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Stability of Planar Switched Systems: the Nondiagonalizable Case

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Abstract Consider the planar linear switched system $\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t)$, where A and B are two 2×2 real matrices, $x \in \mathbb{R}^2$, and $u(\cdot) : [0, \infty[\rightarrow \{0, 1\}$ is a measurable function. In this paper we consider the problem of finding a (coordinate-invariant) necessary and sufficient condition on A and B under which the system is asymptotically stable for arbitrary switching functions $u(\cdot)$.

This problem was solved in previous works under the assumption that both A and B are diagonalizable. In this paper we conclude this study, by providing a necessary and sufficient condition for asymptotic stability in the case in which A and/or B are not diagonalizable.

To this purpose we build suitable normal forms for A and B containing coordinate invariant parameters. A necessary and sufficient condition is then found without looking for a common Lyapunov function but using “worst-trajectory” type arguments.

Keywords: switched systems, planar, arbitrary switchings, worst-trajectory.

AMS: 93D20, 37N35

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1 Introduction

By a switched system, we mean a family of continuous-time dynamical systems and a rule that determines at each time which dynamical system is responsible of the time evolution. More precisely, let $\{f_u : u \in U\}$ (where U is a subset of \mathbb{R}^m) be a finite or infinite set of sufficiently regular vector fields on a manifold M , and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \quad (1)$$

The rule is given by assigning the so-called switching function, i.e., a function $u(\cdot) : [0, \infty[\rightarrow U \subset \mathbb{R}^m$. Here, we consider the situation in which the switching function is not known a priori and represents some phenomenon (e.g., a disturbance) that is not possible to control. Therefore, the dynamics defined in (1) also fits into the framework of uncertain systems (cf. for instance [5]). In the sequel, we use the notations $u \in U$ to label a fixed individual system and $u(\cdot)$ to indicate the switching function. These kind of systems are sometimes called “n-modal systems”, “dynamical polysystems”, “input systems”. The term “switched system” is often reserved to situations in which the switching function $u(\cdot)$ is piecewise continuous or the set U is finite. For the purpose of this paper, we only require $u(\cdot)$ to be a measurable function. For a discussion of various issues related to switched systems, we refer the reader to [4, 9, 11, 13].

A typical problem for switched systems goes as follows. Assume that, for every fixed $u \in U$, the dynamical system $\dot{x} = f_u(x)$ satisfies a given property (P). Then one can investigate conditions under which property (P) still holds for $\dot{x} = f_{u(t)}(x)$, where $u(\cdot)$ is an arbitrary switching function.

In [1, 6, 9, 12], the case of linear switched systems was considered:

$$\dot{x}(t) = A_{u(t)}x(t), \quad x \in \mathbb{R}^n, \quad \{A_u\}_{u \in U} \subset \mathbb{R}^{n \times n}, \quad (2)$$

where $U \subset \mathbb{R}^m$ is a compact set, $u(\cdot) : [0, \infty[\rightarrow U$ is a (measurable) switching function, and the map $u \mapsto A_u$ is continuous (so that $\{A_u\}_{u \in U}$ is a compact set of matrices). For these systems, the problem of asymptotic stability of the origin, uniformly with respect to switching functions was investigated.

Let us recall the notions of stability that are used in the following.

Definition 1 For $\delta > 0$ let B_δ be the unit ball of radius δ , centered in the origin. Denote by \mathcal{U} the set of measurable functions defined on $[0, \infty[$ and taking values on U . Given $x_0 \in \mathbb{R}^n$, we denote by $\gamma_{x_0, u(\cdot)}(\cdot)$ the trajectory of (2) based in x_0 and corresponding to the control $u(\cdot)$. The accessible set from x_0 , denoted by $\mathcal{A}(x_0)$, is

$$\mathcal{A}(x_0) = \cup_{u(\cdot) \in \mathcal{U}} \text{Supp}(\gamma_{x_0, u(\cdot)}(\cdot)).$$

We say that the system (2) is

- **unbounded** if there exist $x_0 \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}$ such that $\gamma_{x_0, u(\cdot)}(t)$ goes to infinity as $t \rightarrow \infty$;
- **uniformly stable** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{A}(x_0) \subset B_\varepsilon$ for every $x_0 \in B_\delta$;
- **globally uniformly asymptotically stable (GUAS, for short)** if it is uniformly stable and globally uniformly attractive, i.e., for every $\delta_1, \delta_2 > 0$, there exists $T > 0$ such that $\gamma_{x_0, u(\cdot)}(T) \in B_{\delta_1}$ for every $u(\cdot) \in \mathcal{U}$ and every $x_0 \in B_{\delta_2}$;

Remark 1 Under our hypotheses (linearity and compactness) there are many notions of stability equivalent to the ones of Definition 1. More precisely since the system is linear, local and global notions of stability are equivalent. Moreover, since $\{A_u\}_{u \in U}$ is compact, all notions of stability are automatically uniform with respect to switching functions (see for instance [3]). Finally, thanks to the linearity, the GUAS property is equivalent to the more often quoted property of GUES (global exponential stability, uniform with respect to switching), see for example [2] and references therein.

Let us recall some results about stability of systems of type (2).

In [1, 12], it is shown that the structure of the Lie algebra generated by the matrices A_u :

$$\mathfrak{g} = \{A_u : u \in U\}_{L.A.},$$

is crucial for the stability of the system (2). The main result of [12] is the following:

Theorem 1 (Hespanha, Morse, Liberzon) *If \mathfrak{g} is a solvable Lie algebra, then the switched system (2) is GUAS.*

In [1] a generalization was given. Let $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{s}$ be the Levi decomposition of \mathfrak{g} in its radical (i.e., the maximal solvable ideal of \mathfrak{g}) and a semi-simple sub-algebra, where the symbol \ltimes indicates the semidirect sum.

Theorem 2 (Agrachev, Liberzon) *If \mathfrak{s} is a compact Lie algebra then the switched system (2) is GUAS.*

Theorem 2 contains Theorem 1 as a special case. Anyway the converse of Theorem 2 is not true in general: if \mathfrak{s} is non compact, the system can be stable or unstable. This case was also investigated. In particular, if \mathfrak{g} has dimension at most 4 as Lie algebra, the authors were able to reduce the problem of the asymptotic stability of the system (2) to the problem of the asymptotic stability of an auxiliary bidimensional system. We refer the reader to [1] for details. For this reason the bidimensional problem assumes particular interest. In [6] (see also [10]) the single input case was investigated,

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \quad (3)$$

where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part (Hurwitz in the following), $x \in \mathbb{R}^2$ and $u(\cdot) : [0, \infty[\rightarrow \{0, 1\}$ is an arbitrary measurable switching function.

Under the assumption that A and B are both diagonalizable (in real or complex sense) a complete solution was found. In the following we refer to this case as to the *diagonalizable case*. More precisely a necessary and sufficient condition for GUAS was given in terms of three coordinate-invariant parameters: one depends on the eigenvalues of A , one on the eigenvalues of B and the last contains the interrelation among the two systems and it is in 1-1 correspondence with the cross ratio of the four eigenvectors of A and B in the projective line $\mathbb{C}P^1$. A remarkable fact is that if the system $\dot{x} = u(t)Ax + (1 - u(t))Bx$ has a given stability property, then the system $\dot{x} = u(t)(\tau_1 A)x + (1 - u(t))(\tau_2 B)x$ has the same stability property, for every $\tau_1, \tau_2 > 0$. This is a consequence of the fact that the stability properties of the system (3), depend only on the shape of the integral curves of Ax and Bx and not on the way in which they are parameterized.

The stability conditions for (3) were obtained with a direct method without looking for a common Lyapunov function, but analyzing the locus in which the two vector fields are collinear, to build the “worst-trajectory”. This method was successful also to study a nonlinear generalization of this problem (see [8]). One of the most important step to obtain these stability conditions was to find good normal forms for the matrices, containing explicitly the coordinate invariant parameters.

Remark 2 It is interesting to notice that common Lyapunov functions do not seem to be the most efficient tool to study the stability of switching systems for arbitrary switchings. In fact, beside providing sufficient conditions for GUAS, like those of Theorems 1 and 2, the concept of common Lyapunov function is useful when one can prove that, if a Lyapunov function exists, then it is possible to find it in a class of functions parameterized by a finite number of parameters. Indeed, once such a class of functions is identified, then in order to verify GUAS, one could use numerical algorithms to check (by varying the parameters) whether a Lyapunov function exists (in which case the system is GUAS) or not (meaning that the system is not GUAS). In the case of stability of switching systems under arbitrary switchings, this is in general a very hard task. For instance for systems of type (3) one can prove that the GUAS property is equivalent to the existence of a common *polynomial* Lyapunov function, but the degree of such common polynomial Lyapunov function is not uniformly bounded over all the GUAS systems [10].

