because the electric discharge excites the neon atoms, which subsequently decay to the ground state, thereby emitting ultraviolet light (subsequently converted into visible light) by the process of spontaneous emission, similar to what happens in thermal sources, but with a narrower spectrum. The so-called Helium-Neon laser, radiating light at a free-space wavelength of $0.63 \mu\text{m}$, differs from conventional neon tubes in that two mirrors located at both ends, and facing each others, force the emitted light to move back and forth in the tube. Light gets amplified by the process of stimulated emission discovered by Einstein, and damped by the process of stimulated absorption. The former exceeds the latter when there are more atoms in the higher state than in the lower state (population inversion). To achieve this condition the lower-level population must be reduced through spontaneous decay to even-lower levels (3-levels lasers). Eventually a steady state of oscillation is reached. The emitted light spreads out in free space as little as is allowed by the laws of diffraction, and the laser light is nearly monochromatic (single frequency). The laser linewidth, though small, is of major importance in some applications. Laser diodes (also called injection lasers) employ a semi-conductor with a doping that deliver electrons (n-type) and a doping that absorbs electrons (p-type)

To summarize, lasers essentially consist of single-mode resonators containing three-level atoms or other forms of matter with a supply of energy called the pump and a sink of energy, perhaps an optical detector. As said before, the latter converts the light energy into a sequence of electrical pulses corresponding to photo-detection events. When the pump is non-fluctuating the emitted light does not fluctuate much. Precisely, this means that the variance of the number of photo-detection events observed over a sufficiently long period of time is much smaller than the average number of events. Light having that property is said to be "sub-Poissonian", or below the "standard quantum level" (SQL).

4.6 Optical-mechanical analogy

Leontovitch and Fock (see, e. g. [30]) where probably the first to point out the analogy that exists between light beams at some frequency ω_o propagating near an axis z and the evolution of Quantum Mechanical wave-functions in the course of time t . We thus have the correspondence $z \rightarrow t$. In that analogy, polarization (resp. spin) effects are neglected. The optical refractive index $n(x, z)$ corresponds to the potential $u(x, t)$. For example, in (possibly z -dependent) square-law media the complex-ray representation of gaussian beams, first proposed by one of us (J.A.) in 1968, is useful. This representation

may be employed also in Quantum Mechanics [33]. As a second example, the coupling of an optical waveguide to a substrate is analogous to the tunneling of an electron, initially in a stationary state, into a continuum of states [30].

The interpretations, however, are different. In Optics, the field

$$\mathcal{E}(x, t; \omega_o) \equiv \Re\{\sqrt{2}\mathcal{E}_c(x, t) \exp(-i\omega_o t)\}$$

is supposed to be a real measurable function of space x and time t , ω_o playing the role of a parameter. In contradistinction, the Quantum Mechanical wave-function $\psi(x, t)$ is fundamentally a complex quantity that has the following significance, due to Born: the real positive quantity $|\psi(x, t)|^2 dx$ is the probability of finding the electron between x and $x + dx$ at time t if a measurement is performed at that time. Furthermore, while in Optics losses may be accounted for by ascribing an imaginary part to the refractive index, such a simple procedure is not allowed in Quantum Mechanics. It is easy to solve the Schrödinger equation with a complex potential, but the physical interpretation of the wave-function is then obscure. Note however that some complex potentials may lead to real eigenvalues.

5 Current-sources spectral density

The present theory is based on two fundamental concepts. One is the introduction of Nyquist-like current sources, the other is the law of conservation of average energy. In the present section we consider the link that exists between fluctuations and dissipation, usually referred to as the "fluctuation-dissipation theorem". An equivalent formulation is in terms of rates (or normalized energy flows). In the rate equation approach, fluctuation rates are found to be uncorrelated to one another and equal to the average values.

Brown was the first to observe the random motion of small particles in viscous fluids (Brownian motion) and Johnson later on measured the electrical noise associated with conductances (Johnson noise). The first interpretation of Brownian motion was offered by Einstein and the first interpretation of electrical noise was offered by Nyquist. The two phenomena are closely related. The Einstein and Nyquist interpretations rest on Classical Statistical Mechanics ($k_B T \gg \hbar\omega$). For our purposes we need consider instead the situation where $k_B T \ll \hbar\omega$. The latter "quantum" situation was treated by Callen and Welton in 1951, see for example [34]. We will first sketch the Quantum Optics viewpoint concerning fluctuations, and subsequently turn to simpler, but admittedly partly heuristic, explanations that do not involve optical-field quantization. These arguments tend to prove that the spectral density of the Nyquist-like noise sources associated with two-level atoms exhibiting a peak conductance G at frequency $\omega = \omega_o$ is equal to $\hbar\omega_o |G|$. In the first argument we show that there is a (perhaps unique) way of generalizing the classical expression $\langle E \rangle = k_B T$ for the average energy of an oscillator resonating at frequency ω_o that avoids divergences. The expression obtained differs from the 1901 Planck formula by an additional energy $\hbar\omega_o/2$ called the vacuum energy. This expression does not lead to

”ultraviolet” divergences if only measurable quantities are being considered. In the second approach we consider an oscillator in a state of thermal equilibrium with a small negative conductance and a slightly larger positive conductance (in absolute value). The ratio of these two conductances is equal to the ratio of lower and higher-state populations. The quantum formula is obtained by requiring agreement with the Classical Mechanical results in the limit $k_B T \gg \hbar\omega_o$. The third approach is quite different from the previous one since we consider an *isolated* oscillator containing two-level atoms in a state of *non-thermal* equilibrium. Statistical mechanics tells us that all states of equal energy are equally likely to occur. The noise terms (Langevin forces) must be such that the variance of the photon number derived from the above Statistical Mechanical law obtains in the long-time limit.

5.1 Quantum-optics approach

In Quantum Optics treatments loss-less optical oscillators oscillating at frequency ω_o are viewed as being akin to loss-less mechanical oscillators with quantized energy levels $E_m = \hbar\omega_o(m + \frac{1}{2})$, $m = 0, 1, \dots$. If we employ Boltzmann’s result that the probability that a level of energy E be occupied is proportional to $\exp(-E/k_B T)$ when the system is in contact with a bath at absolute temperature T , the Planck law of black-body radiation is obtained through a summation over m .

The fluctuation-dissipation theorem (FDT) applies to systems having a linear causal response, in thermal equilibrium. One first evaluates the symmetrized correlation $C_{i,q}(\tau) \equiv \langle \Delta i(\tau)\Delta q(0) + \Delta q(0)\Delta i(\tau) \rangle / 2$, where the current $i = dq/dt$ and the charge q are conjugate operators. The end result of the calculation is that the spectral density of the current, that is the Fourier transform of the correlation $C_{i,i}(\tau)$ reads [34]

$$\mathcal{S}_i(\omega) \equiv \mathcal{S}_{i,i}(\omega) = 2\left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/k_B T) - 1}\right)\mathcal{R}\{Y(\omega)\}. \quad (67)$$

where $\mathcal{R}\{Y(\omega)\} \equiv G(\omega)$ denotes the system conductance, a function of the frequency ω . In the following, operators are not employed, that is, $i(t)$ is a classical function of time.

5.2 Heuristic approach

A simple heuristic derivation of the black-body formula is suggested here. In the classical regime, single-mode oscillators at frequency ω_o have an average energy $\langle E \rangle = k_B T$, independent of ω_o , according to Classical Statistical Mechanics. The action f of an oscillator is the ratio of its average energy and frequency, and accordingly $f(x) = 1/x$, setting $x \equiv \omega/k_B T$. Thus $f(x)$ obeys the Riccati equation

$$\frac{df(x)}{dx} + f(x)^2 = 0. \quad (68)$$

However, as was noted at the end of the 19th century, the expression $\langle E \rangle = k_B T$ leads to infinite radiated heat since the number of electromagnetic modes is infinite in a cavity with

perfectly reflecting walls. It apparently did not occur to scientists at the time that this difficulty is resolved simply by adding a constant $(\hbar/2)^2$ on the right-hand side of (68), that is, supposing

$$\frac{df(x)}{dx} + f(x)^2 = \left(\frac{\hbar}{2}\right)^2, \quad (69)$$

where \hbar is a universal constant with the dimension of action. The solution of this modified equation reads

$$x = \int \frac{df}{\left(\frac{\hbar}{2}\right)^2 - f^2} = \frac{1}{\hbar} \log\left(\frac{2f+1}{2f-1}\right) + x_o, \quad (70)$$

where x_o denotes an arbitrary constant. For $x_o=0$ we obtain that

$$f(x) = \frac{\hbar \exp(\hbar x) + 1}{2 \exp(\hbar x) - 1} \quad (71)$$

which coincides with the Planck formula except for the vacuum energy mentioned above. The arbitrary constant x_o on the right-hand-side of (70) must vanish to obtain agreement with the classical result. It was noted by Einstein and Stern in 1913 that $f(x) - 1/x \rightarrow 0$ if $x \rightarrow 0$, that is, the expansion of $f(x)$ is of the form

$$f(x) = \frac{1}{x} + ax + \dots, \quad (72)$$

where a is a constant, without an x -independent term. One may conjecture that there are no other differential equation but (69) which possesses only one solution satisfying the above classical limit. To conclude, the classical expression of the average energy generalizes to

$$\langle E \rangle = \frac{\hbar\omega_o \exp(\hbar\omega_o/k_B T) + 1}{2 \exp(\hbar\omega_o/k_B T) - 1}. \quad (73)$$

If we next consider a cavity with perfectly-conducting walls, solutions of the Maxwell equation exist only for a series of real resonating frequencies $\omega_1, \omega_2, \omega_3, \dots$. Each of these modes of resonance is ascribed an average energy given by the above expression with ω_o replaced by $\omega_1, \omega_2, \omega_3, \dots$. For a d -dimensional cavity the mode density $\rho(\omega)$, where $\rho(\omega)d\omega$ denotes the number of modes whose frequencies are between ω and $\omega+d\omega$, grows in proportion of ω_o^{d-1} . It follows that the total energy is apparently infinite. This is perhaps why, in his original work, Planck subtracted the vacuum energy $\hbar\omega_o/2$ from the expression given in (73). The total energy in a cavity of volume \mathcal{V} in thermal equilibrium at absolute temperature T is then found to be finite and proportional to the fourth power of T . If the cavity is pierced with a small hole that does not perturb the state of thermal equilibrium, the measured output-power spectral density is supposed to be proportional to the internal

energy spectrum, that is, to the product of $\langle E(\omega) \rangle - \hbar\omega/2$ and the mode density $\rho(\omega)$. This is the famous black-body spectrum formula conjectured by Planck. This formula agrees well with measurements. In our model the power is collected by a small external conductance g at $T=0\text{K}$, having its own noise source. A finite result is then obtained from the expression of $\langle E(\omega) \rangle$ as given above.

5.3 Johnson-Nyquist noise

When a frequency-independent conductance G is in equilibrium with a bath at absolute temperature T , there is associated with it a random current source $j(t)$ whose (double-sided) spectral density \mathcal{S}_j equals $2k_{\text{B}}TG$, as discovered by Johnson and Nyquist in 1927, 1928. This expression holds in the so-called "classical regime", that is, at frequencies much smaller than $k_{\text{B}}T/\hbar$. In the present section currents are denoted by j instead of i because we are mostly concerned with low frequencies.

To justify the Nyquist expression in the classical regime, let us recall that, according to Statistical Mechanics, the average energy must equal $k_{\text{B}}T/2$ per degree of freedom. As a consequence, the energy stored in a capacitance C with a conductance G in parallel must equal $k_{\text{B}}T/2$. The modulus square $|V/I|$ of the impedance of the circuit considered at frequency ω is $1/(G^2 + C^2\omega^2)$. Since the energy stored in a capacitance C submitted to an (root-mean-square) voltage V is $C|V|^2/2$, as we have seen in section 4.1, we must have

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{C\mathcal{S}_j}{2(G^2 + C^2\omega^2)} = \frac{k_{\text{B}}T}{2} \quad (74)$$

from which it follows that $\mathcal{S}_j = 2k_{\text{B}}TG$. What we have given above is essentially the Nyquist argument.

The Nyquist result may be generalized to any (time-invariant, linear) circuit consisting of any number of conductances all of them being at the same temperature T , capacitances and inductances. If the conductance between any two terminals of the circuit is $G(\omega)$, the circuit may be considered noiseless provided a current source $j(t)$ be applied to the two terminals with spectral density $\mathcal{S}_j(\omega) = 2k_{\text{B}}TG(\omega)$.

For later use note that when a white (i.e., with a frequency-independent spectrum) Nyquist current $i(t)$ is applied to an ideal narrow-band filter whose response is centered at $\omega = \pm\omega_o$, the filter output may be written in the form

$$i(t) = \sqrt{2}C'(t) \cos(\omega_o t) + \sqrt{2}C''(t) \sin(\omega_o t) \equiv \Re\{\sqrt{2}C(t) \exp(-i\omega_o t)\} \quad (75)$$

where the real random functions of time $C'(t)$ and $C''(t)$ vary slowly, are uncorrelated, and have (double-sided) spectral density $\mathcal{S}_{C'} = \mathcal{S}_{C''} = 2k_{\text{B}}TG$.

5.4 Thermal equilibrium approach

In the present section we generalize the previous Johnson-Nyquist expression to arbitrary frequencies. Consider an inductance-capacitance ($L-C$) oscillator resonating at frequency

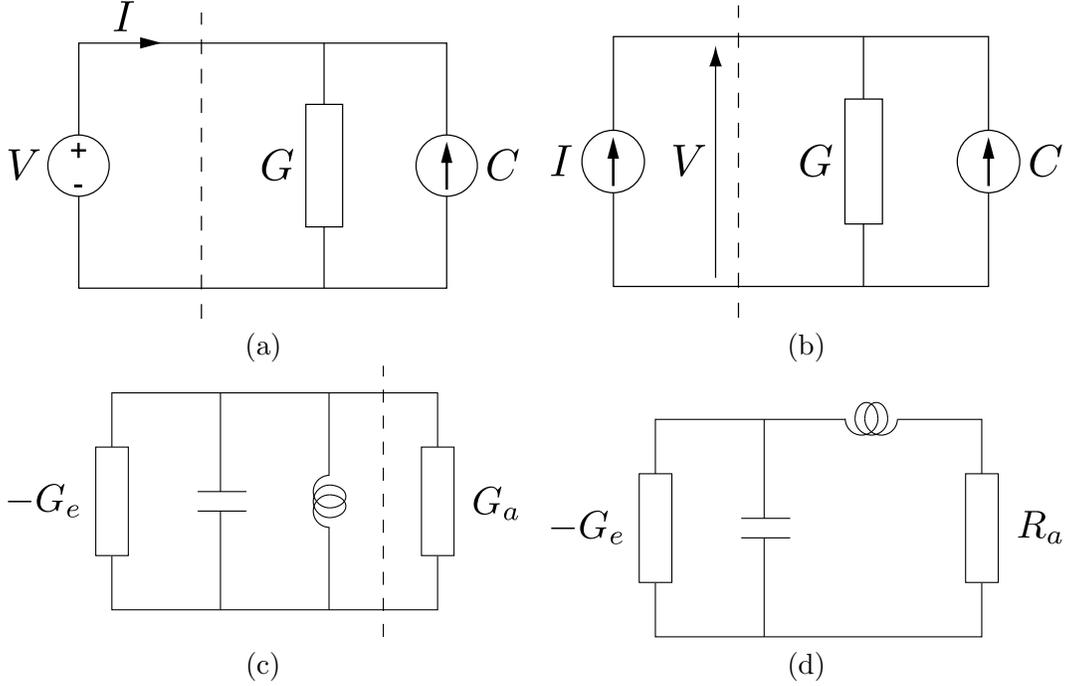


Figure 5: a) represents a potential source V applied to a conductance G . C represents the Nyquist-like noise current source associated with G . We are mostly concerned with the power flowing from left to right through the dashed line, b) represents a conductance driven by a current source I , c) is a laser model with a negative conductance $-G_e$, a resonating circuit, and a positive conductance G_a , representing the detector of radiation, connected in parallel, d) represents a laser model that exhibits a Petermann-like linewidth-enhancement factor.

ω_o with a small positive conductance G_a and a negative conductance $-G_e$ (subscript "e" for "emitting") in parallel, with G_a (subscript "a" for "absorbing") exceeding G_e so that there is a small net loss. We set $G_a - G_e \equiv G > 0$. The conductance G_a is supposed to represent the absorption by two-level atoms in the lower state (energy E_a), while G_e represents the emission from atoms in the upper state (energy E_e). A near-resonance condition $\hbar\omega_o \approx E_e - E_a$ holds. According to the Schrödinger equation the conductances are proportional to the corresponding numbers of atoms n_a and n_e , respectively.

