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Nature of the deconfining phase transition in the 2+1-dimensional SU(N) Georgi-Glashow model

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The nature of the deconfining phase transition in the 2+1-dimensional SU(N) Georgi-Glashow model is investigated. Within the dimensional-reduction hypothesis, the properties of the transition are described by a two-dimensional vectorial Coulomb gas models of electric and magnetic charges. The resulting critical properties are governed by a generalized SU(N) sine-Gordon model with self-dual symmetry. We show that this model displays a massless flow to an infrared fixed point which corresponds to the Z_N parafermions conformal field theory. This result, in turn, supports the conjecture of Kogan, Tekin, and Kovner that the deconfining transition in the 2+1-dimensional SU(N) Georgi-Glashow model belongs to the Z_N universality class.

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1 Introduction

The 2+1 dimensional Georgi Glashow (GG) model has attracted a lot of interest in the past since it is a much simpler theory than QCD but still retains some common interesting features like the existence of a confinement phase. The confinement phase of the GG model appears in the weak-coupling limit and can be investigated analytically. In particular, it has been shown by Polyakov [1] that the resulting phase is a Coulomb plasma of monopoles and antimonopoles and the photon acquires a mass from the Debye screening by monopoles. The resulting phase is a confinement phase since a probe charge inserted in the vacuum will be screened by monopoles [2]. The finite-temperature effect is an important issue since confining gauge theories generally become deconfined at high temperatures [3]. The nature of the confinement-deconfinement phase transition in the 2+1 dimensional GG model has been analysed in detail [4, 5, 6, 2]. In particular, as first stressed by the authors of Ref. [5], the massive $W^{\pm}$ gauge bosons play a crucial role for the deconfinement transition. This phase transition stems from the competition between the monopoles and $W^{\pm}$ gauge bosons which act as U(1) vortices. Within the dimensional-reduction hypothesis, it has been shown that this competition results in an Ising critical behavior [5, 6].

The deconfinement transition in the SU(N)-generalization of the GG model has also been investigated similarly [7]. As it will be briefly reviewed in Section 2, within the dimensional-reduction hypothesis, the low-energy Hamiltonian density, which governs the resulting phase

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transition, takes the form of a generalized two-dimensional sine-Gordon model:

$$\begin{aligned}\mathcal{H}_N &= \frac{1}{2}\left[\left(\partial_x\vec{\Phi}\right)^2+\left(\partial_x\vec{\Theta}\right)^2\right] \\ &- g\sum_{\vec{\alpha}\epsilon\Delta_+}\left[:\cos\left(\sqrt{4\pi}\vec{\alpha}\cdot\vec{\Phi}\right):+:\cos\left(\sqrt{4\pi}\vec{\alpha}\cdot\vec{\Theta}\right):\right],\end{aligned} \tag{1}$$

where the summation over $\vec{\alpha}$ is taken over the positive roots of SU(N) normalized to unity ($\vec{\alpha}^2 = 1$), and $:\,:$ denotes the normal ordering symbol. The bosonic vector field $\vec{\Phi} = (\Phi_1, \ldots, \Phi_{N-1})$ is made of $N-1$ free boson fields with chiral components $\Phi_{aR,L}$: $\Phi_a = \Phi_{aL}+\Phi_{aR}$, $(a = 1, .., N-1)$. The dual vector field $\vec{\Theta} = (\Theta_1, \ldots, \Theta_{N-1})$ is defined by: $\Theta_a = \Phi_{aL} - \Phi_{aR}$. Model (1) is a generalization of the sine-Gordon for the boson vector field $\vec{\Phi}$ with an additional pertubation depending on the dual field $\vec{\Theta}$. This field theory has been introduced in Ref. [8] for exploring critical properties of vectorial Coulomb gas models of electric and magnetic charges. The interacting part of model (1) is a strongly relevant perturbation with scaling dimension one and its special structure makes it invariant under the Gaussian duality symmetry: $\vec{\Phi} \leftrightarrow \vec{\Theta}$, i.e. the exchange of electric and magnetic charges in the Coulomb gas context. In what follows, such model will be referred to as the SU(N) self-dual sine-Gordon (SDSG) model. This self-duality symmetry opens a possibility for the existence of a critical point in the infrared (IR) limit which governs the deconfinement transition in the 2+1 dimensional SU(N) GG model. From the renormalization group (RG) point of view, the SU(N) SDSG model (1) will then be characterized by a massless flow from the ultraviolet (UV) fixed point with central charge $c_{UV} = N-1$ to a conformally invariant IR fixed point with a smaller central charge $c_{IR} < N-1$ according to the c-theorem [9]. The perturbative study of model (1) has been done in Refs. [8, 10] and a fixed point has been found whose nature is beyond the scope of these investigations. In Ref. [7] in connection to the deconfinement transition in the 2+1 dimensional SU(N) GG model, it has been conjected that the IR fixed point belongs to the Z_N parafermions universality class which is a conformally invariant theory (CFT) with central charge $c = 2(N-1)/(N+2)$ [11]. In the following, we shall prove this conjecture in the general N case and show that the SU(N) SDSG model (1) with $g > 0$ displays a massless flow to a Z_N parafermionic fixed point.

The letter is organized as follows. In the next section, we briefly review the connection between the SU(N) SDSG model (1) and the phase transition in the SU(N) GG model. In Section 3, the emergence of the Z_2 criticality in Eq. (1) for $N = 2$ is reviewed for completeness and to fix the notations. We present then a proof of the conjecture for the first non-trivial case i.e. $N = 3$ in Section 4. Finally, we consider the general N case in the last section.