In this paper we provide a necessary and sufficient condition for GUAS in the nongeneric cases omitted in [6]. In particular we study the stability of the system (3) assuming that at least one of the matrices (say A) is not diagonalizable. In the following we refer to this case as to the *nondiagonalizable case*. We also assume that A and B are both Hurwitz and that $[A, B] \neq 0$ otherwise the problem is trivial. These conditions are gathered in the following assumption **(H0)**, which is often recalled in the following:

(H0) A and B are two 2×2 real Hurwitz matrices. We assume that A is nondiagonalizable and $[A, B] \neq 0$.

We also study the cases in which the system is just uniformly stable.

A very useful fact is that the stability properties of systems of kind (2) depend only on the convex hull of the set $\{A_u\}_{u \in U}$ (see for instance [10]). As a consequence we have

Lemma 1 *The system (3) with $u(\cdot) : [0, \infty[\rightarrow \{0, 1\}$ is GUAS (resp. uniformly stable, resp. unbounded) if and only the system (3) with $u(\cdot) : [0, \infty[\rightarrow [0, 1]$ is.*

In the following we refer to the switched system with $u(\cdot)$ taking values in $[0, 1]$ as to the *convexified system*. Sometimes we will take advantage of studying the convexified system.

The techniques that we use to get the stability conditions for (3) under **(H0)** are similar to those of [6]. However new difficulties arise. The first is due to the fact that since A is not diagonalizable then the eigenvectors of A and B are at most 3 noncoinciding points on $\mathbb{C}P^1$. As a consequence the cross ratio is not anymore the right parameter describing the interrelation among the systems. It is either not defined or completely fixed. For this reason new coordinate-invariant parameters should be identified and new normal forms for A and B should be constructed. These coordinate invariant parameters are the three real parameters defined in Definition 2 below. One (η) is the (only) eigenvalue of A , the second (ρ) depends on the eigenvalues of B and the third (k) plays the role of the cross ratio of the diagonalizable case. Section 2 is devoted to the computation of the normal forms.

Once suitable normal forms are obtained, we look for stability conditions, studying the set \mathcal{Z} where the two vector fields are linearly dependent, using a technique coming from optimal control (see [7]). This set is the set of zeros of the function $Q(x) := \det(Ax, Bx)$. Since Q is a quadratic form, we have the following cases (depicted in Figure 1):

- A.** $\mathcal{Z} = \{0\}$ (i.e., Q is positive or negative definite). In this case one vector field points always on the same side of the other and the system is GUAS. This fact can be proved in several way (for instance building a common quadratic Lyapunov function) and it is true in much more generality (even for nonlinear systems, see [8]).
- B.** \mathcal{Z} is the union of two noncoinciding straight lines passing through the origin (i.e., Q is sign indefinite). Take a point $x \in \mathcal{Z} \setminus \{0\}$. We say that \mathcal{Z} is *direct* (respectively, *inverse*) if Ax and Bx have the same (respectively, opposite) versus. One can prove that this definition is independent of the choice of x on \mathcal{Z} . See Proposition 1 below. Then we have the two subcases:
 - B1.** \mathcal{Z} is inverse. In this case one can prove that there exists $u_0 \in]0, 1[$ such that the matrix $u_0Ax + (1 - u_0)Bx$ has an eigenvalue with positive real part. In this case the system is unbounded since it is possible to build a trajectory of the convexified system going to infinity with constant control. (This type of instability is called *static instability*.)
 - B2.** \mathcal{Z} is direct. In this case one can reduce the problem of the stability of (3) to the problem of the stability of a single trajectory called *worst-trajectory*. Fixed $x_0 \in \mathbb{R}^2 \setminus \{0\}$, the worst-trajectory γ_{x_0} is the trajectory of (3), based at x_0 , and having the following property. At each time t , $\dot{\gamma}_{x_0}(t)$ forms the smallest angle (in absolute value) with the (exiting) radial direction (see Figure 2). Clearly the worst-trajectory switches among the two vector fields on the set \mathcal{Z} . If it does not rotate around the origin (i.e., if it crosses the set \mathcal{Z} a finite number of times) then the system is GUAS. This case is better described projecting the system on $\mathbb{R}P^1$ (see Lemma 7 below). On the other side, if it rotates around the origin, the system is GUAS if and only if after one turn the distance from the origin is decreased. (see Figure 1, Case B2). If after one turn the distance from the origin is increased then the system is unbounded (in this case, since there are no trajectories of the convexified system going to infinity with constant control, we call this instability *dynamic instability*). If γ_{x_0} is periodic then the system is uniformly stable, but not GUAS.
- C.** In the degenerate case in which the two straight lines of \mathcal{Z} coincide (i.e., when Q is sign semi-definite), one see that the system is GUAS (resp. uniformly stable, but not GUAS) if and only if \mathcal{Z} is direct (resp. inverse). We call these cases respectively **C2** and **C1**.

The main point of the paper is to translate conditions **A.**, **B.** and **C.** in terms of the coordinate invariant parameters appearing in the normal forms. Several cases (parametrized by the parameter k) should be studied separately:

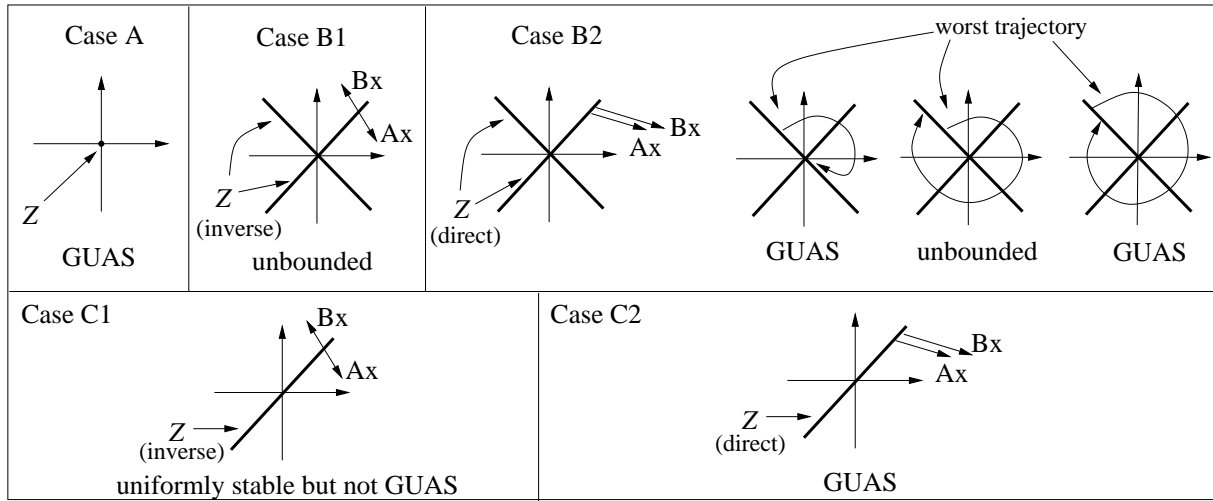


Figure 1:

- $k = 0$, called singular case (**S**-case in the following),
- $k \neq 0$ and B has non real eigenvalues, called “regular -1 ” case (**R**₋₁ case),
- $k \neq 0$ and B has real noncoinciding eigenvalues called “regular $+1$ ” case (**R**₁ case),
- $k \neq 0$ and B is not diagonalizable, called “regular 0 ” case (**R**₀ case).

The case in which B has two coinciding eigenvalues and it is diagonalizable is not considered since in this case B is proportional to the identity and, therefore, (**H0**) is not satisfied.

The structure of the paper is the following. In Section 2 we compute the normal forms for A and B in which the coordinate-invariant parameters appear explicitly. In Section 3 we state the stability conditions, that are proved in the next sections.

Remark 3 As in the diagonalizable case (since the way in which the integral curves of Ax and Bx are parametrized is not important), if the system $\dot{x} = u(t)Ax + (1 - u(t))Bx$ has a given stability property, then the system $\dot{x} = u(t)(\tau_1 A)x + (1 - u(t))(\tau_2 B)x$ has the same stability property, for every $\tau_1, \tau_2 > 0$.

In section 4 we start by studying the set \mathcal{Z} where the two vector fields are linearly dependent. In Section 5 we state and prove some general stability conditions (in particular cases **A**, **B1**, **C1**, **C2** of Figure 1). In Section 6 we build the worst-trajectory (i.e., we study case **B2**).