Because the conductances considered are constant the circuit is linear. The potential V across the circuit is therefore equal to $\mathcal{C}/Y(\omega)$, where \mathcal{C} denotes the driving current, assumed to be independent of frequency, and the resonating circuit admittance $Y(\omega) = G + iB(\omega)$ where $B(\omega)$ represents the sum of the L and C susceptances. Referring to (37)

the oscillator energy is

$$E = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{C |C|^2}{G^2 + 4C^2(\omega - \omega_o)^2} = \frac{|C|^2}{4G} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{|C|^2}{4G}. \quad (76)$$

Let this oscillator be in thermal contact with a bath at absolute temperature T . We replace the deterministic current C by a complex random function of time²⁵. Because the processes are stationary we expect that the statistical properties of the random source $C(t) \equiv C'(t) + iC''(t)$ are unaffected by an arbitrary phase change, that is, we require that $C(t)\exp(i\phi)$ has the same statistical property as $C(t)$ for any phase ϕ . This entails that $C'(t)$ and $C''(t)$ are uncorrelated and have the same statistical density. We thus set $\mathcal{S}'_C = \mathcal{S}''_C \equiv \mathcal{S}$. We have seen that for n independent atoms in some state both G and \mathcal{S} are proportional to n . We therefore expect that $\mathcal{S} = \alpha G$, where α is a constant to be determined.

Because of the symmetry between stimulated absorption and stimulated emission implied by the Schrödinger equation the spectral densities have the same form for positive conductances G_a and negative conductances $-G_e$, namely $\mathcal{S}_a = \alpha G_a$ and $\mathcal{S}_e = \alpha G_e$, with the same constant of proportionality α . Note that C' and C'' contribute equally and that double-sided spectral densities for C' , C'' in the Fourier Ω -domain are employed. If the conductances G_a and $-G_e$ are connected in parallel the total conductance is $G = G_a - G_e$, as said above, and the total spectral density is $2\alpha(G_a + G_e)$. The average resonator energy follows from (76)

$$\langle E \rangle = \frac{2\alpha(G_a + G_e)}{4G} = \frac{\alpha}{2} \frac{G_a/G_e + 1}{G_a/G_e - 1}. \quad (77)$$

Classical Statistical Mechanics tells us that at thermal equilibrium $n_a/n_e = \exp((E_e - E_a)/k_B T)$. It follows that

$$\frac{G_a}{G_e} = \frac{n_a}{n_e} = \exp\left(\frac{E_e - E_a}{k_B T}\right) = \exp\left(\frac{\hbar\omega_o}{k_B T}\right) \quad (78)$$

Classical Statistical Mechanics also tells us that in the classical limit the average energy equals $k_B T/2$ per degree of freedom, and thus $\langle E \rangle = k_B T$ for the oscillator considered when $k_B T \gg \hbar\omega_o$. According to (77) this is the case if and only if

$$\alpha = \hbar\omega_o \quad (79)$$

²⁵Note that in the present linear regime the regulation mechanism at work in above-threshold lasers does not occur, and the fluctuations of V are comparable to average values. Supposing that the current source is gaussian distributed, this is also the case for the optical potential V and optical current I . Power should in general be evaluated as the real part of $V^*(I + C(t))$, but in the linear regime presently considered the term $C(t)$, much smaller than the fluctuations of I , may be neglected. The power $\Re\{V^*I\}$ is Rayleigh-distributed.

At low-temperatures, $k_B T \ll \hbar\omega_o$, the resonator energy is $\hbar\omega_o/2$. Since a resonator may exchange energy resonantly with atoms only in units of $\hbar\omega_o$, we conclude that the energy of a single-mode loss-less resonator may be written as

$$E_T = (m + 1/2)\hbar\omega_o, \quad (80)$$

where m is an integer that one may call "number of photons in the resonator".

5.5 Isolated cavity approach

A single-mode optical cavity resonating at angular frequency ω_o may be modeled as an inductance-capacitance (L, C) circuit with $LC\omega_o^2 = 1$. The active atoms, located between the capacitor plates, interact with a spatially uniform optical field through their electric dipole moment. The 2-level atoms (with the lower level labeled "a" and the upper level labeled "e") are resonant with the field. This means that the atomic levels a and e are separated in energy by $\hbar\omega_o$.

(Non-thermal) equilibrium Consider N identical two-level atoms. For each atom, the zero of energy is taken at the lower level and the unit of energy at the upper level (typically, 1 eV). The atoms are supposed to be at any time in either the upper or lower state. The number of atoms that are in the upper state is denoted by n , and the number of atoms in the lower level is therefore $N - n$. With the convention $\hbar\omega_o = 1$, the atomic energy is equal to n . Its maximum value N occurs when all the atoms are in the upper state. There is population inversion when the atomic energy $n > N/2$. The atoms are supposed to reach a state of equilibrium before other parameters have changed significantly. The strength of the atom-atom coupling, however, needs not be specified.

The statistical weight $W(n)$ of the atomic collection is the number of distinguishable configurations corresponding to some total energy n . For two atoms ($N = 2$), for example, $W(0) = W(2) = 1$ because there is only one possible configuration when both atoms are in the lower state ($n = 0$), or when both are in the upper state ($n = 2$). But $W(1) = 2$ because the energy $n = 1$ obtains with *either one* of the two (distinguishable) atoms in the upper state. For N identical atoms, the statistical weight (number of ways of picking up n atoms out of N) is

$$W(n) = \frac{N!}{n!(N-n)!}. \quad (81)$$

Note that $W(0) = W(N) = 1$ and that $W(n)$ reaches its maximum value at $n = N/2$ (supposing N even), with $W(N/2)$ approximately equal to $2^N \sqrt{2/\pi N}$. Note further that

$$Z \equiv \sum_{n=0}^N W(n) = 2^N. \quad (82)$$

Consider next an isolated single-mode optical cavity containing N resonant two-level atoms. The atoms perform jumps from one state to another in response to the optical field so that the number of atoms in the upper state is a function $n(t)$ of time. If $m(t)$ denote the number of light quanta at time t , the sum $n(t) + m(t)$ is a conserved quantity (essentially the total atom+field energy). Thus, m jumps to $m - 1$ when an atom in the lower state gets promoted to the upper state, and to $m + 1$ in the opposite situation. If N atoms in their upper state are introduced at $t = 0$ into the empty cavity ($m = 0$), part of the atomic energy gets converted into field energy as a result of the atom-field coupling and eventually an equilibrium situation is reached.

The basic principle of Statistical Mechanics asserts that in isolated systems all states of equal energy are equally likely. Accordingly, the probability $P(m)$ that some m value occurs at equilibrium is proportional to $W(N - m)$, where $W(n)$ is the statistical weight of the atomic system. As an example, consider two (distinguishable) atoms ($N=2$). A microstate of the isolated (matter+field) system is specified by telling whether the first and second atoms are in their upper (1) or lower (0) states and the value of m . Since the total energy is $N = 2$, the complete collection of microstates (first atom state, second atom state, field energy), is: (1,1,0), (1,0,1), (0,1,1) and (0,0,2). Since these four microstates are equally likely, the probability that $m = 0$ is proportional to 1, the probability that $m = 1$ is proportional to 2, and the probability that $m = 2$ is proportional to 1. This is in agreement with the fact stated earlier that $P(m)$ is proportional to $W(N - m)$. After normalization, we obtain for example that $P(0)=1/4$.

The normalized probability reads in general

$$P(m) = \frac{W(N - m)}{Z} = \frac{N!}{2^N m!(N - m)!} \quad (83)$$

The moments of m are defined as usual as

$$\langle m^r \rangle \equiv \sum_{m=0}^N m^r P(m) \quad (84)$$

where brackets denote averagings. It is easily shown that $\langle m \rangle = N/2$ and $\text{var}(m) \equiv \langle m^2 \rangle - \langle m \rangle^2 = N/4$. Thus the number m of light quanta in the cavity fluctuates, but the statistics of m is sub-Poissonian, with a variance less than the mean. For example, one may readily deduce from that principle that, if an isolated single-mode cavity initially contains no photons but N two-level resonating atoms in the upper state, the system evolves to an equilibrium state in which the variance of the number of photons in the cavity is half the average number of photons, that is, the photon statistics is sub-poissonian. More generally, for atoms with B evenly-spaced levels, the variance of the photon number is $(B + 1)/6$ times the average photon number, a result that coincides with the previous one if we set $B = 2$.

The expression of $P(m)$ just obtained has physical and practical implications. Suppose indeed that the equilibrium cavity field is allowed to escape into free space, thereby

generating an optical pulse containing m quanta. It may happen, however, that no pulse is emitted when one is expected, causing a counting error. From the expression in (83) and the fact that $\langle m \rangle = N/2$, the probability that no quanta be emitted is seen to be $P(0) = 4^{-\langle m \rangle}$. For example, if the average number of light quanta $\langle m \rangle$ is equal to 20, the communication system suffers from one counting error (no pulse received when one is expected) on the average over approximately 10^{12} pulses. Light pulses of equal energy with Poissonian statistics are inferior to the light presently considered in that one counting error is recorded on the average over $\exp(\langle m \rangle) = \exp(20) \approx 0.5 \cdot 10^9$ pulses.

Time evolution of the number of light quanta in isolated cavities Let us now evaluate the probability $P(m, t)$ that the number of light quanta be m at time t . Note that here m and t represent two independent variables. A particular realization of the process was denoted earlier $m(t)$. It is hoped that this simplified notation will not cause confusion.

Let $E(m)dt$ denote the probability that, given that the number of light quanta is m at time t , this number jumps to $m + 1$ during the infinitesimal time interval $[t, t + dt]$, and let $A(m)dt$ denote the probability that m jumps to $m - 1$ during that same time interval (the letters "E" and "A" stand respectively for "emission" and "absorption"). The probability $P(m, t)$ obeys the relation

$$P(m, t + dt) = P(m + 1, t)A(m + 1)dt + P(m - 1, t)E(m - 1)dt + P(m, t)[1 - A(m)dt - E(m)dt]. \quad (85)$$

Indeed, the probability of having m quanta at time $t + dt$ is the sum of the probabilities that this occurs via states $m + 1$, $m - 1$ or m at time t . All other possible states are two or more jumps away from m and thus contribute negligibly in the small dt limit. After a sufficiently long time, one expects $P(m, t)$ to be independent of time, that is $P(m, t + dt) = P(m, t) \equiv P(m)$. It is easy to see that the "detailed balancing" relation

$$P(m + 1)A(m + 1) = P(m)E(m) \quad (86)$$

holds true because m cannot go negative. When the expression of $P(m)$ obtained in (83) is introduced in (86), one finds that

$$\frac{E(m)}{A(m + 1)} = \frac{P(m + 1)}{P(m)} = \frac{N - m}{m + 1}. \quad (87)$$

E must be proportional to the number $n = N - m$ atoms in the upper state while A must be proportional to the number $N - n = m$ of atoms in the lower state. We therefore set $E(m) = (N - m)f(m)$, $A(m) = mg(m)$, where $f(m)$ and $g(m)$ are two functions to be determined. Substituting in (87) we find that

$$f(m) = g(m + 1). \quad (88)$$

Because we assume that atoms emit or absorb a single light quantum at a time ("1-photon" process) the two functions $f(m)$ and $g(m)$ must be of the linear form $f(m) = am + b$ and $g(m) = cm + d$, where a, b, c, d are constants. But A is required to vanish for $m = 0$ since, otherwise, m could go negative, and thus $d = 0$. Substituting into (88), we find the relation $am + b = c(m + 1)$ which must hold for any m -value. Therefore, $a = b = c$. Setting for brevity $a = b = c = 1$ amounts to fixing up a time scale. Then $E(n, m) = n(m + 1)$, $A(n, m) = (N - n)m$. We note here a lack of symmetry between the rate of stimulated emission (proportional to $m + 1$) and the rate of stimulated absorption (proportional to m). Since according to the Schrödinger equation the two processes should be similar, we are led to define the field energy as $m + 1/2$, and to express E and A in terms of the field energy at jump time, $\mathcal{E}_{jump\ time}$, defined as the arithmetic average of the field energy just before and just after the jump. If we do so, we finally obtain, setting $n = n_e$ and $N - n = n_a$

$$E(n_e, m) = n_e \mathcal{E}_{jump\ time} \quad (89)$$

$$A(n_a, m) = n_a \mathcal{E}_{jump\ time}, \quad (90)$$

and the symmetry is indeed restored.

Let us now restrict our attention to the steady-state regime and large values of N . Since m is large, it may be viewed as a continuous function of time with a well-defined time-derivative. Because the standard deviation $\sqrt{N/4}$ of m is much smaller than the average value, the so-called "weak-noise approximation" is permissible. Within that approximation, the average value of any smooth function $f(n, m)$ may be taken as approximately equal to $f(\langle n \rangle, \langle m \rangle)$.

The evolution in time of a particular realization $m(t)$ of the process obeys the classical Langevin equation

$$\frac{dm}{dt} = \mathcal{E} - \mathcal{A}, \quad (91)$$

where

$$\mathcal{E} \equiv E(m) + e(t) \quad \mathcal{A} \equiv A(m) + a(t). \quad (92)$$

In these expressions, $e(t)$ and $a(t)$ represent uncorrelated white-noise processes whose spectral densities are set equal to $\beta E \equiv \beta E(\langle m \rangle)$ and $\beta A \equiv \beta A(\langle m \rangle)$, respectively, where β is a constant to be determined.

Let us show that the variance of m obtained from the above Langevin equation coincides with the result obtained directly from Statistical Mechanics only if $\beta = 1$. Without the noise sources, the evolution of m in (91) would be deterministic, with a time-derivative equals to the drift term $E(m) - A(m)$. If the expressions (88) are employed, the Langevin

equation (91) reads

$$\begin{aligned}\frac{dm}{dt} &= Nm - 2m^2 + e - a \\ \mathcal{S}_{e-a} &= \beta(E + A) = \beta N \langle m \rangle = \beta N^2/2,\end{aligned}\tag{93}$$

where the approximation $N \gg 1$ has been made, and β is a constant to be determined.

Let $m(t)$ be expressed as the sum of its average value $\langle m \rangle$ plus a small deviation $\Delta m(t)$, and $Nm - 2m^2$ in (93) be expanded to first order. A Fourier transformation of $\Delta m(t)$ with respect to time amounts to replacing d/dt by $j\Omega$. The Langevin equation now reads

$$j\Omega\Delta m = -N\Delta m + e - a \quad \mathcal{S}_{e-a} = \beta N^2/2,\tag{94}$$

where m has been replaced by its average value $N/2$.

Since the spectral density of some random function of time $z(t) = ax(t)$, where $a \equiv a' + ja''$ is a complex number and $x(t)$ a stationary process, reads $\mathcal{S}_z(\Omega) = |a|^2 \mathcal{S}_x(\Omega)$, see Section 3.4, one finds from (94) that the spectral density of the $\Delta m(t)$ process is

$$\mathcal{S}_{\Delta m}(\Omega) = \frac{\beta N^2/2}{N^2 + \Omega^2}.\tag{95}$$

The variance of m is the integral of $\mathcal{S}_{\Delta m}(\Omega)$ over frequency ($\Omega/2\pi$) from minus to plus infinity, that is $\text{var}(m) = \beta N/4$. There is agreement with the previous result derived from the basic Statistical Mechanics rule only if $\beta = 1$. It follows that the spectral density of fluctuation rates such as $r(t)$ must be equal to the average rates, say R .

To conclude, we have offered independent (but, admittedly, partly heuristic) methods of showing that the spectral density of the real part of the fundamental current source associated with a conductance G of a resonator at frequency ω_o is equal to $\hbar\omega_o |G|$, or, equivalently, that the spectral density of the real part of the fluctuation-rate equals the absolute value of the average rate, $|R|$.

In general the admittance $Y(\omega, n, R)$ of a circuit depends on frequency ω , on the number n of atoms or electrons in the conduction band, and on the emitted rate R of electromagnetic radiation. In the above discussion we have implicitly assumed that Y does not depend on R , that is, no spectral-hole burning occurs. In laser diodes, the number n of electrons in the conduction band is a monotonic function of the static electric potential U applied, provided some unessential internal resistance be neglected. It follows that Y does not depend on n if the applied potential U is held constant²⁶.

Under those conditions with Y a function of frequency only, the Nyquist-like noise source spectral density equals $\hbar\omega_o |G(\omega_o)|$. In writing this expression we have assumed

²⁶There are other circumstances where n is approximately constant: there may exist an equilibrium between the electron injection rate and the spontaneous decay rate. As long as the departure from this equilibrium condition, introduced by oscillation or amplification, remains small, we may assume that n is approximately constant. If this is the case, the admittance Y is a function of ω only and system is linear.

that the frequency ω remains close to some average oscillation frequency ω_o . In the simplest cases $dG/d\omega = 0$ at the oscillation frequency. But for some circuits this is not the case, as we discuss in a later section. Within the linear condition considered, the circuit response provides the optical field spectral density and oscillators linewidths. In the linear approximation sources and responses are normal (i.e., $V(t_1), V(t_2), \dots$ are jointly gaussian). The potential spectrum is simply the response of the linear circuit to white noise. The detector spectrum, viewed as resulting from the beat between narrow frequency bands, sometimes called the "noise-noise beat", is an auto-convolution of the optical spectrum. It follows that observation of the detector spectrum provides some information concerning the signal optical spectrum.