2 Deconfinement transition in the SU(N) GG model

The SU(N) GG model describes a SU(N) gauge theory which interacts with a Higgs field transforming in the adjoint representation. Its Lagrangian density in the Euclidean reads as follows:

$$\mathcal{L}_N = \frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}^2\right) + \mathrm{Tr}\left((D_\mu\Phi)^2\right) + V(\Phi), \tag{2}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu]$, $D_\mu \Phi = \partial_\mu \Phi + g\,[A_\mu, \Phi]$, $A_\mu = A_\mu^A T^A$ and $\Phi = \Phi^A T^A$ ($A = 1, 2, \ldots, N^2 - 1$), T^A being the generators of the Lie algebra of SU(N) normalized as: $\mathrm{Tr}(T^A T^B) = \delta^{AB}/2$. In Eq. (2), the Higgs potential is supposed to be such that the SU(N) gauge symmetry is spontaneously broken down to $\mathrm{U}(1)^{N-1}$. In addition to the Higgs field, the perturbative spectrum consists of $N-1$ massless photons and $N(N-1)$ massive gauge bosons in correspondence with the ladder operator $E_{\vec{\alpha}}$ of the Cartan-Weyl basis of SU(N) ($\vec{\alpha}$ being the roots of SU(N) normalized to one) [7]. In this basis, the Higgs vacuum-expectation value is diagonal: $\langle \Phi \rangle = \vec{h} \cdot \vec{H}$, $\vec{H}$ being the Cartan generators of SU(N). The W bosons have the mass: $m_{\mathrm{W}} \equiv m_{\vec{\alpha}} = g|\vec{h} \cdot \vec{\alpha}|$, and carry the $\mathrm{U}(1)^{N-1}$ (electric) charge: $\vec{e}_{\vec{\alpha}} = g\,\vec{\alpha}$. In the weak-coupling regime, which is defined by $m_{\mathrm{W}} \sim m_{\mathrm{H}} \gg g^2$, the massive-gauge bosons and the Higgs field decouple at low-energy and a massless free gauge theory remains. However, non-perturbative configurations (monopoles or instantons) give a mass m_γ to these photons. Indeed, as is well known, model (2) admits stable classical solutions with finite action ($\Pi_2(SU(N)/U(1)^{N-1}) = Z_N$ for $N > 2$). The magnetic field of these monopoles is: $B^\mu = \vec{g} \cdot \vec{H} x^\mu / 4\pi r^3$, where the magnetic charge ($\vec{g}$) satisfies the condition [12]: $\vec{g} = 4\pi \sum_{a=1}^{N-1} n_a \vec{\beta}_a^* / g$, $\vec{\beta}_a^*$ being the dual simple roots ($\vec{\beta}_a^* = \vec{\beta}_a / |\vec{\beta}_a| = \vec{\beta}_a$, $\vec{\beta}_a$ being the simple roots of SU(N)) and n_a are integers.

The weak-coupling phase corresponds to a confined phase and a deconfinement transition should occur at sufficiently high temperature. The central question is the universality class of this transition in the general N case. In the regime $m_\gamma \ll T \ll m_{\mathrm{W}}$, one can adopt the dimensional-reduction hypothesis for exploring the phase transition since the size of the compactified direction (T^{-1}) is much smaller than the average distance between the monopoles. The monopoles form thus a two-dimensional Coulomb gas with vectorial magnetic charges. The contribution of the massive gauge bosons is crucial for the physics of the deconfinement as first stressed in Refs. [5, 7]. In the following, we shall neglect the contribution of the Higgs field for investigating the deconfinement transition (see Ref. [13] for the influence of a Higgs-boson mass). The partition function, which describes the two-dimensional vectorial Coulomb gas of monopoles and massive gauge bosons, reads then as follows:

$$\begin{aligned} \mathcal{Z} &= \sum_{M,N=0}^{\infty} \frac{1}{M!N!} \prod_{i=1}^{M} \prod_{j=1}^{N} \sum_{\vec{\alpha}_i} \sum_{\vec{\alpha}_j} \zeta_{\vec{\alpha}_i} \tilde{\zeta}_{\vec{\alpha}_j} \\ &\quad \int d^2\vec{x}_i d^2\vec{y}_j \;\; \exp\left(-\mathcal{S}\left(\vec{x}_i, \vec{g}_{\vec{\alpha}_i}; \vec{y}_j, \vec{e}_{\vec{\alpha}_j}\right)\right), \end{aligned} \tag{3}$$

where $\mathcal{S}\left(\vec{x}_i, \vec{g}_{\vec{\alpha}_i}; \vec{y}_j, \vec{e}_{\vec{\alpha}_j}\right)$ is the effective action of M monopoles located at $\vec{x}_i$ with magnetic charges $\vec{g}_{\vec{\alpha}_i}$, fugacity $\zeta_{\vec{\alpha}_i}$ and N W bosons at positions $\vec{y}_j$ with electric charges $\vec{e}_{\vec{\alpha}_j}$, fugacity $\tilde{\zeta}_{\vec{\alpha}_j}$. This effective action has been carefully derived in Ref. [7] but can also be found in a phenomenological manner as in the SU(2) case [6]. It separates into three different parts: the actions of two-dimensional Coulomb gas for the monopoles and W bosons with an UV cut-off T^{-1} and an interaction between them which takes the form of an Aharonov-Bohm phase factor:

