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# Remark on the Kato smoothing effect for Schrödinger equation with superquadratic potentials

Luc Robbiano and Claude Zuily

## Résumé

The aim of this note is to extend recent results of Yajima-Zhang [Y-Z1, Y-Z2] on the  $\frac{1}{2}$ -smoothing effect for Schrödinger equation with potential growing at infinity faster than quadratically.

## 1 Introduction

The aim of this note is to extend a recent result by Yajima-Zhang [Y-Z1, Y-Z2]. In this paper these authors considered the Hamiltonian  $H = -\Delta + V(x)$  where  $V$  is a real and  $C^\infty$  potential on  $\mathbb{R}^n$  satisfying for some  $m > 2$  and  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ ,

$$(1.1) \quad |\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{m-|\alpha|}, \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{N}^n,$$

$$(1.2) \quad \text{for large } |x|, \quad V(x) \geq C_1 |x|^m, \quad C_1 > 0,$$

and they proved the following. For any  $T > 0$  and  $\chi \in C_0^\infty(\mathbb{R}^n)$  one can find  $C > 0$  such that for all  $u_0$  in  $L^2(\mathbb{R}^n)$ ,

$$(1.3) \quad \int_0^T \|\chi(I - \Delta)^{\frac{1}{2m}} e^{-itH} u_0\|_{L^2(\mathbb{R}^n)}^2 dt \leq C \|u_0\|_{L^2(\mathbb{R}^n)}^2$$

where  $\Delta$  is the flat Laplacian. In this note, using the ideas contained in Doï [D3] we shall show that one can handle variable coefficients Laplacian with time dependent potentials, one can remove the condition (1.2), one can replace the cut-off function  $\chi$  in (1.3) by  $\langle x \rangle^{-\frac{1+\nu}{2}}$  with any  $\nu > 0$  and finally that the weight  $\langle x \rangle^{-\frac{1}{2}}$  is enough for the tangential derivatives. When  $V = 0$  the estimate (1.3) goes back to Constantin-Saut [C-S], Sjölin [S], Vega [V], Yajima [Y] who extended to the Schrödinger equation a phenomenon discovered by T. Kato [K] on the KdV equation. Later on their results were extended to the variable coefficients operators by Doï in a series of papers [D1, D2, D3, D4] which contained the case  $m = 2$  of Theorem 1.1 below.

Let us describe more precisely our result. It will be convenient to introduce the Hörmander's metric

$$(1.4) \quad g = \frac{dx^2}{\langle x \rangle^2} + \frac{d\xi^2}{\langle \xi \rangle^2}$$

to which we associate the usual class of symbols  $S(M, g)$  if  $M$  is a weight. Recall that  $q \in S(M, g)$  iff  $q \in C^\infty(\mathbb{R}^{2n})$  and

$$\forall \alpha, \beta \in \mathbb{N}^n \exists C_{\alpha\beta} > 0, |\partial_x^\beta \partial_\xi^\alpha q(x, \xi)| \leq C_{\alpha\beta} M(x, \xi) \langle x \rangle^{-|\beta|} \langle \xi \rangle^{-|\alpha|}, \quad \forall (x, \xi) \in T^*(\mathbb{R}^n)$$

If  $T > 0$  we shall set

$$(1.5) \quad S_T(M, g) = L^\infty([0, T], S(M, g)).$$

We shall consider here an operator  $P$  of the form

$$(1.6) \quad P = \sum_{j,k=1}^n (D_j - a_j(t, x)) g^{jk}(x) (D_k - a_k(t, x)) + V(t, x)$$

and we shall denote by  $p$  the principal symbol of  $P$ , namely

$$(1.7) \quad p(x, \xi) = \sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k.$$

We shall make the following structure and geometrical assumptions.

Structure assumptions. We shall assume the following,

$$(1.8) \quad \left\{ \begin{array}{l} (i) \text{ the coefficients } a_j, g^{jk}, V \text{ are real valued for } j, k = 1, \dots, n, \\ (ii) p \in S(\langle \xi \rangle^2, g) \text{ and } \nabla g^{jk}(x) = o(|x|^{-1}), |x| \rightarrow +\infty \quad 1 \leq j, k \leq n \\ (iii) a_j \in S_T(\langle x \rangle^{\frac{m}{2}}, g), \quad 1 \leq j \leq n, \quad V \in S_T(\langle x \rangle^m, g) \quad m \geq 2 \end{array} \right.$$

$$(1.9) \quad \exists \delta > 0, \quad p(x, \xi) \geq \delta |\xi|^2, \quad \forall (x, \xi) \in T^*(\mathbb{R}^n).$$

$$(1.10) \quad \text{For any fixed } t \text{ in } [0, T] \text{ the operator } P \text{ is essentially self adjoint on } L^2(\mathbb{R}^n)$$