In a more general situation Y may depend on both ω and n . However, because the relative fluctuations of ω and n are small, we may take for the Nyquist-like noise source the expression $\hbar\omega_o |G(\omega_o, n_o)|$, where n_o represents the average electron number. Finally, we discuss below the phenomenon of loss or gain compression that occurs when the rates are so high that the condition of thermal equilibrium within bands of states may not be fulfilled. What we wish to establish is that the usual Nyquist-like expression holds under some conditions provided it is supposed to depend on the emitted *rate*. In contradistinction, the assumption that the conductance depends on the (ill-defined) photon number may lead to inaccurate results.

5.6 Gain compression

In the present section we seek an appropriate treatment of noise when non-linear effects are present, particularly those resulting from spectra-hole burning. We show that the Nyquist-like formula remains applicable when the conductance G depends on the absorbed or emitted *rate* R , but does not, even to first order, if G is considered a function of the cavity photon number. As a first step, we consider the dependence of conservative elements such as a capacitance C on the applied potential modulus $|V|$, the reason being that no noise source is associated with such elements. As a second step we consider circuits involving *linear* conductances endowed with the usual noise source and non-linear capacitances. We show that the Nyquist-like formula holds provided the terminals admittance is a frequency-independent (but R -dependent) conductance. We restrict ourselves to small non-linearities.

In previous sections it was assumed that the capacitances and inductances involved in the circuit considered are constant. In the present section we relax this assumption and consider a capacitance C , in particular, that depends on the modulus of the potential V applied to it. A physical model is a capacitance whose plates are maintained separated with the help of a spring. The two plates being oppositely charged they attract each other at all time, this attraction being balanced on the average by the spring force. As $|V|$ increases, the force of attraction increases and therefore the plate spacing decreases, leading to an increased value of the capacitance. It is assumed that the relative plate displacement is small so that $\Delta C \propto \Delta |V|$, and further that the plate inertia is such that the capacitance

variations occur on a time scale much larger than the optical period but much smaller than the Fourier frequency period considered. In the radio frequency range a solid-state device, called a "varactor", exhibits a capacitance that depends on the applied potential. In the microwave range of frequency the permittivity of a medium may be modified by an applied electric field that modifies the orientation of molecules. In the optical range of frequency other physical effects are at work that entail a dependence of the refractive index n (not to be confused with the number of electrons) on the modulus square of the applied field (Kerr effect). This relation is usually written as $n = n_o + n_2 I$, where I denotes the light intensity and n_2 represents the Kerr effect. It is not our intention to discuss further the mechanisms behind the dependence of capacitances in one form or another on the field intensity. At the high-intensity fields considered the concept of light "intensity", or power, makes sense.

The phenomenon we are interested in is the dependence of *conductances* G on the emitted or absorbed rate R . It is not obvious that the Nyquist-like formula for the noise current source given earlier still holds when G depends on R . We therefore employ an indirect method. We consider a circuit consisting of inductances, non-linear capacitances and (linear) conductances such that the admittance "seen" at the terminals is a non-linear conductance. Because conservative elements such as capacitances and inductances (whether linear or non-linear) do not generate fluctuations of their own, we are able to obtain Nyquist-like noise sources for non-linear conductances, at least to first order. The specific application we have in mind is the following. In some circumstances negative conductances $-G$ describing active media depend not only on the number n of electrons in the upper state (or in the conduction band for a semiconductor) but also on the rate R of photonic absorption, so that one should consider in general conductances of the form $G(n, R)$. At usual temperatures G depends strongly on n and not much on R , so that gain compression may be neglected. But at $T = 0\text{K}$, the dependence of G on n in semiconductors may be neglected and G is a function of R only. The non-linearity may in that case be expressed as $R = G(R) |V|^2 + r$, where V denotes the potential applied to the conductance, and r is a zero-mean fluctuation (note that the same R appears on both sides of this relation).

A specific model is considered in [35] which involves a 1/2-1/2 beam splitter, whose ports are labelled 1,2,3,4, respectively. There is a linear conductance on port 3 and an optical fiber with Kerr constant κ on port 1 with full reflection at its end, an optical fiber of equal length with Kerr constant $-\kappa$ on port 2 with full reflection at its end. Beam-splitter and fiber losses are neglected. Since the conductance in port 3 is linear, the usual Nyquist-like noise source is ascribed to it. But the full-device non-linearities lead to the following conclusions concerning the admittance "seen" looking into port 4:

- (1) If $\kappa = 0$ (no Kerr effect) the admittance is a pure conductance equal to G , independent of frequency to first order, that is, $dG(\omega)/d\omega = 0$.
- (2) When the light intensity increases the admittance remains a pure conductance whose value depends on the light intensity.
- (3) The output noise wave from port 4 coincides with the wave emitted by the linear

absorber in port 3.

This means that in the circumstances presently considered the Nyquist-like noise formula is applicable to conductances that exhibit a small dependence on the absorbed or emitted rate R .

All the quantities considered are supposed to be real for reasons of mathematical simplicity. For brevity we set $|V|^2 \equiv \mathcal{U}$. If G denoted the conductance at some reference light level, the rate $R = G\mathcal{U}$ at that level. We next consider small departures from that level. According to the previous discussion we postulate that the absorbed rate R is in general of the form

$$R = G(R)\mathcal{U} + r \quad \mathcal{S}_r = R. \quad (96)$$

Expansion up to first order in R of the above expression gives

$$\Delta R = \frac{dG}{dR} \Delta R \mathcal{U} + G \Delta \mathcal{U} + r \quad (97)$$

since for the average values $R = G\mathcal{U}$. When the photonic flow R into the material increases, the conductance G is expected to decrease. It is therefore appropriate to define the compression factor as

$$\kappa \equiv -\frac{R}{G} \frac{dG}{dR} \quad (98)$$

which is usually positive. With this notation (97) may be written as

$$\Delta R = \frac{R}{1 + \kappa} \frac{\Delta \mathcal{U}}{\mathcal{U}} + \frac{r}{1 + \kappa}. \quad (99)$$

In the field of semiconductor technology, most authors express instead the non-linearity (called "non-linear gain") in the form $R = G(m)m + r$, where m denotes the number of photons in the laser cavity, is supposed to be proportional to \mathcal{U}^{27} . That is, we set $R = G(\mathcal{U})\mathcal{U} + r_u$. To first order,

$$\Delta R = R(1 + \beta) \frac{\Delta \mathcal{U}}{\mathcal{U}} + r_u \quad \beta \equiv \frac{\mathcal{U}}{G} \frac{dG}{d\mathcal{U}}. \quad (100)$$

The first terms on the right-hand-side of (98) and (99) may be identified setting $1 + \beta \equiv 1/(1 + \kappa)$. However, the spectral density of the fluctuating rate r_u is equal to $R/(1 + \kappa)^2$. The spectral density of the fluctuation r_u in (99) is therefore not equal to the average rate

²⁷One may doubt that the concept of photon number be generally well-defined since most media incorporated in a laser cavity are lossy and dispersive. In such media it is difficult to decide which part of the total energy belongs to the field, and which part to matter. To the contrary, power, being the flux of the Poynting vector, is always well-defined in electromagnetism. Thus the concept of photon rate is well defined, at least classically, while the concept of photon number is not always well-defined, even classically.

R except, approximately, when κ or β are small compared with unity, but this needs not be the case in practice. Likewise we could suppose that the conductance depends on $\mathcal{I} \equiv |I|^2$, where I denotes the current flowing through the conductance. Similar conclusions would be reached, namely that the Nyquist noise formula is no longer applicable.

In the next two sections results based on the previously exposed principles are listed. As we discussed in the introduction, realistic models are often discussed in the literature, accounting for example for spontaneous decay from various electronic levels, optical feed-backs, non-uniform current distributions and the detailed properties of semiconductors involving heavy-holes and light-holes, Auger spontaneous recombinations, and so on. Furthermore, other laser properties besides the relative noise and linewidth of autonomous single-mode lasers may be considered. For example multi-mode operation, injection locking, large-amplitude modulations, instabilities.... Some of these effects occur in realistic lasers and a precise comparison between our theoretical results, based on idealized models, and experimental results may not be straightforward. Our purpose thus is not to describe accurately real-life lasers, but instead to proceed by steps, illustrating first simple configurations for the sake of conceptual clarity. The basic concepts lead to physically-significant but not necessarily practical results, in analytical forms. To proceed further, computers may be required either using symbolic algebraic calculus (see the multi-level atomic lasers section) or keeping track of the motion of individual electrons (see the small-size laser-diodes section).

5.7 Frequency noise

We have indicated above that below threshold (linear approximation) laser-oscillator linewidths simply derive from the linear system response. We are mostly concerned, however, with above-threshold lasers, in which case the relative amplitude fluctuations are small. Above threshold, one must consider the deviation $\Delta V(t) \equiv \Delta V'(t) + i\Delta V''(t)$ of optical potentials from their steady-state values $\langle V \rangle$, taken as real for simplicity. Because the fluctuations are small, we may write the phase as $\phi(t) = \Delta V''(t)/V$. Classically, the fluctuating frequency $\Delta\omega = d\phi(t)/dt$. The spectral density $\mathcal{S}_{\Delta\omega} = \Omega^2 \mathcal{S}_\phi$, and the (full-width at half-power) laser linewidth $\delta\omega = \mathcal{S}_{\Delta\omega}$, because $\Delta\omega(t)$ is a white gaussian process.

From the point-of-view of the present paper one should preferably consider specific experimental arrangements and their outcome in terms of average detector outputs. Frequency fluctuations can indeed be measured by introducing a de-tuned cavity before detection. This device acts as a "discriminator" that converts frequency fluctuations into measurable amplitude fluctuations. A better device consists of replacing the frequency-insensitive beam splitter of the dual detector arrangement by a de-tuned Fabry-Perot resonator. The difference between the two detector-output currents does not depend on amplitude fluctuations to first order.

6 Linear regime

In the present section we consider a few applications of the previously described principles to the linear regime. In that regime, the circuit considered involves only constant (positive and negative) conductances G , besides capacitances and inductances. That is, the conductances are independent of the potential applied to them and do not depend on any other parameter. It suffices in principle to evaluate the response of a linear system to current sources $C(t) \equiv C'(t) + iC''(t)$ of known spectral density. We suppose that the response spectrum is sharply peaked near some frequency ω_o , so that the spectral density density of the current sources may be written as $\mathcal{S}_{C'} = \mathcal{S}_{C''} = \hbar\omega_o |G|$, $\mathcal{S}_{C'C''} = 0$. The linear regime is usually applicable when laser-diodes are driven by constant electrical potentials, or when the oscillating or amplified signals are small.

We will first define what we mean when we say that optical beams are in the "C-state". Such optical beams are often called "constant-intensity" optical beams. We avoid this expression because, from our view-point, the concept of light "intensity" is not well-defined.

6.1 C-state

Light waves are said to be in the C-state²⁸ if they generate Poissonian photo-electrons irrespectively of the carrier phase²⁹. One can show that potential or current sources radiate light in the C-state.

Let us first recall well-known observations. The current emitted by a cathode whose emission is temperature-limited consists of independently emitted electrons. Mathematically, this electronic emission process is referred to as a Poisson process. Let the emitted current be denoted by $\mathcal{J} \equiv J + \Delta J$, where J denotes the time-averaged emitted current. The spectral density of the fluctuation ΔJ is given by the formulas

$$\mathcal{S}_{\Delta J} = eJ, \tag{101}$$

$$\mathcal{S}_{\Delta D} = D, \tag{102}$$

²⁸C-state beams resemble the so-called "coherent" states of light employed in quantum optics. However, C-states are fundamentally states of propagating light while coherent states are primarily states of optical resonators. In the context of Quantum Optics, Glauber has shown in 1963 that a classical prescribed current (which we call a current source) radiates light in the so-called "coherent state". When coherent states are incident on a photo-detector the statistics of the photo-electrons is Poissonian. The results therefore are similar.

²⁹Concretely, the carrier phase may be changed by inserting on the optical beam (i.e., before detection) a second-order all-pass filter, which is a conservative (i.e., lossless, gainless) resonating device that changes the carrier phase without changing its amplitude. We suppose that this circuit bandwidth is very small compared with the Fourier frequencies of interest. If this is the case, the carrier phase may be changed arbitrarily from 0 to 2π simply through a very small detuning. The fluctuations, on the other hand, are essentially unaffected by that all-pass filter. The description of second-order all-pass filters may be found in Circuit-Theory textbooks.

where e denotes the absolute value of an electron charge and double-sided spectral densities are employed. The letter "D" stands for "detection". The electronic rate is defined as $\mathcal{D} \equiv \mathcal{J}/e = D + \Delta D$, with $D = J/e$ and $\Delta D \equiv \Delta J/e$. Relation (101) says that the average power dissipated in a 1Ω resistance following a 1 Hz band-pass filter centered at any low frequency (white noise) is given by the shot-noise formula

$$2\mathcal{S}_{\Delta J} = 2eJ, \quad (103)$$

If a light source, for some reason, is supposed to emit photons independently each photon carrying an energy $\hbar\omega_o$, where \hbar denotes the Planck constant and ω_o the light frequency, we are dealing again with a Poisson process. The spectral density of the light-power fluctuation ΔP is thus

$$\mathcal{S}_{\Delta P} = \hbar\omega_o P, \quad (104)$$

$$\mathcal{S}_{\Delta Q} = Q, \quad (105)$$

where the average photonic rate is $Q \equiv P/\hbar\omega_o$ and the fluctuation is $\Delta Q \equiv \Delta P/\hbar\omega_o$. If the optical beam is incident on an ideal photo-detector, the light is converted into an electron rate identical to the photon rate. Thus, the electron rate fluctuation spectral density $\mathcal{S}_{\Delta D} = D = Q$.

In the present theory complex random current sources $C(t) \equiv C'(t) + iC''(t)$ are associated with any conductance G . As recalled above, the real and imaginary parts of C' and C'' of C , respectively, are uncorrelated and have spectral density

$$\mathcal{S}_{C'} = \mathcal{S}_{C''} = \hbar\omega_o |G|, \quad (106)$$

where vertical bars denote absolute values. Let us show that this formalism agrees with the previously-quoted shot-noise formulas. Potential and current sources emit light in the C-state, that is, the photo-electrons generated in a (cold, ideal) photo-detector submitted to light radiated by potential or current sources are Poisson distributed, irrespectively of the light carrier phase.

Consider indeed a potential source V applied to a conductance G . the current delivered by the source consists of two parts. First the current GV flowing through the conductance, and secondly the noise current $C(t)$. It follows that the power delivered by the potential source (and received by the detector according to the law of average-energy conservation) reads

$$P(t) = \Re\{V^*(GV + C)\} = G|V|^2 + V'C' + V''C'', \quad (107)$$

The first term is the average power $P = G|V|^2$. The second terms are the fluctuating terms. It follows that the spectral density of the fluctuation reads

$$\mathcal{S}_{\Delta P} = V'^2 \mathcal{S}_{C'} + V''^2 \mathcal{S}_{C''} = G|V|^2 \hbar\omega_o = P\hbar\omega_o. \quad (108)$$

Thus, setting $D = P/\hbar\omega_o$, $\Delta D = \Delta P/\hbar\omega_o$, we recover the relation $\mathcal{S}_{\Delta D} = D$. The photo-current is Poissonian, irrespectively of the phase of V . A similar calculation can be made for current sources. Thus we have shown that potential and current sources radiate light in the C-state.

6.2 Balanced detection

The relative noise \mathcal{N} can in principle be measured with a single (ideal) detector. If the average photo-electron rate is measured to be D , a measurement of the rate spectral density $\mathcal{S} \equiv \mathcal{S}_{\Delta D}$ gives \mathcal{N} from its very definition $\mathcal{N} \equiv (\mathcal{S}/D - 1)/D$. The balance-detection scheme discussed below allows one to measure \mathcal{N} -values close to unity more accurately than it would be possible with a single detector. The measuring apparatus involves a 50%-50% loss-less beam splitter with outputs labeled 1 and 2, respectively. Under ideal conditions, the relative noise \mathcal{N} is

$$\mathcal{N} = \frac{\mathcal{S}_{\Delta D_1 \Delta D_2}}{D_1 D_2}. \quad (109)$$

That is, the detector outputs are uncorrelated when light is in the C-state ($\mathcal{N} = 0$), and negatively correlated for quiet light ($\mathcal{N} < 0$).

6.3 Splitting light into many beams

It is often the case that light is attenuated before detection. The light beam perhaps expands as a result of diffraction and only a fraction of the light beam is collected by the detector. Alternatively there is perhaps a piece of (cold, linear, non-dispersive and reflection-less) absorbing matter located between the laser and the detector. More generally we consider a light beam split into N beams. This can be achieved by connecting N conductances in parallel. As was indicated in Section 3.1, the relative noise is unaffected.

Let the total absorbed rate \mathcal{Q} split into N absorbed rates \mathcal{D}_k , with $k = 1, 2, \dots, N$. We obtain the cross-spectral density between the rate at absorber (detector) k and the rate at absorber (detector) l

$$\mathcal{S}_{\Delta D_k \Delta D_l} = \delta_{kl} D_k + D_k D_l \mathcal{N} \quad (110)$$

where δ_{kl} equal 1 if $k = l$ and 0 otherwise. It follows from this expression that if the incident beam is Poissonian ($\mathcal{N} = 0$) distinct secondary beams are uncorrelated. In Optics, the configuration presently discussed is more often modeled by a series of beam splitters, with matched cold loads at the unused ports ("vacuum").