$$\mathcal{S} = -\frac{T}{2\pi} \sum_{i<j} \vec{g}_{\vec{\alpha}_i} \cdot \vec{g}_{\vec{\alpha}_j} \ln\left(T|\vec{x}_i - \vec{x}_j|\right)$$

$$- \quad \frac{1}{2\pi T}\sum_{i<j}\vec{e}_{\vec{\alpha}_i}\cdot\vec{e}_{\vec{\alpha}_j}\ln\left(T|\vec{y}_i-\vec{y}_j|\right)-\frac{i}{2\pi}\sum_{i=1}^{M}\sum_{j=1}^{N}\vec{g}_{\vec{\alpha}_i}\cdot\vec{e}_{\vec{\alpha}_j}\theta\left(\vec{x}_i-\vec{y}_j\right), \tag{4}$$

with the neutral condition: $\sum_i \vec{e}_{\vec{\alpha}_i} = \sum_j \vec{g}_{\vec{\alpha}_j} = \vec{0}$, and $\theta\left(\vec{x}_i-\vec{y}_j\right)$ is the angle between the vector connecting the monopole at $\vec{x}_i$ and the W boson at $\vec{y}_j$ and a chosen spatial direction. We then introduce a free massless bosonic vector field $\vec{\Phi} = (\Phi_1,\ldots,\Phi_{N-1})$ and its dual field $\vec{\Theta} = (\Theta_1,\ldots,\Theta_{N-1})$ to express partition function (3) in terms of these bosonic fields:

$$\begin{aligned}\mathcal{Z} \;\sim\; & \sum_{M,N=0}^{\infty}\frac{1}{M!N!}\prod_{i=1}^{M}\prod_{j=1}^{N}\sum_{\vec{\alpha}_i}\sum_{\vec{\alpha}_j}\zeta_{\vec{\alpha}_i}\tilde{\zeta}_{\vec{\alpha}_j}\int d^2\vec{x}_i d^2\vec{y}_j \\ & \left\langle\prod_{i=1}^{M}\exp\left(i\sqrt{T}\vec{g}_{\vec{\alpha}_i}\cdot\vec{\Phi}\left(\vec{x}_i\right)\right)\prod_{j=1}^{N}\exp\left(iT^{-1/2}\vec{e}_{\vec{\alpha}_j}\cdot\vec{\Theta}\left(\vec{y}_j\right)\right)\right\rangle.\end{aligned} \tag{5}$$

The effective Hamiltonian density, which describes the deconfinement-confinement transition of the SU(N) GG model (2), can then be deduced by performing the summations in Eq. (5):

$$\begin{aligned}\mathcal{H}_{\text{eff}} \;=\; & \frac{1}{2}\left[\left(\partial_x\vec{\Phi}\right)^2+\left(\partial_x\vec{\Theta}\right)^2\right] \\ & -\sum_{\vec{\alpha}\epsilon\Delta_+}\left[\zeta_{\vec{\alpha}}\cos\left(\frac{4\pi\sqrt{T}}{g}\,\vec{\alpha}\cdot\vec{\Phi}\right)+\tilde{\zeta}_{\vec{\alpha}}\cos\left(\frac{g}{\sqrt{T}}\,\vec{\alpha}\cdot\vec{\Theta}\right)\right],\end{aligned} \tag{6}$$

where the summation is taken over the positive roots of SU(N) and the fugacities have been rescaled. This low-energy effective theory has been first derived in Ref. [7] by means of a similar Coulomb-gas analysis and also by using the magnetic Z_N symmetry which is spontaneously broken in the confinement phase [14]. Model (6) is a generalization of the sine-Gordon model for multi-boson fields and describes the competition between monopoles and vortices in this SU(N) GG model. The one-loop RG equations for this model have been investigated in Refs. [8, 10] and they are quite complex in general. However, as in Ref. [7], we shall consider here a simpler case, which is stable under the RG flow, where all monopole fugacities are equal $\zeta_{\vec{\alpha}} = \zeta$ and similarly for the vortex fugacities: $\tilde{\zeta}_{\vec{\alpha}} = \tilde{\zeta}$. A non-trivial stable IR fixed point has been found perturbatively within this manifold [8, 10] which should govern the deconfinement transition in the SU(N) GG model (2) [7]. The low-energy physics of the resulting model can be deduced qualitatively by simple scaling arguments. The scaling dimensions of the two vertex operators in Eq. (6) are: $\Delta = T4\pi/g^2$ and $\tilde{\Delta} = g^2/T4\pi$, i.e. $\Delta\tilde{\Delta} = 1$. When $T < g^2/8\pi$, one has $\Delta < 1/2$ and $\tilde{\Delta} > 2$: the perturbation depending on the dual vector field is irrelevant whereas the monopole term is a strongly relevant perturbation. The low-energy theory reduces to a sine-Gordon model for the $\vec{\Phi}$ field and a mass-gap is induced. It corresponds to the confinement phase where the massive W gauge bosons can be neglected. At high-temperature $T > g^2/2\pi$, the monopole term is now irrelevant $\Delta > 2$ and model (6) reduces again to a sine-Gordon model but for the dual vector field $\vec{\Theta}$ with a relevant perturbation ($\tilde{\Delta} < 1/2$). A mass-gap is still present but it corresponds to the deconfined phase with the unbinding of the W gauge bosons. The transition between these two phases takes place in the regime: $g^2/8\pi < T < g^2/2\pi$ where both pertubations of Eq. (6) are relevant. On general grounds, the transition is expected to

appear along the self-dual line where model (6) is invariant under the duality transformation $\vec{\Phi} \leftrightarrow \vec{\Theta}$, i.e. the symmetry between electric and magnetic charges. This self-dual symmetry is realized when $\zeta = \tilde{\zeta}$ and $T = g^2/4\pi$. In this case, the confinement-deconfinement transition is thus governed by the SU(N) SDSG model (1). In Ref. [7], it has been conjectured that this phase transition belongs to the Z_N universality class corresponding to the Z_N parafermionic CFT [11]. We now turn to the proof of this conjecture.

