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► **To cite this version:**

Mohamed Darouach. Solution to Sylvester equation associated to linear descriptor systems. *Systems and Control Letters*, 2006, 55 (10), pp.835-838. 10.1016/j.sysconle.2006.04.004 . hal-00105181

HAL Id: hal-00105181

<https://hal.science/hal-00105181>

Submitted on 10 Oct 2006

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Solution to Sylvester equation associated to linear descriptor systems

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Abstract

This paper presents the solution of the constrained Sylvester equation associated to linear descriptor systems. This problem has been recently studied in [1], where sufficient conditions for the existence of the solution are given. In the present paper, a simple and direct method is developed to solve this problem. This method shows that the conditions given in [1] are necessary and sufficient.

Keywords: Descriptor systems; Sylvester equation; Existence conditions; Observers; Regional pole placement

1 Introduction

Many problems in control and systems theory are solved by computing the solution of Sylvester equations. As it is well known, these equations have important applications in stability analysis, in observers design, in output regulation with internal stability, and in the eigenvalue assignment (see, e.g [1], [2], [3], [4]). Sylvester equations associated with descriptor systems (or generalized Sylvester equations) have received wide attention in the literature, see [1], [2], [3], [4]. Recently, sufficient conditions for the existence of the solution to these equations under a rank constraint have been given in [1]. The present paper considers the problem studied in [1]. A new approach to solve these equations is developed, necessary and sufficient conditions are presented.

Consider the linear time-invariant multivariable system described by

$$E\dot{x} = Ax + Bu, \quad (1a)$$

$$y = Cx, \quad (1b)$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ the output vector, and $u \in \mathbb{R}^m$ the input vector. The matrices $E \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ are known constant matrices, with $\text{rank}(E) = q < n$, $\text{rank}(B) = m$ and $\text{rank}(C) = p < q$.

Consider the Problem 1 studied in [1], which can be formulated as follows:

Let \mathcal{D} be a region in the open left half complex plane, $\mathcal{D} \subseteq \mathbb{C}^-$, symmetric with respect to the real axis. The problem is to find matrices $T \in \mathbb{R}^{(q-p) \times n}$, $Z \in \mathbb{R}^{(q-p) \times n}$, and $H_T \in \mathbb{R}^{(q-p) \times (q-p)}$, such that

$$TA - H_T TE = -ZC, \quad (2)$$

with $\sigma(H_T) \subset \mathcal{D}$, under the rank constraint

$$\text{rank} \begin{bmatrix} TE \\ LA \\ C \end{bmatrix} = n, \quad (3)$$

where $\sigma(H_T)$ is the spectrum of H_T and $L \in \mathbb{R}^{(n-q) \times n}$ is any full row rank matrix satisfying $LE = 0$.

The Sylvester equation (2) has a close relation with many problems in linear control theory of descriptor systems, such as the eigenstructure assignment [4], [5], and the state observer design. The observer design problem can be formulated as follows [1]:

For the descriptor system (1), consider a reduced-order observer of the form

$$\dot{z}(t) = H_T z(t) + T B u(t) - Z y(t), \quad (4a)$$

$$\hat{x}(t) = S z(t) + \bar{N} \bar{y} + N y(t), \quad (4b)$$

where $z \in \mathbb{R}^{(q-p)}$ is the state of the observer and $\bar{y} \in \mathbb{R}^{(n-q)}$ is a fictitious output. As shown in [1], if Problem 1 is solved for some matrices T , Z and H_T and if we compute the matrices S , \bar{N}

and N such that $\begin{bmatrix} S & \bar{N} & N \end{bmatrix} \begin{bmatrix} TE \\ LA \\ C \end{bmatrix} = I$, then, for observer (4) we have:

i) $\lim_{t \rightarrow \infty} (z(t) - T E x(t)) = 0$,

ii) for $\bar{y}(t) = -L B u(t)$, the estimated state $\hat{x}(t)$ satisfies $\lim_{t \rightarrow \infty} (x(t) - \hat{x}(t)) = 0$.