Let us recall the photon-model explanation of this effect. Light at frequency ω_o is supposed to consist of tiny particles called photons, each carrying an energy $\hbar\omega_o$. Attenuation of light is supposed to imply that photons are randomly deleted. For the case of an ideal

detector each photon generates a photo-electron. It follows that the photo-electron statistics is "thinned", and as recalled in earlier sections, thinning does not affect the normalized correlation function or relative noise. It is then permissible to suppose that the incident light consists of a stream of tiny particles called "photons" incident on the beam splitters. These photons are independent and are ascribed some probability of being transmitted or reflected. The statistics of the output light beams obtained in that manner are presently accurate. But this naive photon picture fails if we decide, perhaps years later, to let the exiting light beams interfere.

6.4 Linear attenuators

Previous results relating to the noise properties of cold linear attenuators may be expressed in terms of propagating waves. Let us recall that the Ohm law reads $I = Y(\omega)V$, where I denotes the electrical current, V the electrical potential at optical frequency ω , and $Y(\omega)$ the circuit admittance. It is sometimes convenient to describe circuits in terms of forward-propagating waves of complex amplitude a and backward-propagating waves of amplitude b instead of voltages V and currents I , see Fig. 10 in a). Taking the transmission line characteristic conductance as unity for simplicity these quantities are related by

$$a = \frac{V + I}{2} \quad b = \frac{V - I}{2}, \quad (111)$$

$$V = a + b \quad I = a - b. \quad (112)$$

In this formalism, $|a|^2$ and $|b|^2$ represent respectively the forward and backward propagating powers or, setting for convenience $\hbar\omega_o = 1$, forward and backward photonic flows. The difference $|a|^2 - |b|^2 = (V^*I + I^*V)/2 = \Re\{I^*V\}$ represents the power dissipated in the load admittance Y .

We will suppose that forward-propagating and backward-propagating waves are separated physically from one another with the help of the circulator shown in Fig. 2, see Fig. 10 in b). It is then appropriate to call the a -wave the input wave and the b -wave the output wave. $Q \equiv |a|^2$ represents the input rate. If the output wave is detected by an ideal cold detector $D \equiv |b|^2$ represents the photo-electron rate.

Let us first suppose that the transmission-line load is a cold positive conductance, that is, $Y = G$ with $0 \leq G < \infty$. The noise-source $C \equiv C'$ spectral density equals G . For simplicity the quantities introduced in the present section are supposed to be real. The ratio of the average output to input power is therefore (the letter \mathcal{G} stands for "gain" even though in the present situation this quantity is less than unity and in fact expresses a loss)

$$\mathcal{G} \equiv \mathfrak{g}^2 = \left(\frac{1 - G}{1 + G}\right)^2 \leq 1. \quad (113)$$

Consider now a small variation Δa of a from its average value $\langle a \rangle$, corresponding to an input rate fluctuation $\Delta Q = 2a\Delta a$ (when no confusion may arise we omit the brackets

representing average values). The output b -wave fluctuation reads

$$\Delta b = \frac{1-G}{1+G}\Delta a + \frac{C}{1+G}. \quad (114)$$

If, as supposed above, the reflected b -wave is incident on a cold detector delivering an electron rate $\mathcal{D} \equiv D + \Delta D = (\langle b \rangle + \Delta b)^2 \approx b^2 + 2b\Delta b$. The relative-intensity noise \mathcal{N} reads

$$\mathcal{N} \equiv S_{\Delta D/D} - \frac{1}{D} = S_{\Delta Q/Q} - \frac{1}{Q} \equiv \mathcal{N}_{in}. \quad (115)$$

We thus observe once more that cold linear attenuators preserve the relative-intensity noise, that is, $\mathcal{N} = \mathcal{N}_{in}$.

Generalizing (114) to the complex form we obtain, setting $2\Delta a \equiv x_{in} + iy_{in}$, $2\Delta b \equiv x_{out} + iy_{out}$ and $2C/(1+G) \equiv u' + iu''$, we may write

$$x_{out} = \mathfrak{g}x_{in} + u', \quad (116)$$

$$y_{out} = \mathfrak{g}y_{in} + u'', \quad (117)$$

$$S_{u'u''} = 0 \quad S_{u'} = S_{u''} = 1 - \mathcal{G} \geq 0, \quad (118)$$

Let X, Y denote the spectral densities of the x, y -noise terms, respectively, with subscripts "in" and "out" appended where needed. Since the internal u noise and the input noise are independent of one another, the input and output noise spectral densities are related by

$$X_{out} = \mathcal{G}X_{in} + 1 - \mathcal{G}, \quad (119)$$

$$Y_{out} = \mathcal{G}Y_{in} + 1 - \mathcal{G}. \quad (120)$$

We say that a light beam is in the C-state whenever $X = Y = 1$. It follows that when a C-state beam ($X_{in} = Y_{in} = 1$) suffers from any cold attenuation it remains in the C-state, i.e., $X_{out} = Y_{out} = 1$.

The conductance G receives a power $P = \Re\{I^*V\} = a^*a - b^*b$. If this conductance describe an ideal cold photo-detector, by conservation of energy P equals the photo-detection rate J and ΔP equals ΔJ . In terms of the x -fluctuations we have therefore

$$\Delta J = \Delta(a^*a - b^*b) \quad (121)$$

$$= ax_{in} - bx_{out} \quad (122)$$

$$= a(x_{in} - \mathfrak{g}x_{out}) \quad (123)$$

where we have supposed $\langle a \rangle, \langle b \rangle$ real for simplicity. The noise source associated with G enters in the expression of ΔJ if x_{out} is expressed in terms of x_{in} .

6.5 Linear amplifiers

We considered above a collection of 2-level atoms, all of them residing most of the time in the lower state, which we called a cold absorber (or cold detector) and described it in terms of a positive conductance G . We now consider a collection of 2-level atoms that reside most of the time in the upper state, and call it an amplifying medium. Full population inversion is assumed for simplicity, and spontaneous electronic decay from upper to lower states is neglected. Such an atomic collection is characterized by a constant negative conductance denoted $-G$. The conductance $-G$ terminates a transmission-line whose characteristic conductance is unity. We require that $0 < G < 1$, the restriction $G < 1$ being needed to avoid a singularity.

As in the previous sub-section, the input a -wave and the amplified b -wave are supposed to be separated from one another with the help of a circulator, as shown in Fig. 10 in b). Because in-phase as well as quadrature fluctuations are equally amplified in the present set-up the system is called a "phase-insensitive" amplifier. The power gain \mathcal{G} is given by

$$\mathcal{G} = \mathfrak{g}^2 = \left(\frac{1+G}{1-G} \right)^2 \quad 1 \leq \mathcal{G} < \infty. \quad (124)$$

Using the notation $x_{in} + iy_{in} \equiv 2\Delta a$, $x_{out} + iy_{out} \equiv 2\Delta b$, where Δa , Δb denote the input and output wave fluctuations, denoting by X, Y the spectral densities of the x, y -noise terms, respectively, with appropriate subscripts "in" and "out", and remembering that the internal noise and the input noise are independent, the above expressions show that the input and output noise spectral densities are related by

$$X_{out} = \mathcal{G}X_{in} + \mathcal{G} - 1, \quad (125)$$

$$Y_{out} = \mathcal{G}Y_{in} + \mathcal{G} - 1. \quad (126)$$

For a C-state input beam in particular ($X_{in} = Y_{in} = 1$) the spectral densities of the in-phase and quadrature fluctuations are both equal to $X_{out} = Y_{out} = 2\mathcal{G} - 1$.

If two amplifiers of gains \mathcal{G}_1 and \mathcal{G}_2 respectively are placed in sequence, it is straightforward to show on the basis of (125) and (126) that the output noise is the same as for a single amplifier of gain $\mathcal{G} = \mathcal{G}_1\mathcal{G}_2$ as one expects, a result that generalizes to any sequence of (linear) amplifiers. The case where the active medium inversion is incomplete may be treated by the same method.

6.6 The Schawlow-Townes linewidth formula.

The oscillator model may be employed to obtain the linewidth of below-threshold (but highly resonant) lasers, for a given power P transferred from the laser to the detector.

Consider a laser consisting of a capacitance C and inductance L and a conductance $-G_e$ in parallel, where G_e represents the laser positive action. Full population inversion in the laser material is assumed. This circuit is connected to a positive conductance G_a

describing the detector. We suppose that $G = G_a - G_e$ is positive but small, so that the laser-detector assembly is stable but highly resonant.

The expression of the FWHP (full width at half power) $\delta\omega$ was given in (35) as $\delta\omega = G/C$. We now wish to evaluate the power P delivered by the laser, or equivalently the power absorbed by the conductance G_a . The calculation is similar to the one made earlier for calculating the total energy $\langle E \rangle$. We obtain the celebrated Schawlow-Townes (ST) linewidth formula

$$\delta\omega P = \frac{\hbar\omega_o}{\tau_p^2}, \quad (127)$$

where $\tau_p = C/G_a$ is the photon lifetime, that is, the average time that a photon would spent in the circuit if no laser action were taken place (i.e., if $G_e = 0$). This formula was established by Schawlow and Townes in a somewhat different physical context. These authors consider a maser whose atomic linewidth is much narrower than the cavity linewidth, while the opposite assumption is made here. It should also be pointed out that the ST formula is often confused with the line-width formula applicable to well-above threshold lasers that rests on entirely different physical principles (For simple configurations, though, the above-threshold line-width is just half the below-threshold line-width given above, the power P transferred from the laser to the detector being the same.).

To summarize, we considered in the present section the case where the circuit elements (capacitances, inductances and conductances) are constant, so that the potential between any two terminals is a linear function of the current sources. The isolated system is supposed to have only one nearly real resonant frequency ω_o , the other resonance being strongly damped. We have shown that it is appropriate to ascribe to each conductance G a random complex current source $C(t) = C'(t) + iC''(t)$ where the random functions $C'(t)$, $C''(t)$ are uncorrelated and have each spectral density $\hbar\omega_o |G|$. The current sources for two different conductances are also uncorrelated. If the potential between the terminals of a conductance G is V and the power is $G|V|^2$, so that, by the law of power conservation, we may evaluate both the average value and the spectral density of the electrical power delivered by the laser and of the electrical power received by detectors. Note that the fluctuations of V are of the same order as the average value. This is the reason why, in the linear regime, the noise sources can be neglected in writing the power-conservation law.

6.7 Linear oscillators with incomplete population inversion

We have represented in Fig. 5 in c) a simple laser model consisting of frequency-independent negative conductance $-G_e$ and a positive conductance $G_o > G_e$ representing the detector, in parallel with a $L - C$ -circuit resonating at frequency ω_o . Complete population inversion was assumed. The noise sources associated with these conductances led us to the celebrated Shawlow-Townes (ST) linewidth formula, which we reproduce below from (127) for the

reader convenience. The spectrum is Lorentzian with full-width at half-power (FWHP)

$$\delta\omega_{ST}P = \frac{\hbar\omega_o}{\tau_p^2}, \quad (128)$$

where $\tau_p = C/G_o$ is the photon lifetime, that is, the average time that a photon would spent in the circuit if no laser action were taken place, i.e., if $G_e = 0$. P is the power dissipated in the detector G_o .

In the above expression, (128), it was assumed that the conductance G_o is at temperature $T = 0\text{K}$. It was further assumed that the population inversion in the laser material is complete. We now relax the latter assumption. If the population inversion is incomplete the laser material contains electrons in the lower state and a greater number of electrons in the upper state. The laser material is represented by a negative conductance $-G_e$ in parallel with a positive conductance G_a . We define the conductance-inversion factor

$$\eta = \frac{G_e + G_a}{G_e - G_a} \approx \frac{n_e + n_a}{n_e - n_a} > 1. \quad (129)$$

The second expression holds for isolated atoms, the conductances being in that case proportional to the number of atoms. In parallel with the laser conductances there is a load conductance that we denoted G_o , and we have $G_o \approx G_e - G_a$ that is $G_a + G_o \approx G_e$, so that the circuit is again highly resonant. The linewidth for the circuit presently considered is the same as before if we replace G_a in the previous expression by $G_a + G_o$. However, what we now call radiated power P is the power dissipated in G_o *alone*, not in $G_a + G_o$. Furthermore the photon lifetime τ_p obtained by suppressing the laser material, that is both G_a and G_e being set equal to 0, is now defined as $\tau_p = C/G_o$. The end result of the calculation is that the ST linewidth in (128) should be multiplied by a population-inversion factor

$$n_p = \frac{1 + \eta}{2} = \frac{G_e}{G_e - G_a}. \quad (130)$$

This n_p factor is often denoted by n_{sp} and called "spontaneous-emission factor". This latter denomination is inappropriate since spontaneous emission is presently neglected.

6.8 Dispersive linear oscillators

We consider again linear oscillators, the optical potentials and currents being simply the responses of a linear circuit to current sources associated with conductances. For simplicity, we maintain that the active laser material is represented by a frequency-independent conductance $-G_e$ and assume complete population inversion. We first consider a simple $L - C - R$ oscillator, and subsequently give a more general line-width formula. Finally we consider specifically the so-called "gain-guided" laser diodes.

The circuit consists of a capacitance C and an inductance L in *series* with a cold positive resistance R_o , as represented in Fig. 5 in d). Let $Y(\omega)$ represent the admittance of the circuit

$$\begin{aligned} Y(\omega) &\equiv G(\omega) + iB(\omega) = -G_e - iC\omega + \frac{1}{R_o - iL\omega} \\ &= \frac{R_o}{R_o^2 + L^2\omega^2} - G_e + i\left(\frac{L\omega}{R_o^2 + L^2\omega^2} - C\omega\right) \end{aligned} \quad (131)$$

The resonant frequency ω_o corresponds to a vanishing susceptance $B(\omega_o) = 0$, that is $\omega_o = \sqrt{1/(LC) - (R_o/L)^2}$, which implies that $R_o < \sqrt{L/C}$. We also assume that the circuit is only very slightly damped, which implies that G_e is almost equal to $R_o C/L$. At the resonant frequency, the positive conductance is frequency-dependent. Derivating the conductances and susceptances given above with respect to ω or using (38) we may evaluate the h parameter defined as

$$h \equiv \left(\frac{dG/d\omega}{dB/d\omega}\right)_{\omega=\omega_o}. \quad (132)$$

The non-zero value of the h factor may be viewed as a consequence of the fact that gain and loss do not occur at the same location. That is, if the gain and loss are represented by two conductances in parallel the h -factor vanishes. But in the present situation the conductance $-G_e$ and the resistance R_o are separated from one another in the circuit by the inductance L and the h -factor is non-zero.

When the negative conductance is suppressed, the resonance frequency acquires a negative imaginary part that may be employed to define the photon lifetime

$$\tau_p \equiv \frac{dB/d\omega}{2G_e}. \quad (133)$$

In the case of a classical L - C -circuit, we have $dB/d\omega = 2C$, the inductance and capacitance contributing equally, and the previous formula $\tau_p \equiv C/G_e$ is recovered from (133).

Detailed calculations show that the h factor defined above results in a linewidth-enhancement factor $K = 1 + h^2$ with respect to the ST result. For the C, L, R_o circuit presently considered we obtain [36]

$$K = \frac{1}{1 - R_o^2 C/L}. \quad (134)$$

According to (134), the K -factor is unity when $R_o \ll \sqrt{L/C}$ but tends to infinity when R_o approaches $\sqrt{L/C}$.

To conclude, simple oscillators are subjected to linewidth enhancement. Similar conclusions concerning various linear circuits have been reached. A related linewidth-enhancement

factor was discovered by Petermann in relation to the so-called "gain-guiding" lasers. Using an appropriate simplified schematics one can show that the above linewidth-enhancement effect is the same as the one given by Petermann. Since linewidth enhancement occurs for single-mode resonators, as we have just seen, this effect does not seem to be fundamentally related to mode non-orthogonality as other treatments have suggested.

6.9 Cavity linear oscillators

An expression for the linewidth can be obtained for an arbitrary cavity containing (linear, time-independent) dispersive dielectrics $\epsilon(\omega)$ (or even more generally bi-anisotropic media) in terms of the resonating fields. Let us consider a cavity with perfectly conducting walls containing a medium with permittivity $\epsilon(x, \omega) \equiv \epsilon'(x, \omega) + i\epsilon''(x, \omega)$, where x stands for the three coordinates x, y, z . The imaginary part of the permittivity may be positive or negative. For simplicity, we assume complete population inversion, that is, for positive ϵ'' all the electrons are in the lower state (zero temperature) and for negative ϵ'' all the electrons are in the upper state. We set $\epsilon'' \equiv \epsilon''_a - \epsilon''_e$, where ϵ''_a expresses the loss and ϵ''_e expresses the gain. Its spatial distribution is such that there is one mode whose frequency is only slightly damped, the other modes, being strongly damped, can be ignored. Under those conditions the power generated by the gain medium is equal to the power absorbed by the loss medium. Furthermore, the resonating mode frequency ω_o is nearly real in the limit considered. For simplicity we let the medium permeability be a real constant μ . The product of the (full-width at half power) linewidth $\delta\omega$ and power P dissipated in the positive ϵ'' -part of the medium (or generated by the negative ϵ'' -part of the medium), reads

$$\delta\omega P = \frac{4\left(\int dx \omega \epsilon''_e(\omega) |E|^2\right)^2}{\left|\int dx \left[\frac{d(\omega\epsilon(\omega))}{d\omega} E^2 - \mu H^2\right]\right|^2}. \quad (135)$$

where E, H denotes the (complex) resonating electrical and magnetic fields. All the quantities are evaluated at the real frequency ω_o . Here dx stands for $dx dy dz$ and the integrals are over the full cavity volume.