Geometrical assumptions. Let  $\phi_t$  be the bicharacteristic flow of  $p$ . It is easy to see that under the conditions (1.8), (1.9) it is defined for all  $t \in \mathbb{R}$ . Let us set  $S^*(\mathbb{R}^n) = \{(x, \xi) \in T^*(\mathbb{R}^n) : p(x, \xi) = 1\}$ . Then we shall assume that,

$$(1.11) \quad \forall K \text{ compact } \subseteq S^*(\mathbb{R}^n) \exists t_K > 0 \text{ such that } \Phi_t(K) \cap K = \emptyset, \quad \forall t \geq t_K.$$

This is the so-called "non trapping condition" which is equivalent to the fact that if  $\Phi_t(x; \xi) = (x(t), (\xi(t)))$  then  $\lim_{t \rightarrow +\infty} |x(t)| = +\infty$ .

We shall consider  $u \in C^1([0, T], \mathcal{S}(\mathbb{R}^n))$  and we set

$$(1.12) \quad f(t) = (D_t + P)u(t)$$

For  $s \in \mathbb{R}$  let  $e_s(x, \xi) = (1 + |\xi|^2 + |x|^m)^{\frac{s}{2}}$  and  $E_s$  be the Weyl quantized pseudo-differential operator with symbol  $e_s$ .

Our first result is the following.

**Theorem 1.1** *Let  $T > 0$ . Let  $P$  be defined by (1.6) which satisfies (1.8), (1.9), (1.10), (1.11). Then for any  $\nu > 0$  one can find  $C = C(\nu, T) > 0$  such that for any  $u \in C^1([0, T], \mathcal{S}(\mathbb{R}^n))$  and all  $t$  in  $[0, T]$  we have,*

$$\|u(t)\|_{L^2}^2 + \int_0^T \|\langle x \rangle^{-\frac{1+\nu}{2}} E_{\frac{1}{m}} u(t)\|_{L^2}^2 dt \leq C (\|u(0)\|_{L^2}^2 + \int_0^T \|\langle x \rangle^{\frac{1+\nu}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2 dt).$$

Here  $L^2 = L^2(\mathbb{R}^n)$  and  $f(t)$  is defined by (1.12).

Now even when  $P$  is the flat Laplacian it is known that the estimate in the above Theorem does not hold with  $\nu = 0$ . However we have the following result. Let us set

$$(1.13) \quad \ell_{jk} = \frac{x_j \xi_k - x_k \xi_j}{\langle x \rangle \langle \xi \rangle}, \quad 1 \leq j, k \leq n,$$

and let us denote by  $\ell_{jk}^w$  its Weyl quantization.

**Theorem 1.2** *Let  $T > 0$ . Let  $P$  be defined by (1.6) with real coefficients satisfying (1.9), (1.10), (1.11) and*

$$(1.14) \quad \begin{cases} (i) & g_{jk} = \delta_{jk} + b_{jk}, \quad b_{jk} \in S(\langle x \rangle^{-\sigma_0}, g), \text{ for some } \sigma_0 > 0, \\ (ii) & a_j \in S_T(\langle x \rangle^{\frac{m}{2}}, g), \quad V \in S_T(\langle x \rangle^m, g). \end{cases}$$

*Then for any  $\nu > 0$  one can find  $C = C(\nu, T)$  such that for any  $u \in C^1([0, T], \mathcal{S}(\mathbb{R}^n))$  and  $f(t) = (D_t + P)u(t)$  we have*

$$\sum_{j,k=1}^n \int_0^T \|\langle x \rangle^{-\frac{1}{2}} E_{\frac{1}{m}} \ell_{jk}^w u(t)\|_{L^2}^2 dt \leq C (\|u(0)\|_{L^2}^2 + \int_0^T \|\langle x \rangle^{\frac{1+\nu}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2 dt).$$

Here are some remarks and examples.