2 Basic Definitions and Normal Forms

For $x \in \mathbb{R}$ define

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

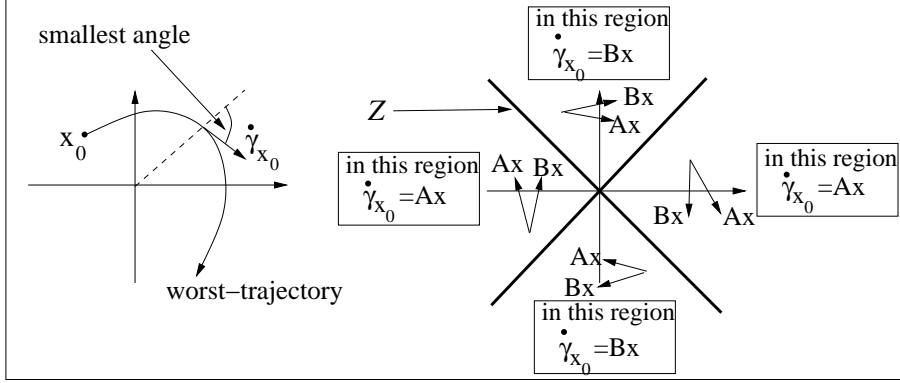


Figure 2:

Definition 2 Assume **(H0)** and let δ be the discriminant of the equation $\det(B - \lambda Id) = 0$. Define the following invariant parameters:

$$\eta = \begin{cases} \frac{Tr(A)}{\sqrt{|\delta|}} & \text{if } \delta \neq 0 \\ \frac{Tr(A)}{2} & \text{if } \delta = 0, \end{cases} \quad (4)$$

$$\rho = \begin{cases} \frac{Tr(B)}{\sqrt{|\delta|}} & \text{if } \delta \neq 0 \\ \frac{Tr(B)}{2} & \text{if } \delta = 0, \end{cases} \quad (5)$$

$$k = \begin{cases} \frac{4}{|\delta|} \left(Tr(AB) - \frac{1}{2} Tr(A)Tr(B) \right) & \text{if } \delta \neq 0 \\ Tr(AB) - \frac{1}{2} Tr(A)Tr(B) & \text{if } \delta = 0. \end{cases} \quad (6)$$

Remark 4 Notice that $\delta = (\lambda_1 - \lambda_2)^2 \in \mathbb{R}$, where λ_1 and λ_2 are the eigenvalues of B . Notice moreover that B has non real eigenvalues if and only if $\delta < 0$. Finally observe that $\eta, \rho < 0$ and $k \in \mathbb{R}$.

Definition 3 In the following, under the assumption **(H0)**, we call regular case (**R-case for short**), the case in which $k \neq 0$ and singular case (**S-for short**), the case in which $k = 0$.

Lemma 2 (R-case) Assume **(H0)** and $k \neq 0$. Then it is always possible to find a linear change of coordinates and a constant $\tau > 0$ such that A/τ and B/τ (that we still call A and B) have the following form:

$$A = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \quad (7)$$

$$B = \begin{pmatrix} \rho & \text{sign}(\delta)/k \\ k & \rho \end{pmatrix}. \quad (8)$$

Moreover in this case $[A, B] \neq 0$ is automatically satisfied.

Remark 5 In the following, for the regular case, we call **R₁-case**, **R₋₁-case**, **R₀-case** the cases corresponding respectively to $\text{sign}(\delta) = 1$, $\text{sign}(\delta) = -1$, $\text{sign}(\delta) = 0$. See Lemma 4 below for the discussion of the eigenvalues and eigenvectors of B in these three cases.

Lemma 3 (S-case) Assume **(H0)** and $k = 0$. Then $\delta > 0$ and it is always possible to find a linear change of coordinates and a constant $\tau > 0$ such that A/τ and B/τ (that we still call A and B) have the following form,

$$A = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \quad B = \begin{pmatrix} \rho - 1 & 0 \\ 0 & \rho + 1 \end{pmatrix}, \quad \text{called } \mathbf{S}_1\text{-case}, \quad (9)$$

or the form,

$$A = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \quad B = \begin{pmatrix} \rho+1 & 0 \\ 0 & \rho-1 \end{pmatrix}, \quad \text{called } \mathbf{S}_{-1}\text{-case.} \quad (10)$$

Proof of Lemma 2 and Lemma 3. We can always find a system of coordinates such that,

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}. \quad (11)$$

In this case the discriminant of B is $\delta = (a-d)^2 + 4bc$. We have

$$[A, B] = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}. \quad (12)$$

Notice moreover that $c = \text{Tr}(AB) - \frac{1}{2}\text{Tr}(A)\text{Tr}(B)$. Hence, according with (6), $c = 0$ iff $k = 0$.

Case $c \neq 0$. First notice that in this case $[A, B] \neq 0$. Consider the transformation

$$T = \begin{pmatrix} 1 & \frac{a-d}{2c} \\ 0 & 1 \end{pmatrix}. \quad (13)$$

Then

$$A' := T^{-1}AT = A, \quad B' := T^{-1}BT = \begin{pmatrix} \frac{a+d}{2} & \frac{\delta}{4c} \\ c & \frac{a+d}{2} \end{pmatrix}. \quad (14)$$

we have the following:

- If $\delta = 0$, according to Definition 2, then $c = k$, $\lambda = \eta$, $\rho = (a+d)/2$, and with the change of notation $B' \rightarrow B$ we get the normal forms (7) and (8).
- If $\delta \neq 0$, we define

$$A'' := \frac{2}{\sqrt{|\delta|}}A' = \begin{pmatrix} \lambda \frac{2}{\sqrt{|\delta|}} & \frac{2}{\sqrt{|\delta|}} \\ 0 & \lambda \frac{2}{\sqrt{|\delta|}} \end{pmatrix}, \quad B'' := \frac{2}{\sqrt{|\delta|}}B' = \begin{pmatrix} \frac{a+d}{\sqrt{|\delta|}} & \frac{\sqrt{|\delta|}}{2c} \\ \frac{2c}{\sqrt{|\delta|}} & \frac{a+d}{\sqrt{|\delta|}} \end{pmatrix}. \quad (15)$$

Consider the transformation

$$T' = \begin{pmatrix} \frac{\sqrt{2}}{|\delta|^{1/4}} & 0 \\ 0 & \frac{|\delta|^{1/4}}{\sqrt{2}} \end{pmatrix}. \quad (16)$$

We have

$$A''' := (T')^{-1}A''T' = \begin{pmatrix} \lambda \frac{2}{\sqrt{|\delta|}} & 1 \\ 0 & \lambda \frac{2}{\sqrt{|\delta|}} \end{pmatrix}, \quad B''' := (T')^{-1}B''T' = \begin{pmatrix} \frac{a+d}{\sqrt{|\delta|}} & \frac{\delta}{4c} \\ \frac{4c}{|\delta|} & \frac{a+d}{\sqrt{|\delta|}} \end{pmatrix}. \quad (17)$$

According to Definition 2, and with the change of notations $A''' \rightarrow A$, $B''' \rightarrow B$ we get the normal forms (7) and (8). Lemma 2 is proved.

Case $c = 0$. In this case **(H0)** implies $\delta = (a-d)^2 > 0$. Consider the transformation:

$$T = \begin{pmatrix} 1 & \frac{-b}{a-d} \\ 0 & 1 \end{pmatrix}. \quad (18)$$

We have:

$$A' := T^{-1}AT = A, \quad B' := T^{-1}BT = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}. \quad (19)$$

We have two cases

- $d > a$. In this case from $a + d = \text{Tr}(B)$ and $d - a = \sqrt{\delta}$, we get

$$a = \frac{1}{2}(\text{Tr}(B) - \sqrt{\delta}), \quad d = \frac{1}{2}(\text{Tr}(B) + \sqrt{\delta}).$$

Define

$$A'' := \frac{2}{\sqrt{\delta}}A' = \begin{pmatrix} \lambda \frac{2}{\sqrt{\delta}} & \frac{2}{\sqrt{\delta}} \\ 0 & \lambda \frac{2}{\sqrt{\delta}} \end{pmatrix}, \quad B'' := \frac{2}{\sqrt{\delta}}B = \begin{pmatrix} \rho - 1 & 0 \\ 0 & \rho + 1 \end{pmatrix}. \quad (20)$$

Define now T' as in (16). Then

$$A''' := (T')^{-1}A''T = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \quad B''' := (T')^{-1}B''T = B''. \quad (21)$$

With the change of notations $A''' \rightarrow A$, $B''' \rightarrow B$ we get the normal forms (9).