3 Ising criticality

We start with the simplest case, i.e. $N = 2$, where the SU(2) SDSG model takes the form:

$$\mathcal{H}_2 = \frac{1}{2}\left[(\partial_x\Phi)^2 + (\partial_x\Theta)^2\right] - g\left[\,\vdots\, \cos\left(\sqrt{4\pi}\,\Phi\right) \,\vdots\, + \,\vdots\, \cos\left(\sqrt{4\pi}\,\Theta\right) \,\vdots\,\right]. \tag{7}$$

This model is well known (see e.g. Ref. [15]) and can be exactly diagonalized even in a more general case when the two cosine terms in Eq. (7) have independent amplitudes. Assuming that the boson field Φ is compactified with radius $R = 1/\sqrt{4\pi}$ or $R = 1/\sqrt{\pi}$, i.e. the Dirac point (see the book [16] for a review) in our notation, the model can be refermionized by introducing two Majorana fermions fields $\xi^{1,2}$. This procedure is nothing but the standard bosonization of two Ising models [17, 18, 19]. The bosonization rules are given by

$$\begin{aligned} \xi^1_R + i\xi^2_R &= \frac{1}{\sqrt{\pi}} :\exp\left(i\sqrt{4\pi}\Phi_R\right):, \\ \xi^1_L + i\xi^2_L &= \frac{1}{\sqrt{\pi}} :\exp\left(-i\sqrt{4\pi}\Phi_L\right):, \end{aligned} \tag{8}$$

where $\Phi_{R,L}$ are the chiral components of the Bose field: $\Phi_L = (\Phi + \Theta)/2$ and $\Phi_R = (\Phi - \Theta)/2$. They satisfy $[\Phi_R, \Phi_L] = i/4$ to insure the anticommutation relation between right and left fermions. One can easily check that the bosonic representation (8) is consistent with the defining operator product expansion (OPE) for the Majorana fields:

$$\xi^a_L(z)\,\xi^b_L(w) \sim \frac{\delta^{ab}}{2\pi(z-w)}, \tag{9}$$

with a similar OPE for the right Majorana fermion. The self-dual Hamiltonian (7) can then be expressed in terms of these Majorana fermions:

$$\mathcal{H}_2 = -\frac{i}{2}\sum_{a=1}^{2}\left(\xi^a_R\partial_x\xi^a_R - \xi^a_L\partial_x\xi^a_L\right) - im\xi^2_R\xi^2_L, \tag{10}$$

with $m = 2\pi g$. The Hamiltonian of the SU(2) SDSG model separates thus into two commuting pieces. One of the decoupled degrees of freedom corresponds to an effective off-critical Ising model described by the massive Majorana fermion $\xi^2_{R,L}$, whereas the second Majorana field $\xi^1_{R,L}$ remains massless. The Gaussian self-dual symmetry $\Phi \leftrightarrow \Theta$ coincides to the Kramers-Wannier (KW) duality symmetry of the Ising model associated to the Majorana fermion ξ^1: no mass term $i\xi^1_R\xi^1_L$, which is odd under the KW duality, can appear in the effective Hamiltonian. The existence of this massless Majorana mode signals the Z_2 (Ising) criticality of the SU(2) SDSG model (7) and the Ising nature of the deconfining transition in the SU(2) GG model [5, 6].

4 Z_3 criticality

We now turn to the $N = 3$ case which is much more complex than the previous case. In this section, we shall argue that the SU(3) SDSG model displays in the IR limit a Z_3 critical behavior with central charge $c = 4/5$. To this end, let us first rewrite model (1) explicitely in terms of the two bosonic fields using the roots of SU(3):

$$\begin{aligned}\mathcal{H}_3 &= \frac{1}{2}\left[\left(\partial_x\vec{\Phi}\right)^2+\left(\partial_x\vec{\Theta}\right)^2\right]-g\left[:\cos\left(\sqrt{4\pi}\Phi_s\right)+\cos\left(\sqrt{\pi}\Phi_s\right)\cos\left(\sqrt{3\pi}\Phi_f\right):\right.\\ &+ \left.:\cos\left(\sqrt{4\pi}\Theta_s\right)+\cos\left(\sqrt{\pi}\Theta_s\right)\cos\left(\sqrt{3\pi}\Theta_f\right):\right],\end{aligned} \tag{11}$$

