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The degree of polarization (DOP) is an important tool in many optical measurement and imaging applications. We address the problem of its estimation in images that are perturbed with both speckle and photon noise, by determining the Cramer–Rao lower bounds (CRLBs) when the illuminated materials are purely depolarizing. We demonstrate that the CRLBs are simply the sum of the CRLBs due to speckle noise and Poisson noise. We use this result to analyze the influence of different optical parameters on DOP estimation.

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Polarization imaging is increasingly used in medical imaging,[1] remote sensing,[2] and industrial control.[3] For example, this technique can reveal contrasts between regions of a scene that have the same intensity reflectivity but different polarimetric properties.[2,3] These systems often measure the degree of polarization (DOP) of laser light backscattered by a scene. Images are thus corrupted with speckle noise, which is inherent to coherent imaging.[4] Moreover, in some configurations of practical interest, the number of detected photons is so low that photon noise must also be taken into account.

The influence of the speckle noise on intensity and DOP estimation was analyzed in Ref. 5. The influence of coupled speckle and photon noise on target detection on low-flux-intensity images was also studied in Ref. 6. Our purpose in this Letter is to address estimation of the DOP in the presence of both speckle and photon noise. We will determine the Cramer–Rao lower bounds (CRLBs) and use them to analyze the influence of the mean photon flux and the speckle order on DOP estimation. We will assume throughout the Letter that the observed materials are purely depolarizing.

Consider an active polarimetric imaging system in which the scene is illuminated with collimated and purely polarized laser light. Two images are thus obtained: $\mathbf{X}=\{X_i, i\in[1,N]\}$ (N being the number of pixels in the image) is formed with the fraction of the light polarized parallel to the incident state, and $\mathbf{Y}=\{Y_i, i\in[1,N]\}$ is formed with the fraction of the light in the orthogonal state. In the following mathematical developments, one-dimensional notation will be used for simplicity, and bold symbols will denote N-dimensional vectors.

The problem we address is the following. One assumes to have observed a sample χ of size N pixels, defined by $\chi=\{n_{X,1}, n_{Y,1}, n_{X,2}, n_{Y,2}, \ldots n_{X,N}, n_{Y,N}\}$, where $n_{X,j}, n_{Y,j}$ represent, respectively, the number of photons measured at pixel j in images $\mathbf{X}$ and $\mathbf{Y}$. One will assume that all the elements of the sample are statistically independent. The $n_{X,j}$ are assumed to have the same average value I_X, and the $n_{Y,j}$ the average value I_Y. Since the illuminated materials are assumed to be purely depolarizing, these values can be expressed as a function of the total intensity I_0 (expressed in number of photons) and of the DOP P as[4] $I_X=I_0(1+P)/2$ and $I_Y=I_0(1-P)/2$. Our goal will be to estimate the two parameters I_0 and P from the sample χ.

The probability distribution function (PDF) of the sample values is classically determined by using a semiclassical model of light. Since illumination is coherent, in the absence of photon noise, the reflected intensity I measured in one pixel of image $\mathbf{U}=\mathbf{X},\mathbf{Y}$ is modeled as a Gamma-distributed random variable with mean I_U and order L, whose PDF is

$$P_U(I)=\frac{L^L I^{L-1}}{\Gamma[L] I_U^L}\exp\left(-\frac{LI}{I_U}\right) \quad (1)$$

with $U=X,Y$. For a given realization of the intensity I, the number of detected photons is an integer-valued random variable n distributed with a Poisson PDF: $P(n|I)=\exp(-I)I^n/n!$. The PDF of the number of photons averaged over the possible realizations of I can be expressed as[7] $P_U(n)=\int_0^{+\infty}P(n|I)P_U(I)\mathrm{d}I$ with $U=X,Y$. An explicit expression of this integral can be computed[7]:

$$P_U(n)=\frac{\Gamma(L+n)}{\Gamma(L)\Gamma(n+1)}\left(1+\frac{L}{I_U}\right)^{-n}\left(1+\frac{I_U}{L}\right)^{-L}. \quad (2)$$

It represents the PDF of the photon number measured for a light of average intensity I_U in the presence of speckle noise of order L.