- Similarly if $d < a$, we get the normal forms (10).

Lemma 3 is proved. ■

The following Lemma can be directly checked.

Lemma 4 *Under the condition (H0), we have the following:*

- A has a unique eigenvalue $\eta < 0$ and its corresponding eigenvector is $(1, 0)$.
- if $k \neq 0$ and $\delta > 0$ (i.e., in the \mathbf{R}_1 -case), then the eigenvalues of B are $\rho + 1$ and $\rho - 1$, corresponding respectively to the eigenvectors $(1, k)$ and $(1, -k)$, and we have $\rho < -1$.
- if $k \neq 0$ and $\delta < 0$, (i.e., in the \mathbf{R}_{-1} -case) then the eigenvalues of B are $\rho + i$ and $\rho - i$, and we have $\rho < 0$. In this case the integral curves of the vector field Bx are elliptical spirals rotating counter-clockwise if $k > 0$ (clockwise if $k < 0$).
- if $k \neq 0$ and $\delta = 0$ (i.e., in the \mathbf{R}_0 -case), then B has a unique eigenvalue $\rho < 0$, and it corresponds to the eigenvector $(0, 1)$.
- if $k = 0$, (i.e., in the \mathbf{S} -case) then the eigenvalues of B are $\rho + 1$ and $\rho - 1$, with $\rho < -1$.
 - In the case of the normal form (9) (i.e., in the \mathbf{S}_1 -case) they correspond respectively to the eigenvectors $(0, 1)$ and $(1, 0)$.
 - In the case of the normal form (10) (i.e., in the \mathbf{S}_{-1} -case), they correspond respectively to the eigenvectors $(1, 0)$ and $(0, 1)$.

3 Main Results

In this section we state our stability conditions. First we need to define some functions of the invariants η , ρ , k defined in Definition 2. Set

$$\Delta = k^2 - 4\eta\rho k + \text{sign}(\delta)4\eta^2.$$

By direct computation one gets that if $k = 2\eta\rho$ then $\Delta = -4 \det A \det B < 0$. It follows

Lemma 5 *Assume (H0). Then $\Delta \geq 0$ implies $k \neq 2\eta\rho$.*

Moreover, when $\Delta > 0$ and $k < 0$, define

$$\mathcal{R} = \begin{cases} \left| \frac{(-k + \sqrt{\Delta})}{(-k - \sqrt{\Delta})} \frac{2k\rho^2 - \text{sign}(\delta)(\text{Tr}(AB) + \sqrt{\Delta})}{2k\sqrt{\det B}(\rho - \text{sign}(\delta)\frac{\eta}{k})} \right| \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_{\text{sign}(\delta)}\right), & \text{if } \rho - \text{sign}(\delta)\frac{\eta}{k} \neq 0, \\ \frac{-2\eta}{\sqrt{k^2 + \eta^2}} \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_{-1}\right), & \text{if } \rho - \text{sign}(\delta)\frac{\eta}{k} = 0 \text{ (which implies } \text{sign}(\delta) = -1). \end{cases} \quad (22)$$

where

$$\begin{aligned} \theta_{-1} &= \begin{cases} \arctan \frac{\sqrt{\Delta}}{k(\rho + \frac{\eta}{k}) + \eta} & \text{if } k(\rho + \frac{\eta}{k}) + \eta \neq 0 \\ \pi/2 & \text{if } k(\rho + \frac{\eta}{k}) + \eta = 0, \end{cases} \\ \theta_1 &= \text{arctanh} \frac{\sqrt{\Delta}}{k(\rho - \frac{\eta}{k}) - \eta}, \\ \theta_0 &= \frac{\sqrt{\Delta}}{k\rho}. \end{aligned}$$

Notice that when $k < 0$, then $k(\rho - \frac{\eta}{k}) - \eta > 0$ and $k\rho > 0$. Hence θ_1 and θ_0 are well defined.

The following Theorem states the main result of the paper. The letters **A.**, **B.**, and **C.** refer again to the cases described in the introduction and in Figure 1. Recall Lemma 5.

Theorem 3 *Assume (H0). We have the following stability conditions for the system (3).*

A. *If $\Delta < 0$, then the system is GUAS.*

B. *If $\Delta > 0$, then:*

B1. *if $k > 2\eta\rho$, then the system is unbounded,*

B2. *if $k < 2\eta\rho$, then*

- *in the regular case ($k \neq 0$), the system is GUAS, uniformly stable (but not GUAS) or unbounded respectively if*

$$\mathcal{R} < 1, \mathcal{R} = 1, \mathcal{R} > 1.$$

- *In the singular case ($k = 0$), the system is GUAS.*

C. *If $\Delta = 0$, then:*

C1 *If $k > 2\eta\rho$, then the system is uniformly stable (but not GUAS),*

C2 *if $k < 2\eta\rho$, then the system is GUAS.*

4 The set where the two vector fields are parallel

As explained in the introduction, the stability conditions are obtained by studying the locus in which the two vector fields Ax and Bx are linearly dependent. This set is the set \mathcal{Z} of zeros of the quadratic form $Q(x) := \text{Det}(Ax, Bx)$. Set $x = (x_1, x_2)$. We have the following:

$$Q = \begin{cases} (\rho - \text{sign}(\delta)\frac{\eta}{k})x_2^2 + kx_1x_2 + (\eta k)x_1^2 & \text{in the } \mathbf{R}\text{-case} \\ x_2((\rho + 1)x_2 + 2\eta x_1) & \text{in the } \mathbf{S}_1 \text{ case} \\ x_2((\rho - 1)x_2 - 2\eta x_1) & \text{in the } \mathbf{S}_{-1} \text{ case.} \end{cases} \quad (23)$$

The discriminant of this quadratic form is the quantity Δ defined in (22). Hence we have the following:

- if $\Delta < 0$ then $\mathcal{Z} = \{0\}$,
- if $\Delta > 0$ then \mathcal{Z} is a pair of (noncoinciding) straight lines passing through the origin,

- if $\Delta = 0$ then \mathcal{Z} is a single straight line passing through the origin (the two straight lines of the previous case coincide).

Remark 6 Notice that, under the condition **(H0)**, in the regular case, the sign of Δ can be positive or negative, while in the singular case, since $k = 0$ and $\delta > 0$ (cf. Lemma 3), we have $\Delta = 4\eta^2 > 0$. In the singular case \mathcal{Z} is the union of a pair of straight lines one of them coinciding with the x_1 axis (as it is clear from the expression of Q).

In the following, under the assumption $\Delta \geq 0$, we give the explicit expression of the angular coefficients of the two (possibly coinciding) straight lines whose union is \mathcal{Z} .

R₋₁-case (i.e., the case in which B has non real eigenvalues)

$$\text{if } \rho + \eta/k \neq 0 \text{ then } m^\pm = \frac{-k \pm \sqrt{\Delta}}{2(\rho + \eta/k)} = \frac{-k \pm \sqrt{k^2 - 4\eta k(\rho + \eta/k)}}{2(\rho + \eta/k)}, \quad (24)$$

$$\text{if } \rho + \eta/k = 0 \text{ then } m^+ = \infty, \quad m^- = -\eta. \quad (25)$$

R₁-case (i.e., the regular case in which B is diagonalizable and has real eigenvalues)

$$\text{if } \rho - \eta/k \neq 0 \text{ then } m^\pm = \frac{-k \pm \sqrt{\Delta}}{2(\rho - \eta/k)} = \frac{-k \pm \sqrt{k^2 - 4\eta k(\rho - \eta/k)}}{2(\rho - \eta/k)}, \quad (26)$$

$$\text{if } \rho - \eta/k = 0 \text{ then } m^- = \infty, \quad m^+ = -\eta. \quad (27)$$

R₀-case (i.e., the regular case in which B is nondiagonalizable)

$$m^\pm = \frac{-k \pm \sqrt{\Delta}}{2\rho} = \frac{-k \pm \sqrt{k^2 - 4\eta k\rho}}{2\rho}. \quad (28)$$

S-case (i.e., the singular case)

$$m^+ = 0, \quad m^- = -\frac{2\eta}{\rho + 1} < 0 \text{ in the } \mathbf{S}_1\text{-case}, \quad (29)$$

$$m^+ = 0, \quad m^- = \frac{2\eta}{\rho - 1} > 0 \text{ in the } \mathbf{S}_{-1}\text{-case}. \quad (30)$$

The following Proposition says that if $\Delta \geq 0$ and the two vector fields have the same versus on a point of $\mathcal{Z} \setminus \{0\}$, then this is the case all along $\mathcal{Z} \setminus \{0\}$. More precisely it says that if $k < 2\eta\rho$ (resp. $k > 2\eta\rho$) then they have the same (resp. opposite) versus. Recall Lemma 5.

