



HAL
open science

Upper limits of Sinai's walk in random scenery

Olivier Zindy

► **To cite this version:**

| Olivier Zindy. Upper limits of Sinai's walk in random scenery. 2006. hal-00105215v1

HAL Id: hal-00105215

<https://hal.science/hal-00105215v1>

Preprint submitted on 10 Oct 2006 (v1), last revised 8 Nov 2007 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Upper limits of Sinai's walk in random scenery

by

Olivier Zindy

Université Paris VI

Summary. We consider Sinai's walk in i.i.d. random scenery and focus our attention on a conjecture of Révész [17] concerning the upper limits of Sinai's walk in random scenery when the scenery is bounded from above. A close study of the competition between the concentration property for Sinai's walk and negative values for the scenery enables us to prove that the conjecture is true if the scenery has "thin" negative tails and is false otherwise.

Keywords. Random walk in random environment, random scenery, localization, concentration property.

2000 Mathematics Subject Classification. 60K37, 60G50, 60J55.

1 Introduction

1.1 Random walk in random environment

Problems involving random environments arise in different domains of physics and biology. Originally, one-dimensional random walk in random environment appeared as a simple model for DNA transcription. In the following, we consider the elementary model of one-dimensional Simple Random Walk in Random Environment (SRWRE), defined as follows.

Let $\omega := (\omega_i, i \in \mathbb{Z})$ be a family of independent and identically distributed (i.i.d.) random variables defined on Ω , which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation.

Conditioning on ω (i.e., choosing an environment), we define the SRWRE $(X_n, n \geq 0)$ as a nearest-neighbor random walk on \mathbb{Z} with transition probabilities given by ω : $(X_n, n \geq 0)$ is a Markov chain satisfying $X_0 = 0$ and

$$P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} \omega_i & \text{if } j = i + 1, \\ 1 - \omega_i & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by P_ω the law of $(X_n, n \geq 0)$, by E_ω the corresponding expectation, and by \mathbb{P} the joint law of $(\omega, (X_n)_{n \geq 0})$. We refer to Zeitouni [22] for an overview of random walks in random environment.

Throughout the paper, we make the following assumptions on ω :

$$(1.1) \quad \exists \delta \in (0, 1/2) : \quad P(\delta \leq \omega_0 \leq 1 - \delta) = 1,$$

$$(1.2) \quad E[\log(\frac{1 - \omega_0}{\omega_0})] = 0,$$

$$(1.3) \quad \sigma^2 := \text{Var}[\log(\frac{1 - \omega_0}{\omega_0})] > 0.$$

Assumption (1.1) implies that $|\log(\frac{1 - \omega_0}{\omega_0})|$ is, P -a.s., bounded by the constant $L := \log(\frac{1 - \delta}{\delta})$. It is a technical assumption, which can be replaced by an exponential moment of $\log(\frac{1 - \omega_0}{\omega_0})$. According to a recurrence-transience result due to Solomon [21], assumption (1.2) ensures that $(X_n)_{n \geq 0}$ is \mathbb{P} -almost surely recurrent, i.e., the random walk hits any site infinitely often. Assumption (1.3) excludes the case of deterministic environment, which corresponds to the homogeneous symmetric random walk.

Under assumptions (1.1)–(1.3), the SRWRE is referred to as Sinai's walk. Sinai [20] proves that $X_n/(\log n)^2$ converges in law, under \mathbb{P} , toward a non-degenerate random variable, whose distribution is explicitly computed by Kesten [14] and Golosov [8]. This result contrasts with the usual central limit theorem which gives the convergence in law of X_n/\sqrt{n} .

Let

$$(1.4) \quad \begin{aligned} L(n, x) &:= \#\{0 \leq i \leq n : X_i = x\}, \quad n \geq 0, x \in \mathbb{Z}, \\ L(n, A) &:= \sum_{x \in A} L(n, x), \quad n \geq 0, A \subset \mathbb{Z}. \end{aligned}$$

In words, the quantity $L(n, A)$ measures the number of visits at set A by the walk in the first n steps.

The maximum of local time is studied by Révész ([17], p. 337) and Shi [18]: under assumptions (1.1)–(1.3), there exists $c_0 > 0$ such that

$$\limsup_{n \rightarrow \infty} \frac{\max_{x \in \mathbb{Z}} L(n, x)}{n} \geq c_0, \quad \mathbb{P}\text{-a.s.}$$

It means that the walk spends, infinitely often, a positive part of its life on a single site. A concentration property is obtained by Theorem 1.3 of Andreatti [1], which says that, under assumptions (1.1)–(1.3) and for any $0 < \beta < 1$, there exists $\ell(\beta) > 0$ such that

$$(1.5) \quad \limsup_{n \rightarrow \infty} \frac{\sup_{x \in \mathbb{Z}} L(n, [x - \ell(\beta), x + \ell(\beta)])}{n} \geq \beta, \quad \mathbb{P}\text{-a.s.}$$

In words, for any β close to 1, it is possible to find a length $\ell(\beta)$ such that, \mathbb{P} -almost surely, the particle spends, infinitely often, more than a β -fraction of its life in an interval of length $2\ell(\beta)$.

1.2 Random walk in random scenery

Random Walk in Random Scenery (RWRS) is a simple model of diffusion in disordered media, with long-range correlations. It is a class of stationary random processes exhibiting rich behavior. It can be described as follows: given a Markov chain on a state space, there may be a random field associated to the state space, called random scenery. As the random walk moves on this state space, he observes the scenery at his location. For a survey of recent results about RWRS, we refer to den Hollander and Steif [12], and to Asselah and Castell [2] for large deviations results in dimension $d \geq 5$.

Let us now define the model of one-dimensional RWRS: consider $S = (S_n, n \geq 0)$ a random walk on \mathbb{Z} and $\xi := (\xi(x), x \in \mathbb{Z}) = (\xi_x, x \in \mathbb{Z})$, a family of i.i.d. random variables defined on a probability space Ξ . We refer to ξ as the random scenery and denote by Q its law. Then, define the process $(Y_n, n \geq 0)$ by

$$Y_n := \sum_{i=0}^n \xi(S_i),$$

called RWRS or the Kesten-Spitzer Random Walk in Random Scenery. An interpretation is the following: if a random walker has to pay ξ_x each time he visits x , then Y_n stands for the total amount he has paid in the time interval $[0, n]$.

The model is introduced and studied by Kesten and Spitzer [15] in dimensions $d = 1$ and $d \geq 3$. They prove in dimension $d = 1$ that, when S and ξ belong to the domains of

attraction of stable laws of indices α and β respectively, then there exists δ , depending on α and β , such that $n^{-\delta} Y_{[nt]}$ converges weakly. Bolthausen [3] studies the case $d = 2$. He proves that, if S is a recurrent random walk and ξ_0 has zero expectation and finite variance, then $(n \log n)^{-1/2} Y_{[nt]}$ satisfies a functional central limit theorem. Independently, Borodin analyzes the case of one-dimensional simple random walk, see [4] and [5].

1.3 Random environment and random scenery

In this paper, we consider Sinai's Walk in Random Scenery. Problems combining random environment and random scenery have been examined for more general models. Replacing \mathbb{Z} by a more general countable state space, Lyons and Schramm [16] exhibit, under certain conditions, a stationary measure for Random Walks in a Random Environment with Random Scenery (RWRERS) from the viewpoint of the random walker. Häggström [9], Häggström and Peres [10] treat the case where the scenery arises from percolation on a graph. In this particular case, the scenery determines the random environment of the associated RWRE, which is used by the authors to obtain information about the scenery.

Let us first describe the model of Sinai's walk in random scenery. We consider Sinai's walk $(X_n, n \geq 0)$ under assumptions (1.1)–(1.3), and recall that the environment ω is defined on (Ω, P) . For the scenery, we consider a family of i.i.d. random variables $\xi := (\xi(x), x \in \mathbb{Z}) = (\xi_x, x \in \mathbb{Z})$, defined on (Ξ, Q) , independent of ω and $(X_n, n \geq 0)$. We denote by Q the distribution of ξ . To translate independence between ω and ξ , we consider the probability space $(\Omega \times \Xi, P \otimes Q)$, on which we define (ω, ξ) . Moreover, we denote by $\mathbb{P} \otimes Q$ the law of $(\omega, (X_n)_{n \geq 0}, \xi)$. Then we define as Sinai's walk in random scenery the process $(Z_n, n \geq 0)$:

$$Z_n := \sum_{i=0}^n \xi(X_i).$$

Observe that Z_n can be written with local time notation:

$$(1.6) \quad Z_n = \sum_{x \in \mathbb{Z}} \xi(x) L(n, x), \quad n \geq 0,$$

where $L(n, x)$ stands for the local time of the random walk at site x until time n , see (1.4).

We are interested in the upper limit of Z_n in the case where $a := \text{ess sup } \xi_0$ is finite. We consider the concentration property of order β for Sinai's walk with β close to 1 (see (1.5)), which enables us to formulate the conjecture of Révész ([17], p. 353): does the assumption

that $a := \text{ess sup } \xi_0$ is finite implies that, $\mathbb{P} \otimes Q$ -almost surely,

$$\limsup_{n \rightarrow \infty} \frac{Z_n}{n} = a ?$$

It turns out that the conjecture holds only under some additional assumptions on the distribution of the random scenery. A close study of the competition between the concentration property for Sinai's walk and negative values for the scenery enables us to obtain the following theorem, which gives a solution to this problem, depending on the tail decay of $\xi_0^- := \max\{-\xi_0, 0\}$.