Remark 1 Notice that for $L = 0$, Problem 1 is reduced to finding matrices $T \in \mathbb{R}^{(n-p) \times n}$, $Z \in \mathbb{R}^{(n-p) \times n}$, and $H_T \in \mathbb{R}^{(n-p) \times (n-p)}$, such that

$$TA - H_T TE = -ZC, \quad (5)$$

with $\sigma(H_T) \subset \mathcal{D}$, under the rank constraint

$$\text{rank} \begin{bmatrix} TE \\ C \end{bmatrix} = n. \quad (6)$$

These conditions are those required for the observer design with order $(n-p)$, see [5]. \square

2 Main results

In this section we will present a new and simple solution to Problem 1. Necessary and sufficient conditions to solve this problem are also given.

Since matrix $E \in \mathbb{R}^{n \times n}$ is singular and $\text{rank}(E) = q < n$, there exist two nonsingular matrices P and Q of appropriate dimensions such that

$$E_c = P E Q = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, A_c = P A Q = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, C Q = [C_1 \ C_2], T P^{-1} = [T_1 \ T_2],$$

and $L P^{-1} = [L_1 \ L_2]$.

According to this partitioning, equation $LE = 0$ becomes $[L_1 \ L_2] E_c = 0$, which leads to $L_1 = 0$ and $L_2 \in \mathbb{R}^{(n-q) \times (n-q)}$ is an arbitrary nonsingular matrix, since L is of full row rank.

Therefore, equation (3) can be written as

$$\text{rank} \begin{bmatrix} TE \\ LA \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} TP^{-1}E_c \\ LP^{-1}A_c \\ CQ \end{bmatrix} = \text{rank} \begin{bmatrix} T_1 & 0 \\ L_2A_{21} & L_2A_{22} \\ C_1 & C_2 \end{bmatrix} = n, \quad (7)$$

or equivalently

$$\text{rank} \begin{bmatrix} TE \\ LA \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} T_1 & 0 \\ A_{21} & A_{22} \\ C_1 & C_2 \end{bmatrix} = n, \quad (8)$$

since L_2 is a nonsingular matrix.

Now equation (2) can be written as

$$TP^{-1}A_c - H_T TP^{-1}E_c = -ZCQ, \quad (9)$$

then using the above results we obtain

$$T_1A_{11} - H_T T_1 = \mathbf{J}C_1, \quad (10a)$$

$$T_1A_{12} = \mathbf{J}C_2, \quad (10b)$$

with $\sigma(H_T) \subset \mathcal{D}$, and where $\mathbf{J} = -[T_2 \quad Z]$, $C_1 = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}$, and $C_2 = \begin{bmatrix} A_{22} \\ C_2 \end{bmatrix}$.

Define the following matrices $\mathbf{C} = [C_1 \quad C_2]$ and $\mathbf{T} = TEQ$, then equation (10) becomes

$$\mathbf{T}A_c - H_T \mathbf{T} = \mathbf{J}C, \quad (11)$$

and equation (8) can be written as

$$\text{rank} \begin{bmatrix} \mathbf{T} \\ \mathbf{C} \end{bmatrix} = n, \quad (12)$$

where $\mathbf{T} = TEQ = [T_1 \quad 0]$, which can also be written as

$$\mathbf{T}F = 0, \quad (13)$$

with $F = \begin{bmatrix} 0 \\ I_{n-q} \end{bmatrix}$.

Then Problem 1 reduces to finding matrices \mathbf{T} , \mathbf{J} , and H_T , with $\sigma(H_T) \subset \mathcal{D}$, such that (11) - (13) are satisfied.

Remark 2 From the above results it is easy to see that condition $\text{rank} \begin{bmatrix} LA \\ C \end{bmatrix} = n - q + p$ of theorem 1 of [1] is equivalent to \mathbf{C} being a full row rank matrix. \square

Before proceeding, let us give the following definition which is useful for the sequel.