Proposition 1 Assume **(H0)**, $\Delta \geq 0$, and let $\mathcal{Z} = D^+ \cup D^-$ where $D^\pm = \{(h, m^\pm h) \in \mathbb{R}^2, h \in \mathbb{R}\}$. Let us define α^\pm by $Bx = \alpha^\pm Ax$ for $x \in D^\pm \setminus \{0\}$. Then

- $\alpha^+ \alpha^- = \frac{\det B}{\det A} > 0$,

- $\alpha^+ + \alpha^- = \frac{2\eta\rho - k}{\det A}$.

In other words $\text{sign}(\alpha^+) = \text{sign}(\alpha^-) = \text{sign}(2\eta\rho - k)$.

Proof. Let us start with the regular case and $\Delta > 0$. In this case it is easy to check that $\alpha^\pm = \frac{k + \rho m^\pm}{\eta m^\pm}$. Hence $\alpha^+ \alpha^- = \frac{k^2 + k\rho(m^+ + m^-) + \rho m^+ m^-}{\eta^2 m^+ m^-}$. Using the fact that $m^+ + m^- = \frac{-k}{\chi}$ and $m^+ m^- = \frac{\eta k}{\chi}$, where $\chi := \rho - \text{sign}(\delta) \frac{\eta}{k}$, we get

$$\alpha^+ \alpha^- = \frac{k(\chi - \rho) + \rho^2 \eta}{\eta^3} = \frac{\rho^2 \eta - \text{sign}(\delta) \eta}{\eta^3} = \frac{\rho^2 - \text{sign}(\delta)}{\eta^2} = \frac{\det B}{\det A}.$$

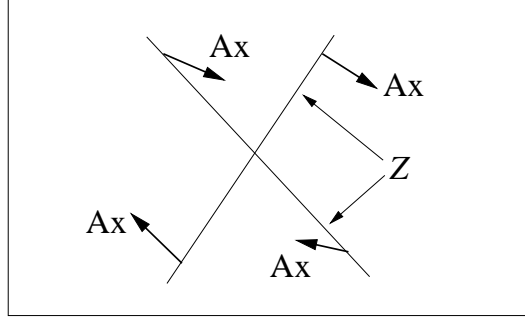


Figure 3:

Similarly we have

$$\alpha^+ + \alpha^- = \frac{k(m^+ + m^-) + 2\rho m^+ m^-}{\eta m^+ m^-} = \frac{-k^2 + 2\rho\eta k}{\eta^2 k} = \frac{2\eta\rho - k}{\det A}.$$

In the regular case, with $\Delta = 0$, we have $D^+ = D^-$, $m^+ = m^- =: m^0$ and $\alpha^+ = \alpha^- =: \alpha^0$. Now $Bx_1^0 = \alpha^0 Ax_1^0$ implies that

$$\alpha^0 = \frac{k + \rho m^0}{\eta m^0} = \frac{k + \rho \frac{-k}{2\chi}}{\eta \frac{-k}{2\chi}} = \frac{\rho - 2\chi}{\eta}.$$

Since $\Delta = k^2 - 4\eta k \chi = 0$ we have $2\chi = \frac{k}{2\eta}$ and $\alpha^0 = \frac{\rho - \frac{k}{2\eta}}{\eta} = \frac{2\eta\rho - k}{2\eta^2}$. Thus

$$2\alpha^0 = \frac{2\eta\rho - k}{\det A} \quad \text{and} \quad (\alpha^0)^2 = \frac{\rho^2 - \text{sign}(\delta)}{\eta^2} = \frac{\det B}{\det A}$$

In the singular case $k = 0$, we have $\Delta > 0$ and

$$\begin{aligned} \mathcal{Z} &= \{(x_1, x_2) \mid x_2 = 0\} \cup \{(x_1, x_2) \mid x_2 = \frac{-2\eta}{\rho + 1} x_1\} \quad \text{or} \\ \mathcal{Z} &= \{(x_1, x_2) \mid x_2 = 0\} \cup \{(x_1, x_2) \mid x_2 = \frac{2\eta}{\rho - 1} x_1\}. \end{aligned}$$

An easy computation show that

$$\alpha^+ \alpha^- = \frac{(\rho - 1)(\rho + 1)}{\eta^2} = \frac{\det B}{\det A} \quad \text{and} \quad \alpha^+ + \alpha^- = \frac{2\rho}{\eta} = \frac{2\eta\rho}{\eta^2} = \frac{2\eta\rho}{\det A}.$$

Definition 4 If $\Delta \geq 0$ and $k < 2\eta\rho$ (resp. $k < 2\eta\rho$) we say that \mathcal{Z} is “direct” (resp. “inverse”).

The following Proposition says, roughly speaking, that on $\mathcal{Z} \setminus \{0\}$ the vector field Ax points “clockwise” (see Figure 3).

Proposition 2 Assume **(H0)** and $\Delta \geq 0$. Let ∂_θ be the vector field defined by $\partial_\theta(x_1, x_2) = (-x_2, x_1)$. Then in the normal forms of Section 2, we have $Ax \cdot \partial_\theta \leq 0$ for every $x \in \mathcal{Z} \setminus \{0\}$.

Proof. Let $x^+ \in D^+$, where D^+ is defined in Proposition 1. Then $x^+ = (h, m^+h)$ for some $h \in \mathbb{R} \setminus \{0\}$. It follows $Ax^+ \cdot \partial_\theta(x^+) = -(m^+)^2 h^2$. Similarly if $x^- = (\bar{h}, m^- \bar{h})$ for some $\bar{h} \in \mathbb{R} \setminus \{0\}$, we have $Ax^- \cdot \partial_\theta(x^-) = -(m^-)^2 \bar{h}^2$. ■

5 General Stability Conditions

In this Section we study some basic stability conditions.

The following proposition (from which it follows **A.** of Theorem 3) expresses the idea that if the two vector fields are linearly dependent just at the origin, then the system is always GUAS. For the proof one can follow the same steps used in the case of two diagonalizable matrices [6]. It is also a particular case of the same result proved for nonlinear systems in [8].

Proposition 3 *Assume H0. If $\Delta < 0$ then the system is GUAS.*

5.1 The inverse case

The next proposition (from which it follows **B1.** of Theorem 3) says that if the vector fields are parallel along two (noncoinciding) straight lines and point in opposite directions, then there are trajectories going to infinity.

Proposition 4 *Assume (H0). If $\Delta > 0$ and $k > 2\eta\rho$ then the system is unbounded. In the degenerate case in which $\Delta = 0$ and $k > 2\eta\rho$ then the system is uniformly stable but not GUAS.*

Proof. To prove the first statement, recall that we can work with the convexified system. Consider the following constant control

$$u_0 = \frac{\alpha^+\alpha^- - \frac{1}{2}(\alpha^+ + \alpha^-)}{1 + \alpha^+\alpha^- - (\alpha^+ + \alpha^-)}.$$

From Proposition 1 we have $\alpha^+ + \alpha^- < 0$ and $\alpha^+\alpha^- > 0$, which implies that $u_0 \in]0, 1[$. A simple computation show that:

$$\det M(u_0) = \det(u_0A + (1 - u_0)B) = \frac{-\Delta}{4 \det A(1 + \alpha^+\alpha^- - (\alpha^+ + \alpha^-))} < 0,$$

since $\Delta > 0$. Hence $M(u_0)$ has a positive real eigenvalue and the system is unbounded.