Theorem 1.1 *Assume (1.1)–(1.3) and $a := \text{ess sup } \xi_0 < \infty$.*

(i) *If $Q\{\xi_0^- > \lambda\} \leq \frac{1}{(\log \lambda)^{2+\varepsilon}}$, for some $\varepsilon > 0$ and all large λ , then*

$$\mathbb{P} \otimes Q \left\{ \limsup_{n \rightarrow \infty} \frac{Z_n}{n} = a \right\} = 1.$$

(ii) *If $Q\{\xi_0^- > \lambda\} \geq \frac{1}{(\log \lambda)^{2-\varepsilon}}$, for some $\varepsilon > 0$ and all large λ , then*

$$\mathbb{P} \otimes Q \left\{ \lim_{n \rightarrow \infty} \frac{Z_n}{n} = -\infty \right\} = 1.$$

The paper is organized as follows: in Section 2, we quote useful results about environment, and about Sinai's walk when the environment is fixed (i.e., quenched results). In Section 3, we define the notion of “good” environment-scenery and prove Theorem 1.1 by accepting two intermediate propositions. These two propositions are proved in Sections 4 and 5, respectively.

We use c_i ($1 \leq i \leq 34$) to denote finite and positive constants.

2 Preliminaries

In this section, we collect some basic properties of random walk in random environment that will be useful in the forthcoming sections.

2.1 About the environment

In the study of one-dimensional SRWRE, an important role is played by a function of the environment ω , called the potential. This process, noted $V = (V(x), x \in \mathbb{Z})$, is defined on (Ω, P) by:

$$V(x) := \begin{cases} \sum_{i=1}^x \log\left(\frac{1-\omega_i}{\omega_i}\right) & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^0 \log\left(\frac{1-\omega_i}{\omega_i}\right) & \text{if } x \leq -1. \end{cases}$$

By (1.1), we observe that $|V(x) - V(x-1)| \leq L$ for any $x \in \mathbb{Z}$ and, for technical reason concerning Section 5, we define $P_z(\cdot) := P(\cdot | V(0) = z)$, for any $z \in \mathbb{R}$.

We recall the following result, whose proof is given by a simple martingale argument:

$$(2.1) \quad P_y \{d^+([z, \infty)) < d^+((-\infty, x])\} \geq \frac{y-x}{z-x+L}.$$

Moreover, we recall a result of Hirsch [11], which, under assumptions (1.1)–(1.3), takes the following simplified form: for any $0 < \varepsilon' < \frac{1}{34}$,

$$(2.2) \quad P \left\{ \max_{0 \leq x \leq N} V(x) < N^{\frac{1}{2}-\varepsilon'} \right\} \sim c_1 N^{-\varepsilon'}, \quad N \rightarrow \infty.$$

2.2 Quenched results

We define, for any $x \in \mathbb{Z}$,

$$\tau(x) := \min \{n \geq 1 : X_n = x\}, \quad \min \emptyset := \infty.$$

(Note in particular that when $X_0 = x$, then $\tau(x)$ is the first *return* time to x .) Throughout the paper, we write $P_\omega^x(\cdot) := P_\omega(\cdot | X_0 = x)$ (thus $P_\omega^0 = P_\omega$) and denote by E_ω^x the expectation with respect to P_ω^x .

It is known (Zeitouni [22], formula (2.1.4)) that for $r < x < s$,

$$(2.3) \quad P_\omega^x \{\tau(r) < \tau(s)\} = \sum_{j=x}^{s-1} e^{V(j)} \left(\sum_{j=r}^{s-1} e^{V(j)} \right)^{-1}.$$

The next result, which gives a simple bound for the expectation of $\tau(r) \wedge \tau(s)$ when the walk starts from a site $x \in (r, s)$, is essentially contained in Golosov [7], its proof can be found in [19]. For any integers $r < s$, we have

$$(2.4) \quad \max_{x \in (r, s) \cap \mathbb{Z}} E_\omega^x [\tau(s) \mathbf{1}_{\{\tau(s) < \tau(r)\}}] \leq (s-r)^2 \exp \left[\max_{r \leq i \leq j \leq s} (V(j) - V(i)) \right].$$

We will also use the following estimate borrowed from Lemma 7 of Golosov [7]: for $\ell \geq 1$ and $x < y$,

$$(2.5) \quad P_\omega^x \{\tau(y) < \ell\} \leq \ell \exp \left(- \max_{x \leq i < y} [V(y-1) - V(i)] \right).$$

Looking at the environment backwards, we get: for $\ell \geq 1$ and $w < x$,

$$(2.6) \quad P_\omega^x \{\tau(w) < \ell\} \leq \ell \exp \left(- \max_{w < i \leq x} [V(w+1) - V(i)] \right).$$

Finally we quote an important result about excursions of Sinai's walk. Let $b \in \mathbb{Z}$ and $x \in \mathbb{Z}$, and consider $L(\tau(b), x)$ under P_ω^b . In words, we look at the number of visits at site x by the random walk (starting from b) until the first return to b . Then there exist constants c_2 and c_3 such that

$$(2.7) \quad c_2 e^{-[V(x)-V(b)]} \leq E_\omega^b [L(\tau(b), x)] \leq c_3 e^{-[V(x)-V(b)]}.$$

3 Good environment-scenery and proof of Theorem 1.1

For any $j > 0$, we define

$$\begin{aligned} d^+(j) &:= \min \{n \geq 0 : V(n) \geq j\}, \\ b^+(j) &:= \min \left\{ n \geq 0 : V(n) = \min_{0 \leq x \leq d^+(j)} V(x) \right\}. \end{aligned}$$

These variables enable us to consider the valley $(0, b^+(j), d^+(j))$. Similarly, we define

$$\begin{aligned} d^-(j) &:= \max \{n \leq 0 : V(n) \geq j\}, \\ b^-(j) &:= \max \left\{ n \leq 0 : V(n) = \min_{d^-(j) \leq x \leq 0} V(x) \right\}. \end{aligned}$$

In the next sections, we will be frequently using the following elementary estimates: for any $\varepsilon > 0$, P -almost surely for all large j ,

$$(3.1) \quad j^{2-\varepsilon} \leq |b^\pm(j)| < |d^\pm(j)| \leq j^{2+\varepsilon}.$$

More generally, it will be useful to define, for any Borel set $A \subset \mathbb{R}$,

$$\begin{aligned} d^+(A) &:= \min \{n \geq 0 : V(n) \in A\}, \\ d^-(A) &:= \max \{n \leq 0 : V(n) \in A\}. \end{aligned}$$

To introduce the announced “good” environment-scenery, we fix $\varepsilon > 0$ such that assumption of part (i) of Theorem 1.1 is verified. For $\alpha \in (0, 1)$ (which will depend on ε), $c_4 > 0$, and $j \geq 100$, we define

$$(3.2) \quad \begin{aligned} \gamma_0 = \gamma_0(j) &:= j, \\ \gamma_i = \gamma_i(j) &:= j^{(1-\alpha)^i} = \gamma_{i-1}^{1-\alpha}, \quad i \geq 1, \end{aligned}$$

$$(3.3) \quad \varepsilon_i = \varepsilon_i(j) := \exp \left\{ -c_4 \gamma_{i+2}(j) \right\}.$$

For convenience of notation we define $\varepsilon_{-1} = \varepsilon_{-1}(j) := \varepsilon_0(j)$. In words, $(\gamma_i)_{i \geq 0}$ represents a decreasing sequence of distances, which enables us to classify the sites according to the value of $V(x) - V(b)$.

Write \log_j for the j th iterative logarithmic function. Fix $\varepsilon' := \min\{1/35, \varepsilon/2\} > 0$, and introduce, for $j \geq 100$,

$$(3.4) \quad M = M(j) := \inf \left\{ n \geq 0 : \gamma_n \leq (\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}} \right\}.$$

It is easy to observe that

$$(3.5) \quad M(j) \sim \frac{1}{|\log(1-\alpha)|} \log_2 j, \quad j \rightarrow \infty,$$

and that

$$\gamma_{M-1} \in \left[(\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}}, (\log_2 j)^{\frac{1}{2+\varepsilon'}} \right].$$

Note that we choose α small enough such that

$$(3.6) \quad \beta := (1-\alpha)^2(2+\varepsilon) - (2+\varepsilon') > 0,$$

$$(3.7) \quad \beta' := \frac{\varepsilon'}{2} - \alpha > 0.$$

Then we introduce the set

$$\bar{\Theta}_{M-1} = \bar{\Theta}_{M-1}(j) := \left[b^+(j) - c_5 \gamma_{M-1}^{2+\varepsilon'}, b^+(j) + c_5 \gamma_{M-1}^{2+\varepsilon'} \right],$$

and, for $i = M-2, \dots, 1, 0$, the sets

$$\bar{\Theta}_i = \bar{\Theta}_i(j) := \left[b^+(j) - c_6 \gamma_i^{2+\varepsilon'}, b^+(j) + c_6 \gamma_i^{2+\varepsilon'} \right] \setminus \bigcup_{p=i+1}^{M-1} \bar{\Theta}_p.$$