**Remark 1.3** *1) We know that one can find  $\psi \in C_0^\infty(|x| < 1)$  and  $\phi \in C_0^\infty(\frac{1}{2} \leq |x| \leq 2)$  positive such that  $\psi(x) + \sum_{j=0}^{+\infty} \phi(2^{-j}x) = 1$ , for all  $x$  in  $\mathbb{R}^n$ . Let  $V = |x|^m \sum_{j \text{ even}} \phi(2^{-j}x) - |x|^2 \sum_{j \text{ odd}} \phi(2^{-j}x)$ . Then  $V \in S(\langle x \rangle^m, g)$  and since  $V \geq -|x|^2$  the operator  $P = -\Delta + V$  is essentially self adjoint on  $C_0^\infty(\mathbb{R}^n)$ . It follows that (1.9), (1.10), (1.11) and (1.14) are satisfied, therefore Theorem 1.1 and 1.2 apply. However the lower bound (1.2) assumed in [Y-Z2] is not satisfied.*

*2) Assume that  $p(x, \xi) = |\xi|^2 + \varepsilon \sum_{j,k=1}^n b_{jk}(x) \xi_j \xi_k$  with  $b_{jk} \in S(\langle x \rangle^{-\sigma_0}, g)$  for some  $\sigma_0 > 0$ .*

*Then if  $\varepsilon$  is small enough the non trapping condition (1.11) is satisfied.*

## 2 Proofs of the results

Let us consider the symbol  $a_0(x, \xi) = \frac{x \cdot \xi}{\langle \xi \rangle}$ . A straightforward computation shows that under condition (1.8) (ii) one can find  $C_0, C_1, R$  positive such that

$$(2.1) \quad H_p a_0(x, \xi) \geq C_0 |\xi| - C_1, \text{ if } (x, \xi) \in T^*(\mathbb{R}^n) \text{ and } |x| \geq R.$$

where  $H_p$  denotes the Hamiltonian field of the symbol  $p$ .

Then we have the following result due to Doi [D3].

**Lemma 2.1** *Assume moreover that (1.11) is satisfied then there exist  $a \in S(\langle x \rangle, g)$  and positive constants  $C_2, C_3$  such that*

- (i)  $H_p a(x, \xi) \geq C_2 |\xi| - C_3, \quad \forall (x, \xi) \in T^*(\mathbb{R}^n),$
- (ii)  $a(x, \xi) = a_0(x, \xi), \quad \text{if } |x| \text{ is large enough.}$

The symbol  $a$  is called a global escape function for  $p$ . Here is the form of this symbol. Let  $\chi \in C_0^\infty(\mathbb{R}^n)$  be such that  $\chi(x) = 1$  if  $|x| \leq 1$ ,  $\chi(x) = 0$  if  $|x| \geq 2$  and  $0 \leq \chi \leq 1$ . With  $R$  large enough and  $M \geq 2R$  we have,

$$a(x, \xi) = a_0(x, \xi) + M^{\frac{1}{2}} \chi\left(\frac{x}{M}\right) a_1\left(x, \frac{\xi}{\sqrt{p(x, \xi)}}\right) (1 - \theta(\sqrt{p(x, \xi)}))$$

where

$$a_1(x, \xi) = - \int_0^{+\infty} \chi\left(\frac{1}{R} \pi(\Phi_t(x, \xi))\right) dt$$

and  $\pi(\Phi_t(x, \xi)) = x(t; x, \xi)$ ,  $\theta(t) = 1$  if  $0 \leq t \leq 1$ ,  $\theta(t) = 0$  if  $t \geq 2$ ,  $0 \leq \theta \leq 1$ . Details can be found in [D3].

Proof of Theorem 1.1

Let  $\psi \in C^\infty(\mathbb{R}^n)$  be such that  $\text{supp } \psi \in [\varepsilon, +\infty[$ ,  $\psi(t) = 1$  in  $[2\varepsilon, +\infty[$  (where  $\varepsilon > 0$  is a small constant chosen later on) and  $\psi'(t) \geq 0$  for  $t \in \mathbb{R}$ . Following Doi [D3] we set,

$$(2.2) \quad \begin{cases} \psi_0(t) = 1 - \psi(t) - \psi(-t) = 1 - \psi(|t|) \\ \psi_1(t) = \psi(-t) - \psi(t) = -\text{sgn } t \psi(|t|) \end{cases}$$

Then  $\psi_j \in C^\infty(\mathbb{R})$ , for  $j = 0, 1$  and we have

$$(2.3) \quad \psi_0'(t) = -\text{sgn } t \psi'(|t|) \quad \text{and} \quad \psi_1'(t) = -\psi'(|t|).$$