To prove the second statement, observe that if $\Delta = 0$ and $k > 2\eta\rho$, then $\mathcal{Z} = D^+ = D^- = \{(x_1, x_2) \mid x_2 + 2\eta x_1 = 0\}$, and thanks to Proposition 1, $\alpha^+ = \alpha^- = \alpha^0 < 0$. Consider again the matrix

$$M(u_0) = u_0A + (1 - u_0)B \quad \text{where} \quad u_0 = \frac{(\alpha^0)^2 - \alpha^0}{1 - 2\alpha^0 + \alpha^{02}} = \frac{\alpha^0}{\alpha^0 - 1}.$$

Then $Tr(M(u_0)) = \frac{2(\alpha^0\eta - \rho)}{\alpha^0 - 1} < 0$ and $\det M(u_0) = 0$. Therefore for u_0 the system is not asymptotically stable. To prove that the system is uniformly stable we show that Ax and Bx admit the following common (non strict) Lyapunov function,

$$V(x) = V(x_1, x_2) = x_1^2 + \frac{x_2^2}{4\eta^2}.$$

Let us prove that

$$\nabla V(x)Ax = \frac{1}{2\eta}(x_2 + 2\eta x_1)^2 \leq 0 \tag{31}$$

$$\nabla V(x)Bx = \frac{\rho}{2\eta^2}(x_2 + 2\eta x_1)^2 \leq 0. \tag{32}$$

To prove (31) observe that $\nabla V(x)Ax = 2\eta x_1^2 + 2x_1x_2 + \frac{1}{2\eta}x_2^2 = \frac{1}{2\eta}(x_2 + 2\eta x_1)^2$. To prove (32), observe that

$$\nabla V(x)Bx = 2\rho x_1^2 + \left(\frac{2\text{sign}(\delta)}{k} + \frac{k}{2\eta^2}\right)x_1x_2 + \frac{\rho}{2\eta^2}x_2^2 = 2\rho x_1^2 + \left(\frac{4\text{sign}(\delta)\eta^2 + k^2}{2\eta^2k}\right)x_1x_2 + \frac{\rho}{2\eta^2}x_2^2.$$

But $\Delta = k^2 - 4\eta k\rho + 4\eta^2\text{sign}(\delta) = 0$ implies that $k^2 + 4\eta^2\text{sign}(\delta) = 4\eta k\rho$. Then

$$\nabla V(x)Bx = 2\rho x_1^2 + \frac{4\eta k\rho}{2\eta^2k}x_1x_2 + \frac{\rho}{2\eta^2}x_2^2 = 2\rho x_1^2 + \frac{2\rho}{\eta}x_1x_2 + \frac{\rho}{2\eta^2}x_2^2 = \frac{\rho}{2\eta^2}(x_2 + 2\eta x_1)^2.$$

It follows that $V(x)$ is a common Lyapunov function and the system is uniformly stable. ■

Notice that in the **S**-case, Proposition 4 never apply because \mathcal{Z} is always “direct”.

5.2 The direct case: the worst-trajectory

In the case in which $\Delta \geq 0$ and $k < 2\eta\rho$, the stability of (3) can be reduced to the study of a single trajectory called the “worst-trajectory”.

Definition 5 *Assume (H0), $\Delta \geq 0$ and $k < 2\eta\rho$. Fix $x_0 \in \mathbb{R}^2 \setminus \{0\}$. The worst-trajectory γ_{x_0} is the trajectory of (3), based at x_0 , and having the following property. At each time t , $\dot{\gamma}_{x_0}(t)$ forms the smallest angle (in absolute value) with the (exiting) radial direction (see Figure 2).*

As explained in the introduction, (see also [6, 10]) the worst-trajectory switches among the two vector fields on the set \mathcal{Z} . The following Lemma (whose proof is a consequence of the arguments used in [6]) reduce the problem of stability of (3) to the problem of the stability of the worst-trajectory.

Lemma 6 *Assume (H0), $\Delta > 0$, and $k < 2\eta\rho$. Fix $x_0 \in \mathbb{R}^2 \setminus \{0\}$. Then the system (3) is GUAS (resp. uniformly stable but not GUAS, resp. unbounded) if and only if γ_{x_0} tends to the origin (resp. is periodic, resp. tends to infinity). Moreover when γ_{x_0} is periodic or tends to infinity then it rotates around the origin switching an infinity number of times. In the degenerate case in which $\Delta = 0$, and $k < 2\eta\rho$, then the system is GUAS.*

Remark 7 The last statement can be be proved either by using worst-trajectory type arguments or by building a common quadratic Lyapunov function and it implies statement **C2.** of Theorem 3.

Remark 8 The explicit construction of the worst trajectory in the case $\Delta > 0$ and $k < 2\eta\rho$ (i.e., the proof of case **B2.** of Theorem 3) is done in the next section. The worst-trajectory can rotate or not around the origin. As it will be clear next, if B has complex eigenvalues then it always rotates. One of the statement of Lemma 6 is that if the worst-trajectory does not rotate then the system is always GUAS. Notice that, if the worst-trajectory rotates around the origin, then it rotates clockwise, thanks to Proposition 2.

6 Construction of the worst-trajectory

In this section we prove **B2.** of Theorem 3, separately for the cases **R**₋₁, **R**₁, **R**₀, and **S**.

6.1 The **R**₋₁-case

In the **R**₋₁-case we have $\delta < 0$, therefore the matrix B has non real eigenvalues.

Notice that under the conditions $\delta < 0$ and $0 < k < 2\eta\rho$ we have $\Delta = k^2 - 4\eta k(\rho + \eta/k) = k(k - 2\eta\rho) - 2k\eta\rho - 4\eta^2 < 0$. Therefore, since we are interested to the case in which $\Delta > 0$, we may assume $k < 0$. Recall formulas (24) and (25). We have the following cases depicted in Figure 4, for $x_1 \geq 0$.

- $k < 0$ and $\rho + \eta/k > 0$. In this case $\sqrt{\Delta} < |k|$. It follows that $m^+ > m^- > 0$.
- $k < 0$ and $\rho + \eta/k < 0$. In this case $\sqrt{\Delta} > |k|$. It follows that $m^- > 0 > m^+$.
- $k < 0$ and $\rho + \eta/k = 0$. In this case $m^+ = \infty$, $m^- = -\eta > 0$.

Fix $x_0 = (1, m^+) \in D^+$. In all cases, the worst-trajectory γ_{x_0} rotates clockwise (cf. Remark 8) and is a concatenation of integral curves of Ax (from the line D^+ of equation $x_2 = m^+x_1$ to line D^- of equation $x_2 = m^-x_1$.) and integral curves of Bx otherwise. See Figure 5.

Let t_1 be the time needed to an integral curve of Ax to steer a point from D^+ to D^- . Since

$$\exp(At) = \exp(\eta t) \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad (33)$$

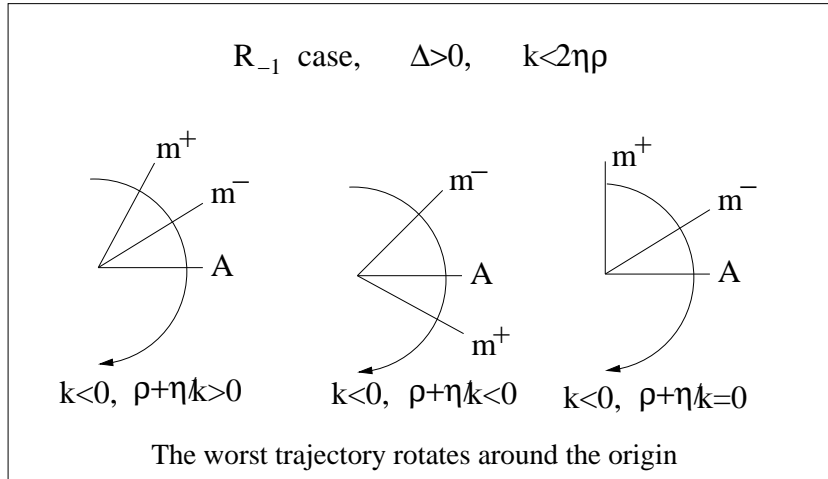


Figure 4:

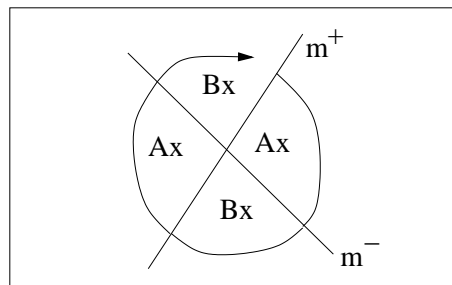


Figure 5:

we have

$$t_1 = \frac{m^+ - m^-}{m^+ m^-} = \frac{\sqrt{\Delta}}{\eta k}.$$

Let t_2 be the time needed to an integral curve of Bx to steer a point from D^- to D^+ . Since

$$\exp(Bt) = \exp(\rho t) \begin{pmatrix} \cos t & \frac{-\sin t}{k} \\ k \sin t & \cos t \end{pmatrix},$$

we have

$$t_2 = \begin{cases} \arctan \frac{\sqrt{\Delta}}{k(\rho + \frac{\eta}{k}) + \eta} & \text{if } k(\rho + \frac{\eta}{k}) + \eta \neq 0 \\ \frac{\pi}{2} & \text{if } k(\rho + \frac{\eta}{k}) + \eta = 0. \end{cases}$$