Observe that the sets $(\bar{\Theta}_i)_{0 \leq i \leq M-1}$ form a partition of the interval $[b^+(j) - c_6 j^{2+\varepsilon'}, b^+(j) + c_6 j^{2+\varepsilon'}]$. The final sets we consider are given, for $0 \leq i \leq M(j) - 1$, by

$$\Theta_i := \bar{\Theta}_i \cap I(j),$$

where $I(j) := [d^+((-\infty, -j]), d^+(j)]$. Note that $d^+((-\infty, -j]) < d^+(j)$ on $A(j)$ which will be defined in (3.16). In this case, the sets $(\Theta_i)_{0 \leq i \leq M-1}$ form a partition of $I(j)$ in annulus. Loosely speaking, the set Θ_i contains the sites x satisfying $V(x) - V(b) \approx \gamma_i$. To cover $[d^-(j), d^+(j)]$, we define

$$(3.8) \quad \Theta_{-1} = \Theta_{-1}(j) := \left[-c_7 j^{2+\varepsilon'}, c_7 j^{2+\varepsilon'}\right] \cap [d^-(j), d^+((-\infty, -j])].$$

Moreover, for the environment on \mathbb{Z}^+ , we introduce the events

$$(3.9) \quad A_1^{env}(j) := \{-4j \leq V(b^+(j)) \leq -3j\},$$

$$(3.10) \quad A_2^{env}(j) := \left\{ \max_{0 \leq x \leq y \leq b^+(j)} [V(y) - V(x)] \leq \frac{j}{4} \right\}.$$

The first event ensures that the valley considered is “deep enough” and the second one that the particle reaches the bottom of the valley “fast enough”. To control the time spent by the particle in different Θ_i during an excursion from $b^+(j)$ to $b^+(j)$, we define

$$(3.11) \quad A_{ann}^{env}(j) := \bigcap_{i=0}^{M-2} \left\{ \sum_{x \in \Theta_i(j)} e^{-[V(x) - V(b^+(j))]} \leq \varepsilon_i^2 \right\} =: \bigcap_{i=0}^{M-2} A_{ann,i}^{env}(j).$$

For the environment on \mathbb{Z}^- , let

$$(3.12) \quad B^{env}(j) := \left\{ V(b^-(j)) \leq -\frac{j}{6}, \max_{d^-(j) \leq x \leq y \leq 0} [V(y) - V(x)] \leq \frac{j}{3} \right\},$$

which ensures that the particle will not spent too much time on \mathbb{Z}^- .

Recalling that $\xi_x^- = \max\{-\xi_x, 0\}$, we define for the scenery

$$(3.13) \quad A_{-1}^{sce}(j) := \left\{ \max_{x \in \Theta_{-1}(j)} \xi_x^- < (\varepsilon_{-1})^{-1/2} \right\},$$

$$(3.14) \quad A_i^{sce}(j) := \left\{ \max_{x \in \Theta_i(j)} \xi_x^- < (\varepsilon_i)^{-1/2} \right\}, \quad 0 \leq i \leq M-2,$$

which ensures that the scenery does not reach excessive negative value in each Θ_i . In order to force the scenery in a neighborhood of the bottom (where the particle is concentrated), to be close to $a = \text{ess sup } \xi_0$, we fix $\rho \in (0, 1)$ and introduce

$$(3.15) \quad A_{M-1}^{sce}(j) := \left\{ \min_{x \in \Theta_{M-1}(j)} \xi_x \geq a - \rho \right\}.$$

We set

$$A^{env}(j) := A_1^{env}(j) \cap A_2^{env}(j) \cap A_{ann}^{env}(j), \quad A^{sce}(j) := \bigcap_{i=-1}^{M-1} A_i^{sce}(j).$$

Moreover, we note

$$(3.16) \quad A(j) := A^{env}(j) \cap B^{env}(j) \cap A^{sce}(j).$$

A pair (ω, ξ) is a “good” environment-scenery if $(\omega, \xi) \in A(j)$ for infinitely many j .

For future use, let us note that for $\omega \in B^{env}(j) \cap A_2^{env}(j)$, we have

$$(3.17) \quad \max_{d^-(j) \leq x \leq y \leq b^+(j)} [V(y) - V(x)] \leq \frac{2j}{3}.$$

To prove Theorem 1.1, we need two propositions, whose proofs are respectively postponed until Sections 5 and 4. The first one ensures that almost all pair (ω, ξ) is a “good” environment-scenery, while the second one describes the behavior of the particle in a “good” environment.

Proposition 3.1 *Under assumptions (1.1)–(1.3), we have that $P \otimes Q$ -almost all (ω, ξ) is a “good” environment-scenery. More precisely, there exists a sequence $(m_k)_{k \geq 1}$ such that $m_k \geq k^{3k}$ and (ω, ξ) is a good environment-scenery along $(m_k)_{k \geq 1}$, i.e., $(\omega, \xi) \in A(m_k)$, for all $k \geq 1$.*

Let $(m_k)_{k \geq 1}$ be the sequence introduced in Proposition 3.1. For each realization of (ω, ξ) , we extract $(n_k)_{k \geq 1}$ from $(m_k)_{k \geq 1}$ such that

$$(3.18) \quad \sum_{k \geq 1} \varepsilon_M(n_k) < \infty.$$

To ease notations, we write throughout the paper, $d_k^+ := d^+(n_k)$, $\tau_k^+ := \tau(d_k^+)$, $b_k^+ := b^+(n_k)$ and $d_k^- := d^-(n_k)$, $\tau_k^- := \tau(d_k^-)$. Moreover, we define, for all $k \geq 1$,

$$(3.19) \quad t_k := \lfloor e^{n_k} \rfloor.$$

Proposition 3.2 *For $P \otimes Q$ almost all (ω, ξ) , we have that, P_ω -a.s., for all large k ,*

$$(3.20) \quad L(t_k, \Theta_{-1}(n_k)) \leq \varepsilon_{-1}(n_k) t_k,$$

$$(3.21) \quad L(t_k, \Theta_i(n_k)) \leq \varepsilon_i(n_k) t_k, \quad 0 \leq i \leq M-2,$$

$$(3.22) \quad \tau_k^+ \wedge \tau_k^- > t_k.$$

Proof of Theorem 1.1.

Proof of part (i). For any $\delta > 0$, we define $\varepsilon^{(\delta)}(j) := \sum_{i=-1}^{M-2} \varepsilon_i^{(\delta)}(j)$. Recalling (1.6), we use Proposition 3.2 and (3.1) to obtain, for $\mathbb{P} \otimes Q$ -almost all realization of ω , ξ and $(X_j)_{j \geq 0}$,

$$\sum_{j=1}^{t_k} \xi(X_j) \geq (1 - \varepsilon^{(1)}(n_k)) t_k \left(\min_{x \in \Theta_{M-1}(n_k)} \xi_x \right) - \sum_{i=-1}^{M-2} \varepsilon_i(n_k) t_k \left(\max_{x \in \Theta_i(n_k)} \xi_x^- \right).$$

Then, Proposition 3.1 implies

$$\begin{aligned} \sum_{j=1}^{t_k} \xi(X_j) &\geq (1 - \varepsilon^{(1)}(n_k)) t_k (a - \rho) - \sum_{i=-1}^{M-2} \sqrt{\varepsilon_i(n_k)} t_k \\ (3.23) \qquad \qquad &\geq (1 - \varepsilon^{(1)}(n_k)) t_k (a - \rho) - \varepsilon^{(1/2)}(n_k) t_k, \end{aligned}$$

for all large k .

We claim that, for any $\delta > 0$ and all large j ,

$$(3.24) \qquad \varepsilon^{(\delta)} = \varepsilon^{(\delta)}(j) := \sum_{i=-1}^M \varepsilon_i^\delta \leq 2 \left(1 + \frac{1}{\delta} \right) \varepsilon_M^\delta.$$

To prove (3.24), we observe that

$$\varepsilon^{(\delta)} = \sum_{i=-1}^M \varepsilon_i^\delta \leq 2 \varepsilon_M^\delta + \sum_{i=0}^{M-1} \int_{\varepsilon_i}^{\varepsilon_{i+1}} \frac{\varepsilon_i^\delta}{\varepsilon_{i+1} - \varepsilon_i} dx.$$

Recalling (3.3), we have that $\varepsilon_{i+1} - \varepsilon_i = \varepsilon_{i+1} (1 - e^{-c_4(\gamma_{i+2} - \gamma_{i+3})})$. Recalling (3.2) we get that $\gamma_{i+2} - \gamma_{i+3} = \gamma_{i+2}(1 - \gamma_{i+2}^{-\alpha})$. Since (3.2) and (3.4) imply $\gamma_{i+2} \geq \gamma_{M+2} \geq (\log_2 j)^{\frac{(1-\alpha)^3}{2+\varepsilon}}$ for $2 \leq i \leq M+2$, we obtain that $\gamma_{i+2} - \gamma_{i+3} \geq \gamma_{i+2}/2$, for all large j and for $2 \leq i \leq M-1$. Therefore, we get $\varepsilon_{i+1} - \varepsilon_i \geq \varepsilon_{i+1}/2$, implying that

$$\varepsilon^{(\delta)} \leq 2 \varepsilon_M^\delta + 2 \sum_{i=0}^{M-1} \int_{\varepsilon_i}^{\varepsilon_{i+1}} \frac{\varepsilon_i^\delta}{\varepsilon_{i+1}} dx.$$

Moreover, $\sum_{i=0}^{M-1} \int_{\varepsilon_i}^{\varepsilon_{i+1}} \frac{\varepsilon_i^\delta}{\varepsilon_{i+1}} dx \leq \sum_{i=0}^{M-1} \int_{\varepsilon_i}^{\varepsilon_{i+1}} x^{\delta-1} dx = \int_{\varepsilon_0}^{\varepsilon_M} x^{\delta-1} dx \leq \varepsilon_M^\delta / \delta$, which implies (3.24).