Let  $\chi \in C^\infty(\mathbb{R})$  be such that  $\chi(t) = 1$  if  $t \leq \frac{1}{2}$ ,  $\chi(t) = 0$  if  $t \geq 1$  and  $\chi(t) \in [0, 1]$ . With  $a$  given by Lemma 2.1 we set

$$(2.4) \quad \begin{cases} \theta(x, \xi) = \frac{a(x, \xi)}{\langle x \rangle}, & (x, \xi) \in T^*(\mathbb{R}^n), \\ r(x, \xi) = \frac{\langle x \rangle^{\frac{m}{2}}}{\sqrt{p(x, \xi)}}, & (x, \xi) \in T^*(\mathbb{R}^n) \setminus 0. \end{cases}$$

Finally we set

$$(2.5) \quad -\lambda = \left( \frac{a}{\langle x \rangle} \psi_0(\theta) - (M_0 - \langle a \rangle^{-\nu}) \psi_1(\theta) \right) p^{\frac{1}{m} - \frac{1}{2}} \chi(r),$$

where  $\nu > 0$  is an arbitrary small constant and  $M_0$  a large constant to be chosen. The main step of the proof is the following Lemma.

**Lemma 2.2** (i) *One can find  $M_0 > 0$  such that for any  $\nu > 0$  there exist positive constants  $C, C'$  such that*

$$(2.6) \quad -H_p \lambda(x, \xi) \geq C \langle x \rangle^{-1-\nu} (|\xi|^2 + |x|^m)^{\frac{1}{m}} - C', \quad \forall (x, \xi) \in T^*(\mathbb{R}^n),$$

$$(ii) \quad \lambda \in S(1, g),$$

$$(iii) \quad [P, \lambda^w] - \frac{1}{i} (H_p \lambda)^w \in Op^w S_T(1, g).$$

Proof

First of all on the support of  $\chi(r)$  we have  $\langle x \rangle^{\frac{m}{2}} \leq \sqrt{p(x, \xi)} \leq C|\xi|$ . It follows that  $|\xi| \sim \langle \xi \rangle$  and  $|\xi| \leq |\xi| + \langle x \rangle^{\frac{m}{2}} \leq C'|\xi|$ . Now

$$(2.7) \quad -H_p \lambda = \sum_{j=1}^6 A_j$$

where the  $A_j$ 's are defined below.

1)  $A_1 = (H_p \langle x \rangle^{-1}) p^{\frac{1}{m} - \frac{1}{2}} a \psi_0(\theta) \chi(r)$ . Since on the support of  $\psi_0(\theta)$  we have  $|a| \leq 2\varepsilon \langle x \rangle$ , it is easy to see that

$$(2.8) \quad |A_1| \leq C_1 \varepsilon \langle x \rangle^{-1} |\xi|^{\frac{2}{m}} (1 - \psi(|\theta|)) \chi(r).$$

2)  $A_2 = \langle x \rangle^{-1} p^{\frac{1}{m} - \frac{1}{2}} (H_p a) \psi_0(\theta) \chi(r)$ . By Lemma 2.1 (i) we have

$$(2.9) \quad A_2 \geq C_2 \langle x \rangle^{-1} (|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} (1 - \psi(|\theta|)) \chi(r) - C'_2.$$

3)  $A_3 = \langle x \rangle^{-1} p^{\frac{1}{m} - \frac{1}{2}} a \psi'_0(\theta) (H_p \theta) \chi(r)$ . It follows from (2.3), (2.4) that

$$(2.10) \quad A_3 = -p^{\frac{1}{m} - \frac{1}{2}} |\theta| (H_p \theta) \psi'(|\theta|) \chi(r)$$

4)  $A_4 = p^{\frac{1}{m} - \frac{1}{2}} (H_p \langle a \rangle^{-\nu}) \psi_1(\theta) \chi(r)$ . Here we have  $H_p \langle a \rangle^{-\nu} = -\nu \langle a \rangle^{-2-\nu} a H_p a$ . It follows from (2.2) that  $A_4 = \nu p^{\frac{1}{m} - \frac{1}{2}} |a| \langle a \rangle^{-2-\nu} (H_p a) \psi(|\theta|) \chi(r)$ . Now on the support of  $\psi(|\theta|)$  we have  $\varepsilon \langle x \rangle \leq |a|$  and since  $a \in S(\langle x \rangle, g)$  we have  $|a| \leq C \langle x \rangle$ . It follows from Lemma 2.1 (i) that

$$(2.11) \quad A_4 \geq C_3 \langle x \rangle^{-1-\nu} (|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} \psi(|\theta|) \chi(r) - C'_3.$$