Notice that in the case $\rho + \frac{\eta}{k} = 0$, we have the following simple formulas for the switching time: $t_1 = \frac{-1}{\eta}$ and $t_2 = \arctan \frac{-k}{\eta}$. Fix $x_0 \in D^+$ and let \mathcal{R} be the ratio between the norm of γ_{x_0} after half turn (i.e., after time $t_1 + t_2$) and the norm of $\gamma_{x_0}(0)$, i.e.,

$$\mathcal{R} = \frac{\|\gamma_{x_0}(t_1 + t_2)\|}{\|\gamma_{x_0}(0)\|} \quad (34)$$

We have the following cases

- $\rho + \frac{\eta}{k} \neq 0$. In this case

$$\mathcal{R} = \exp\left(\frac{\sqrt{\Delta}}{k}\right) \exp(\rho t_2) \left| \left(1 + \frac{m^+ \sqrt{\Delta}}{k\eta}\right) \left(\cos t_2 - \frac{m^-}{k} \sin t_2\right) \right|,$$

after simplification we get

$$\mathcal{R} = \left| \left(\frac{-k + \sqrt{\Delta}}{-k - \sqrt{\Delta}} \right) \frac{2k\rho^2 + \text{Tr}(AB) + \sqrt{\Delta}}{2k\sqrt{\det B}(\rho + \frac{\eta}{k})} \right| \exp\left(\frac{\sqrt{\Delta}}{k}\right) \exp(\rho t_2),$$

i.e.,

$$\mathcal{R} = \left| \left(\frac{-k + \sqrt{\Delta}}{-k - \sqrt{\Delta}} \right) \frac{2k\rho^2 - \text{sign}(\delta)(\text{Tr}(AB) + \sqrt{\Delta})}{2k\sqrt{\det B}(\rho - \text{sign}(\delta)\frac{\eta}{k})} \right| \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_{-1}\right).$$

- $\rho + \frac{\eta}{k} = 0$. In this case

$$\mathcal{R} = \left| \exp\left(\frac{\sqrt{\Delta}}{k}\right) \exp(\rho t_2) \left(\cos t_2 + \frac{m^-}{k} \sin t_2\right) \right| = \frac{-2\eta}{\sqrt{k^2 + \eta^2}} \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_{-1}\right).$$

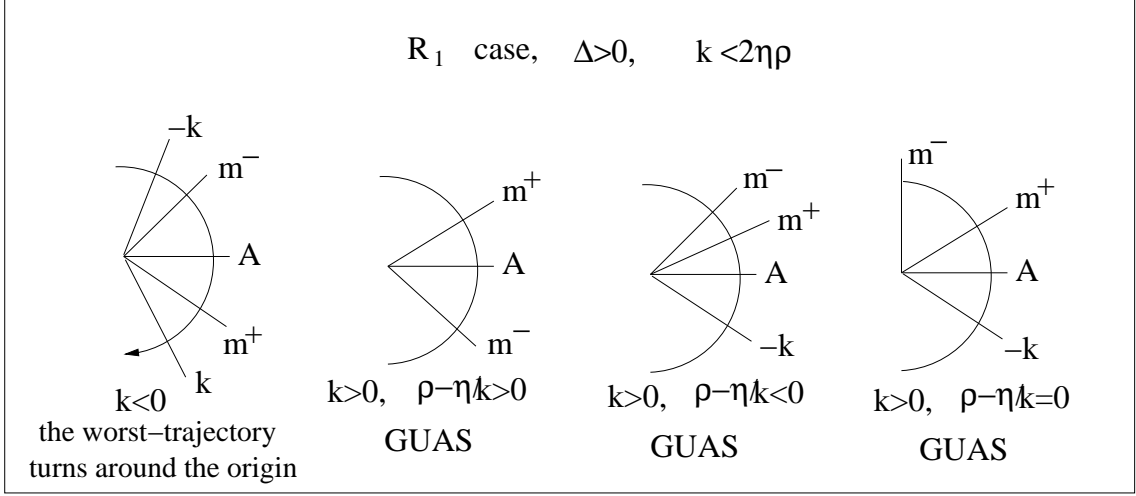
The system is GUAS, uniformly stable (but not GUAS) or unbounded respectively if $\mathcal{R} < 1$, $\mathcal{R} = 1$, $\mathcal{R} > 1$.

6.2 The \mathbf{R}_1 -case

In this section we study the \mathbf{R}_1 -case i.e., the case in which the matrix B has real eigenvalues. In this case since $\delta > 0$, we have $\Delta = k^2 - 4\eta k(\rho - \eta/k)$. Again we restrict to the case $\Delta > 0$, and $k < 2\eta\rho$. Set

$$\chi := \rho - \frac{\eta}{k}. \quad (35)$$

Recall formulas (26) and (27). We have the following cases depicted in Figure 6, for $x_1 > 0$.



i) $k < 0$. In this case we have $\chi < 0$ and $\sqrt{\Delta} > |k|$. It follows that $m^- > 0 > m^+$ and we have:

Claim 1. $k < m^+ < m^- < -k$.

Proof of Claim 1. We have

$$m^+ > k \iff \frac{-k + \sqrt{\Delta}}{2\chi} > k \iff -k + \sqrt{\Delta} < 2\chi k \iff \sqrt{\Delta} < k(1 + 2\chi). \quad (36)$$

Since $1 + \rho < 0$ (cf. Lemma 4), we have $k(1 + 2\chi) = k(1 + 2\rho - 2\eta k) = k((1 + \rho) + \rho - 2\eta k) > 0$. Hence

$$m^+ > k \iff \Delta < k^2(1 + 2\chi)^2 \iff -4\eta k\chi < 4\chi k(k + \chi k) \iff -\eta < k + \chi k \iff 0 < k(\rho + 1). \quad (37)$$

The last inequality holds since $\rho + 1 < 0$. Similarly we prove that $m^- < -k$. ■

ii) $k > 0$ and $\chi > 0$. In this case $\sqrt{\Delta} > |k|$. It follows that $m^- < 0 < m^+$ and we have:

Claim 2. $m^- < -k < m^+ < k$.

Proof of Claim 2. We have

$$\begin{aligned} m^+ > k &\iff \frac{-k + \sqrt{\Delta}}{2\chi} > k \iff -k + \sqrt{\Delta} > 2\chi k \iff \sqrt{\Delta} > k + 2k\chi \iff \Delta > k^2(1 + 2\chi)^2 \\ &\iff -4\eta k\chi > 4\chi k(k + \chi k) \iff -\eta > k + k\rho - \eta \iff 0 > k(\rho + 1). \end{aligned}$$

Since $(\rho + 1) < 0$, the last inequality is always true. In a similar way we can show that $m^- < -k$. ■

iii) $k > 0$ and $\chi < 0$. In this case $\sqrt{\Delta} < |k|$. It follows that $m^- > m^+ > 0$.

iv) $k > 0$ and $\rho - \eta/k = 0$. In this case $m^- = \infty$, $m^+ = -\eta > 0$.

In the case i), the worst-trajectory γ_{x_0} rotates clockwise around the origin (cf. Remark 8) and it is a concatenation of integral curves of Ax (from the line D^+ of equation $x_2 = m^+x_1$ to line D^- of equation

$x_2 = m^-x_1$) and integral curves of Bx otherwise. See again Figure 5. Let t_1 be the time needed to an integral curve of Ax to steer a point from D^+ to D^- . Recall formula (33). We have

$$t_1 = \frac{m^+ - m^-}{m^+m^-} = \frac{\sqrt{\Delta}}{\eta k}.$$

Let t_2 be the time needed to the an integral curve of Bx to steer a point from D^- to D^+ . Since

$$\exp(Bt) = \exp(\rho t) \begin{pmatrix} \cosh t & \frac{\sinh t}{k} \\ k \sinh t & \cosh t \end{pmatrix},$$

we have

$$t_2 = \operatorname{arctanh} \frac{\sqrt{\Delta}}{k(\rho - \frac{\eta}{k}) - \eta}.$$

Defining again \mathcal{R} as in formula (34), we have similarly to the previous case

$$\mathcal{R} = \exp\left(\frac{\sqrt{\Delta}}{k}\right) \exp(\rho t_2) \left| \left(1 + \frac{m^+ \sqrt{\Delta}}{k\eta}\right) (\cosh t_2 + \frac{m^-}{k} \sinh t_2) \right|.$$

After simplification we get

$$\mathcal{R} = \left| \left(\frac{-k + \sqrt{\Delta}}{-k - \sqrt{\Delta}}\right) \frac{2k\rho^2 - \operatorname{Tr}(AB) - \sqrt{\Delta}}{2k\sqrt{\det B}(\rho - \frac{\eta}{k})} \right| \exp\left(\frac{\sqrt{\Delta}}{k}\right) \exp(\rho\theta_1),$$

i.e.,

$$\mathcal{R} = \left| \left(\frac{-k + \sqrt{\Delta}}{-k - \sqrt{\Delta}}\right) \frac{2k\rho^2 - \operatorname{sign}(\delta)(\operatorname{Tr}(AB) + \sqrt{\Delta})}{2k\sqrt{\det B}(\rho - \operatorname{sign}(\delta)\frac{\eta}{k})} \right| \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_1\right).$$

As in the previous case, the system is GUAS, uniformly stable (but not GUAS) or unbounded respectively if $\mathcal{R} < 1$, $\mathcal{R} = 1$, $\mathcal{R} > 1$.