Recalling (3.23) and using (3.24) which tells us that $\varepsilon^{(\delta)}(j) \rightarrow 0$ when $j \rightarrow \infty$, we have

$$(3.25) \qquad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \xi(X_i) \geq a - \rho, \qquad \mathbb{P} \otimes Q\text{-a.s.}$$

It remains only to observe that (3.25) is true for all $\rho > 0$ and that the definition of a implies that $\mathbb{P} \otimes Q$ -a.s., $\frac{1}{n} \sum_{i=0}^n \xi(X_i) \leq a$, for all $n \geq 0$. \square

Proof of part (ii). Using Theorem 1.5 of [13], we have that, for any $\varepsilon' > 0$, \mathbb{P} -almost surely, $\max_{0 \leq i \leq n} X_i \geq (\log n)^{2-\varepsilon'} + 1$, for all large n . This implies

$$(3.26) \quad \sum_{i=0}^n \xi(X_i) \leq a n - \max_{0 \leq x \leq \lceil (\log n)^{2-\varepsilon'} \rceil} \xi_x^-.$$

By assumption, there exists $\varepsilon > 0$ such that $Q \{ \xi_0 < -\lambda \} \geq (\log \lambda)^{-2+\varepsilon}$. Therefore, fixing $\varepsilon' < \varepsilon$, we get for $k \geq 1$ and all $N \geq 1$,

$$(3.27) \quad Q \left\{ \max_{0 \leq x \leq N} \xi_x^- < k a e^{N^{\frac{1}{2-\varepsilon'}}} \right\} \leq \exp \{ -c_8 N^\delta \},$$

where $\delta := 1 - \frac{2-\varepsilon}{2-\varepsilon'} > 0$.

We choose $N_p := \lfloor (\log p)^T \rfloor$ for $p \geq 1$ with T large enough such that $T\delta > 1$. Therefore, using (3.27) and applying the Borel–Cantelli lemma, we get that Q -almost surely, for $p \geq p_0$,

$$(3.28) \quad \max_{0 \leq x \leq N_p} \xi_x^- \geq k a e^{N_p^{\frac{1}{2-\varepsilon'}}}.$$

Fixing the realization of ξ , we define $p(n)$ such that

$$(3.29) \quad N_{p(n)} \leq \lceil (\log n)^{2-\varepsilon'} \rceil \leq N_{p(n)+1},$$

for all n such that $p(n) \geq p_0$. This yields

$$\max_{0 \leq x \leq \lceil (\log n)^{2-\varepsilon'} \rceil} \xi_x^- \geq \max_{0 \leq x \leq N_{p(n)}} \xi_x^- \geq k a e^{N_{p(n)}^{\frac{1}{2-\varepsilon'}}},$$

the last inequality being a consequence of (3.28). Therefore, we obtain

$$\begin{aligned} \max_{0 \leq x \leq \lceil (\log n)^{2-\varepsilon'} \rceil} \xi_x^- &\geq k a \exp \left\{ \lceil (\log n)^{2-\varepsilon'} \rceil^{\frac{1}{2-\varepsilon'}} \right\} \exp \left\{ - \left(\lceil (\log n)^{2-\varepsilon'} \rceil^{\frac{1}{2-\varepsilon'}} - N_{p(n)}^{\frac{1}{2-\varepsilon'}} \right) \right\} \\ &\geq k a n \exp \left\{ - \left(N_{p(n)+1}^{\frac{1}{2-\varepsilon'}} - N_{p(n)}^{\frac{1}{2-\varepsilon'}} \right) \right\}, \end{aligned}$$

the second inequality being a consequence of (3.29). Moreover, we easily get that $N_{p(n)+1}^{\frac{1}{2-\varepsilon'}} - N_{p(n)}^{\frac{1}{2-\varepsilon'}} \rightarrow 0$, when $n \rightarrow \infty$, implying that for all large n ,

$$(3.30) \quad \max_{0 \leq x \leq \lceil (\log n)^{2-\varepsilon'} \rceil} \xi_x^- \geq \frac{k}{2} a n.$$

Assembling (3.26) and (3.30), we get that $\mathbb{P} \otimes Q$ -almost surely, $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \xi(X_i) \leq a(1 - \frac{k}{2})$. This inequality being true for all $k \geq 1$, it implies part (ii) of Theorem 1.1. \square

4 Proof of Proposition 3.2

To get (3.22), we observe that

$$P_\omega \{ \tau_k^+ \wedge \tau_k^- \leq t_k \} \leq P_\omega \{ \tau_k^+ \leq t_k \} + P_\omega \{ \tau_k^- \leq t_k \}.$$

Then using (2.5), (2.6) and (3.9), (3.12) we obtain

$$P_\omega \{ \tau_k^+ \wedge \tau_k^- \leq t_k \} \leq t_k (e^{-4n_k} + e^{-7n_k/6}) \leq 2e^{-n_k/6},$$

Then, since $n_k \geq k$, we get

$$\sum_{k \geq 0} P_\omega \{ \tau_k^+ \wedge \tau_k^- \leq t_k \} \leq 2 \sum_{k \geq 0} e^{-n_k/6} < \infty.$$

We conclude by using the Borel–Cantelli lemma.

To prove (3.21), we apply the strong Markov property at $\tau(b_k^+)$ and get for $0 \leq i \leq M-2$,

$$\begin{aligned} & P_\omega \{ L(t_k, \Theta_i(n_k)) \geq \varepsilon_i(n_k) t_k \} \\ & \leq P_\omega^{b_k^+} \{ L(t_k, \Theta_i(n_k)) \geq \varepsilon_i(n_k) t_k - \lambda_k \} + P_\omega \{ \lambda_k \leq \tau(b_k^+) \leq \tau_k^- \} + P_\omega \{ \tau_k^- \leq \tau(b_k^+) \}. \end{aligned}$$

By (2.3), (3.10) and (3.1), we get, for all large k ,

$$P_\omega \{ \tau_k^- \leq \tau(b_k^+) \} \leq \frac{b_k^+ e^{n_k/3}}{e^{n_k}} \leq n_k^{2+\varepsilon'} e^{-2n_k/3}.$$

Chebyshev's inequality, (2.4), (3.17) and (3.1) yield

$$P_\omega \{ \lambda_k \leq \tau(b_k^+) \leq \tau_k^- \} \leq \frac{(b_k^+ - d_k^-)^2}{\lambda_k} e^{2n_k/3} \leq \frac{2n_k^{2(2+\varepsilon')}}{\lambda_k} e^{2n_k/3}.$$

Choosing $\lambda_k := e^{5n_k/6}$, we obtain, for all large k ,

$$(4.1) \quad P_\omega \{ \lambda_k \leq \tau(b_k^+) \leq \tau_k^- \} + P_\omega \{ \tau_k^- \leq \tau(b_k^+) \} \leq e^{-n_k/7}.$$

To treat $P_{k,i} := P_\omega^{b_k^+} \{ L(t_k, \Theta_i(n_k)) \geq \varepsilon_i(n_k) t_k - \lambda_k \}$, we recall (3.19) and observe that $\lambda_k \leq 2e^{-n_k/6} t_k$, to obtain

$$P_{k,i} \leq P_\omega^{b_k^+} \{ L(t_k, \Theta_i(n_k)) \geq (\varepsilon_i(n_k) - 2e^{-n_k/6}) t_k \}.$$

Then, by Chebyshev's inequality, we get

$$P_{k,i} \leq \frac{1}{(\varepsilon_i(n_k) - 2e^{-n_k/6})t_k} E_\omega^{b_k^+} [L(t_k, \Theta_i(n_k))].$$

Since Sinai's walk can not make more than t_k excursions from b_k^+ to b_k^+ before t_k , and since these excursions are i.i.d., we obtain

$$P_{k,i} \leq \frac{t_k}{(\varepsilon_i(n_k) - 2e^{-n_k/6})t_k} E_\omega^{b_k^+} [L(\tau(b_k^+), \Theta_i(n_k))].$$

Observe that (2.7) implies $E_\omega^{b_k^+} [L(\tau(b_k^+), \Theta_i(n_k))] \leq c_3 \sum_{x \in \Theta_i(n_k)} e^{-[V(x) - V(b_k^+)]}$, for all $0 \leq i \leq M-2$. Recalling (3.11), we get for all large k , and for $0 \leq i \leq M-2$,

$$P_{k,i} \leq \frac{c_3 (\varepsilon_i(n_k))^2}{(\varepsilon_i(n_k) - 2e^{-n_k/6})} \leq c_9 \varepsilon_i(n_k),$$

the second inequality being a consequence of $\varepsilon_i(n_k) \geq \varepsilon_0(n_k)$ and the fact that $c_4 < 1/6$ implies $e^{-n_k/6} = o(\varepsilon_0(n_k))$.