Usually, the power delivered by the active medium is not dissipated internally as we supposed above. Instead, part of the cavity wall transmits power to some external detector. Formally, we may in that case apply the previous expression to a large cavity enclosing both the part of the medium considered as being the laser and the part of the medium considered as the detector. One expects intuitively that the linewidth measured by some cold reflexion-less detector does not depend on the distance between the laser and the detector, the laser being separated from the detector by a loss-less, dispersion-less transmission line. Let us show that this is indeed the case. Changing the laser-detector distance clearly does not affect the numerator of (135) since ϵ in the intermediate region is real. The denominator is not affected either because ϵ does not depend on ω in the intermediate region, and thus $d(\omega\epsilon(\omega))/d\omega = \epsilon$. For a matched transmission line we have $\epsilon E^2 + \mu H^2 = 0$. It follows

that the additional term in the denominator of (135) corresponding to the transmission-line volume, that is, the space comprised between the laser and the detector, does not contribute to the above expression.

6.10 Propagating wave oscillators with gain.

In the present section we consider linear propagating-wave oscillators. That is, we consider a dielectric wave-guide exhibiting gain and terminated at planes $z = 0$ and $z = L$ by partially-transmitting mirrors. When a medium is uniform in the propagation direction z , with permittivity $\epsilon(x)$, and a source at some fixed frequency ω , the medium may be characterized by the "free wave-number" $k(x)$ with $k(x)^2 = \epsilon(\omega)\mu\omega^2$. Because ω is a constant this argument is omitted. For simplicity we take the permeability μ as being a real constant. Because the medium is z -invariant we may define "transverse modes" as solutions of the wave equation of the form $\psi(x)\exp(ik_z z)$, where $\psi(x)$ may represent the electric field within the scalar or weakly-guiding approximation, and the propagation constant k_z may be complex valued. We suppose that the real part of k_z is positive and that accordingly the wave propagates in the positive z -direction. If the imaginary part of k_z is positive the wave amplitude decays as it propagates, and conversely the wave amplitude grows if the imaginary part of k_z is negative. The function $\psi(x)$ is some complex function of x .

If the permittivity is real (that is, the medium is loss-less, gain-less) and decreases as $|x|$ increases, there exist a number of solutions (trapped modes) corresponding to real k_z -values and real $\psi(x)$ -functions decaying exponentially in the outer medium. In that case the wave is said to be "index-guided". Some higher-order modes, however, may be "leaky", in which case the wave amplitude decays along the z -axis while $\psi(x)$ grows exponentially as a function of $|x|$ and is no longer real³⁰. As a matter of fact, when the permittivity *increases* as a function of $|x|$, only leaky-modes can be found. Consider a leaky mode and suppose that the medium has gain for $|x| < d/2$, but is loss-less, gain-less, for $|x| > d/2$. Intuitively, we feel that the inner-medium gain is, figuratively speaking, fighting against the leaky-mode loss, and that, provided the gain is large enough, the wave amplitude may grow along the z -axis, instead of decaying. There is indeed a threshold gain when this does occur. What has not always be recognized is that the wave then ceases to be a leaky wave, that is, $\psi(x)$ *decays* exponentially at large $|x|$ values. There exists therefore no basic difference between index-guided and gain-guided configurations. In both cases, as long as there is a net gain along the z -axis, the field decays exponentially in the outer medium and the modes may be viewed as being "trapped". The wave may be perfectly matched to an incident beam at plane $z = 0$, and perfectly matched to an outgoing beam at the

³⁰To avoid a confusion let us note that by introducing fictitious planes far-away from the guiding structure, where the field is required to vanish, we obtain the so-called "radiation modes". These modes are needed from a mathematical stand-point to expand the actual field into a complete set of functions. They are related to the method of "box quantization".

output plane $z = L$. In such mode-matching conditions the power delivered by the active medium is simply the difference between the outgoing and ingoing powers. However the $\psi(x)$ function is complex and describes diverging wave-fronts. As a consequence, there is a K -factor larger than unity in both cases, index and gain-guided configurations. The K -factor is likely to be much greater in the case of an inverted real-index profile, though, than in the case of the usual guiding real-index profile.

7 Linearized regime

We give first in this section the relative noise of idealized laser diodes at high power levels. At high power the time derivative of the number n of electrons in the conduction band may be neglected (i.e., $dn/dt = 0$). It follows from that assumption that the net rate of photons entering into the cavity is equal to the pumping rate. But the detected rate is different because photons may be stored for a while in the cavity before exiting. In that limiting situation no relaxation-oscillation appear. Spontaneous emission and gain compression are neglected. The more general theory will be given later on. Reference to a circuit simulating the lasers Fourier-frequency response is cited.

7.1 Rate equations at high power

Lasers are open systems with a source of energy called the pump, and a sink of energy presently viewed as an ideal optical detector. It is natural to suppose that the probabilities of atomic decay or atomic promotion that were found earlier consistent with the laws of statistical mechanics, still hold when there is a supply of atoms in the upper state (the pump), and an absorber of light power (the detector) of conductance G_o . We thus use the expressions derived in Section 5.5, supplemented by a quiet-pump term $\mathcal{J} = J$ and a detector-absorption term $\mathcal{Q}(t)$ as follows

$$\frac{dm(t)}{dt} = J - \mathcal{Q}(t) \quad \mathcal{Q}(t) = \frac{m(t)}{\tau_p} + q(t) \quad \mathcal{S}_q = \langle \mathcal{Q} \rangle = Q, \quad (136)$$

where m denotes the number of photons in the $L - C$ cavity ($LC\omega_o^2 = 1$) and $\tau_p = C/G_o$ denotes the photon lifetime. For simplicity we set the absolute value of the electron charge $e = 1$ and the photon energy $\hbar\omega_o = 1$, so that the electrical current J coincides with the electron injection rate, and the output optical power coincides with the photon rate \mathcal{Q} , incident on the ideal photo-detector, whose output rate $\mathcal{D} = \mathcal{Q}$. We omit averaging signs when no confusion is likely to arise. As far as average values are concerned we have $Q = J$, the prescribed pump rate, a relation that follows from the fact that no loss is considered.

Considering small deviations from steady-state (or average) values, denoted by Δ , and replacing d/dt by $j\Omega$, we obtain

$$j\Omega\Delta m = -\frac{\Delta m}{\tau_p} - q \quad \mathcal{S}_q = Q. \quad (137)$$

It follows from the above expression that

$$(j\Omega\tau_p + 1) \frac{\Delta m}{\tau_p} = -q \quad (138)$$

$$\Delta Q = \frac{\Delta m}{\tau_p} + q = q \frac{j\Omega^\circ}{1 + j\Omega^\circ} \quad \text{with } \Omega^\circ \equiv \Omega\tau_p \quad (139)$$

$$\mathcal{S}_{\Delta Q} = \frac{\Omega^{\circ 2}}{1 + \Omega^{\circ 2}} Q. \quad (140)$$

Since the photo-current spectrum is equal to that of ΔQ , we finally obtain

$$\mathcal{S}_{\Delta D} = \frac{\Omega^{\circ 2}}{1 + \Omega^{\circ 2}} D. \quad (141)$$

We observe that the photo-current is quiet ($\mathcal{S}_{\Delta D} = 0$) at small Fourier frequencies, and reach the shot-noise level ($\mathcal{S}_{\Delta D} = D$) at high Fourier frequencies.

7.2 Fourier-frequency model

Since, from our view-point, only photo-currents are measurable, it is appropriate to provide a circuit at the low Fourier frequencies Ω that gives the same result as the actual laser-detector configuration. A simple circuit is given in [37] which accounts for gain compression. The laser part is represented by a negative resistance and the detector part by a positive resistance. Such a circuit has been constructed and tested (the negative resistance has been realized with the help of an operational amplifier).

7.3 Laser rate equations

The present section provides a more general form of the results given in Section 7.1. In particular, dn/dt is no longer neglected. But pump fluctuations, spontaneous decay and gain compression are still neglected. The detector is supposed to be ideal and at zero temperature. Strong relaxation peaks now appear at low or moderate output powers. The previous result is recovered in the large-output-power limit.

Relative noise is evaluated with the help of the rate equations given in Sections 7.4 and 5.5. Recall that we associate to any rate of average value R a fluctuating rate $r(t)$ whose spectral density equals R , rates of different origins being uncorrelated. Atomic collections containing electrons in both the lower and upper states are represented respectively by a positive conductance G_a and a negative conductance $-G_e$, considered as independent. The fluctuating rates, say r_a and r_e , associated with these two conductances being independent their spectral densities add up. The detector is represented by a conductance G_o .

We consider an ideal single-mode $L - C$ cavity resonating at frequency ω_o containing N two-level atoms, $n(t)$ of them being in the upper state. That is, n electrons are in the upper state of energy $\hbar\omega_o \equiv 1$ and $N - n$ are in the lower state of zero energy. In the

present situation there is a clear separation between matter energy and field energy so that the photon-concept may be meaningfully employed. The number n of electrons in the upper state equals the matter energy while m represents the field energy. If the cavity were an isolated system we would have $m + n = \text{constant}$. This is not presently the case because of (prescribed) pumping and (linear) detection. A time unit is selected such that stimulated emission or absorption rates are equal (rather than just proportional to) the photon number m .

When the equations are linearized and Fourier transformed, one obtains the detection rate fluctuation $\Delta D = \Delta Q = \Delta m / \tau_p + q$, where $\tau_p = C / G_o$. The relative noise $\mathcal{N}(\Omega)$ is given by

$$Q\mathcal{N}(\Omega) = \frac{\mathcal{S}_{\Delta Q}(\Omega)}{Q} - 1 = \frac{(N^\circ + 1) \left(\frac{\Omega^\circ}{2m^\circ}\right)^2 - 1}{\Omega^{\circ 2} + \left(1 - \frac{\Omega^{\circ 2}}{2m^\circ}\right)^2}. \quad (142)$$

where

$$N^\circ \equiv N\tau_p \quad \Omega^\circ \equiv \Omega\tau_p \quad m^\circ \equiv \langle m \rangle \tau_p. \quad (143)$$

and $D = Q = J = \langle m \rangle / \tau_p$. Thus, given the injected current J , the photon life-time τ_p and the total number N of atoms in the cavity, we obtain the relative noise \mathcal{N} as a function of the Fourier frequency Ω .

The results in (142) is exemplified in Fig. 6 for $N^\circ = 5$ and $m^\circ = N^\circ / 2$ (plain line). The spectral density of the photo-detection process again goes to zero at zero Fourier frequency. Note also that, for the parameter values considered, a small relaxation oscillation peak appears. In the large optical power limit the above expression reduces to

$$Q\mathcal{N}(\Omega) = -\frac{1}{(\Omega\tau_p)^2 + 1}. \quad (144)$$

Fig. 6 also shows (dashed line) the result that would be obtained if the correlation between q and Δm were ignored. In that case the photo-current spectral density does not vanish at $\Omega = 0$.

7.4 Above-threshold lasers from the circuit-theory view-point

In the linearized regime capacitances and inductances remain constant but conductances are allowed to depend on a parameter (e.g., the number of electrons, n , in the conduction band for semi-conductors). This number, which may depend on time, is written as $n(t)$. We consider variations of n which, though small in relative value, are significant. They are responsible for the fact that the intensity of the light emitted by lasers does not fluctuate much.

The starting point is the steady state corresponding to potentials $\langle V \rangle$ and currents $\langle I \rangle$. Ignoring all noise sources, the system is supposed to oscillate at some real frequency ω_o .

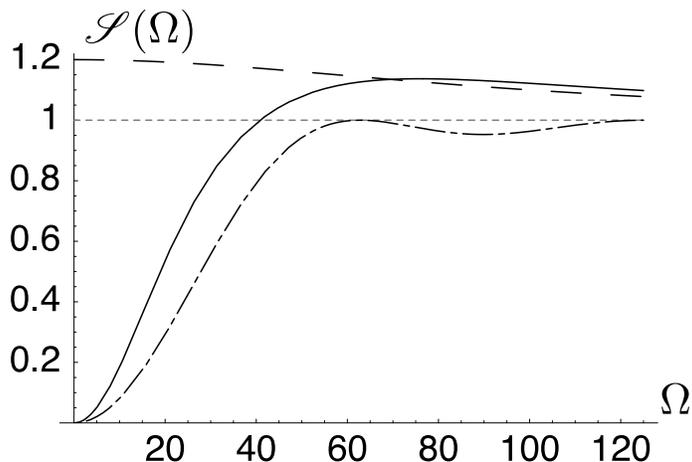


Figure 6: The plain line represents the laser normalized spectrum, (143). The dashed line would hold if there were no correlation between the output light and the photon number. The dashed-dot line corresponds to the dark room picture in section 3.8 for $\tau_r = 2\tau_p = 0.1$. The gray dotted line figures the Poissonian level.

Next, we consider small deviations from steady-state values, that is, the optical potential and current are written as the sum of average values and small fluctuations $\Delta V, \Delta I$, respectively. The latter express the response of the linearized circuit to the Nyquist-like current-sources $C(t)$ associated with conductances. It is essential to include the current sources in the expression of the law of energy conservation.

More specifically, consider n electrons that may reside in either an upper or a lower level, with an energy difference $E_e - E_a = \hbar\omega_o$, submitted to a potential source $u(t) = U + v(t)$, where U denotes a static potential and $v(t) \ll U$ varies at the optical frequency ω_o and is described by a potential V . The (quantum mechanical) average induced current $i(t)$ consists of a slowly varying part $J(t) = J + \Delta J(t)$ (photo-current or current driving a laser diode) plus a current varying at the optical frequency ω . The latter current is represented by the complex function of time $I(t) = G(n(t))V(t) + C(t)$, where the conductance $G(n)$ is positive for absorbers and negative for emitters (in the latter case we find it convenient to denote it by $-G$). For atoms that are coupled to one another only through the applied optical field, $G(n)$ is proportional to n . Statistical Mechanics told us that the real and imaginary parts, $C'(t)$ and $C''(t)$ respectively, of $C(t)$ are uncorrelated and have spectral densities $\hbar\omega_o |G|$, where vertical bars denote absolute values.

For slow variations, conservation of energy entails, in a simplified form, that the power $\langle u(t)i(t) \rangle \approx UJ(t) + V[G(n)V + C'(t)] = 0$. This relation provides the average value J and the fluctuations $\Delta J(t)$ about this average value, called "noise". To avoid a confusion, let us emphasize that $J(t)$ is a slowly-varying electrical current, involving frequencies $\Omega/2\pi$ of the order of, say, 1 MHz, while the slow variations of the complex function $C(t)$ refer

to fluctuations, in amplitude and phase, of the *envelope* of an optical signal at a carrier frequency of, say, 300 THz.

Because the system considered is stationary, the electronic energy does not vary much in the course of time. Therefore the electrical *power* delivered by the potential source to the electron vanishes. We have

$$UJ(t) + \Re\{V^*(I + C(t))\} = 0, \quad (145)$$

If we take V as real for convenience, the above relation reduces to

$$UJ(t) + V(I' + C'(t)) = 0. \quad (146)$$

If $I = GV$ (linear resonant operation) we obtain

$$UJ(t) + V(GV + C'(t)) = 0. \quad (147)$$

The slowly-varying current $J(t)$ is measurable and its fluctuations are called "noise". This is the law of energy conservation as it should be written in the linearized regime when at least some of the circuit conductances depend on a parameter such as n . If the number n of electrons in some state fluctuates as a function of time, G and thus I vary even if V is a constant. Furthermore, we have so-far considered only potential sources, that is, potentials that are not influenced by the currents delivered. But we now relax this condition and allow the potential applied to the atoms to vary. If this is the case the law of conservation of energy reads more generally

$$UJ(t) + \Re\{V(t)^*(I(t) + C(t))\} = 0. \quad (148)$$

where we have emphasized that V may depend on time.

Equation (148) is the law of conservation of energy as it is employed in the present paper. As said above, we set $V(t) = V + \Delta V(t)$ and $I(t) = I + \Delta I(t)$, where $\Delta V(t), \Delta I(t)$ are fluctuations that play only an intermediate role. Because $\Delta V(t) \ll V$, $\Delta I(t) \ll I$ and $\Delta n(t) \ll n$ and thus $\Delta G(t) \ll G$ for above-threshold lasers, the statistical properties of $C(t)$ are negligibly affected by the $V(t)$ and $I(t)$ variations. Further, we set $J(t) = J + \Delta J(t)$. Equation (148) then splits into an expression for the steady-state or average values $UJ + \Re\{V^*I\} = 0$, and an expression for the fluctuations

$$U\Delta J(t) + \Re\{\Delta V(t)^*I + V(t)^*(\Delta I(t) + C(t))\} = 0. \quad (149)$$

Let us further observe that the frequency response of the system considered is very narrow. The assumption that the conductance G is independent of frequency, at least in some range, made earlier may require that atomic transition frequencies be spread out, either because of the environment as is often the case in crystals, or because of thermal motion through the Doppler effect, as is generally the case in gases. Such atomic collections are said to be "in-homogeneously" broadened.

