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We suggest a scalar model for deformation and flow of an amorphous material such as a foam or an emulsion. To describe elastic, plastic and viscous behaviours, we use three scalar variables: elastic deformation, plastic deformation rate and total deformation rate; and three material specific parameters: shear modulus, yield deformation and viscosity. We obtain equations valid for different types of deformations and flows slower than the relaxation rate towards mechanical equilibrium. In particular, they are valid both in transient or steady flow regimes, even at large elastic deformation. We discuss why viscosity can be relevant even in this slow shear (often called "quasi-static") limit. Predictions of the storage and loss moduli agree with the experimental literature, and explain with simple arguments the non-linear large amplitude trends.

I. INTRODUCTION

Elastic materials deform reversibly [1]; plastic materials can be sculpted, that is, they can be deformed into a new shape and keep it [2]; and viscous materials flow [3]. A wide variety of materials display a combination of these properties, such as elasto-plastic metals and rocks, visco-elastic polymer solutions or visco-plastic mineral suspensions [4–6].

Liquid foams, that is gas bubbles separated by liquid walls, are visco-elasto-plastic [7–9]: they are elastic at low strain, plastic at high strain and flow under high shear rate. This is also the case for other concentrated suspensions of deformable objects in a liquid [4, 10, 11], such as droplets in emulsions, vesicles suspensions, or red blood cells in blood.

Despite a large literature on experiments and simulations (see [9] for a review), we lack an unified theoretical description of foams. There is no consensus yet on a central question: what are the physically relevant variables? A series of statistical models focus on fluctuations and their correlations [12–16]. Conversely, recent contributions [17–23] focus on average macroscopic quantities to obtain a more classical continuous description.

Here we choose to group three macroscopic quantities which are measurable as averages on microscopical details [17]: (i) Elastic deformation is a state variable [24] reversibly stored by the foam's microstructure, that is, the shape of bubbles [25, 26]; it determines the elastic contribution to the stress. (ii) Plastic deformation results in energy dissipation analogous to solid friction. (iii) Large scale velocity gradients are associated with a viscous friction. Each of the three mechanical behaviors is associated with a material specific parameter: elastic modulus, yield deformation and viscosity.

For simplicity, we assume here that these parameters are constant and the equations are linear. We consider here homogeneous deformation of a material, not depending on space coordinates. We consider only the magnitude of deformation, but not spatial orientation: the material state variables are all scalars. This represents an incompressible liquid foam, where the deformation is a pure shear. We assume that this shear is slow enough so that the foam is always close to mechanical equilibrium, but quick enough to neglect coarsening such as due to gas diffusion between bubbles, or bubble coalescence due to soap film breakage. Although this model is minimal, it is written with enough generality to enable for extensions to higher dimensions using tensors (the correspondance with tensors introduces a factor 1/2, see section IV C), to higher shear rates, and to other ingredients such as external forces (to be published).

This article is organised as follows. Section II introduces a visco-elasto-plastic model (eqs. 3,7) based on two scalar variables: the elastic deformation and the (slow) shear rate (Fig. 3). The rate of plastic deformation is determined by both the applied shear rate, and the current state of the elastic deformation (or equivalently the elastic part of the stress) rather than by the total stress [27, 28]. Section III presents scalar predictions of creep and oscillatory responses. The storage and loss moduli predicted as a function of the strain amplitude agree with experimental data without any adjustable parameters, using only the three model-independent parameters determined by experiments (yield point, shear modulus, viscosity). The agreement becomes very good if we describe the plastic yielding as a gradual transition spreading between an onset value of deformation and a saturation value (eq. 5). Section IV summarises and discusses our model, and opens some perspectives.

II. MODEL

A. Kinetics

1. *Elastic and plastic strain*

The elastic deformation U is a *state variable*, that is an intrinsic property of the foam's current deformation state. We note its time derivative dU/dt. Conversely, we use a dot for the total strain rate $\dot{\varepsilon}$ and the plastic strain rate $\dot{\varepsilon}_P$, emphasising that they are not the time derivative of a state variable. For instance, the time integral $\varepsilon = \int \dot{\varepsilon}\ \mathrm{d}t$ of the velocity gradient is the gradient of displacement (more generally, for large deformations, it is a function of the displacement): it is extrinsic and *explicitly depends* on the sample's past history.

The total applied deformation rate is shared between elastic deformation U and the plastic deformation rate:

$$\dot{\varepsilon} = \frac{\mathrm{d}U}{\mathrm{d}t} + \dot{\varepsilon}_P. \quad (1)$$

In the particular case of an elastic regime, $\dot{\varepsilon}_P = 0$, the elastic deformation U is equal to the total applied deformation on the material ε. Thus, in an elastic regime, no intrinsic definition of U is necessary.