To analyze the precision of estimation of parameters I_0 and P from this sample, we determine the CRLBs[8] that represent the lowest variance that can be reached by any unbiased estimator. It is an efficient way to characterize the intrinsic difficulty of an estimation task. To determine the CRLBs, one first needs to calculate the Fisher information matrix

$$J = \begin{bmatrix} -\left\langle \frac{\partial^2}{\partial P^2} l(\chi) \right\rangle & -\left\langle \frac{\partial^2}{\partial P \partial I_0} l(\chi) \right\rangle \\ -\left\langle \frac{\partial^2}{\partial P \partial I_0} l(\chi) \right\rangle & -\left\langle \frac{\partial^2}{\partial I_0^2} l(\chi) \right\rangle \end{bmatrix}, \quad (3)$$

where $l(\chi)=\sum_{i=1}^{N} \log[P_X(n_{X,i})P_Y(n_{Y,i})]$ is the loglikelihood[8] of the sample and $\langle\ \rangle$ corresponds to statistical averaging. Let $\hat{P}(\chi)$ and $\hat{I}_0(\chi)$ be some estimators of, respectively, the polarization P and the intensity I_0. They are unbiased if $\langle \hat{P}(\chi)\rangle = P$ and $\langle \hat{I}_0(\chi)\rangle = I_0$, and one can define their covariance matrix Γ as

$$\Gamma = \begin{bmatrix} \langle(\hat{P}(\chi)-P)^2\rangle & \langle(\hat{P}(\chi)-P)(\hat{I}_0(\chi)-I_0)\rangle \\ \langle(\hat{P}(\chi)-P)(\hat{I}_0(\chi)-I_0)\rangle & \langle(\hat{I}_0(\chi)-I_0)^2\rangle \end{bmatrix}. \quad (4)$$

The diagonal elements of this matrix are the variances σ_P^2 of the DOP and $\sigma_{I_0}^2$ of the average intensity. The Cramer–Rao theorem[8] states that for unbiased estimators the covariance matrix Γ and the inverse of the Fisher information matrix J^{-1} are related by the following inequality: $\mathbf{v}^\dagger \Gamma \mathbf{v} \geqslant \mathbf{v}^\dagger J^{-1} \mathbf{v}$, where $\mathbf{v}$ can be any vector. From this inequality, one gets $\sigma_P^2 \geqslant \kappa_{PP}$ and $\sigma_{I_0}^2 \geqslant \kappa_{I_0 I_0}$, where

$$J^{-1} = \begin{bmatrix} \kappa_{PP} & \kappa_{PI_0} \\ \kappa_{PI_0} & \kappa_{I_0 I_0} \end{bmatrix}. \quad (5)$$

These values are called the Cramer–Rao lower bounds[8] (CRLBs). We first determine the CRLBs when only photon noise or Gamma noise is present. We then address the case where the image is perturbed by both types of noise.

In the absence of speckle noise, the number of photons is distributed with a Poisson PDF. A direct application of Eq. (3) leads, after some calculus, to $\kappa_{PI_0}^{\pi} = 0$ and to

$$\kappa_{PP}^{\pi} = \frac{1-P^2}{NI_0}, \qquad \kappa_{I_0 I_0}^{\pi} = \frac{I_0}{N}, \quad (6)$$

where the superscript π stands for Poisson. On the other hand, in the absence of photon noise, the measurements are distributed with the Gamma PDF defined in Eq. (1). A direct application of Eq. (3) yields, after some calculus,

$$\kappa_{PP}^{S} = \frac{(1-P^2)^2}{2LN}, \qquad \kappa_{I_0 I_0}^{S} = \frac{I_0^2(1+P^2)}{2LN}, \quad (7)$$

and $\kappa_{PI_0}^{S} = 1/(2LN) I_0 P(1-P^2)$, where the superscript S stands for speckle. In contrast to the Poisson noise case, one can expect correlation in the fluctuations of the estimation of P and I_0, since the nondiagonal element of J^{-1} is nonzero.