Summing from 0 to $M-2$ and using (3.24), we get, with $c_{10} := 2(1 + \frac{1}{\delta})c_9$,

$$(4.2) \quad \sum_{i=0}^{M-2} P_{k,i} \leq c_9 \sum_{i=0}^{M-2} \varepsilon_i(n_k) \leq c_{10} \varepsilon_M(n_k).$$

Assembling (4.1), (4.2) and recalling (3.5), (3.18) we obtain

$$\sum_{k \geq 1} \sum_{i=0}^{M-2} P_\omega \{L(t_k, \Theta_i(n_k)) \geq \varepsilon_i(n_k) t_k\} \leq \sum_{k \geq 1} (c_{10} \varepsilon_M(n_k) + M(n_k) e^{-n_k/7}) < \infty.$$

This implies (3.21) by an application of the Borel–Cantelli lemma.

We get (3.20) by an argument very similar to the one used in the proof of (3.21), the main ingredients being the facts that $V(x) - V(b^+(n_k)) \geq 2n_k$, for $x \in \Theta_{-1}(n_k)$ (which is a consequence of (3.12), (3.9) and the definition of $d^+(j)$), and that $\Theta_{-1}(n_k)$ contains less than $2c_7 n_k^{2+\varepsilon'}$ sites (by (3.8)). We feel free to omit the details.

5 Proof of Proposition 3.1

We now prove that, for $P \otimes Q$ -almost all (ω, ξ) , there exists a sequence (m_k) such that $(\omega, \xi) \in A(m_k)$, $\forall k \geq 1$, where $A(m_k)$ is defined in (3.16).

Let $j_k := k^{3k}$ ($k \geq 1$) and $\mathcal{F}_{j_{k-1}} := \sigma\{V(x), \xi_z, d^-(j_{k-1}) \leq x, z \leq d^+(j_{k-1})\}$.

If we are able to show that

$$(5.1) \quad \sum_k P \otimes Q \{A(j_k) \mid \mathcal{F}_{j_{k-1}}\} = \infty, \quad P \otimes Q\text{-a.s.},$$

then Lévy's Borel–Cantelli lemma ([6], p. 237) will tell us that $P \otimes Q$ -almost surely there are infinitely many k such that $(\omega, \xi) \in A(j_k)$.

To bound $P \otimes Q\{A(j_k) \mid \mathcal{F}_{j_{k-1}}\}$ from below, we start with the trivial inequality $A(j_k) \supset A(j_k) \cap C(j_{k-1})$, for any set $C(j_{k-1})$. We choose $C(j_{k-1}) := C^{env}(j_{k-1}) \cap D^{env}(j_{k-1}) \cap C^{sce}(j_{k-1})$, where

$$\begin{aligned} C^{env}(j_{k-1}) &:= \left\{ \inf_{0 \leq y \leq d^+(j_{k-1})} V(y) \geq -j_{k-1} \log^2 j_{k-1} \right\}, \\ D^{env}(j_{k-1}) &:= \left\{ \inf_{d^-(j_{k-1}) \leq y \leq 0} V(y) \geq -j_{k-1} \log^2 j_{k-1} \right\}, \\ C^{sce}(j_{k-1}) &:= \left\{ \max_{d^-(j_{k-1}) \leq x \leq d^+(j_{k-1})} \xi_x^- < (\varepsilon_{-1}(j_k))^{-1/2} \right\}. \end{aligned}$$

Clearly, $C(j_{k-1})$ is $\mathcal{F}_{j_{k-1}}$ -measurable. Moreover on $C^{env}(j_{k-1}) \cap A^{env}(j_k)$, we have $d^+(j_{k-1}) \leq d^+((-\infty, -j_k]) \leq b^+(j_k)$.

Let

$$E_{-1}^{sce}(j_k) := \left\{ \max_{x \in \Theta_{-1} \setminus [d^-(j_{k-1}), d^+(j_{k-1})]} \xi_x^- < (\varepsilon_{-1}(j_k))^{-1/2} \right\},$$

and consider

$$E^{sce}(j_k) := \bigcap_{i=0}^{M-1} A_i^{sce}(j_k) \cap E_{-1}^{sce}(j_k).$$

Since $C^{sce}(j_{k-1}) \cap E_{-1}^{sce}(j_k) \subset A_{-1}^{sce}(j_k)$, it follows that

$$\begin{aligned} &P \otimes Q \{A(j_k) \mid \mathcal{F}_{j_{k-1}}\} \\ &\geq P \otimes Q \{P \otimes Q \{A^{env}(j_k), B^{env}(j_k), E^{sce}(j_k), C(j_{k-1}) \mid \mathcal{F}_{j_{k-1}}, \omega\} \mid \mathcal{F}_{j_{k-1}}\} \\ &\geq P \otimes Q \{\mathbf{1}_{\{A^{env}(j_k), B^{env}(j_k), C(j_{k-1})\}} P \otimes Q \{E^{sce}(j_k) \mid \omega\} \mid \mathcal{F}_{j_{k-1}}\}. \end{aligned}$$

Now, we suppose for the moment that we are able to prove that there exists $c_{11} > 0$ such that, for P -almost all ω ,

$$(5.2) \quad P \otimes Q \{E^{sce}(j_k) \mid \omega\} \geq \frac{c_{11}}{k^{1/4}}.$$

We get

$$\begin{aligned}
P \otimes Q \{A(j_k) \mid \mathcal{F}_{j_{k-1}}\} &\geq \frac{c_{11}}{k^{1/4}} P \otimes Q \{A^{env}(j_k), B^{env}(j_k), C(j_{k-1}) \mid \mathcal{F}_{j_{k-1}}\} \\
(5.3) \qquad \qquad \qquad &\geq \frac{c_{11}}{k^{1/4}} P_k^+ P_k^- \mathbf{1}_{C^{scc}(j_{k-1})},
\end{aligned}$$

where we use the fact that $(V(x), x \geq 0)$ and $(V(x), x < 0)$ are independent processes, and note

$$\begin{aligned}
P_k^+ &:= P \{A^{env}(j_k), C^{env}(j_{k-1}) \mid \mathcal{F}_{j_{k-1}}\}, \\
P_k^- &:= P \{B^{env}(j_k), D^{env}(j_{k-1}) \mid \mathcal{F}_{j_{k-1}}\}.
\end{aligned}$$

To bound P_k^+ from below, we introduce

$$E_2^{env}(j_k) := \left\{ \max_{0 \leq x \leq y \leq b^+(j_k)} [V(y) - V(x)] \leq \frac{j_k}{4} - j_{k-1} \log^2 j_{k-1} - j_{k-1} - L \right\},$$

and consider

$$E^{env}(j) := A_1^{env}(j) \cap A_{ann}^{env}(j) \cap E_2^{env}(j_k).$$

Observe that $C^{env}(j_{k-1}) \cap \{\max_{d^+(j_{k-1}) \leq x \leq y \leq b^+(j_k)} [V(y) - V(x)] \leq \frac{j_k}{4} - j_{k-1} \log^2 j_{k-1} - j_{k-1} - L\} \subset A_2^{env}(j_k)$. Thus, since $V(d^+(j_{k-1})) \in I_{j_{k-1}} := [j_{k-1}, j_{k-1} + L]$, we have, by applying the strong Markov property at $d^+(j_{k-1})$,

$$(5.4) \qquad P_k^+ \geq \left(\inf_{z \in I_{j_{k-1}}} P_z \{E^{env}(j_k)\} \right) \mathbf{1}_{C^{env}(j_{k-1})}.$$

To bound P_k^- from below, we observe the following inclusion

$$B^{env}(j_k) \supset \left\{ \max_{d^-(j_k) \leq x \leq y \leq d^-(j_{k-1})} [V(y) - V(x)] \leq \frac{j_k}{3} \right\} \cap D^{env}(j_{k-1}).$$

Then since $V(d^-(j_{k-1}))$ belongs to $I_{j_{k-1}}$, the strong Markov property applied at $d^-(j_{k-1})$ yields

$$(5.5) \qquad P_k^- \geq \left(\inf_{z \in I_{j_{k-1}}} P_z \{B^{env}(j_k)\} \right) \mathbf{1}_{D^{env}(j_{k-1})},$$

Observing that $\mathbf{1}_{C(j_{k-1})} = 1$, $P \otimes Q$ -almost surely for all large k , and recalling (5.3), (5.4) and (5.5), the proof of (5.1) boils down to showing that

$$(5.6) \qquad \liminf_{k \rightarrow \infty} \inf_{z \in I_{j_{k-1}}} P_z \{E^{env}(j_k)\} > 0,$$

$$(5.7) \qquad \liminf_{k \rightarrow \infty} \inf_{z \in I_{j_{k-1}}} P_z \{B^{env}(j_k)\} > 0.$$

The rest of the section is devoted to the proof of (5.2) and (5.6), whereas (5.7) is an immediate consequence of Donsker's theorem.

5.1 Proof of (5.2)

Since the sets $\{\Theta_i\}_{-1 \leq i \leq M-1}$ are disjoint, the events $E_{-1}^{scc}(j_k)$ and $\{A_i^{scc}(j_k)\}_{0 \leq i \leq M-1}$ are mutually independent. We write

$$P \otimes Q \{E^{scc}(j_k) | \omega\} = \prod_{i=0}^{M-1} P \otimes Q \{A_i^{scc}(j_k) | \omega\} \times P \otimes Q \{E_{-1}^{scc}(j_k) | \omega\}.$$

Thus, (5.2) will be a consequence of the two following lemmas.