Cases **ii)**, **iii)**, **iv)**, are easily studied projecting the system on $\mathbb{R}P^1$. This is the purpose of the next section.

6.2.1 The system on the projective space

In the \mathbf{R}_1 -case, it may happen that worst trajectory goes to the origin without rotating. In this case the system is GUAS. The description of this fact is contained in the following Lemma, where we project the system on $\mathbb{R}P^1$ using its linearity. Recall Lemma 6.

Lemma 7 *Assume (H0), $\Delta > 0$, $\delta > 0$, $k < 2\eta\rho$. Consider the projective space $\mathbb{R}P^1$ represented by the semicircle $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \text{ and } x_1 \geq 0\}$ with the points $(0, 1)$ and $(0, -1)$ identified. On $\mathbb{R}P^1$ represent the eigenvector of A as the point $p_A = (1, 0)$, the eigenvectors of B as the points*

$$p_B^+ = \left(\frac{1}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}}\right), \quad p_B^- = \left(\frac{1}{\sqrt{1+k^2}}, -\frac{k}{\sqrt{1+k^2}}\right), \quad (38)$$

and the set $\mathcal{Z} \cap \mathbb{R}P^1$ as the couple of points

$$p_m^+ = \left(\frac{1}{\sqrt{1+(m^+)^2}}, \frac{m^+}{\sqrt{1+(m^+)^2}}\right), \quad p_m^- = \left(\frac{1}{\sqrt{1+(m^-)^2}}, \frac{m^-}{\sqrt{1+(m^-)^2}}\right). \quad (39)$$

See Figure 7. Let C_A be the connected component to p_A of the set $\mathbb{R}P^1 \setminus \{p_m^+, p_m^-\}$. If p_B^+ or p_B^- belongs to C_A , then the system is GUAS.

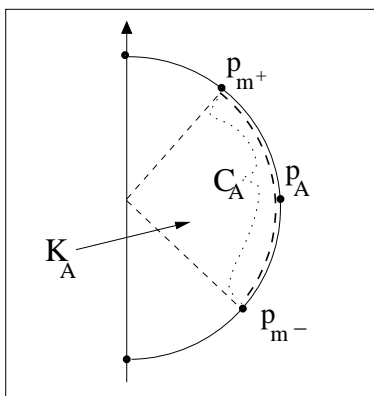


Figure 7:

Sketch of the proof of Lemma 7. By contradiction, if the system is not GUAS, then the worst trajectory rotates around the origin and it switches an infinity number of times on the set \mathcal{Z} . Under the hypotheses of the Lemma, once a trajectory enters the cone

$$K_A = \{x \in \mathbb{R}^2 \setminus \{0\} : \frac{x}{|x|} \in C_A\} \quad (40)$$

it cannot leave it. Indeed, if in K_A the worst trajectory corresponds to the vector field B (resp. A), it cannot cross the eigenvector of B (resp A) lying in C_A , since p_A , p_B^+ and p_B^- are stable points for the system projected on $\mathbb{R}P^1$). Hence this trajectory cannot rotate around the origin. Contradiction. ■

In the cases **ii)**, **iii)**, **iv)**, the system is GUAS thanks to Lemma 7.

6.3 The R_0 Case

In this section we study the R_0 -case, i.e., the case in which the matrix B is also nondiagonalizable. In this case $\Delta = k^2 - 4\eta k\rho$. Again we assume $\Delta > 0$, and $k < 2\eta\rho$. The set \mathcal{Z} is the union of a pair of straight lines with angular coefficients given by formula (28). Notice that $\Delta > 0$, and $k < 2\eta\rho$ implies that $k < 0$ and $m^+ < 0 < m^-$. Indeed if $k > 0$, then $\Delta = k^2 - 4k\eta\rho = k(k - 2\eta\rho) - 2\eta\rho k < 0$. From Lemma 4 we know that B has the unique eigenvector $(0, 1)$. The relative position of m^+ , m^- and of the eigenvectors of A and B is shown in Figure 8.

In this case the worst-trajectory rotates clockwise around the origin and it is a concatenation of integral curves of Ax from the line $D^+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = m^+x_1\}$ to the line $D^- = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = m^-x_1\}$, and integral curves of Bx otherwise. In this case we have

$$\exp(Bt) = \exp(\rho t) \begin{pmatrix} 1 & 0 \\ kt & 1 \end{pmatrix},$$

and the switching times are respectively:

$$t_1 = \frac{m^+ - m^-}{m^+m^-} = \frac{\sqrt{\Delta}}{\eta k} \quad \text{and} \quad t_2 = \frac{\sqrt{\Delta}}{k\rho}.$$

Defining again \mathcal{R} as in formula (34), we have similarly to the previous cases

$$\begin{aligned} \mathcal{R} &= \left| \exp\left(\frac{2\sqrt{\Delta}}{k}\right) \left(1 + \frac{m^+\sqrt{\Delta}}{k\eta}\right) \right| = \left| \left(\frac{-k + \sqrt{\Delta}}{-k - \sqrt{\Delta}}\right) \exp\left(\frac{2\sqrt{\Delta}}{k}\right) \right| \\ &= \left| \left(\frac{-k + \sqrt{\Delta}}{-k - \sqrt{\Delta}}\right) \frac{2k\rho^2 - \text{sign}(\delta)(\text{Tr}(AB) + \sqrt{\Delta})}{2k\sqrt{\det B}(\rho - \text{sign}(\delta)\frac{\eta}{k})} \right| \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_0\right). \end{aligned}$$

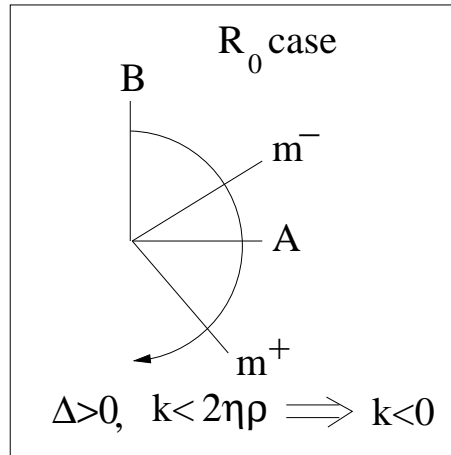


Figure 8:

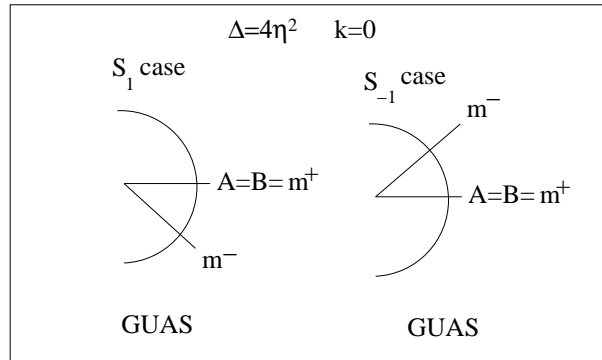


Figure 9:

Again the system is GUAS, uniformly stable (but not GUAS) or unbounded respectively if $\mathcal{R} < 1$, $\mathcal{R} = 1$, $\mathcal{R} > 1$.

6.4 The S-case

In the singular case, the worst-trajectory is always tending to the origin (see Figure 9). This is due to the fact that the straight line $D^+ = \{(h, 0) \in \mathbb{R}^2, h \in \mathbb{R}\}$ (belonging to \mathcal{Z}) coincide with the eigenvector of A and with one eigenvector of B . Using arguments similar to those of Lemma 7, one sees that the worst-trajectory can never cross D^+ and it cannot rotate around the origin. Hence the system is GUAS. One can also check that in this case $V(x) = x_1^2 + x_2^2$ is a common Lyapunov function.

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