In the presence of both speckle and Poisson noise, the data are distributed with the PDF defined in Eq. (2). By application of Eq. (3), a somewhat involved, yet direct calculus yields

$$\kappa_{PP}^{M} = \frac{1}{2LN}(1-P^2)(1-P^2+2L/I_0),$$

$$\kappa_{I_0 I_0}^{M} = \frac{I_0^2}{2LN}(1+P^2+2L/I_0), \quad (8)$$

and $\kappa_{PI_0}^{M} = I_0 P(1-P^2)$.

Several remarks can be made about this expression. Let us first consider the case $L/I_0 \gg 1$, in which light intensity is very low and Poisson noise is dominant. In this case, it is seen that Eqs. (8) lead to the CRLBs of the Poisson noise case [see Eqs. (6)]. On the other hand, if $L/I_0 \ll 1$, speckle noise is dominant (the photon flux is high), and it is seen that the CRLBs reduce to that of the speckle-only case. More unexpectedly, one can notice the following property:

$$\kappa_{PP}^{M} = \kappa_{PP}^{\pi} + \kappa_{PP}^{S}, \qquad \kappa_{I_0 I_0}^{M} = \kappa_{I_0 I_0}^{\pi} + \kappa_{I_0 I_0}^{S}. \quad (9)$$

The CRLBs in the presence of mixed speckle and Poisson noise are simply the sum of the CRLBs that result from each source of fluctuations. It can also be noticed that the nondiagonal term of J^{-1} in the mixed case is due only to speckle.

Let us first consider estimation of intensity I_0. The signal-to-noise ratio (SNR) can be defined as $\eta_I = I_0/\sqrt{\kappa_{I_0 I_0}^{M}}$. When only photon noise is present, the SNR is proportional to $\sqrt{I_0}$ (a well-known property of the Poisson noise) and is independent of P. On the other hand, when only speckle noise is present, the SNR is independent of I_0 but decreases as P increases. It is higher for totally depolarized light. Let us define the ratios

$$\rho_I = \frac{\kappa_{I_0 I_0}^{S}}{\kappa_{I_0 I_0}^{\pi}} = \frac{I_0}{2L}(1+P^2), \qquad \rho_P = \frac{\kappa_{PP}^{S}}{\kappa_{PP}^{\pi}} = \frac{I_0}{2L}(1-P^2). \quad (10)$$

For intensity estimation, the crossover between the two regimes characterized by a dominant Poisson noise or a dominant speckle noise can be defined as $\rho_I = 1$. As expected, this crossover depends on L and I_0 only through the ratio L/I_0. The value of I_0 corresponding to this crossover (in the case of $L=1$), denoted $I_0^C(I)$, has been plotted as a function of P in Fig. 1 (dashed curve). For totally depolarized light, speckle noise overcomes Poisson noise when the

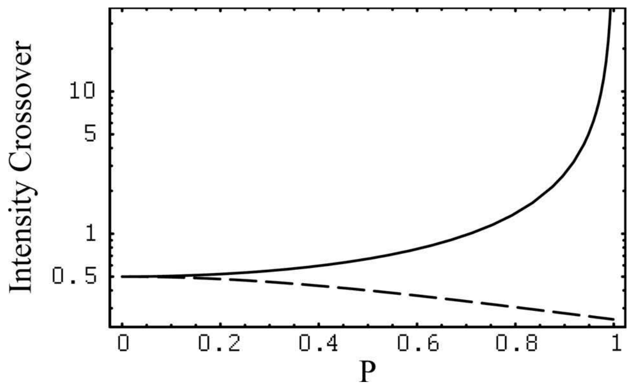
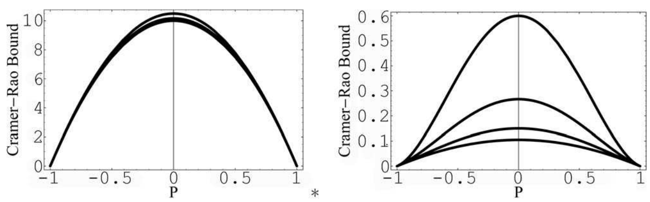


Fig. 1. Base-10 logarithm of the crossover intensity between Poisson-dominant and speckle-dominant regimes plotted as a function of P when $L=1$. The solid curve corresponds to the crossover intensity $I_0^C(P)$ for the estimation of P; the dashed curve corresponds to the crossover intensity $I_0^C(I)$ for the estimation of I_0.