5)  $A_5 = -p^{\frac{1}{m} - \frac{1}{2}} (M_0 - \langle a \rangle^{-\nu}) (H_p \theta) \psi'_1(\theta) \chi(r)$ . It follows from (2.3) that

$$(2.12) \quad A_5 = p^{\frac{1}{m} - \frac{1}{2}} (M_0 - \langle a \rangle^{-\nu}) (H_p \theta) \psi'(|\theta|) \chi(r)$$

We deduce from (2.10) and (2.12) that

$$A_3 + A_5 = p^{\frac{1}{m}-\frac{1}{2}}(M_0 - \langle a \rangle^{-\nu} - |\theta|)(H_p \theta) \psi'(|\theta|) \chi(r)$$

Now  $H_p \theta = \langle x \rangle^{-1} H_p a + a H_p \langle x \rangle^{-1}$ . Since  $|a| \leq 2\varepsilon |\theta|$  on the support of  $\psi'(|\theta|)$  we deduce that  $H_p \theta \geq C_4 \langle x \rangle^{-1} |\xi| - C_5 \geq -C_5$ . Taking  $M_0 \geq 2$  and using the facts that  $\psi' \geq 0$ ,  $\chi \geq 0$  and  $\varepsilon \leq |\theta| \leq 2\varepsilon$  on the support of  $\psi'(|\theta|)$  we obtain

$$(2.13) \quad A_3 + A_5 \geq -C_6$$

6)  $A_6 = (\langle x \rangle^{-1} a \psi_0(\theta) - (M_0 - \langle a \rangle^{-\nu}) \psi_1(\theta)) p^{\frac{1}{m}-\frac{1}{2}} H_p[\chi(r)]$ . We have  $H_p[\chi(r)] = \frac{1}{\sqrt{p}} (H_p \langle x \rangle^{\frac{m}{2}}) \chi'(r)$ .

On the support of  $\chi'(r)$  we have  $\langle x \rangle \sim |\xi|^{\frac{2}{m}}$ ; this implies that

$$p^{\frac{1}{m}-\frac{1}{2}} |H_p[\chi(r)]| \leq C |\xi|^{\frac{2}{m}-1} \frac{|\xi| \langle x \rangle^{\frac{m}{2}-1}}{|\xi|} |\chi'(r)| \leq C_7.$$

Therefore we obtain

$$(2.14) \quad |A_6| \leq C_8.$$

Gathering the estimates obtained in (2.8) to (2.14) we obtain

$$(2.15) \quad -H_p \lambda \geq C_9 \langle x \rangle^{-1-\nu} (|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} \chi(r) - C_{10}.$$

Now on the support of  $1 - \chi(r)$  we have  $|\xi| \leq C_{11} \langle x \rangle^{\frac{m}{2}}$  so  $\langle x \rangle^{-1-\nu} (|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} \leq C_{12}$ . Therefore writing  $1 = 1 - \chi + \chi$  and using (2.15) we obtain (2.6).

(ii) We use the symbolic calculus in the classes  $S(M, g)$ . We have  $\langle x \rangle^{-1} \in S(\langle x \rangle^{-1}, g)$ ,  $a \in S(\langle x \rangle, g)$ ,  $p \in S(\langle \xi \rangle^2, g)$  so  $p^{\frac{1}{m}-\frac{1}{2}} \in S(\langle \xi \rangle^{\frac{2}{m}-1}, g)$  since  $p \geq C > 0$  on  $\text{supp } \chi(r)$ . Moreover  $\chi(r) \in S(1, g)$  and on  $\text{supp } \chi(r)$  we have  $\langle x \rangle^{\frac{m}{2}} \leq C |\xi|$ . It follows that  $\lambda \in S(\langle \xi \rangle^{\frac{2}{m}-1}, g) \subset S(1, g)$ .