Lemma 5.1 *For P -almost all ω , we have*

$$P \otimes Q \{A_{M-1}^{scc}(j_k) | \omega\} \geq \frac{1}{k^{1/4}}.$$

Lemma 5.2 *There exists $c_{12} > 0$ such that, for P -almost all ω ,*

$$(5.8) \quad \liminf_{k \rightarrow \infty} \prod_{i=0}^{M-2} P \otimes Q \{A_i^{scc}(j_k) | \omega\} \times P \otimes Q \{E_{-1}^{scc}(j_k) | \omega\} \geq c_{12}.$$

Proof of Lemma 5.1. Recalling (3.15), (3.8) and (3.6), we get, P -almost surely,

$$P \otimes Q \{A_{M-1}^{scc}(j_k) | \omega\} \geq \exp \{c_5 \log q \log_2 j_k\},$$

where $q := Q \{\xi_0 \geq a - \rho\}$. Definition of a implies $-\infty < \log q < 0$. To conclude, it remains only to observe that $\log_2 j_k = \log k + \log_2 k + \log 3$ and to choose c_5 small enough such that $c_5 \log q > -1/5$. \square

Proof of Lemma 5.2. Recalling (3.14) and that $(\xi_x^-)_{x \in \mathbb{Z}}$ is a family of i.i.d. random variables, we get, P -almost surely, for $0 \leq i \leq M-2$,

$$\begin{aligned} P \otimes Q \{A_i^{scc}(j_k) | \omega\} &\geq \left(Q \{ \xi_0^- \leq (\varepsilon_i)^{-1/2} \} \right)^{2c_6 \gamma_i^{2+\varepsilon'}} \\ &\geq \exp \left\{ 2c_6 \gamma_i^{2+\varepsilon'} \log \left(1 - Q \{ \xi_0^- \geq (\varepsilon_i)^{-1/2} \} \right) \right\}. \end{aligned}$$

Then, since $Q \{ \xi_0^- \geq (\varepsilon_i)^{-1/2} \}$ tends to 0 when k tends to ∞ and using the fact that $\log(1-x) \geq -c_{13}x$ for $x \in [0, 1/2)$ with $c_{13} := 2 \log 2 > 0$, we obtain, for all large k ,

$$P \otimes Q \{A_i^{scc}(j_k) | \omega\} \geq \exp \left\{ -c_{14} \gamma_i^{2+\varepsilon'} Q \{ \xi_0^- \geq (\varepsilon_i)^{-1/2} \} \right\},$$

where $c_{14} := 2c_6c_{13}$. Recalling that $Q\{\xi_0^- \geq \lambda\} \leq (\log \lambda)^{-(2+\varepsilon)}$ for $\lambda \geq \lambda_0 > 0$ and (3.3) we get for k large enough and uniformly in $0 \leq i \leq M-2$,

$$(5.9) \quad P \otimes Q\{A_i^{scc}(j_k) | \omega\} \geq \exp\left\{-c_{15}\gamma_i^{-\beta}\right\},$$

where $\beta := (1-\alpha)^2(2+\varepsilon) - (2+\varepsilon') > 0$ by (3.6), and $c_{15} := c_{14}\left(\frac{2}{c_4}\right)^{2+\varepsilon}$. Then, recalling (3.13), we obtain similarly

$$(5.10) \quad P \otimes Q\{E_{-1}^{scc}(j_k) | \omega\} \geq \exp\left\{-c_{16}\gamma_0^{-\beta}\right\}.$$

Putting together (5.9) and (5.10), we get with $c_{17} := \max\{c_{15}, c_{16}\}$,

$$\prod_{i=0}^{M-2} P \otimes Q\{A_i^{scc}(j_k) | \omega\} \times P \otimes Q\{E_{-1}^{scc}(j_k) | \omega\} \geq \exp\{-c_{17}\sigma_\beta\},$$

where $\sigma_\beta := \gamma_0^{-\beta} + \sum_{i=0}^{M-2} \gamma_i^{-\beta}$. By the same way we proved (3.24), we obtain $\sigma_\beta \leq c_{18}\gamma_{M-1}^{-\beta}$, where $c_{18} \leq 1 + 2/\beta$. Recalling (3.6), it follows that $\sigma_\beta \rightarrow 0$ when $k \rightarrow \infty$, which implies (5.8). \square

5.2 Proof of (5.6)

To prove (5.6), we need some preliminary results.

Lemma 5.3 *For any $\delta > 0$, $k \geq 1$ and any $0 \leq p \leq M$, we have*

$$\sum_{i=p}^M \gamma_i^\delta \leq \left(1 + \frac{2}{\delta}\right) \gamma_p^\delta.$$

Proof. We have

$$\sum_{i=p}^M \gamma_i^\delta \leq \gamma_p^\delta + \sum_{i=p+1}^M \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1} - \gamma_i} dx.$$

Recalling that $\gamma_{i-1} - \gamma_i \geq \gamma_{i-1}/2$, for all large j and for $1 \leq i \leq M$, we get

$$\sum_{i=p}^M \gamma_i^\delta \leq \gamma_p^\delta + 2 \sum_{i=p+1}^M \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1}} dx.$$

Moreover, $\sum_{i=p+1}^M \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1}} dx \leq \sum_{i=p+1}^M \int_{\gamma_i}^{\gamma_{i-1}} x^{\delta-1} dx = \int_{\gamma_M}^{\gamma_p} x^{\delta-1} dx \leq \gamma_p^\delta / \delta$, which yields Lemma 5.3. \square

We now proceed to prove (5.6). Let

$$a_\ell := -3j_k - \ell \gamma_M, \quad F_1^{env}(j_k, \ell) := \{a_{\ell+1} \leq V(b^+(j_k)) < a_\ell\}.$$

Noting $\theta_k := \lfloor j_k / \gamma_M \rfloor - 1$, we have

$$(5.11) \quad P_z \{E^{env}(j_k)\} \geq \sum_{\ell=0}^{\theta_k} P_z \{F_1^{env}(j_k, \ell), A_{ann}^{env}(j_k), E_2^{env}(j_k)\} =: \sum_{\ell=0}^{\theta_k} P_{k,\ell}^+.$$

To bound $P_{k,\ell}^+$ by below for $0 \leq \ell \leq \theta_k$, we define recursively the following levels,

$$(5.12) \quad \eta_i = \eta_i(j_k, \ell) := a_\ell + \gamma_i, \quad 0 \leq i \leq M,$$

$$(5.13) \quad \eta_{M+1} = \eta_{M+1}(j_k, \ell) := a_\ell.$$

In the following, we introduce stopping times for the potential, which will enable us to consider a ‘‘typical’’ event that exhibits a valley having ‘‘good’’ properties. Let us write

$$T = T(j_k, \ell) := d^+((-\infty, a_\ell]),$$

$$\tilde{T} = \tilde{T}(j_k, \ell) := d^+((-\infty, a_{\ell+1}]).$$

Then, we define the stopping times for the potential, for $0 \leq i \leq M$,

$$T_i = T_i(j_k, \ell) := d^+((-\infty, \eta_i]),$$

$$T'_i = T'_i(j_k, \ell) := \min \{n \geq T : V(n) \geq \eta_{M-i}\},$$

$$R_i = R_i(j_k, \ell) := \min \{n \geq T'_i : V(n) \leq \eta_{M-i+1}\}.$$

We introduce the events

$$G_i(j_k) := \left\{ T_{i+1} - T_i \leq c_{19}(\gamma_i)^{2+\varepsilon'}, \max_{T_i \leq x \leq y \leq T_{i+1}} [V(y) - V(x)] \leq \frac{j_k}{5} \right\}, \quad 0 \leq i \leq M-1,$$

$$G_M(j_k) := \left\{ T - T_M \leq c_{19}(\gamma_M)^{2+\varepsilon'}, \max_{T_M \leq x \leq y \leq T} [V(y) - V(x)] \leq \frac{j_k}{5} \right\},$$

and

$$G'_0(j_k) := \left\{ T'_0 - T \leq c_{20}(\gamma_M)^{2+\varepsilon'}, T'_0 < \tilde{T} \right\},$$

$$G'_i(j_k) := \left\{ T'_i - T'_{i-1} \leq c_{20}(\gamma_{M-i})^{2+\varepsilon'}, T'_i < R_{i-1} \right\}, \quad 1 \leq i \leq M.$$