Definition 1. *System (1) is \mathcal{D} -strongly detectable if and only if the following conditions are satisfied*

$$1) \text{rank} \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} = n, \forall \lambda \in \mathbb{C}, \lambda \notin \mathcal{D},$$

$$2) \text{rank} \begin{bmatrix} E \\ LA \\ C \end{bmatrix} = n.$$

For the standard systems, where $E = I$, Definition 1 becomes.

Definition 2. Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$ be constant matrices, then the pair (C, A) is said to be \mathcal{D} -detectable if and only if

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \forall \lambda \in \mathbb{C}, \lambda \notin \mathcal{D}. \quad (14)$$

Before we give our main results, we establish the following lemma.

Lemma 1. The following statements are equivalent

- 1) System (1) is \mathcal{D} -strongly detectable,
- 2) $\text{rank} \begin{bmatrix} \lambda I - A_c & F \\ C & 0 \end{bmatrix} = 2n - q, \forall \lambda \in \mathbb{C}, \lambda \notin \mathcal{D}$, and $\text{rank} \begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} = n - q$.

Proof. By using the above transformations we obtain:

$$\text{rank} \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} P(A - \lambda E)Q \\ CQ \end{bmatrix} = \text{rank} \begin{bmatrix} \lambda I - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \\ C_1 & C_2 \end{bmatrix}, \quad (15)$$

and

$$\text{rank} \begin{bmatrix} E \\ LA \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} PEQ \\ LP^{-1}PAQ \\ CQ \end{bmatrix} = \text{rank} \begin{bmatrix} I_q & 0 \\ L_2A_{21} & L_2A_{22} \\ C_1 & C_2 \end{bmatrix} = q + \text{rank} \begin{bmatrix} A_{22} \\ C_2 \end{bmatrix}, \quad (16)$$

which prove the lemma. \square

The necessary and sufficient conditions for the existence of the solution to Problem 1 are then given by the following theorem.

Theorem 1. Under the assumption that matrix \mathbf{C} is of full row rank, or $\text{rank } \mathbf{C} = n - q + p$, the necessary and sufficient condition for the existence of the solution to Problem 1 is that system (1) must be \mathcal{D} -strongly detectable.

Proof. Under condition (12), there exists a matrix \mathbf{R} , such that $\mathbf{T} = \mathbf{R} - K\mathbf{C}$ and $\text{rank} \begin{bmatrix} \mathbf{T} \\ \mathbf{C} \end{bmatrix} = \text{rank} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix} = n$, or $\text{rank} \begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix} = n$. In this case (11) becomes equivalent to

$$(\mathbf{R} - K\mathbf{C})A_c - H_T(\mathbf{R} - K\mathbf{C}) = \mathbf{J}\mathbf{C}, \quad (17)$$

or

$$H_T\mathbf{R} + K_1\mathbf{C} + K\mathbf{C}A_c = \mathbf{R}A_c, \quad (18)$$

where $K_1 = \mathbf{J} - H_TK$. From equation (18) it is easy to see that the knowledge of matrices \mathbf{R} , H_T , K_1 , and K is necessary and sufficient to determine the matrices T and Z of the initial problem.

Now let $\begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix}^{-1} = [M_1 \ N_1]$, then post multiplying the two sides of equation (18) by the nonsingular matrix $[M_1 \ N_1]$ leads to

$$H_T = \mathbf{R}A_cM_1 - K\mathbf{C}A_cM_1, \quad (19)$$

and

$$K_1 = \mathbf{R}A_cN_1 - K\mathbf{C}A_cN_1. \quad (20)$$

On the other hand, equation (13) can be written as $(\mathbf{R} - K\mathbf{C})F = 0$, or

$$K\mathbf{C}F = \mathbf{R}F. \quad (21)$$

The necessary and sufficient condition for the existence of a solution to (21) is that $\text{rank } \mathbf{C}F = \text{rank} \begin{bmatrix} \mathbf{R}F \\ \mathbf{C}F \end{bmatrix} = \text{rank } F = n - q$, which is equivalent to $\begin{bmatrix} A_{22} \\ C_2 \end{bmatrix}$ being of full column rank. Under this condition, the solution to (21) is given by