However, as soon as $\dot{\varepsilon}_P \neq 0$, the situation changes. U and $\dot{\varepsilon}$ become independent variables, and $\varepsilon = \int \dot{\varepsilon} dt$ does not define the elastic deformation. In the extreme example of a steady flow, $\mathrm{d}U/\mathrm{d}t = 0$, then $\dot{\varepsilon} = \dot{\varepsilon}_P$: U and $\dot{\varepsilon}$ are no longer correlated.

These variables are macroscopic: U is related to the elastic contribution to macroscopic stress and $\dot{\varepsilon}_P$ to the irreversibility of the stress *versus* total strain curve. In the specific case of foams, they can be traced back to detailed properties of the bubbles pattern: independent, intrinsic definition [24] based on geometry (shape of bubbles [26]) for U; and topological rearrangements called "T1 processes" [14, 29, 30] (using their rate and orientation [17]) for $\dot{\varepsilon}_P$.

2. *Sharing the total strain*

The problem now is to express how, in eq. (1), $\dot{\varepsilon}$ is shared between $\mathrm{d}U/\mathrm{d}t$ and $\dot{\varepsilon}_P$. We must write a closure relation between these variables, for instance by expressing how $\dot{\varepsilon}_P$ depends on the current state of elastic deformation and on the applied deformation rate: $\dot{\varepsilon}_P(U, \dot{\varepsilon})$. We use the following three hypotheses leading to eq. 2.

First we describe an abrupt transition from elastic to plastic regime, as could be the case for an ordered foam [31]. To indicate that T1s appear when the absolute value of deformation $|U|$ exceeds the yield deformation U_Y, we introduce the discontinuous Heaviside function $\mathcal{H}$ (which is zero for negative numbers, and 1 for numbers greater than or equal to zero). This hypothesis can be relaxed in the section II A 3, introducing a more progressive transition.

Secondly, we account for the hysteresis. Plastic rearrangements occur when the deformation rate $\dot{\varepsilon}$ and the current deformation U have the same sign, and again we express it using $\mathcal{H}$. Else, the deformation rate results in elastic unloading, and the deformation gets smaller than the yield deformation.

Thirdly, we use the fact that, in a slowly sheared motion, the only relevant time scale to fix the rate of plastic rearrangements is $\dot{\varepsilon}$.

Eventually, the plasticity equation writes:

$$\dot{\varepsilon}_P = \mathcal{H}(|U| - U_Y)\ \mathcal{H}(U\dot{\varepsilon})\ \dot{\varepsilon}. \quad (2)$$

Eq. (2) can be used to close the system of equations. Injecting it in eq. (1) yields an evolution equation of U as a function of the applied shear rate $\dot{\varepsilon}$:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \dot{\varepsilon}\ \left[1 - \mathcal{H}(|U| - U_Y)\ \mathcal{H}(U\dot{\varepsilon})\right]. \quad (3)$$

In eq. (3) U_Y appears as the stable value for U, that is, a fixed point, at least if $\dot{\varepsilon} > 0$; else, the stable fixed point is $-U_Y$.

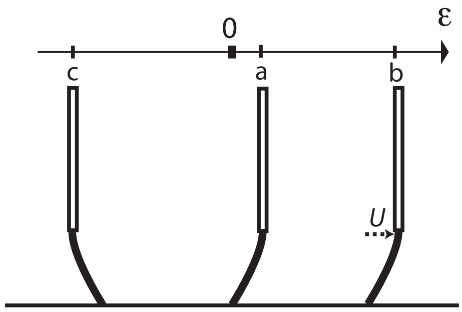

FIG. 1: Analog scalar system: an elastic brush whose flexion is U stick/slipping on wall. We represent several states, for an imposed oscillatory "painting-like" motion of the handle ε, from rest position 0: (a) onset of sliding to the right, (b) far-right position, (c) far-left position.

To visualise the direction and the amplitude of the deformation U, we suggest an analogy with the motion of a brush on a wall (Fig. 1). The handle of the brush moves with an oscillatory position ε parallel to the wall (analog of the imposed scalar deformation of the material), while the displacement of the handle with respect to the brush tip is U (the analog of the internal elasticity of the material). The sliding velocity of the contact point is therefore $\dot{\varepsilon}_P$ according to equation (1) and is the analog of plasticity in a material.

3. *Gradual transition to plasticity*

In a disordered foam, for instance with a wide distribution of bubble sizes, topological rearrangements do not necessarily occur for the same value of deformation.

We therefore distinguish two different yield deformations. First, a *plasticity yield* U_y, where deformation ceases to be reversible, as defined in material sciences. It is the highest deformation for which there is no T1. It is characteristic of the microstructure, and can even be close to zero for a very disordered foam.

Second, a *saturation yield* U_Y, the saturation value of elastic deformation at which the material can flow with arbitrary large total deformations (for instance in Bingham fluids). It is the lowest deformation for which the T1s convert the whole total strain into plastic strain. That is, U_Y is the collapse limit at which a material structure cannot sustain stress.