We set

$$G(j_k, \ell) := \bigcap_{i=0}^M G_i(j_k), \quad G'(j_k, \ell) := \bigcap_{i=0}^{M-1} G'_i(j_k),$$

and

$$H(j_k, \ell) := \left\{ \max_{0 \leq x \leq y \leq T_0} [V(y) - V(x)] \leq \frac{j_k}{5} \right\},$$

$$H'(j_k, \ell) := \{d^+(j_k) < R_M\}.$$

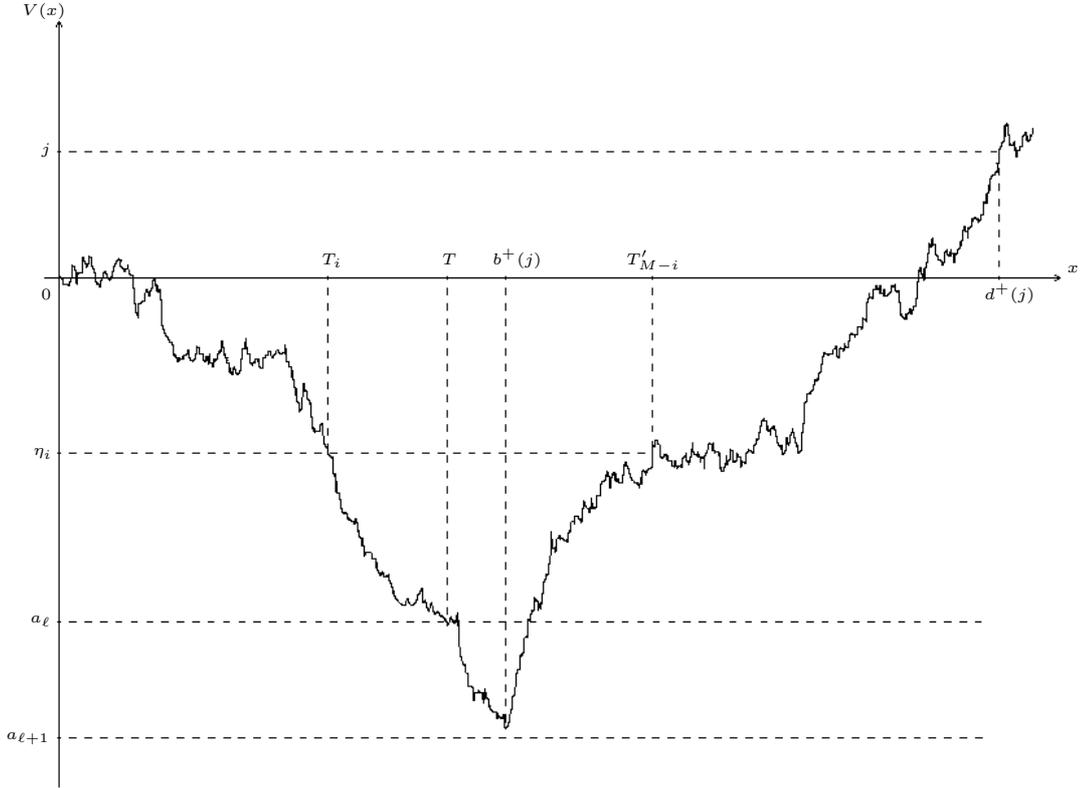


Figure 1: Example of $\omega \in G(j_k, \ell) \cap G'(j_k, \ell) \cap H(j_k, \ell) \cap H'(j_k, \ell)$

See Figure 1 for an example of $\omega \in G(j_k, \ell) \cap G'(j_k, \ell) \cap H(j_k, \ell) \cap H'(j_k, \ell)$.

Observe that on $G(j_k, \ell) \cap G'(j_k, \ell) \cap H'(j_k, \ell)$, we have, for $0 \leq i \leq M - 1$,

$$(5.14) \quad [T_i, T'_{M-i}] \supset \{x \in [0, d^+(j_k)] : V(x) - V(b^+(j_k)) \leq \gamma_{i+1}\}.$$

Moreover, on $G(j_k, \ell) \cap G'(j_k, \ell)$,

$$T'_{M-i} - T_i \leq (c_{20} + c_{19}) \sum_{p=i}^M (\gamma_p)^{2+\varepsilon'}, \quad 0 \leq i \leq M.$$

If we choose c_6 such that $c_6 \geq (c_{20} + c_{19})(1 + \frac{2}{2+\varepsilon'})$, we get by Lemma 5.3 that

$$(5.15) \quad [T_i, T'_{M-i}] \subset [b^+(j_k) - c_6 \gamma_i^{2+\varepsilon'}, b^+(j_k) + c_6 \gamma_i^{2+\varepsilon'}], \quad 0 \leq i \leq M-2.$$

Recall now (3.8), so that, by assembling (5.14) and (5.15), we have on $G(j_k, \ell) \cap G'(j_k, \ell) \cap H'(j_k, \ell)$,

$$\Theta_i(j_k) \subset \{x \in \mathbb{Z} : V(x) - V(b^+(j_k)) \geq \gamma_{i+2}\}, \quad 0 \leq i \leq M-2.$$

An easy calculation gives that, $\sum_{x \in \Theta_i(j_k)} \exp\{-[V(x) - V(b^+(j_k))]\} \leq 2c_6 \gamma_i^{2+\varepsilon'} e^{-\gamma_{i+2}}$, for all $0 \leq i \leq M-2$, on $G(j_k, \ell) \cap G'(j_k, \ell) \cap H'(j_k, \ell)$. On the other hand, if we choose c_4 small enough such that $3c_4 < 1$, we get $2c_6 \gamma_i^{2+\varepsilon'} e^{-\gamma_{i+2}} \leq \varepsilon_i^2$, for all large k and uniformly in $0 \leq i \leq M-2$. This implies that $G(j_k, \ell) \cap G'(j_k, \ell) \cap H'(j_k, \ell) \subset F_1^{env}(j_k, \ell) \cap A_{ann}^{env}(j_k)$. We easily observe that $G(j_k, \ell) \cap H(j_k, \ell) \subset E_2^{env}(j_k)$, for all large k . Thus we obtain

$$G(j_k, \ell) \cap G'(j_k, \ell) \cap H(j_k, \ell) \cap H'(j_k, \ell) \subset F_1^{env}(j_k, \ell) \cap A_{ann}^{env}(j_k) \cap E_2^{env}(j_k).$$

Recalling (5.11), we get

$$P_{k,\ell}^+ \geq P_z \{G(j_k, \ell), G'(j_k, \ell), H(j_k, \ell), H'(j_k, \ell)\}.$$

To bound $P_z \{G(j_k, \ell), G'(j_k, \ell), H(j_k, \ell), H'(j_k, \ell)\}$ from below, we will apply the strong Markov property several times.

First, we apply the strong Markov property at T'_M . Since $V(T'_M) \in I_{\eta_0} := [\eta_0, \eta_0 + L]$, we have for $z \in I_{j_{k-1}}$,

$$P_{k,\ell}^+ \geq P_z \{G(j_k, \ell), G'(j_k, \ell), H(j_k, \ell)\} \inf_{y \in I_{\eta_0}} P_y \{d^+(j_k) \leq d^+((-\infty, \eta_1])\}.$$

To bound by below $P_y \{\dots\}$ on the right hand side, observe that $P_y \{\dots\}$ is greater than $P_{\eta_0} \{\dots\}$. Observe that $\eta_1 \geq -4j_k$ implies $j_k - \eta_1 \leq 5j_k$. Therefore using (2.1), we obtain

$$P_{\eta_0} \{d^+(j_k) \leq d^+((-\infty, \eta_1])\} \geq \frac{\eta_0 - \eta_1}{5j_k + L} = \frac{\eta_0(1 - \eta_0^{-\alpha})}{5j_k + L} \geq c_{21},$$

for all large k , which implies

$$P_{k,\ell}^+ \geq c_{21} P_z \{G(j_k, \ell), G'(j_k, \ell), H(j_k, \ell)\}.$$

We now apply the strong Markov property successively at $(T'_{M-i})_{1 \leq i \leq M}$ and T , such that

$$(5.16) \quad P_{k,\ell}^+ \geq c_{21} P_z \{G(j_k, \ell), H(j_k, \ell)\} \inf_{y \in I_{a_\ell}} Q'_{0,y} \prod_{p=1}^M \inf_{y \in I_{\eta_{M-p+1}}} Q'_{p,y},$$

where

$$\begin{aligned} Q'_{0,y} &:= P_y \left\{ d^+(\eta_M) \leq c_{20} \gamma_M^{2+\varepsilon'}, d^+(\eta_M) < d^+((-\infty, a_{\ell+1}]) \right\}, \\ Q'_{p,y} &:= P_y \left\{ d^+(\eta_{M-p}) \leq c_{20} \gamma_{M-p}^{2+\varepsilon'}, d^+(\eta_{M-p}) < d^+((-\infty, \eta_{M-p+2}]) \right\}, \quad 1 \leq p \leq M. \end{aligned}$$

First, observe that $\inf_{y \in I_{\eta_{M-p+1}}} Q'_{p,y} \geq Q'_{p,\eta_{M-p+1}}$, for all $0 \leq p \leq M$. Therefore we only have to bound from below $Q'_p := Q'_{p,\eta_{M-p+1}}$, for $0 \leq p \leq M$. Clearly,

$$Q'_p \geq P_{\eta_{M-p+1}} \left\{ d^+(\eta_{M-p}) < d^+((-\infty, \eta_{M-p+2}]) \right\} - P_{\eta_{M-p+1}} \left\{ d^+(\eta_{M-p}) \leq c_{20} \gamma_{M-p}^{2+\varepsilon'} \right\}.$$

By (2.1), we obtain, for $1 \leq p \leq M$,

$$P_{\eta_{M-p+1}} \left\{ d^+(\eta_{M-p}) < d^+((-\infty, \eta_{M-p+2}]) \right\} \geq \frac{\eta_{M-p+1} - \eta_{M-p+2}}{\eta_{M-p} - \eta_{M-p+2} + L},$$

and

$$P_{\eta_{M+1}} \left\{ d^+(\eta_M) < d^+((-\infty, a_{\ell+1}]) \right\} \geq \frac{\eta_{M+1} - a_{\ell+1}}{\eta_M - a_{\ell+1} + L}.$$