where $\vec{\Phi} = (\Phi_s, \Phi_f)$ and $\vec{\Theta} = (\Theta_s, \Theta_f)$ for $N = 3$. In the following, we shall assume that Φ_s (respectively Φ_f) is a bosonic field compactified with radius $R_s = 1/\sqrt{\pi}$ (respectively $R_f = \sqrt{3/\pi}$). As in Section 3, at this free-fermion point for the bosonic field Φ_s, one can refermionize model (11) by introducing two Majorana fermions $\xi^{1,2}$ using Eq. (8). We need also the refermionization of the vertex operators $\cos(\sqrt{\pi}\Phi_s)$ and $\cos(\sqrt{\pi}\Theta_s)$ with scaling dimension 1/4. They can be expressed in terms of the Ising order ($\sigma_{1,2}$) and disorder ($\mu_{1,2}$) parameters of the two underlying Ising models [18, 19]: $:\cos(\sqrt{\pi}\Phi_s):\ \sim \mu_1\mu_2$ and $:\cos(\sqrt{\pi}\Theta_s):\ \sim \sigma_1\mu_2$. Model (11) can then be recasted in terms of this Ising^2 CFT and the free massless bosonic field Φ_f:

$$\begin{aligned}\mathcal{H}_3 &= \frac{1}{2}\left[(\partial_x\Phi_f)^2+(\partial_x\Theta_f)^2\right]-\frac{i}{2}\sum_{a=1}^{2}\left(\xi_R^a\partial_x\xi_R^a-\xi_L^a\partial_x\xi_L^a\right)\\ &- im\xi_R^2\xi_L^2-g\left[\mu_1\mu_2:\cos\left(\sqrt{3\pi}\Phi_f\right):+\sigma_1\mu_2:\cos\left(\sqrt{3\pi}\Theta_f\right):\right],\end{aligned} \tag{12}$$

with $m = 2\pi g$. Clearly due to the mass term in Eq. (12), the Ising model, corresponding to the fermion ξ^2, is out of criticality and since $m > 0$ ($g > 0$) it belongs to its high-temperature phase in our convention ($m \sim T - T_c$). In this case, the Ising disorder operator condenses: $\langle\mu_2\rangle \neq 0$. One can formally integrate out over this massive degrees of freedom and rewrite model (12) in this low-energy limit ($E \ll m$):

$$\begin{aligned}\mathcal{H}_3 &= \frac{1}{2}\left[(\partial_x\Phi_f)^2+(\partial_x\Theta_f)^2\right]-\frac{i}{2}\left(\xi_R^1\partial_x\xi_R^1-\xi_L^1\partial_x\xi_L^1\right)\\ &-\lambda\ \left[\mu_1:\cos\left(\sqrt{3\pi}\Phi_f\right):+\sigma_1:\cos\left(\sqrt{3\pi}\Theta_f\right):\right],\end{aligned} \tag{13}$$

where λ is a non-universal constant that results from the integration of the massive degrees of freedom. The effective Hamiltonian (13) describes a critical Ising model and a free massless boson field that interact with a strongly relevant perturbation with scaling dimension 7/8.

The next step of the approach is to switch on a different basis to determine the main effect of the relevant perturbation of Eq. (13). To this end, it is important to observe that the free massless bosonic field Φ_f has a very special radius $R_f = \sqrt{3/\pi}$ in the classification of the CFT with central charge $c = 1$. At this radius, it displays a CFT with an extended symmetry: a $N = 2$ (respectively $N = 1$) superconformal field theory (SCFT) whether the bosonic field Φ_f is compactified along a circle (respectively an orbifold) [20]. The conformal symmetry of the

UV fixed point of model (13), i.e. Ising $\times$ [$c = 1$ SCFT], with central charge $c = 3/2$ can also be described in terms of the product of TIM $\times$ Potts CFTs, where the TIM and Potts refers respectively to the tricritical Ising and three-state Potts CFTs. The precise conformal embedding has been derived by the authors of Ref. [21] and the result depends on the nature of the compactification of the bosonic field Φ_f:

$$\begin{aligned} \text{Ising} \times (c = 1 \;\; N = 1 \;\; \text{SCFT}) &= P\left[\mathcal{M}_4 \times \mathcal{M}_5\right] && (14) \\ \text{Ising} \times (c = 1 \;\; N = 2 \;\; \text{SCFT}) &= P\left[(c = 7/10 \;\; N = 1 \;\; \text{SCFT}) \times \mathrm{Z}_3\right], && (15) \end{aligned}$$

where $\mathcal{M}_p$ denotes the minimal model series with central charge $c_p = 1-6/p(p+1)$ i.e. the TIM and Potts CFTs for $p = 4$ and $p = 5$ respectively [16]. The $c = 7/10 \;\; N = 1$ SCFT is known to be equivalent to the TIM CFT [22]. In Eqs. (14,15), a projection P is crucial to realize the equivalence as demonstrated in Ref. [21]. For instance, if we denote the primary fields of the $\mathcal{M}_p$ CFT as $\Phi^{(p)}_{r,s}$ $(1 \leq r \leq p-1, 1 \leq s \leq p)$ then the projection P, for the equivalence of Eq. (14), restricts the operators to the subset: $\{\Phi^{(4)}_{r,s}\Phi^{(5)}_{s,q}\}$ [21]. By looking at the dimensions of the fields in this subset and OPEs consistency, we find that the original perturbation with scaling dimension 7/8 of Eq. (13) identifies to the submagnetic operator $\sigma^{'}$ of the TIM CFT with scaling dimension 7/8:

$$\sigma^{'} \sim \; \mu_1 \!:\! \cos\left(\sqrt{3\pi}\Phi_f\right) \!:\; + \sigma_1 \!:\! \cos\left(\sqrt{3\pi}\Theta_f\right) \!:\, . \tag{16}$$