$$K = \mathbf{R}F(\mathbf{C}F)^+ + Y(I - (\mathbf{C}F)(\mathbf{C}F)^+), \quad (22)$$

where A^+ is any generalised inverse of matrix A , satisfying $AA^+A = A$, and Y is an arbitrary matrix of appropriate dimension. Then substituting this expression in (19) yields

$$H_T = \Lambda - Y\Omega, \quad (23)$$

where $\Lambda = \mathbf{R}A_cM_1 - \mathbf{R}F(\mathbf{C}F)^+\mathbf{C}A_cM_1$ and $\Omega = (I - (\mathbf{C}F)(\mathbf{C}F)^+)\mathbf{C}A_cM_1$.

From (23), $\sigma(H_T) \subset \mathcal{D}$ if and only if the pair (Ω, Λ) is \mathcal{D} -detectable, or equivalently $\text{rank} \begin{bmatrix} \lambda I - \Lambda \\ \Omega \end{bmatrix} = q - p, \forall \lambda \in \mathbb{C}, \lambda \notin \mathcal{D}$.

Now, define the following full column matrices $S_1 = \begin{bmatrix} \mathbf{R} & 0 \\ \mathbf{C} & 0 \\ 0 & I \end{bmatrix}$, $S_2 = \begin{bmatrix} I & \mathbf{R}F(\mathbf{C}F)^+ \\ 0 & I - (\mathbf{C}F)(\mathbf{C}F)^+ \\ 0 & \mathbf{C}F^+ \end{bmatrix}$,

and the nonsingular matrix $S_3 = \begin{bmatrix} M_1 & N_1 & 0 \\ 0 & 0 & I \end{bmatrix}$, then we have

$$\begin{aligned} \text{rank} \begin{bmatrix} \lambda I - A_c & F \\ \mathbf{C} & 0 \end{bmatrix} &= \text{rank } S_1 \begin{bmatrix} \lambda I - A_c & F \\ \mathbf{C} & 0 \end{bmatrix} S_3 \\ &= n - q + p + \text{rank } S_2 \begin{bmatrix} \lambda I - \mathbf{R}A_cM_1 & \mathbf{R}F \\ -\mathbf{C}A_cM_1 & \mathbf{C}F \end{bmatrix} \\ &= 2n - 2q + p + \text{rank} \begin{bmatrix} \lambda I - \Lambda \\ \Omega \end{bmatrix} = 2n - q, \end{aligned}$$

if and only if the pair (Ω, Λ) is \mathcal{D} -detectable, and using lemma 1, we obtain the result of the theorem. \square

From the above results we can give the algorithm to solve Problem 1.

2.1 Algorithm

Let the conditions of theorem 1 be satisfied, then the following method computes the solution of Problem 1.

Step1: Compute matrices P and Q such that $PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, then compute $A_c = PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $CQ = [C_1 \ C_2]$.

Step2: Find a matrix \mathbf{R} such that $\begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix}$ is nonsingular and let $[M_1 \ N_1] = \begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix}^{-1}$. Define \mathcal{D} , and compute the gain matrix Y , using the pole placement method (see [7] and [8]), such that $H_T = \Lambda - Y\Omega$, and $\sigma(H_T) \subset \mathcal{D}$, then compute K_1 and K from (20) and (22).

Step3: Compute $\mathbf{J} = K_1 + H_T K$, $\mathbf{T} = \mathbf{R} - K\mathbf{C}$, $T_1 = \mathbf{T} \begin{bmatrix} I_q \\ 0 \end{bmatrix}$, $T_2 = -\mathbf{J} \begin{bmatrix} I_{n-q} \\ 0 \end{bmatrix}$, then deduce $Z = -\mathbf{J} \begin{bmatrix} 0 \\ I_p \end{bmatrix}$ and $T = [T_1 \ T_2] P$.

We can make the following remarks.