(iii) By the symbolic calculus  $\{\lambda, V\} \in S_T(\langle \xi \rangle^{\frac{2}{m}-1} \langle x \rangle^m \langle x \rangle^{-1} \langle \xi \rangle^{-1}, g)$ . Since we have  $\langle x \rangle^{\frac{m}{2}} \leq C |\xi|$  on its support we will have  $\langle x \rangle^{m-1} \langle \xi \rangle^{\frac{2}{m}-2} \leq C |\xi|^{\frac{2}{m}(m-1)} \langle \xi \rangle^{\frac{2}{m}-2} \leq C'$ . Therefore  $\{\lambda, V\} \in S_T(1, g)$ . Now if  $b \in S_T(\langle x \rangle^{\frac{m}{2}}, g)$  we have  $\{\lambda, b \xi_j\} \in S(\langle \xi \rangle^{\frac{2}{m}-1} \langle x \rangle^{\frac{m}{2}} |\xi| \langle x \rangle^{-1} \langle \xi \rangle^{-1}, g)$  and since  $\langle x \rangle^{\frac{m}{2}} \leq C |\xi|$  we have  $\langle x \rangle^{\frac{m}{2}-1} \langle \xi \rangle^{\frac{2}{m}-1} \leq C |\xi|^{\frac{2}{m}(\frac{m}{2}-1)} \langle \xi \rangle^{\frac{2}{m}-1} \leq C'$  so  $\{\lambda, b \xi_j\} \in S_T(1, g)$ .

Finally  $[\text{Op}^w(p), \lambda^w] - \frac{1}{i} (H_p \lambda)^w \in S(\langle \xi \rangle^2 \langle \xi \rangle^{\frac{2}{m}-1} \langle x \rangle^{-2} \langle \xi \rangle^{-2}, g) \subset \text{Op}^w S(1, g)$ . ■

End of the proof of Theorem 1.1.

Since  $\lambda \in S(1, g)$  we can set  $M = 1 + \sup_{(x, \xi) \in \mathbb{R}^{2n}} |\lambda(x, \xi)|$ . Let us introduce  $N(t) = ((M + \lambda^w)u(t), u(t))_{L^2(\mathbb{R}^n)}$ . Then there exist absolute constants  $C_1 > 0$ ,  $C_2 > 0$  such that  $C_1 \|u(t)\|_{L^2}^2 \leq N(u(t)) \leq C_2 \|u(t)\|_{L^2}^2$ . Now

$$\frac{d}{dt} N(t) = ((M + \lambda^w) \frac{\partial u}{\partial t}(t), u(t))_{L^2} + ((M + \lambda^w)u(t), \frac{\partial u}{\partial t}(t))_{L^2}$$

Since  $\frac{\partial u}{\partial t}(t) = -iPu(t) + if(t)$  and  $P^* = P$  we obtain

$$\begin{aligned} \frac{d}{dt}N(t) &= i([P, \lambda^w]u(t), u(t))_{L^2} - 2\operatorname{Im}((M + \lambda^w)f(t), u(t))_{L^2} \\ &= -((-H_p\lambda)^w u(t), u(t))_{L^2} - 2\operatorname{Im}((M + \lambda^w)f(t), u(t))_{L^2} + O(\|u(t)\|_{L^2}^2) \end{aligned}$$

By lemma 2.2 (iii).

Now by Lemma 2.2 (i) and the sharp Gårding inequality, we obtain

$$(2.16) \quad ((-H_p\lambda)^w u(t), u(t))_{L^2} \geq C\|\langle x \rangle^{-\frac{1+\nu}{2}} E_{\frac{1}{m}} u(t)\|_{L^2}^2 - C'\|u(t)\|_{L^2}^2$$

On the other hand we have for any  $\varepsilon > 0$

$$(2.17) \quad |((M + \lambda^w)f(t), u(t))_{L^2}| \leq \varepsilon\|\langle x \rangle^{-\frac{1+\nu}{2}} E_{\frac{1}{m}} u(t)\|_{L^2}^2 + C_\varepsilon\|\langle x \rangle^{\frac{1+\nu}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2$$

Using (2.16) and (2.17) with  $\varepsilon$  small enough, we obtain

$$\frac{d}{dt}N(t) \leq -C_1\|\langle x \rangle^{-\frac{1+\nu}{2}} E_{\frac{1}{m}} u(t)\|_{L^2}^2 + C_2\|\langle x \rangle^{\frac{1+\nu}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2 + C_3N(t)$$

Integrating this inequality between 0 and  $t$  (in  $[0, T]$ ) and using Gronwall's inequality, we obtain the conclusion of Theorem 1.1.  $\blacksquare$

Proof of theorem 1.2.

Let  $\chi \in C_0^\infty(\mathbb{R}^+)$ ,  $\chi(t) = 1$  if  $t \in [0, 1]$ ,  $\chi(t) = 0$  if  $t \geq 2$ . Recall that according to (1.14) we have  $p = |\xi|^2 + q(x, \xi)$  where  $q(x, \xi) = \sum_{j,k=1}^n b^{jk}(x)\xi_j\xi_k$  and  $b^{jk} \in S(\langle x \rangle^{-\sigma_0}, g)$ . Let us set

$$(2.18) \quad A_{jk} = \frac{x_j\xi_k - x_k\xi_j}{\langle \xi \rangle}, \quad 1 \leq j, k \leq n$$

Then we have the following result.