With all these results, we can now express the low-energy effective Hamiltonian (13) in the new basis:

$$\mathcal{H}_3 = \mathcal{H}_0^{\mathrm{Z}_3} + \mathcal{H}_0^{\mathrm{TIM}} - \lambda\sigma^{'}, \tag{17}$$

where $\mathcal{H}_0^{\mathrm{Z}_3}$ (respectively $\mathcal{H}_0^{\mathrm{TIM}}$) denotes the Hamiltonian of the three-state Potts (respectively TIM) CFT. The deformation of the TIM CFT by the subleading magnetization $\sigma^{'}$ (i.e. $\Phi^{\mathrm{TIM}}_{2,1}$) is known to be an integrable massive field theory [23]. Therefore, we deduce that the SU(3) SDSG model flows in the far IR limit towards a fixed point with central charge $c = 4/5$ corresponding to the Z_3 universality class. In this respect, we observe that the original Gaussian self-duality $\vec{\Phi} \leftrightarrow \vec{\Theta}$ of model (11) coincides with the KW symmetry of the three-state Potts model. In addition, it might be interesting to see the emergence of this Potts criticality from another point of view. Indeed, it is possible to rewrite model (12) before the integration of the massive Majorana fermion ξ^2 in the $\mathrm{Z}_3 \times$ TIM basis using the identification (16):

$$\mathcal{H}_3 = -\frac{i}{2}\left(\xi^2_R\partial_x\xi^2_R - \xi^2_L\partial_x\xi^2_L\right) + \mathcal{H}_0^{\mathrm{Z}_3} + \mathcal{H}_0^{\mathrm{TIM}} - im\xi^2_R\xi^2_L - g\mu_2\sigma^{'}. \tag{18}$$

The Ising and TIM CFTs can be combined to form another CFT with extended symmetry [21, 24]: a $\mathcal{W}_3$ CFT with Z_3 symmetry which has been introduced by Fateev and Zamolodchikov [25]. More precisely, the P(Ising $\times$ TIM) CFT corresponds to the $\mathrm{Z}_3^{[5]}$ CFT with central charge $c = 6/5$ [21]. Some character decompositions have been found in Ref. [24] and we have:

$$\chi^{\mathrm{Ising}}_{1/16}\chi^{\mathrm{TIM}}_{7/16} = \chi^{\mathrm{Ising}}_{0}\chi^{\mathrm{TIM}}_{3/2} + \chi^{\mathrm{Ising}}_{1/2}\chi^{\mathrm{TIM}}_{0} = \chi^{\mathcal{W}_3}_{1/2}, \tag{19}$$

where $\chi_h^{\rm CFT}$ is the character of the conformal tower with holomorphic weight h of the underlying CFT. We have checked numerically the identities (19) with Mathematica. The primary fields of the $\mathrm{Z}_3^{[5]}$ CFT can be noted as $\Phi^{\mathrm{W}_3}_{(\lambda,\lambda')}$, where λ and λ' are dominant weights of SU(3) algebra i.e. $\lambda=\sum_{i=1}^{2}(l_i-1)\lambda_i$, $\lambda'=\sum_{i=1}^{2}(l_i'-1)\lambda_i$ (λ_i being the fundamental weights of SU(3) and l_i, l_i' are positive). The $\mathrm{Z}_3^{[5]}$ primary field with scaling dimension one is $\Phi^{\mathrm{W}_3}_{(\lambda_1+\lambda_2,0)}$ which transform in the adjoint and trivial representations of SU(3). The relation (19) leads us to expect that model (18) can be written in the following compact form:

$$\mathcal{H}_3=\mathcal{H}_0^{\mathrm{Z}_3}+\mathcal{H}_0^{\mathrm{W}_3}-\lambda\Phi^{\mathrm{W}_3}_{(\lambda_1+\lambda_2,0)}, \tag{20}$$

where $\mathcal{H}_0^{\mathrm{W}_3}$ is the Hamiltonian of the $\mathrm{Z}_3^{[5]}$ CFT. It is interesting to note that some integrable deformations of the $\mathcal{W}_3$ CFT have been found in Ref. [26]. In particular, a $\mathrm{Z}_3^{[5]}$ CFT perturbed by a primary field which transform in the adjoint and trivial representations of SU(3) is a massive integrable field theory. Therefore, we conclude again that the SU(3) SDSG model flows towards the Z_3 fixed point in the IR limit within this approach. In this respect, we deduce that the deconfining transition in the SU(3) GG model belongs to the Z_3 universality class in full agreement with the conjecture of Kogan *et al.* [7].

5 The general case

We now consider the general N case namely that the SU(N) SDSG flow towards the Z_N fixed point in the IR limit. In this respect, let us introduce N copies of the $\mathrm{SU}(2)_1$ Wess-Zumino-Novikov-Witten (WZNW) CFT and consider the following Hamiltonian density:

$$\mathcal{H}=\frac{2\pi}{3}\sum_{a=1}^{N}\left(:\vec{J}_{aR}^{2}:+:\vec{J}_{aL}^{2}:\right)-\frac{g}{2}\sum_{a<b}\left(\mathrm{Tr}g_a\mathrm{Tr}g_b-\mathrm{Tr}\left(g_a\vec{\sigma}\right)\cdot\mathrm{Tr}\left(g_b\vec{\sigma}\right)\right), \tag{21}$$