Remark 3 Contrary to the algorithm presented in [1], where many transformations are used in addition to the resolution of a Sylvester equation, in our paper the problem is reduced to a pole placement one which can be solved by the existing robust methods. \square

Remark 4 When $\text{rank}(\mathbf{C}) = n - q + p - d$, for some integer $0 < d \leq n - q$, we can apply the above approach to this case, if and only if system (1) is \mathcal{D} -detectable. \square

Remark 5 In the above result, we have introduced a full row rank matrix \mathbf{R} such that $\begin{bmatrix} \mathbf{R} \\ \mathbf{C} \end{bmatrix}$ is nonsingular. The choice of this matrix is non unique, one can see that the results are independent of this choice. In fact, let $\bar{\mathbf{R}}$ be another full row rank matrix such that $\begin{bmatrix} \bar{\mathbf{R}} \\ \mathbf{C} \end{bmatrix}$ is nonsingular, then there exists a regular matrix θ such that $\theta \bar{\mathbf{R}} = \mathbf{R}$. By premultiplying (17) by θ we obtain $(\bar{\mathbf{R}} - \bar{K}\mathbf{C})A_c - \bar{H}_T(\bar{\mathbf{R}} - \bar{K}\mathbf{C}) = \bar{\mathbf{J}}\mathbf{C}$, where $\bar{\mathbf{J}} = \theta\mathbf{J}$, $\bar{H}_T = \theta H_T \theta^{-1}$, and $\sigma(\bar{H}_T) = \sigma(H_T)$, which shows that the solution given above is independent of the choice of matrix \mathbf{R} . \square

2.2 Regional pole placement in LMI regions

This section discusses the solution of Problem 1 by regional pole placement in LMI regions of the complex plane. We shall use the same notations as in [1] and define the LMI type region $\mathcal{D} \subseteq \mathbb{C}^-$ by the existence of two matrices $\Delta = \Delta^T \in \mathbb{R}^{(q-p) \times (q-p)}$ and $\Gamma \in \mathbb{R}^{(q-p) \times (q-p)}$ such that $\mathcal{D} = \{\lambda \in \mathbb{C} : f_{\mathcal{D}}(\lambda) = \Delta + \lambda\Gamma + \bar{\lambda}\Gamma^T < 0\}$, then we propose the following result.

Theorem 2. *Let $\mathcal{D} \subseteq \mathbb{C}^-$ be an LMI type region, and assume that system (1) is \mathcal{D} -strongly detectable, then there exists a solution to Problem 1, with $\sigma(H_T) \subset \mathcal{D}$ if and only if there exist a symmetric positive definite matrix $X \in \mathbb{R}^{(q-p) \times (q-p)}$ and a matrix $\Psi \in \mathbb{R}^{(q-p) \times (q-p)}$ such that*

$$\Delta \otimes X + \Gamma \otimes (X\Lambda) - \Gamma \otimes (\Psi\Omega) + \Gamma^T \otimes (X\Lambda)^T - \Gamma^T \otimes (\Psi\Omega)^T < 0, \quad (24)$$

where $\Lambda = \mathbf{R}A_c M_1 - \mathbf{R}F(\mathbf{C}F)^+ \mathbf{C}A_c M_1$, $\Omega = (I - (\mathbf{C}F)(\mathbf{C}F)^+) \mathbf{C}A_c M_1$, and \otimes is the Kronecker's product. In this case $H_T = \Lambda - Y\Omega$, with $Y = X^{-1}\Psi$.

Proof. The proof is direct, in fact from [6], $\sigma(H_T) \subset \mathcal{D}$ if and only if there exists a symmetric positive definite matrix $X \in \mathbb{R}^{(q-p) \times (q-p)}$ such that

$$\Delta \otimes X + \Gamma \otimes (XH_T) + \Gamma^T \otimes (XH_T)^T < 0. \quad (25)$$

From the above result we have $H_T = \Lambda - Y\Omega$, and by substituting this value in (25), and by putting $\Psi = XY$ we obtain the desired result. \square

3 Conclusion

In this paper, we have presented a simple method to solve the generalized Sylvester equation associated to descriptor systems observers and control design. This problem was considered in [1], where only sufficient conditions were given. We have also given the necessary and sufficient conditions for the existence of the solution. Solution by LMI regional pole placement was also presented.

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