Let the previous results be expressed in terms of *photon rates* instead of current sources, with the word "photon" omitted for brevity. Rates are defined as propagating powers divided by the so-called "photon energy" $\hbar\omega_o$. Thus the rate R is defined as $(V'I' + V''I'')/\hbar\omega_o = G|V|^2/\hbar\omega_o$ for a conductance G , since G is real. This rate is supplemented by a real fluctuating rate $r(t)$ related to the current source $C(t) \equiv C'(t) + iC''(t)$ previously considered by the relation $r(t) = (V'C'(t) + V''C'')/\hbar\omega_o$. As one recalls, $C'(t), C''(t)$ are uncorrelated and have spectral densities $\mathcal{S}_{C'} = \mathcal{S}_{C''} = \hbar\omega_o G$ for a positive G conductance. The spectral density of the fluctuating rate is therefore $\mathcal{S}_r = G|V|^2/\hbar\omega_o = R$. It follows that the spectral density of the fluctuating rate $r(t)$ is simply equal to the average rate R .

7.5 Simple above-threshold laser linewidth

When the linewidth of above-threshold lasers is investigated it is permissible to neglect amplitude fluctuations. We are seeking primarily an expression for the frequency fluctuation $\Delta\omega(t)$ whose spectral density is denoted $\mathcal{S}_{\Delta\omega}$. Assuming that this fluctuation is gaussian-distributed the laser linewidth is $\delta\omega = \mathcal{S}_{\Delta\omega}$. The model of above-threshold oscillator consists of an admittance $-Y_e(n)$ representing the laser material with complete population inversion, where n is a parameter, perhaps the electron number. There is in parallel the $L - C$ circuit, and a linear admittance $Y(\omega)$ representing the load, perhaps a detector. The laser power is defined as the power dissipated in the load. As we discussed in previous sections, any admittance is set as $Y \equiv G + iB$. The load is at $T = 0\text{K}$. Without the noise sources, the total admittance vanishes, that is

$$0 = Y_a(\omega) - Y_e(n) \implies Y_a(\omega_o) = Y_e(n_o) \quad (150)$$

This relation defines the steady-state oscillation frequency ω_o and the steady-state value of the parameter n_o , setting to 0 the real and imaginary parts of the equation. It is assumed that the solution is unique. There is a linewidth enhancement factor

$$A = \frac{1 + \alpha_A^2}{2} K \quad \alpha_A \equiv \frac{\Delta\omega'}{\Delta\omega''} = \frac{\alpha + h}{1 - \alpha h} \quad K = 1 + h^2. \quad (151)$$

Here α is the phase-amplitude coupling factor of the active material employed and h the dispersion factor introduced earlier. The parameter α_A is defined from the complex frequency deviation of the circuit $\Delta\omega \equiv \Delta\omega' + i\Delta\omega''$ corresponding to some departure of n from n_o .

7.6 Laser diodes at temperature $T > 0$

The expression of above-threshold relative noise, given in earlier sections, may be applied to laser diodes. In that case transitions occur between the valence band and the conduction band. We first give the result for the case where the electron distribution is given by the Fermi-Dirac distribution, and subsequently present the results obtained from a numerical

simulation that keeps track of the number of electron in each state and of the number of photons in the optical cavity. At low power it is permissible to suppose that the electron distribution in each band is given by the Fermi-Dirac temperature. If there is a good thermal contact with a large reservoir (or heat-sink) at temperature T , one may assume that the temperature is T in both bands. For the sake of simplicity, we assume that the level spacing is constant and equal to ϵ . The diode driving current is supposed to quiet (non-fluctuating) with average rate J . The relative noise at Fourier frequency Ω is found to be given by the expression

$$J\mathcal{N}(\Omega) = \frac{\frac{\alpha(1+\alpha)}{1-\alpha} \frac{F}{J^*} - 1}{\frac{1-\alpha^2}{2\alpha^2} J^* F + (1-F)^2} \quad (152)$$

where $F \equiv \Omega/\Omega_r$. The relaxation frequency is given by $\Omega_r = \sqrt{(1-\alpha^2)J^*/2}$, $J^* \equiv J\epsilon/k_B T$, and the population inversion factor is related to α by $n_p = (1+\alpha)^2/(4\alpha)$. This α -parameter should not be confused with the previously-defined phase-amplitude coupling parameter.

Thus the relative noise can be evaluated as a function of the Fourier frequency Ω , the parameters being the injected current eJ , the population inversion factor n_p , the temperature T , and the spacing ϵ between adjacent states. For typical values ($\alpha = 0.6$, $J^* = 4.64$) we obtain a relaxation frequency of 1.22 GHz.

7.7 Numerical simulation

With increasing laser-diode power, significant departures from the Fermi-Dirac distributions may occur in the valence and conduction bands, which are referred to as "spectral-hole burnings". Analytically, this effect could be accounted for by introducing a gain-compression term in the rate equations, as was described previously in Section 5.6. In order to verify the validity of this approach, and to go beyond the small-hole-burning approximation, we set up a purely numerical procedure. As in Section 7.6, the semiconductor valence and conduction bands are supposed to involve the same number of evenly-spaced levels, with spacing ϵ . Because of numerical limitations, the number of states considered in each band would be realistic only for small laser diodes. Each level may be occupied by 0 or 1 electron, according to the Pauli principle, the electron spin being not considered. The thermal bath is accounted for by allowing upward and downward electronic transitions between adjacent states at rates specified by the Boltzmann law, corresponding to some heat-sink temperature T_m .

We are particularly interested in regularly-pumped semiconductor laser because of its ability to produce sub-Poissonian laser light. In agreement with the general average-energy conservation concept discussed earlier, the photo-current spectral density is expected to vanish as the Fourier frequency $\Omega \rightarrow 0$. Regular pumping is simulated by introducing

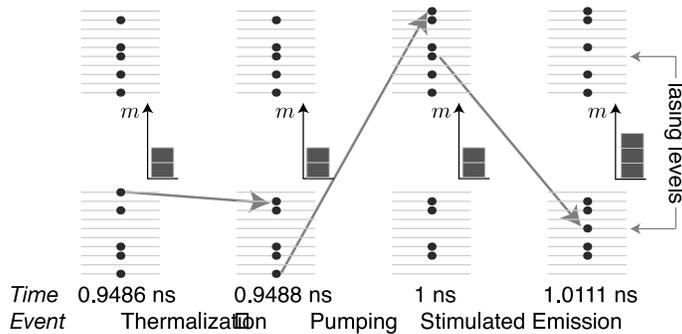


Figure 7: Four-frame sequence illustrating the Monte Carlo simulation of a semiconductor laser. Ten equally spaced energy levels are considered in the valence band and the conduction band. The insets represent the number of photons m stored in the cavity at a given time by stacked blocks. Arrows show electron displacements from one energy level to another.

periodic promotion of electrons from the bottom of the valence band to the top of the conduction band. Finally we keep track of the number m of photons in the optical resonator. The photon number is incremented by radiative decay of electrons and decremented by random transfer of the energy to the detector (absorbing load). Thus, by keeping track of the number of electrons in each level and of the number of photons in the optical cavity, it is possible to describe accurately the steady state and the noise properties of laser diodes. The Fig. 7 illustrates this model with only ten levels per band to make the figure more visible. The occurrence-times of electronic events are easily calculated since there are all given by independent Poisson processes, with the exception of the regular pumping process previously mentioned.

Electrons get thermalized in both bands owing to the prescribed $T_m = 300$ K lattice temperature. As a result, calculated occupancies plotted in Fig. 8a are close to those predicted by the Fermi-Dirac distribution (grand canonical statistical model) when the pumping level is small, i.e., when carrier thermalization within bands is efficient. But they depart from them when the pumping level increases. A dip appears in the electronic distribution at the energy where the lasing effect takes place. This well-known "spectral hole burning" effect is clearly visible when $\langle m \rangle \geq 400$.

The photo-current spectral density is evaluated from Eq. (19). As a matter of fact, this algorithm converges slowly as a function of the averaging time T so the calculated spectra given in Fig. 8b were obtained with $\approx 3 \cdot 10^6$ photon events and $\approx 4 \cdot 10^{11}$ thermalization events per curve. Their comparisons with the theoretical formula of Eq. (152) shows a very good overall agreement and, as expected, \mathcal{S} is below the shot-noise level at low frequencies, irrespectively of pumping, from a small ($\langle m \rangle = 50$) to large ($\langle m \rangle = 400$) average number of photons within the cavity. The main discrepancy is a shift of calculated spectral densities

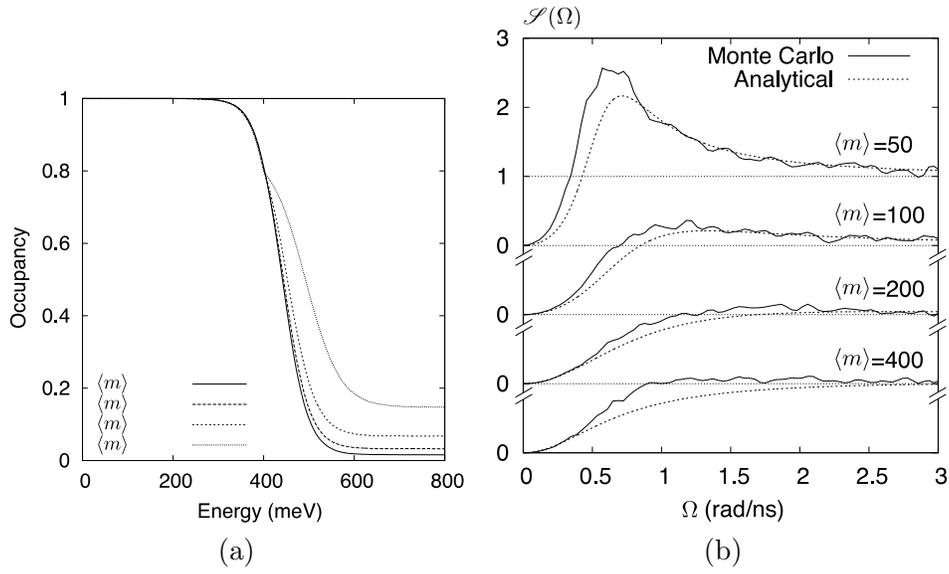


Figure 8: (a) Electron occupancies as a function of the electron energy referred to the bottom of the conduction band. Four values of the average photon number in the cavity, $\langle m \rangle$, are considered. The lasing levels are located at the middle of the bands ($E = 400$ meV here). (b) Normalized photo-current spectra. Solid lines are the results of the Monte Carlo calculations described in the present section, and dashed lines are the result of analytical calculations from Eq. (152).

toward lower Fourier frequencies. This is partly explained by the dip in the electron population occurring at the highest pumping levels ($\langle m \rangle \geq 400$) as a consequence of the spectral hole burning [38]. The displacement nonetheless occurs at a rather low pumping level (e.g., $\langle m \rangle = 50$). This shift at moderate pumping level may result also from an increment of the carrier temperature above the heat-sink temperature, which results from the flow of energy from the pump to the semi-conductor. To summarize, the numerical model described in this section shows combined effects of carrier heating and spectral-hole burning on laser-diodes noise characteristics.

7.8 Multiple active elements

We consider the linewidth of a laser with multiple elements submitted to the same field. This circumstance holds when the laser end mirrors have a reflectivity close to unity, so that the optical field does not vary much along the length. The present model differs from the previous one in that the active admittance $-Y_e$ is replaced by a set of admittances $-Y_k$, $k = 1, 2, \dots$ connected in parallel. These admittances are driven by possibly different electrical currents J_k and they may possess different phase-amplitude coupling factors α_k and population-inversion factor $n_p k$. The dispersion of the load G_a is expressed as in previous section by a factor $h \equiv (dG_a/d\omega)/(dB_a/d\omega)$. The steady-state oscillation condition requires that the sum of the G_k be equal to G_a , and that the total susceptance vanishes. The product of the (FWHP) linewidth $\delta\omega$ and the power P transferred from the active elements to the load is given by a formula in the ST form with multiplicative factors

$$\delta\omega P = \frac{\hbar\omega_o}{\tau_p^2} \frac{\langle n_p/(1 + \alpha^2) \rangle}{(1 - \langle \alpha \rangle h)^2} \quad \tau_p \equiv \frac{dB/d\omega}{2G}. \quad (153)$$

where the averaging sign is presently defined for any quantity a_k as

$$\langle a \rangle \equiv \frac{\sum a_k J_k}{\sum J_k}. \quad (154)$$

7.9 Detuned inhomogeneously-broadened lasers

The present model is similar to the previous one but we consider a collection of de-tuned atoms, with a Lorentzian distribution of resonant frequencies. The origin of this de-tuning may be the environment of rare-earth atoms. For some mirror reflectivity, there is a minimum number N_o of active atoms required to reach the threshold of oscillation. Our first parameter is the ratio N/N_o of the actual number of atoms and N_o . If the atom homogeneous line-width is $1/\tau_o$ and the spectral width of the atom resonant frequency distribution is denoted $1/\tau_i$, we define a second parameter $r \equiv 1 + \tau_o/\tau_i$. Finally, the resonant optical cavity may be detuned with respect to the center of the Lorentzian atomic-frequency distribution. The difference normalized to the Lorentzian spectral width is called

δ . For each element, the pumping rate is supposed to be proportional to the number of atoms in the lower state. Remarkably, the $\delta\omega P$ product may be expressed in closed form for any value of the parameters $n \equiv N/N_o$, r and δ . In the limit $r \rightarrow \infty$. For example

$$\delta\omega P = \frac{\hbar\omega_o}{\tau_p^2} \frac{1/n^2 + 10 + 5n^2}{32}. \quad (155)$$

an expression which reduces to a previous one when $n = 1$.

7.10 Spatially-varying α -factors

We consider in the present section ring-type above-threshold laser oscillators. We will be concerned mainly with the laser linewidth $\delta\omega$. We consider a closed path with coordinate z , implying that after a round-trip (at $z = L$) the field recovers its initial value (at $z = 0$). Of course the linewidth may not depend on the point along the path at which the z -axis origin is selected. The propagating wave experiences both a power gain $\gamma(z)$ (complete population inversion is assumed) and a power loss $\ell(z)$ (a temperature $T = 0\text{K}$ is assumed), distributed arbitrarily along the path. A localized loss ($\ell(z) \propto \delta(z)$) may describe a partially-reflecting mirror. In addition to the gain and the loss, one must take into account the phase-amplitude coupling factor $\alpha(z)$. As one recalls, this factor is defined as follows. We consider a piece of the active medium with permittivity ϵ . Under a small change of the carrier density the real and imaginary parts of ϵ vary and we defined $\alpha \equiv -\delta\epsilon'/\delta\epsilon''$. Frequency dispersion is neglected. The product of the (full-width at half-power point) linewidth $\delta\omega$ and the power P transferred from the gain medium to the loss medium reads

$$\delta\omega P = \frac{\hbar\omega}{2\tau^2} \frac{\int_1^\Omega d\gamma/\ell}{\int_1^\Omega d\gamma(1 + \alpha^2)/\gamma^2}. \quad (156)$$

where τ denotes the round trip transit time and Ω the round-trip gain (or loss). In this expression the power gain (varying from 1 to Ω) is employed as an integration variable instead of the coordinate z (varying from 0 to L). Even though this is not immediately obvious, the above expression does not depend on the selected origin of the z -axis. It reduces to previously known expressions for simpler configurations.

7.11 Multilevel atoms

It is well-known that lasers incorporating two-level atoms (i.e. atoms having a lower or absorbing level, labeled 1, and a higher or emitting level, labeled 2) cannot be pumped optically with light resonant with these two levels. Indeed, the upper working-level population is smaller than the lower working-level population. We may at best equalize the two

populations if pumping is very strong. This implies that no net gain may occur since the stimulated-absorption rate always exceeds the stimulated-emission rate³¹.

It has been recognized very early in the history of Laser Physics that, besides the two working levels, the atoms must exhibit at least one additional level, either higher in energy than the emitting level, and labeled here as 3, or lower in energy than the absorbing level, and labeled here as 0 (three-level lasers). When both extra levels are present (four-level lasers), pumping is resonant with the 0 to 3 transition and spontaneous decay is supposed to occur rather quickly from 3 to 2 and from 1 to 0. The average (or steady-state) levels populations are easily calculated from rate equations, given

- The probability density P that an electron in the lowest level be promoted to the highest level ($0 \rightarrow 3$).
- The probability density ℓP of demotion from 3 to 0.
- The optical-cavity absorption rate α (due here exclusively to the detector).
- The spontaneous-decay probability densities, p_u from 3 to 2, γ from 2 to 1, and p_d from 1 to 0. Other spontaneous decay rates are neglected for simplicity.

All of the above processes are supposed to be Poissonian. The pump and spontaneous-emission rates are of the form $S(t) = n(t)/\tau_s + s(t)$, where $n(t)$ denotes the level population, τ_s denotes the life-time, and the spectral density of $s(t)$ is equal to the average rate $\langle n(t) \rangle / \tau_s$.