where $\vec{J}_{aR,L}$ are the chiral $\mathrm{SU}(2)_1$ current of the ath WZNW CFT ($a=1,\ldots,N$) and $\vec{\sigma}$ are the Pauli matrices. These currents satisfy the $\mathrm{SU}(2)_1$ current algebra defined by [19]:

$$J_{aL}^{\alpha}\left(z\right)J_{bL}^{\beta}\left(\omega\right)\sim\frac{\delta^{\alpha\beta}\delta_{ab}}{8\pi^2\left(z-\omega\right)^2}+\frac{i\delta_{ab}\epsilon^{\alpha\beta\gamma}J_{aL}^{\gamma}\left(\omega\right)}{2\pi\left(z-\omega\right)}, \tag{22}$$

with $\alpha,\beta,\gamma=x,y,z$ and a similar OPE for the right current. The interacting part of model (21), that we denote $\mathcal{H}_{\rm int}$, is a strongly relevant perturbation with scaling dimension one which is made of the WZNW field g_a. This primary field transforms in the fundamental representation of SU(2) and is defined by the following OPEs [16]:

$$\begin{aligned}
J_{aL}^{\alpha}\left(z\right)g_b\left(\omega,\bar{\omega}\right) &\sim -\frac{\delta_{ab}}{2\pi\left(z-\omega\right)}\ \frac{\sigma^{\alpha}}{2}\ g_b\left(\omega,\bar{\omega}\right)\\
J_{aR}^{\alpha}\left(\bar{z}\right)g_b\left(\omega,\bar{\omega}\right) &\sim \frac{\delta_{ab}}{2\pi\left(\bar{z}-\bar{\omega}\right)}\ g_b\left(\omega,\bar{\omega}\right)\ \frac{\sigma^{\alpha}}{2}.
\end{aligned} \tag{23}$$

It might be interesting to note that model (21) appears in the context of coupled electronic chains [27]. The next step of the approach is to consider the following conformal embedding to analyse the main effect of $\mathcal{H}_{\rm int}$:

$$SU(2)_1 \times SU(2)_1 \times .. \times SU(2)_1 \rightarrow SU(2)_N \times \mathcal{G}_N, \tag{24}$$

where $\mathrm{SU}(2)_N$ is the level-N SU(2) WZNW CFT with central charge $c_N = 3N/(N+2)$ and $\mathcal{G}_N$ is a discrete CFT with central charge $c_{\mathcal{G}_N} = N(N-1)/(N+2)$. The latter central charge coincides with the sum of the central charge of the $N-1$ first minimal models:

$$c_{\mathcal{G}_N} = \frac{N\,(N-1)}{(N+2)} = \sum_{m=2}^{N+1} \left(1 - \frac{6}{m\,(m+1)}\right), \tag{25}$$

which leads us to expect that $\mathcal{G}_N$ should be related to the product $\mathcal{M}_3 \times \mathcal{M}_4 \times \ldots \times \mathcal{M}_{N+1}$. The precise identification requires a projection P: $\mathcal{G}_N \sim P(\mathcal{M}_3 \times \mathcal{M}_4 \ldots \times \mathcal{M}_{N+1})$, which has been found in Ref. [21]. In fact, this projection restricts the product of the primary fields of $\mathcal{M}_3 \times \mathcal{M}_4 \times \ldots \times \mathcal{M}_{N+1}$ to the subset:

$$\{\Phi^{(3)}_{r_3,s_3}\Phi^{(4)}_{s_3,s_4}\Phi^{(5)}_{s_4,s_5}\ldots\Phi^{(N)}_{s_{N-1},s_N}\Phi^{(N+1)}_{s_N,s_{N+1}}\}, \tag{26}$$

with $1 \leq r_p \leq p-1$ and $1 \leq s_p \leq p$. This quantum equivalence can be proved by a recursive approach [28]. We would like to express $\mathcal{H}_{\rm int}$ in the new basis i.e. $SU(2)_N \times \mathcal{G}_N$. To this end, we introduce the $\mathrm{SU}(2)_N$ chiral currents $\vec{I}_{R,L}$ which are the sum of the $\mathrm{SU}(2)_1$ currents: $\vec{I}_{R,L} = \sum_{a=1}^{N} \vec{J}_{aR,L}$. The key point of the analysis is that $\mathcal{H}_{\rm int}$ does not depend on the $\mathrm{SU}(2)_N$ CFT but expresses only in terms of the fields of $\mathcal{G}_N$. To show this, it is sufficient to determine the OPE between the $\mathrm{SU}(2)_N$ currents and $\mathcal{H}_{\rm int}$ and it should be zero. This calculation can be done by using the defining OPEs (23) and we find:

$$\begin{aligned} I^{\alpha}_{L}(z)\left(\mathrm{Tr}g_a\mathrm{Tr}g_b - \mathrm{Tr}\left(g_a\vec{\sigma}\right)\cdot\mathrm{Tr}\left(g_b\vec{\sigma}\right)\right)(\omega,\bar{\omega}) &\sim 0 \\ I^{\alpha}_{R}(\bar{z})\left(\mathrm{Tr}g_a\mathrm{Tr}g_b - \mathrm{Tr}\left(g_a\vec{\sigma}\right)\cdot\mathrm{Tr}\left(g_b\vec{\sigma}\right)\right)(\omega,\bar{\omega}) &\sim 0, \end{aligned} \tag{27}$$

so that $\mathcal{H}_{\rm int}$ depends only on the fields of $\mathcal{G}_N$. We found an operator $\prod_{m=3}^{N}\Phi^{(m)}_{m-2,m-1}\Phi^{(N+1)}_{N-1,N+1}$ in the $\mathcal{G}_N \sim P(\mathcal{M}_3 \times \mathcal{M}_4 \ldots \times \mathcal{M}_{N+1})$ CFT which couples all minimal models involved with scaling dimension one:

$$\Delta = \sum_{m=3}^{N} \frac{3}{2m\,(m+1)} + \frac{N+4}{2\,(N+1)} = 1. \tag{28}$$