(a) (b)

Fig. 2. CRLB for the estimation of P plotted as a function of P for different values of the speckle order L. (a) $I_0=0.1$, (b) $I_0=10$. Curves correspond to $L=1,3,10,100$ (from top to bottom).

number of photons is larger than $2L$, whereas for totally polarized light the crossover takes place as soon as the number of photons is L.

Let us now consider estimation of the DOP. Since P is a parameter without dimension, its estimation precision is characterized by the CRLB. As for estimation of the intensity, the CRLB decreases with I_0 and L [see Eqs. (8)]. However, it is seen to decrease as P increases and even to become null when $P=1$. The value $I_0^C(P)$ of I_0 corresponding to the crossover between the speckle and the Poisson regimes for estimation of P, defined as $\rho_P=1$, has been plotted in Fig. 1 (solid curve). It behaves quite differently from $I_0^C(I)$ (dashed curve). Indeed, it increases with P and even tends to infinity as P tends to 1, since in this case the contribution of speckle tends to zero faster than that of Poisson noise [see Eqs. (6) and (7)]. For $P=0$ the crossover occurs for $I_0^C(P)=2L$ photons, and for $P=0.9$ it occurs for $I_0^C(P)\simeq 10.5L$ photons, whereas it remains around L photons for intensity estimation [see Eqs. (10)]. Photon noise thus has a greater influence on the estimation of the DOP than on the estimation of the intensity. A practical consequence of this fact is that even if photon noise is negligible for intensity estimation, it may not be so for DOP estimation, especially when light is highly polarized.

Let us now concentrate on the estimation of the DOP. For a given value of P, the signal parameters that influence the estimation precision are the average number of photons, I_0, and the order of the speckle L. We have plotted in Fig. 2(a) the CRLB κ_{PP}^M as a function of the actual value of P for $I_0=0.1$. The four curves correspond to different values of L. For such a low value of I_0, the photon noise is dominant whatever the value of L. Thus increasing L, which reduces the fluctuations due to speckle noise but not those due to photon noise, does not significantly reduce the CRLB. Figure 2(b) corresponds to $I_0=10$. In this case speckle noise is dominant, and thus increasing L significantly reduces the CRLB.

To get a synthetic view of the respective influence of I_0 and L on the estimation precision of P, let us consider that $P=0$, which corresponds the worst situation for DOP estimation. In this case it is easily seen from Eq. (8) that the DOP estimation precision is constant when $1/I_0+1/2L$ is constant. From this expression it is clearly seen that when photon noise is dominant ($I_0<2L$) performance is more improved by increasing I_0 than L. For example, estimation precision is better for $I_0=0.7$ and $L=1$ than for $I_0=0.5$ and $L=100$. On the other hand, when speckle noise is dominant ($I_0>2L$), it is more efficient to increase the speckle order. For example, estimation precision is better for $I_0=3$ and $L=10$ than for $I_0=100$ and $L=1$. If the average number of photons is large enough to be in the speckle-dominant regime, increasing the speckle order is thus an efficient way of improving the estimation precision of the DOP.

When we have both photon and speckle noise, it has been shown that photon noise has a greater influence on DOP estimation than on intensity estimation, especially when light is highly polarized. Moreover, at low intensity levels, increasing the speckle order is not efficient for improving precision. It will be interesting to analyze different measurement strategies and to compare their performance with those analyzed in this Letter. Taking into account possible correlation of the two components of the reflected light is also a challenging problem.

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