Stimulated emission and absorption rates are of the form $R(t) = R + \Delta R(t) + r(t)$, where R denotes the average rate, $\Delta R(t)$ the variation, and $r(t)$ a rate source whose spectral density equals the average rate R . Because the optical resonator is resonant with the 1-2 transition, and we are presently concerned with relative noise and not with frequency noise or linewidth, all the quantities introduced above are taken as real. We select a time unit such that the stimulated time constant be unity ($g = 1$).

It is not so obvious that the additional levels (0,3) allow sub-Poissonian light to be delivered by the laser, as one can prove on the basis of the present generalized rate-equation theory, since the pump is Poissonian. The additional levels actually play a regulatory role, akin to the large cold resistance discussed earlier in the case of laser diodes driven by non-fluctuating currents.

Consider a single-mode laser incorporating N identical 4-level atoms, see Fig. 9a, resonant with the 1-2 transition. We will consider here mainly the case of Poissonian "incoherent" pumping (i.e., unidirectional, $\ell = 0$). Note that when the levels 0 and 3 are broad, the coupling to a thermal source gets improved.

³¹We do not consider here inversion-less lasers, in which stimulated absorption is, ideally, prevented from occurring.

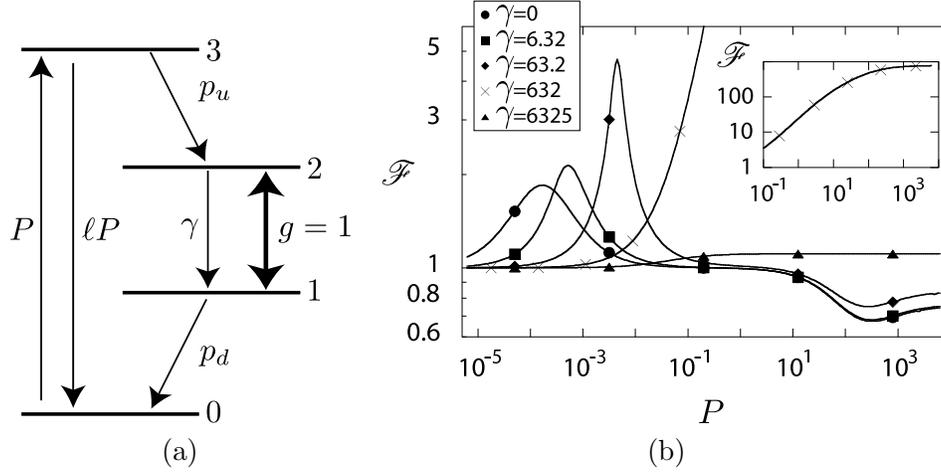


Figure 9: (a) Level schemes for a 4-level atomic laser with uni-directional pumping $\ell = 0$. We select the time unit such that the stimulated-emission gain $g = 1$. (b) Fano factor $\mathcal{F} \equiv \text{variance}(m)/\langle m \rangle$ as a function of the normalized pumping-probability P . The parameters are: $N = 10^5$ atoms, $p_u = 316$, $p_d = 632$, $\alpha = 6.32$. Various values of the spontaneous-decay probability γ from 2 to 1 are considered. Note that for small γ -values and appropriate pumping the Fano factor is below 1.

Analytical formulas are obtained for the spectral density of the output light normalized to the shot-noise level, $\mathcal{S}(\omega)$, and for the Fano factor: $\mathcal{F} = \langle \Delta m^2 \rangle / \langle m \rangle$. For arbitrary values of the parameters defined above analytic expressions have been obtained using symbolic calculus. They are, however, are too lengthy to be written down here. Fig. 9b illustrates the evolution of \mathcal{F} for various values of the spontaneous decay rate γ from 2 to 1. Obviously sub-Poissonian light statistics in the resonator occur at relatively high pumping values and small γ -values.

In the special case where the cavity losses are small ($\alpha \ll N$) and no spontaneous decay occur between the lasing levels ($\gamma = 0$), the zero-frequency spectral density of the detector photo-current reads

$$\mathcal{S} = 1 - \frac{2Pp_dp_u(p_d + 2(P + P\ell + p_u))}{(2Pp_u + p_d(P + P\ell + p_u))^2} \quad (157)$$

This expression coincides with the Quantum Optics results [39]. It allows us to evaluate the absolute minimum \mathcal{S}_{min} of the spectral density and the corresponding parameter values. One may treat similarly the three-level V-type laser (obtained when levels 0 and 1 coincide) and the three-level Λ -type laser (obtained when levels 3 and 2 coincide). The complete picture for 3- and 4-level lasers is summarized in Tab. 1. Whenever a comparison can be made with an existing Quantum Optics result, an exact agreement is obtained [40, 41].

Table 1: Minimum value of the zero-frequency photo-current spectral density \mathcal{S}_{min} and intra-cavity Fano factor \mathcal{F} for three- and four-level atomic lasers. The conditions on P , p_u and p_d are given together with the pumping type, $\ell = 0$ for incoherent and $\ell = 1$ for coherent pumping. Spontaneous decay from the upper working level is neglected and $N \gg \alpha$ is assumed.

Laser	\mathcal{S}_{min}	\mathcal{F}	Conditions
Λ -type 3-level	1/2	3/4	$p_d = 2P, \ell = 0$
Λ -type 3-level	2/3	5/6	$p_d = 3P, \ell = 1$
V-type 3-level	1/2	3/4	$p_u = \frac{1}{2}P, \ell = 0$
V-type 3-level	5/6	11/12	$p_u = \frac{3}{2}P, \ell = 1$
4-level	1/3	2/3	$p_u = P, p_d = 2P, \ell = 0$
4-level	3/7	5/7	$p_u = 2P, p_d = \frac{4}{3}P, \ell = 1$

7.12 Arbitrary media

The early paper [42] enables one to discretize any device into N cell. The noise properties are obtained through the inversion of an $N \times N$ *bi-complex* matrix. Only a simple configuration was treated explicitly. The detailed results concerning relative noise, phase noise, potential noise, and correlations, are given in that reference. The various parameters entering into the theory were evaluated for semi-conductors, and reported in a subsequent issue of the same journal. In this evaluation, the dependence of the semi-conductor band-gap on the electron number (band-gap shrinkage) is accounted for.

8 Amplifier with electrical feedback

The electrical current $J(t)$ originating from a detector or entering into an amplifier may be employed as a signal. It may modulate in amplitude and phase the optical signal either at the input or the output of the device considered. Such (backward or forward) feedbacks profoundly affect the gain and noise properties. We show how a phase-sensitive amplifier (PSA) may be constructed out of a conventional (phase-insensitive) amplifier and a phase modulator driven by a photo-current J . The special noise characteristic of this amplifier are employed in Section 8.1 to realize an amplifier, called a C-amplifier, that leaves unchanged in-phase and quadrature noises. It is supposed that the electrical amplifiers considered do not introduce any noise of their own. This is possible, at least in principle, at $T = 0K$. Remarkably, the rather complicated schematic considered is described exactly by simple relations, even in the case of gain compression. As one recalls, our basic optical-amplifier model at frequency $\omega = \omega_o$ is a negative conductance $-G$ connected at the end of a transmission line of characteristic conductance unity. The reflected b -wave is

separated out from the incident a -wave with the help of a circulator, ideally without any loss. The negative conductance is endowed with a complex current noise source $C(t)$, as we discussed earlier, re-expressed as a complex source $u(t)$. The negative conductance delivers a photo-current J which is essentially the difference between the ingoing and outgoing optical powers ($\hbar\omega_o = 1$). According to the schematic in Fig. 10 in c), The current J is amplified by an electrical amplifier (e.g., a maser) of gain F (electrical feedback factor). The amplified current next modulates the phase of the *input* optical beam. Once modulated, the input beam is transmitted through a phase-shifter that interchanges in-phase and quadrature components. The resulting optical signal coincides with the a -wave entering into the conductance $-G$ mentioned at the beginning of this paragraph, thereby closing the opto-electrical loop. The spectral densities are related by

$$Y_{out} = \mathcal{G}X_{in} + \mathcal{G} - 1 \quad (158)$$

and

$$X_{out} = \frac{\mathcal{G}Y_{in} + f^2(\mathcal{G} - 1)}{[\mathcal{G} + f(\mathcal{G} - 1)]^2}. \quad (159)$$

For a fixed value of \mathcal{G} and an input fluctuation Y_{in} , the quantity X_{out} , as given above, reaches its minimum value when $f = Y_{in}$ ³². For that particular value of the normalized feedback factor f , we find after slight rearranging that

$$\frac{1}{X_{out}} = \frac{\mathcal{G}}{Y_{in}} + \mathcal{G} - 1 \quad (160)$$

as asserted in (164).

In the case of gain compression, the expression of X_{out} as a function of the normalized feedback factor f reads

$$X_{out} = \frac{(\mathbf{g} + \kappa)^2 Y_{in} + f^2(\mathcal{G} - 1)}{(f(\mathcal{G} - 1) + \mathcal{G} + \kappa\mathbf{g})^2}. \quad (161)$$

If we employ the general result of the previous section, with $a = (\mathbf{g} + \kappa)^2 Y_{in}$, $b = \mathcal{G} - 1$, $c = \mathcal{G} + \kappa\mathbf{g}$, we find that the reciprocal of the minimum value of X_{out} reads

$$\frac{1}{X_{out}} = b + \frac{c^2}{a} = \frac{\mathcal{G}}{Y_{in}} + \mathcal{G} - 1, \quad (162)$$

which coincides with the expression obtained in the previous section, applicable to the special case where the gain compression may be neglected, i.e., $\kappa = 0$.

³² $X(f) = (a + bf^2)/(c + bf)^2$, where a, b, c denote constants, is stationary when $f = a/c$. The reciprocal of X then reads: $1/X = b + c^2/a$. In the present situation we have: $c = b + 1 = \mathcal{G}$, $a/c = Y_{in}$.

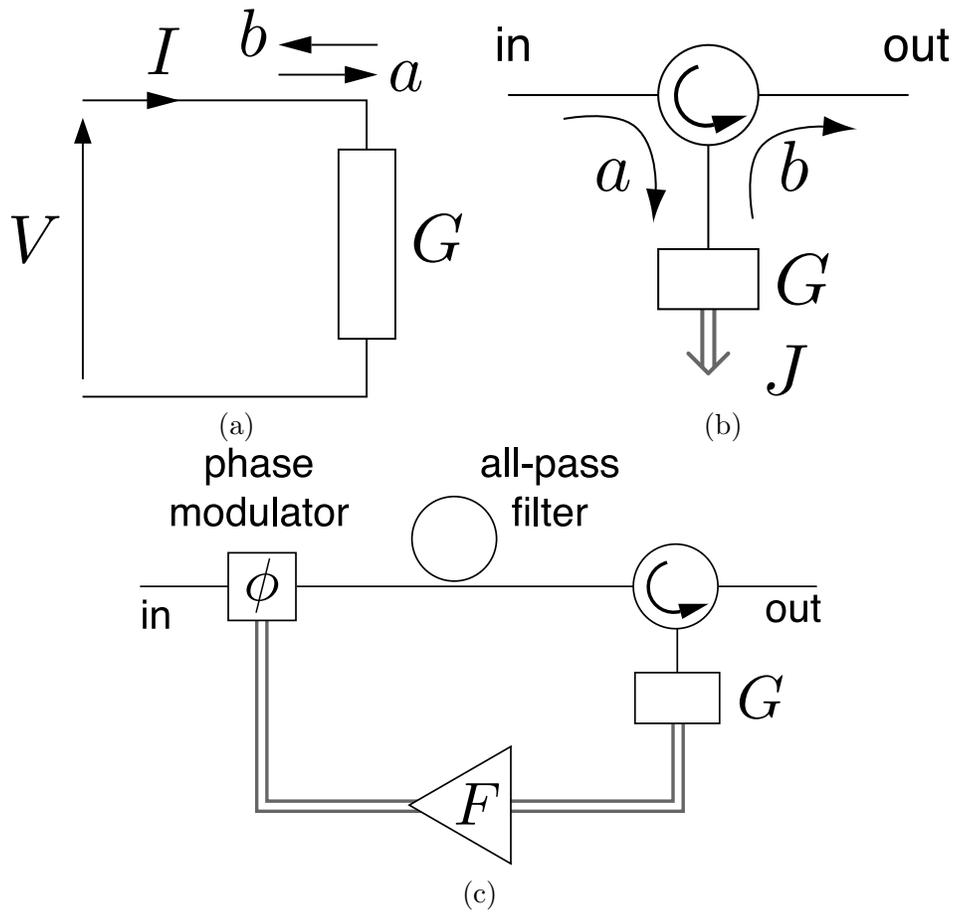


Figure 10: a) defines forward and backward propagating waves, denoted by a and b , respectively, b) shows how forward and backward waves may be separated out with the help of a circulator, in principle without introducing losses, c) represents an optical amplifier involving an electrical feedback F , a phase modulator, and an all-pass filter that interchanges in-phase and quadrature components.

8.1 C-amplifiers

As we have seen, when an optical wave is amplified by a conventional optical amplifier of gain \mathcal{G} followed by an attenuator with an attenuation equal to \mathcal{G} , the average input power is preserved but in-phase and quadrature fluctuations get enhanced. However, for a special kind of amplifier called here a "C-amplifier", fluctuations are preserved. Remarkably, this result holds even if the amplifier suffers from gain compression, that is, if the amplifier gain depends on the emitted optical rate, besides its dependence on the number, n , of atoms. An input beam in the C-state³³ remains in the C-state. Since the same property is known to hold for cold linear attenuators, it follows that such an amplifier followed by an attenuator of gain $1/\mathcal{G}$ recycle an initial C-state beam. That is, both the average intensity and the second-order correlations recycle. This remarkable property holds even if the gain depends on the light-emitted power (gain compression). This conclusion has perhaps a broad thermodynamical significance.

We have shown in the previous section that with the help of a feedback and a phase-shift it is possible to realize an optical amplifier, which we called a feed-back amplifier (F.A), that transforms the noise terms in the following manner

$$Y_{out} = \mathcal{G}X_{in} + \mathcal{G} - 1 \quad (163)$$

$$1/X_{out} = \mathcal{G}/Y_{in} + \mathcal{G} - 1 \quad (164)$$

We notice, incidentally, that for the case of a C-state input beam (i.e., $X_{in} = Y_{in} = 1$), the output spectral densities are $X_{out} = 1/(2\mathcal{G} - 1)$, $Y_{out} = 2\mathcal{G} - 1$, so that the product $X_{out}Y_{out}$ remains unity.

Let us suppose that two such FA amplifiers, with gains \mathcal{G}_1 and \mathcal{G}_2 respectively, are assembled one behind the other. If X, Y denote intermediate noise values, the input-output relations read

$$Y = \mathcal{G}_1X_{in} + \mathcal{G}_1 - 1 \quad (165)$$

$$1/X = \mathcal{G}_1/Y_{in} + \mathcal{G}_1 - 1 \quad (166)$$

and

$$Y_{out} = \mathcal{G}_2X + \mathcal{G}_2 - 1 \quad (167)$$

$$1/X_{out} = \mathcal{G}_2/Y + \mathcal{G}_2 - 1 \quad (168)$$

If we eliminate X, Y and select $\mathcal{G}_2 = 2 - 1/\mathcal{G}_1$, the total gain $\mathcal{G} = \mathcal{G}_1\mathcal{G}_2 = 2\mathcal{G}_1 - 1$. We obtain

$$X_{out} = \frac{X_{in} + \mu}{\mu X_{in} + 1} \quad Y_{out} = \frac{Y_{in} + \mu}{\mu Y_{in} + 1} \quad (169)$$

³³An optical beam is said to be in the C-state if it delivers Poissonian photo-electrons, irrespectively of the carrier phase.

where

$$\mu \equiv \frac{\mathcal{G} - 1}{\mathcal{G} + 1} \quad (170)$$

It follows from these expressions that if the input beam is in the C-state ($X_{in} = Y_{in} = 1$), the output beam is also in the C-state ($X_{out} = Y_{out} = 1$), but is amplified. We call this device a C-amplifier.

If a C-amplifier with gain \mathcal{G} is followed by an attenuator with "gain" $1/\mathcal{G}$, and recycled, the net gain (attenuator followed by an amplifier) is unity. The power delivered to the amplifier, ideally, is the difference between the input and output power, namely $P = (\mathcal{G} - 1)P_o$, if P_o denotes the amplifier input power. This is also the power delivered by the attenuator under the conditions presently considered, namely with gain $1/\mathcal{G}$ and output power P_o . Accordingly, the complete system is energetically neutral, as far as steady-state values are concerned.

As far as fluctuations are concerned, it follows from the above discussion that optical beams in the C-state recycle in the presently considered configuration, so that the whole system is in a state of thermodynamics equilibrium. This, it seems, is a remarkable result.