Therefore, due to the special structure of $\mathcal{H}_{\rm int}$, a mass gap in the $\mathcal{G}_N$ sector will be opened and model (21) displays a massless flow from the UV fixed with central charge $c_{\rm UV} = N$ to an IR fixed point with $\mathrm{SU}(2)_N$ criticality: $c_{\rm IR} = 3N/(N+2)$.

The connection with the SU(N) SDSG model can then be made by exploiting the fact that the $\mathrm{SU}(2)_1$ CFT has a free-field representation in terms of a massless bosonic field compactified on a circle at the self-dual radius: $R = 1/\sqrt{2\pi}$ [19, 16]. For the N copies of the $\mathrm{SU}(2)_1$ theory of the original model (21), we thus introduce the bosonic field φ_a $(a = 1, 2, \ldots, N)$ with its

chiral components $\varphi_{aR,L}$. A faithfull representation of the OPEs (23) for the WZNW tensor field g_a is given by [19]:

$$g_a = \frac{1}{\sqrt{2}} \begin{pmatrix} :e^{-i\sqrt{2\pi}\varphi_a}: & i:e^{-i\sqrt{2\pi}\vartheta_a}: \\ i:e^{i\sqrt{2\pi}\vartheta_a}: & :e^{i\sqrt{2\pi}\varphi_a}: \end{pmatrix}, \tag{29}$$

from which we deduce the bosonic representation of model (21):

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\sum_{a=1}^{N}\left[(\partial_x\varphi_{aR})^2 + (\partial_x\varphi_{aL})^2\right] \\ &- g\sum_{a<b}\left(:\cos(\sqrt{2\pi}(\varphi_a-\varphi_b)): + :\cos(\sqrt{2\pi}(\vartheta_a-\vartheta_b)):\right). \end{aligned} \tag{30}$$

We then perform a canonical transformation on the Bose fields to simplify Eq. (30). To this end, we introduce a bosonic field $\Phi_{cR,L}$ and $N-1$ other bosonic fields $\Phi_{lR,L}$ $(l = 1, \ldots, N-1)$ as follows:

$$\begin{aligned} \Phi_{cR(L)} &= \frac{1}{\sqrt{N}}\left(\varphi_1 + \ldots + \varphi_N\right)_{R(L)} \\ \Phi_{lR(L)} &= \frac{1}{\sqrt{l(l+1)}}\left(\varphi_1 + \ldots + \varphi_l - l\varphi_{l+1}\right)_{R(L)}. \end{aligned} \tag{31}$$

In this new basis, Hamiltonian (30) reads:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\left[(\partial_x\Phi_c)^2 + (\partial_x\Theta_c)^2\right] + \frac{1}{2}\left[\left(\partial_x\vec{\Phi}\right)^2 + \left(\partial_x\vec{\Theta}\right)^2\right] \\ &- g\sum_{\vec{\alpha}\epsilon\Delta_+}\left[:\cos\left(\sqrt{4\pi}\vec{\alpha}\cdot\vec{\Phi}\right): + :\cos\left(\sqrt{4\pi}\vec{\alpha}\cdot\vec{\Theta}\right):\right], \end{aligned} \tag{32}$$

where the summation over $\vec{\alpha}$ is taken over the positive roots of SU(N), $\vec{\Phi} = (\Phi_1, \ldots, \Phi_{N-1})$, and $\vec{\Theta} = (\Theta_1, \ldots, \Theta_{N-1})$. We thus observe that the initial model (21) separates into two commuting pieces: $\mathcal{H} = \mathcal{H}_c^0 + \mathcal{H}_N$, a free-massless boson Hamiltonian $\mathcal{H}_c^0$ for the field Φ_c and the SU(N) SDSG model. Since we know that model (21) admits a massless flow onto the $\mathrm{SU}(2)_N$ fixed point, we deduce that the SU(N) SDSG model flows in the IR limit to a conformally invariant fixed point with central charge $c_{\mathrm{IR}} = 2(N-1)/(N+2)$. The massless free-bosonic field Φ_c, which decouples from the interaction of Eq. (32), is compactified with radius $R_c = \sqrt{N/2\pi}$ and describes an extended $\mathrm{U}(1)_N$ rational $c = 1$ CFT [29]. Moreover, the Z_N parafermionic CFT is known to be equivalent to the coset: $\mathrm{Z}_N \sim \mathrm{SU}(2)_N/\mathrm{U}(1)_N$ [11, 16] so that we conclude that the SU(N) SDSG model admits a massless flow onto the Z_N CFT. The original Gaussian self-duality $\vec{\Phi} \leftrightarrow \vec{\Theta}$ coincides with the KW self-duality symmetry of the Z_N parafermionic model. Finally, the massless flow found here implies that the confinement-deconfinement phase transition of the 2+1 dimensional SU(N) GG model (2) should belong to the Z_N universality class. In this respect, this result demonstrates the conjecture presented in Ref. [7]. Other implications of the quantum equivalence approach (24) will be discussed elsewhere [28].

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