Recalling (5.12) and (5.13), we get, for $1 \leq p \leq M$,

$$P_{\eta_{M-p+1}} \left\{ d^+(\eta_{M-p}) < d^+((-\infty, \eta_{M-p+2}]) \right\} \geq \frac{\gamma_{M-p+1}}{\gamma_{M-p}} (1 - \gamma_{M-p}^{-\alpha(1-\alpha)}),$$

and for all large k

$$P_{\eta_{M+1}} \left\{ d^+(\eta_M) < d^+((-\infty, a_{\ell+1}]) \right\} \geq \frac{\gamma_M}{2\gamma_M + L} > c_{21},$$

Moreover, (2.2) yields for $0 \leq p \leq M$,

$$P_{\eta_{M-p+1}} \left\{ d^+(\eta_{M-p}) \leq c_{20} \gamma_{M-p}^{2+\varepsilon'} \right\} \leq c_{22} \gamma_{M-p}^{-\varepsilon'/2}.$$

Therefore, recalling (3.7), we obtain $Q'_0 \geq c_{24} > 0$ and, for $1 \leq p \leq M$,

$$Q'_p \geq \frac{\gamma_{M-p+1}}{\gamma_{M-p}} (1 - c_{23} \gamma_{M-p}^{-\beta''}),$$

where $\beta'' := \min\{\alpha(1-\alpha), \beta'\} > 0$. Observe that $\gamma_{M-p}^{-\beta''} \leq \gamma_M^{-\beta''}$ for $1 \leq p \leq M$, and that $\gamma_M^{-\beta''} \rightarrow 0$, $k \rightarrow \infty$. Recalling the fact that $\log(1-x) \geq -c_{13}x$, for $x \in [0, 1/2)$, we obtain

$$\inf_{y \in I_{a_\ell}} Q'_{0,y} \prod_{p=1}^M \inf_{y \in I_{\eta_{M-p+1}}} Q'_{p,y} \geq c_{24} \frac{\gamma_M}{\gamma_0} \exp \left\{ -c_{25} \sum_{p=1}^M \gamma_{M-p}^{-\beta''} \right\},$$

where $c_{25} := c_{13}c_{23}$.

Recalling (5.16), and the fact that $\sum_{p=1}^M \gamma_{M-p}^{-\beta''} \leq c_{26} \gamma_M^{-\beta''}$, this yields, for all large k ,

$$(5.17) \quad P_{k,\ell}^+ \geq c_{27} \frac{\gamma_M}{\gamma_0} P_z \{G(j_k, \ell), H(j_k, \ell)\}.$$

To bound $P_z \{G(j_k, \ell), H(j_k, \ell)\}$ from below, we apply successively the strong Markov property at $(T_{M-i})_{0 \leq i \leq M}$ such that

$$P_z \{G(j_k, \ell), H(j_k, \ell)\} \geq P_z \{H(j_k, \ell)\} \prod_{p=0}^M Q_p,$$

where

$$Q_p := P_{\eta_p} \left\{ d^+((-\infty, \eta_{p+1}]) \leq \min \left\{ d^+(\eta_{p+1} + j_k/5), c_{20} \gamma_p^{2+\varepsilon'} \right\} \right\}, \quad 0 \leq p \leq M.$$

Recall that $P\{A, B\} \geq P\{A\} - P\{B^c\}$. Then (2.2) yields, for $1 \leq p \leq M$,

$$P_{\eta_p} \left\{ d^+((-\infty, \eta_{p+1}]) \leq c_{20} \gamma_p^{2+\varepsilon'} \right\} \geq 1 - c_{28} \gamma_p^{-\varepsilon'/2}.$$

Moreover, using (2.1), we get, for $1 \leq p \leq M$,

$$P_{\eta_p} \left\{ d^+(\eta_{p+1} + j_k/5) \leq d^+((-\infty, \eta_{p+1}]) \right\} \leq c_{29} \frac{\gamma_p}{j_k}.$$

Therefore, observing that, for $1 \leq p \leq M$, we have $\frac{\gamma_p}{j_k} \leq \frac{\gamma_1}{j_k} = j_k^{-\alpha} \rightarrow 0$, $k \rightarrow \infty$, and using the fact that $\log(1-x) \geq -c_{13}x$, for $x \in [0, 1/2)$, we get that

$$(5.18) \quad \prod_{p=1}^M Q_p \geq \exp \left\{ -c_{30} \sum_{p=1}^M \left(\gamma_p^{-\varepsilon'/2} + \frac{\gamma_p}{j_k} \right) \right\},$$

where $c_{30} := c_{13} \max\{c_{28}, c_{29}\}$.

Recalling that $\sum_{p=1}^M \gamma_p^{-\varepsilon'/2} \leq c_{26} \gamma_M^{-\varepsilon'/2}$ and $\sum_{p=1}^M \gamma_p \leq c_{26} \gamma_1 = o(j_k)$, (5.18) yields that

$$(5.19) \quad P_z \{G(j_k, \ell), H(j_k, \ell)\} \geq c_{31} Q_0 P_z \{H(j_k, \ell)\}.$$

Observe that by Donsker's theorem, $\min\{P_z \{H(j_k, \ell)\}, Q_0\} \geq c_{32}$. By assembling (5.17) and (5.19),

$$P_{k,\ell}^+ \geq c_{33} \frac{\gamma_M}{\gamma_0},$$

where $c_{33} := c_{27} c_{31} c_{32}^2$.

Recalling (5.11) and $\theta_k = \lfloor j_k/\gamma_M \rfloor - 1$, we get, uniformly in $z \in I_{j_{k-1}}$,

$$P_z \{E^{env}(j_k)\} \geq c_{33} \theta_k \frac{\gamma_M}{\gamma_0} \geq c_{34} > 0,$$

for all large k , which concludes the proof of (5.6). \square

Acknowledgements

I would like to thank my Ph.D. supervisor Zhan Shi for his help and suggestions.

References

- [1] Andreatti, P. (2005+). Almost sure estimates for the concentration neighborhood of Sinai's walk. To appear in *Stoch. Proc. Appl.* [ArXiv math.PR/0501439](#).
- [2] Asselah, A. and Castell, F. (2005+). Random walk in random scenery and self-intersection local times in dimensions $d \geq 5$. To appear in *Probab. Theory Related Fields.* [ArXiv math.PR/0509721](#).
- [3] Bolthausen, E. (1989). A central limit theorem for two-dimensional random walks in random sceneries. *Ann. Probab.* **17**, 108–115.
- [4] Borodin A. N. (1979). Limit theorems for sums of independent random variables defined on a transient random walk. *Investigations in the theory of probability distributions, IV.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **85** , 17–29, 237, 244.
- [5] Borodin A. N. (1979). A limit theorem for sums of independent random variables defined on a recurrent random walk. *Dokl. Akad. Nauk SSSR* **246:4**, 786–787.
- [6] Durrett, R. (2004). *Probability: Theory and Examples* (third edition). Duxbury, Belmont.
- [7] Golosov, A.O. (1984). Localization of random walks in one-dimensional random environments. *Comm. Math. Phys.* **92**, 491–506.
- [8] Golosov, A.O. (1986). Limit distributions for random walks in random environments. *Soviet Math. Dokl.* **28**, 18–22.
- [9] Häggström, O. (1997). Infinite clusters in dependent automorphism invariant percolation on trees. *Ann. Probab.* **25**, 1423–1436.
- [10] Häggström, O. and Peres, Y. (1997). Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously. *Probab. Theory Related Fields* **113**, 273–285.

- [11] Hirsch, W. M.(1965). A strong law for the maximum cumulative sum of independent random variables. *Comm. Pure Appl. Math.* **18**, 109–127.
- [12] den Hollander, F. and Steif, J. (2006). Random walk in random scenery: A survey of some recent results. Dynamics and Stochastics: Festschrift in Honor of Michael Keane. *IMS Lecture Notes-Monograph Series.* **48**, 53–65.
- [13] Hu, Y. and Shi, Z. (1998). The limits of Sinai’s simple random walk in random environment. *Ann. Probab.* **26**, 1477–1521.
- [14] Kesten, H. (1979). The limit distribution of Sinai’s random walk in random environment. *Physica* **138A**, 299–309.
- [15] Kesten, H. and Spitzer, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete* **50**, 5–25.
- [16] Lyons, R. and Schramm, O. (1999). Stationary measures for random walks in a random environment with random scenery. *New York J. Math.* **5**, 107–113.
- [17] Révész, P. (2005). *Random Walk in Random and Non-Random Environments* (second edition). World Scientific, Singapore.
- [18] Shi, Z. (1998). A local time curiosity in random environment. *Stoch. Proc. Appl.* **76**, 231–250.
- [19] Shi, Z. and Zindy, O. (2006+). A weakness in strong localization for Sinai’s walk. To appear in *Ann. Probab.* ArXiv math.PR/0606376.
- [20] Sinai, Ya.G. (1982). The limiting behavior of a one-dimensional random walk in a random medium. *Th. Probab. Appl.* **27**, 256–268.
- [21] Solomon, F. (1975). Random walks in a random environment. *Ann. Probab.* **3**, 1–31.
- [22] Zeitouni, O. (2004). *Random Walks in Random Environment*, XXXI summer school in probability, St Flour (2001), *Lecture Notes in Math.* **1837**, pp. 193–312. Springer, Berlin.

Laboratoire de Probabilités et Modèles Aléatoires
 Université Paris VI
 4 place Jussieu
 F-75252 Paris Cedex 05
 France
 zindy@ccr.jussieu.fr