**Lemma 2.3** *Let  $a$  be defined in Lemma 2.1. One can find positive constants  $C_0, C_1$  and  $C_2$  such that if we set*

$$(2.19) \quad -\lambda = \frac{a}{(1 + a^2 + \sum_{j,k=1}^n A_{jk}^2)^{\frac{1}{2}}} p^{\frac{1}{m}-\frac{1}{2}} \chi\left(\frac{\langle x \rangle^{\frac{m}{2}}}{\sqrt{p(x, \xi)}}\right)$$

then

- (i)  $-H_p\lambda \geq C_0\langle x \rangle^{-3}(|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} \sum_{j,k=1}^n A_{jk}^2 - C_1\langle x \rangle^{-1-\sigma_0}(|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} - C_2,$
- (ii)  $\lambda \in S(\langle \xi \rangle^{\frac{2}{m}-1}, g),$
- (iii)  $[P, \lambda^w] - \frac{1}{i}(H_p\lambda)^w \in \operatorname{Op}^w S_T(1, g).$



Proof

First of all we have

$$(2.20) \quad |H_p A_{jk}(x, \xi)| \leq C_1 \frac{|\xi|}{\langle x \rangle^{\sigma_0}}, \quad 1 \leq j, k \leq n, \quad (x, \xi) \in T^*(\mathbb{R}^n).$$

Indeed we have  $\{|\xi|^2, A_{jk}\} = 0$  and  $|\{q, A_{jk}\}| \leq C_2 \frac{|\xi|}{\langle x \rangle^{\sigma_0}}$ .

Let us set

$$(2.21) \quad D = 1 + a^2 + \sum_{j,k=1}^n A_{jk}^2.$$

We claim that on the support of  $\chi(\langle x \rangle^{\frac{m}{2}} p^{-\frac{1}{2}})$  we have

$$(2.22) \quad C_3 \langle x \rangle^2 \leq D \leq C_4 \langle x \rangle^2$$

for some positive constants  $C_3$  and  $C_4$ .

Indeed a straightforward computation shows that

$$(x \cdot \xi)^2 + \sum_{j,k=1}^n (x_j \xi_k - x_k \xi_j)^2 \geq |x|^2 |\xi|^2.$$

Since by Lemma 2.1 we have  $a(x, \xi) = \frac{x \cdot \xi}{\langle \xi \rangle}$  for  $|x| \geq R_0 \gg 1$  and  $|\xi| \geq C_5 > 0$  on the support of  $\chi$  we deduce that  $D \geq C_6 \langle x \rangle^2$  when  $|x| \geq R_0$ . When  $|x| \leq R_0$  we have  $D \geq 1 \geq \frac{1}{1 + R_0^2} \langle x \rangle^2$ .

Now we can write with  $r(x, \xi) = \langle x \rangle^{\frac{m}{2}} p^{-\frac{1}{2}}$ ,

$$(2.23) \quad \begin{cases} -H_p \lambda = I_1 + I_2 \\ I_1 = D^{-\frac{3}{2}} (D(H_p a) - \frac{1}{2} a(H_p D)) p^{\frac{1}{m} - \frac{1}{2}} \chi(r) \\ I_2 = p^{\frac{1}{m} - \frac{1}{2}} a D^{-\frac{1}{2}} H_p(\chi(r)) \end{cases}$$

We have

$$\begin{aligned} D H_p a - \frac{1}{2} a(H_p D) &= (1 + \sum_{j,k=1}^n A_{jk}^2) H_p a + a^2 H_p a - \frac{1}{2} a(2a H_p a + 2 \sum_{j,k=1}^n A_{jk} H_p A_{jk}) \\ &= (1 + \sum_{j,k=1}^n A_{jk}^2) H_p a - a \sum_{j,k=1}^n A_{jk} H_p A_{jk}. \end{aligned}$$

Using (2.18) and (2.20) we see that,

$$(2.24) \quad |a| \sum_{j,k=1}^n |A_{jk}| |H_p A_{jk}| \leq C_7 |x|^2 \frac{|\xi|}{\langle x \rangle^{\sigma_0}}.$$