8.2 Amplifiers with gain compression

We presently generalize the above noise formulas applicable to linear amplifiers, accounting for the gain compression effect discussed in Section 5.6. Physically, this effect may be due to spectral-hole burning. It is expressed by supposing that the conductance $-G$ depends on the emitted photon rate. We now have

$$\begin{aligned} X_{out} &= \left(\frac{\mathfrak{g} + \kappa}{1 + \kappa\mathfrak{g}} \right)^2 X_{in} + \frac{\mathcal{G} - 1}{(1 + \kappa\mathfrak{g})^2} \\ Y_{out} &= \mathcal{G}Y_{in} + \mathcal{G} - 1 \quad \mathcal{G} \equiv \mathfrak{g}^2 \end{aligned} \quad (171)$$

Note that the amplifier presently considered is *phase-sensitive*.

9 Conclusion

We have shown that the photo-electron spectrum originating from a detector submitted to non-fluctuating-pump laser light may be understood in semi-classical terms. We did not employ the Quantum Optics concept that the optical field should be treated as an operator acting on the state of the light, nor the concept that light consists of point particles called "photons". The expression "photon rate" was used only as another name for electromagnetic power divided by $\hbar\omega_o$, where ω_o denotes the average laser frequency. Photon rates are written in the form $R + \Delta R(t) + r(t)$, where R denotes the average rate, $\Delta R(t)$ is proportional to the fundamental noise sources in the linearized regime, and

the spectral density of the fundamental noise source $r(t)$ is equal to R . This conclusion has been reached from different approaches, essentially requiring agreement with Classical Statistical Mechanics formulas. Results obtained from the present circuit theory for more complicated configurations are mentioned.

As long as one considers only stationary devices in the non-relativistic linearized approximation, it may be that one may consider light as a formal way of ensuring the validity of the law of average energy conservation.

A Point processes

Point processes are (possibly unlimited) sequences of increasing positive real numbers $t_k, k = 1, 2, \dots$, collectively denoted as $\{t_k\}$. Each k value corresponds to an "event", or "point", occurring at time t_k . We consider M such sequences, labeled by $m = 1, 2, \dots, M$, called runs. Averages denoted by the sign $\langle \cdot \rangle$ refer to sums from $m = 1$ to $m = M$ of some quantity defined for each run divided by M , letting M go to infinity. Clearly, averaging is a linear operation, that is $\langle a + b \rangle = \langle a \rangle + \langle b \rangle$. The full specification of a point process requires correlations of all order. However we will limit ourselves to first and second-order quantities.

Let $d(t)$ be the number of events occurring up to time t , that is the number of k values such that $t_k < t$. Obviously $d(0) = 0$ since the t_k are positive numbers. Let us prove that for some measurement time T $\langle d(T) \rangle = DT$, where D is a constant called the intensity of the process. We introduce the (positive) number $D_h(t) := d(t+h) - d(t)$ of events occurring between t and $t+h$. Because the process considered is stationary $\langle D_h(t) \rangle$ does not depend on t . It is convenient to split the measurement time T into time slots of duration $h = T/n$, labeled by $i = 1, 2, \dots, n$. Eventually, we let n go to infinity, so that it is unlikely that more than one event occur within any time slot. Thus, if $D_i \equiv D_h((i-1)h)$ denotes the number of events occurring during slot $i = 1, 2, \dots, n$, we have either $D_i = 1$ or $D_i = 0$ and $\langle D_i \rangle$ does not depend on i . For later use note that $D_i^2 = D_i$. The number $d(T)$ of events occurring during the measurement time T is the sum of the D_i with i running from 1 to n , so that its average reads

$$\langle d(T) \rangle = \left\langle \sum_{i=1}^n D_i \right\rangle = n \langle D_i \rangle = \frac{T}{h} \langle D_i \rangle \equiv TD, \quad (172)$$

where we have set $D \equiv \langle D_i \rangle / h$.

Because the process is stationary its auto-correlation $\langle D_h(t+\tau)D_h(t) \rangle$ does not depend on t for every $h > 0$ and every $\tau > 0$. The degree of second order coherence $g(\tau)$ is the limit of $\langle D_h(t+\tau)D_h(t) \rangle / \langle D_h(t) \rangle^2$ as h goes to 0. Let us set for $j > i$

$$\langle D_i D_j \rangle \equiv \langle D_i \rangle^2 g_n\left(\left(j-i\right)\frac{T}{n}\right), \quad (173)$$

and evaluate

$$\begin{aligned}
\langle d(T)^2 \rangle &= \left\langle \sum_{i=1}^n D_i \sum_{j=1}^n D_j \right\rangle \\
&= n \langle D_i \rangle + 2 \langle D_i \rangle^2 \sum_{i=1}^n \sum_{j=i+1}^n g_n\left((j-i)\frac{T}{n}\right) \\
&= \langle d(T) \rangle + 2 \langle D_i \rangle^2 \sum_{i=1}^n (n-i) g_n\left(\frac{iT}{n}\right) \\
&= \langle d(T) \rangle + 2D^2 \frac{T}{n} \sum_{i=1}^n \left(T - \frac{iT}{n}\right) g_n\left(\frac{iT}{n}\right). \tag{174}
\end{aligned}$$

In the limit $n \rightarrow \infty$ the sum may be replaced by an integral and g_n by g , thus

$$\langle d(T)^2 \rangle = \langle d(T) \rangle + 2D^2 \int_0^T d\tau (T - \tau) g(\tau). \tag{175}$$

After slight rearranging the variance of $d(T)$ may be written in the form

$$\begin{aligned}
\mathcal{V}(T) &\equiv \frac{\text{var}(d(T))}{\langle d(T) \rangle} - 1 = \frac{\langle d(T)^2 \rangle - \langle d(T) \rangle^2}{\langle d(T) \rangle} - 1 \\
&= 2D \int_0^T d\tau \left(1 - \frac{\tau}{T}\right) (g(\tau) - 1) \tag{176}
\end{aligned}$$

since $\int_0^T d\tau (1 - \tau/T) = T/2$. The motivation for introducing $g(\tau) - 1$ in the integral is that this quantity usually tends to 0 quickly as τ tends to infinity. Intuitively, this is because widely separated events tend to be independent and consequently in that limit $\langle D_i D_j \rangle \approx \langle D_i \rangle \langle D_j \rangle = \langle D_i \rangle^2$.

Setting $D = 1$ for brevity, relation (176) may be written as,

$$P_c(\tau) = \sum_{k=0}^{\infty} k^2 \frac{d^2 P(k, \tau)}{d\tau^2}, \tag{177}$$

where $P_c(\tau) dt d\tau$ denotes the probability of having an event between 0 and dt and an event between τ and $\tau + d\tau$ or, equivalently, $P_c(\tau) d\tau$ is the probability of another event being registered during the time interval τ and $\tau + d\tau$, given that an event occurred at $t = 0$. In (177) $P(k, \tau)$ denotes the probability of k events being registered between $t = 0$ and $t = \tau$.

A more complicated formula has been given by Ueda in 1988 [43], namely

$$\sum_{n=0}^{\infty} \sum_{m=0}^n \sum_{k=0}^m \frac{1}{D} \frac{d^2}{d\tau^2} P(k, \tau). \tag{178}$$

The above expression differs from the one given in (177) by the additional term

$$R = \lim_{N \rightarrow \infty} \frac{(N+1)(N+2)}{2} \sum_{k=0}^N \frac{d^2 P(k, \tau)}{d\tau^2} - (2N+3) \sum_{k=0}^N k \frac{d^2 P(k, \tau)}{d\tau^2}. \quad (179)$$

We have verified that $R = 0$ is special cases only.

We define the event rate $\mathcal{D}(t) = D + \Delta D(t)$ as the sum over k of $\delta(t - t_k)$, where $\delta(\cdot)$ denotes the Dirac distribution. Going back to time slots of small duration h , the event rate is the sum over i of D_i/h where, as before, $D_i = 0$ or 1 . The denominator h may be omitted because of the subsequent normalization. The calculations given below parallel the ones given above in relation with the photo-count variance and some details will therefore be omitted. The spectral density is defined in terms of the event times t_k occurring during runs of duration T , to be later tend to infinity, see (17).

$$T \mathcal{S}_{\Delta D}(\Omega) = \left\langle \sum_{i=1}^n D_i \exp(-j\Omega i) \sum_{j=1}^n D_j \exp(j\Omega j) \right\rangle \quad (180)$$

$$= n \langle D_i^2 \rangle + D^2 \left(\frac{T}{n}\right)^2 \sum_{i=1}^n \sum_{j=i+1}^n g_n\left((j-i)\frac{T}{n}\right) \cos\left((j-i)\Omega\right) \quad (181)$$

$$= DT + D^2 \frac{T}{n} \sum_{i=1}^n \left(T - \frac{iT}{n}\right) g_n\left(\frac{iT}{n}\right) \cos(i\Omega). \quad (182)$$

The above expression may be transformed as was done earlier for evaluating the variance of d . On account of the fact that

$$2 \sum_{i=1}^T \left(1 - \frac{i}{T}\right) \cos\left(\frac{2\pi i n}{T}\right) + 1 = 0, \quad (183)$$

where n denotes any non-zero integer, converting the sum into an integral, and in the large T limit, we obtain

$$\mathcal{N}(\Omega) \equiv \mathcal{S}_{\Delta D/D}(\Omega) - \frac{1}{D} = 2 \int_0^{\infty} d\tau (g(\tau) - 1) \cos(\Omega\tau). \quad (184)$$

This relation between the relative noise $\mathcal{N}(\Omega)$ and the normalized correlation function $g(\tau)$ has been established directly for point processes. It is however instructive to show how this relation may be alternatively derived from the Wiener-Khinchin theorem in (17). We consider the event-rate process $\mathcal{D}(t) = \sum_k \delta(t - t_k)$ which, in the limit $h \rightarrow 0$ corresponds to $D_h(t)/h$, where $D_h(t)$ is the number of events between t and $t + h$, as defined earlier. For simplicity we suppose, without loss of generality, that the average rate D is unity. If $R(\tau)$ denotes the correlation of $\mathcal{D}(t)$, the Wiener-Khinchin (WK) theorem tells us that

$$\mathcal{S}_{\mathcal{D}}(\Omega) = \int_{-\infty}^{+\infty} d\tau R(\tau) \exp(j\Omega\tau) \quad (185)$$

But $R(\tau)$ presents a singularity at $\tau = 0$. We are thus led to define $g(\tau) = R(\tau) - \delta(\tau)$. On the other hand, $\mathcal{S}_{\mathcal{D}}(\Omega)$ presents a singularity $2\pi\delta(\Omega)$ but $\mathcal{S}_{\Delta D}(\Omega) = \mathcal{S}_{\mathcal{D}}(\Omega) - 2\pi\delta(\Omega)$ is free from singularity. Substituting these expressions in the above WK equation and using the form of the δ -distribution in (5) we recover (184). Since presently the correlation $R(0) = \infty$, the restriction $-R(0) \leq R(\tau) \leq R(0)$ established in Section 3.1 does not entail any restriction on $g(\tau)$.

Note also that the word "correlation" as employed in this paper refers to the *photo-current* correlation. In contradistinction the word "correlation" as it is employed in Quantum Optics (QO) refers to the so-called optical field intensity. The latter would be classically defined as $I(t) = a^*(t)a(t)$, where a is proportional to the optical-field amplitude, and the un-normalized classical correlation would be $g(\tau) = \langle I(t)I(t + \tau) \rangle$. The QO expression is instead $g(\tau) = \langle a^-(t)a^-(t + \tau)a^+(t + \tau) \rangle a^+(t)$, where a^\pm create or destroy photons in the cavity mode. It is then found that if the optical resonator contains exactly n photons (number state), we have $g(\tau) = 1 - 1/n$, and is therefore equal to 0 if $n = 1$. For a coherent state $g(\tau) = 1$. This formalism, as just given, remains insufficient because of the back action of detector atoms on the field. Baths need be introduced. To go further along these lines, the interested reader is referred to QO text books.

B Useful integrals

A number of integrals from $x = -\infty$ to $x = \infty$ will be needed in subsequent papers, which may be evaluated by contour integration. The method is as follows:

Complex numbers are denoted $z \equiv z' + iz''$, where $i^2 = -1$. The complex conjugate of z is denoted $z^* \equiv z' - iz''$. Let $f(z)$ be a function of z whose only singularities are simple poles at z_1, z_2, \dots . One calls *residue* at z_k the coefficient of $(z - z_k)^{-1}$ in the (Laurent) series expansion of $f(z)$ near z_k . The integral of $f(z)$ along a closed counterclockwise contour is equal to $2\pi i$ times the sum of the enclosed pole residues.

For example, closing the real axis by an upper half-circle of infinite radius we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x+i)} = 2\pi i \frac{1}{i+i} = \pi \quad (186)$$

Here we have a single enclosed pole at $x = i$. The coefficient of $1/(x-i)$ in the integrand is $1/(2i)$ when $x = i$.

We obtain similarly

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{(1-ax^2)^2 + x^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx ax^2}{(1-ax^2)^2 + x^2} = 1 \quad (187)$$

where a denotes a non-zero constant.

Further, let us define a weight function

$$w(x) \equiv \frac{(g-1)/\pi}{(g-1)^2 + x^2} \quad (188)$$

that reduces to the Dirac δ -distribution when g tends to 1, and

$$I_{mn} \equiv 8(g^2 + y^2)^m \int_{-\infty}^{\infty} \frac{dx w(x-y)x^n}{(1+x^2)^m} \quad (189)$$

We obtain

$$I_{10} = 8g \quad (190)$$

$$I_{12} = 8y^2 + 8g(g-1) \quad (191)$$

$$I_{20} = 4y^2(g-1) + 4g^2(g+1) \quad (192)$$

$$I_{21} = 8gy \quad (193)$$

$$I_{30} = 3(g-1)y^4 + 6g(g^2-1)y^2 + g^3(3g^2+3g+2) \quad (194)$$

$$I_{31} = 2y[y^2(g-1) + g^2(g+3)] \quad (195)$$

$$I_{32} = (g-1)y^4 + 2g(g^2+3)y^2 + g^3(g-1)(g+2) \quad (196)$$

$$I_{33} = 2y[(3g+1)y^2 + 3g^2(g-1)] \quad (197)$$

$$I_{34} = (3g+5)y^4 + 6g(g^2-1)y^2 + g^3(3g-2)(g-1) \quad (198)$$

We also have

$$\frac{1}{\pi} \int_0^{\infty} \frac{dx \sqrt{x}}{(x+1)(x-a)} = \frac{1}{1 + \sqrt{-a}} \quad (199)$$

if a is negative, and $1/(1+a)$ if a is positive. In the latter case, the integral is understood in principal value.

C Bi-complex representation of signals

As recalled in Section 3.1 electrical engineers usually factor out a term $\exp(j\Omega t)$ to represent time-harmonic sources. $\Omega \equiv 2\pi f$ is the angular *baseband* (or "Fourier") angular frequency. The real signal is obtained by taking the real part of the product $V \exp(j\Omega t)$, where V denotes some complex number. Only time-invariant linear causal systems are considered in this appendix. It follows that the system response has the same form as the applied source. In Physics, it is usual to factor out a term of the form $\exp(-i\omega t)$ where $\omega \equiv 2\pi\nu$ denotes the *carrier* angular frequency.

When a source at frequency ω is modulated at frequency Ω , the bi-complex representation described below proves useful. To avoid bothering with minus signs, it is convenient to set $i_1 \equiv -i$ and $i_2 \equiv j$. Further, we set $p_1 \equiv i_1\omega$ and $p_2 \equiv i_2\Omega$. The algebra of bi-complex numbers is associative and commutative. A bi-complex number is written as

$$\mathcal{V} = a + bi_1 + ci_2 + di_1i_2 \quad (200)$$

where $i_1^2 = i_2^2 = -1$, $i_1i_2 = i_2i_1 \equiv j$, $j^2 = 1$, and a, b, c, d are real numbers. The algebra of bi-complex numbers was first discussed by Segre in 1892. A modern account of the bi-complex numbers algebra may be found, for example, in [44]. A bi-complex number is invertible if $a^2 + b^2 + c^2 + d^2 \neq \pm 2(ad - bc)$. To prove it, multiply $a + bi_1 + ci_2 + dj$ by $a - bi_1 - ci_2 + dj$ and obtain $A + Bj$, where $A = a^2 + b^2 + c^2 + d^2$ and $B = 2(ad - bc)$ are real numbers. Next, note that $(A + Bj)(A - Bj) = A^2 - B^2$ is real. The condition for a bi-complex number to be invertible is therefore that $A \neq \pm B$, which is the above condition.

The real signal $v(t)$ is equal to $\mathcal{V}(p_1, p_2) \exp[(p_1 + p_2)t] + \text{ccc}$, where "ccc" means that one must add 3 terms to the one written out, one with p_1 changed to $-p_1$, the second with p_2 changed to $-p_2$, and the third with both p_1 and p_2 changed to $-p_1$ and $-p_2$.

Let us now state the basic theorem. If $v(t)$ denotes a modulated voltage represented by $\mathcal{V}(p_1, p_2)$ as said above, $Y(p)$ denotes the usual complex circuit admittance (the ratio of two real polynomials in p), and $i(t)$ the real electrical current flowing through the circuit, we have

$$i(t) = Y(p_1 + p_2) \mathcal{V}(p_1, p_2) \exp[(p_1 + p_2)t] + \text{ccc} \quad (201)$$

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