Moreover by Lemma 2.1 we have on the support of  $\chi(r)$ ,

$$(2.25) \quad p^{\frac{1}{m}-\frac{1}{2}} \left(1 + \sum_{j,k=1}^n A_{jk}^2\right) H_p a \geq \left(1 + \sum_{j,k=1}^n A_{jk}^2\right) (C_8 (|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} - C_9).$$

Therefore (2.21), (2.23), (2.24), (2.25) show that,

$$I_1 \geq \left[ C_{10} \langle x \rangle^{-3} (|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} \sum_{j,k=1}^n A_{jk}^2 - C_{11} \frac{|\xi|}{\langle x \rangle^{1+\sigma_0}} \right] \chi(r).$$

On the support of  $1 - \chi(r)$  we have  $|\xi| \leq \langle x \rangle^{\frac{m}{2}}$  so we obtain,

$$(2.26) \quad I_1 \geq C_{12} \langle x \rangle^{-3} (|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}} \sum_{j,k=1}^n A_{jk}^2 - C_{13} \frac{(|\xi| + \langle x \rangle^{\frac{m}{2}})^{\frac{2}{m}}}{\langle x \rangle^{1+\sigma_0}} - C_{14}.$$

On the other hand we have,

$$|H_p(\chi(r))| = |p^{-\frac{1}{2}} \chi'(r) H_p \langle x \rangle^{\frac{m}{2}}| \leq \frac{C_{15}}{|\xi|} |\chi'(r)| |\xi| \langle x \rangle^{\frac{m}{2}-1}.$$

It follows from (2.22) and the estimate  $|a| \leq C_{16} \langle x \rangle$  that,

$$(2.27) \quad |I_2| \leq C_{17},$$

since  $\langle x \rangle^{\frac{m}{2}-1} |\xi|^{\frac{2}{m}-1} \leq C_{18}$ .

Then (i) in lemma 2.3 follows from (2.23), (2.26) and (2.27). The proofs of (ii) and (iii) are the same as those in the proof of lemma 2.2. ■

End of the proof of Theorem 1.2.

We introduce as before, for  $t$  in  $(0, T)$ .

$$N(t) = ((M_0 + \lambda^w)u(t), u(t))_{L^2}$$

Where  $M_0$  is a large constant. Then  $N(t) \sim \|u(t)\|_{L^2}^2$ .

Now using the equation and Lemma 2.3 (iii) we can write,

$$\frac{d}{dt} N(t) = -((-H_p \lambda)^w u(t), u(t))_{L^2} - 2 \operatorname{Im}((M_0 + \lambda^w) f(t), u(t))_{L^2} + O(\|u(t)\|_{L^2}^2)$$

Since by (1.13) and (2.19) we have  $\langle x \rangle^{-2} A_{jk}^2 = \ell_{jk}^2$ , Lemma 2.3 (i) and the sharp Gårding inequality ensure that

$$\begin{aligned} \frac{d}{dt} N(t) &\leq -C_1 \sum_{j,k=1}^n \|\langle x \rangle^{-\frac{1}{2}} E_{\frac{1}{m}} \ell_{jk}^w u(t)\|_{L^2}^2 + C_2 \|\langle x \rangle^{-\frac{1+\sigma_0}{2}} E_{\frac{1}{m}} u(t)\|_{L^2}^2 \\ &\quad + \|\langle x \rangle^{\frac{1+\sigma_0}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2 + C_3 N(t). \end{aligned}$$

It follows that for  $0 < t < T$ ,

(2.28)

$$\begin{aligned} N(t) + C_1 \int_0^t \sum_{j,k=1}^n \|\langle x \rangle^{-\frac{1}{2}} E_{\frac{1}{m}} \ell_{jk}^w u(s)\|_{L^2}^2 ds &\leq N(0) + C_2 \int_0^T \|\langle x \rangle^{-\frac{1+\sigma_0}{2}} E_{\frac{1}{m}} u(s)\|_{L^2}^2 ds \\ &\quad + \int_0^T \|\langle x \rangle^{\frac{1+\sigma_0}{2}} E_{-\frac{1}{m}} f(s)\|_{L^2}^2 ds + C_3 \int_0^t N(s) ds. \end{aligned}$$

Using Theorem 1.1 to bound the second term in the right hand side and then using the Gronwall inequality we obtain

$$N(t) \leq C(T) (\|u(0)\|_{L^2}^2 + \int_0^T \|\langle x \rangle^{\frac{1+\sigma_0}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2 dt).$$

Using again the inequality (2.28) we obtain the conclusion of Theorem 1.2. The proof is complete. ■

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