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Quantum cohomology of minuscule homogeneous spaces II
Hidden symmetries

P.E. Chaput, L. Manivel, N. Perrin

September 28, 2006

Abstract

We prove that the quantum cohomology ring of any minuscule or cominuscule homogeneous space, once localized at the quantum parameter, has a non trivial involution mapping Schubert classes to multiples of Schubert classes. This can be stated as a strange duality property for the Gromov-Witten invariants, which turn out to be very symmetric.

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1 Introduction

This paper is a sequel to [CMP], where we began a unified study of the quantum cohomology of (co)minuscule homogeneous manifolds. One intriguing feature of quantum cohomology, as already observed by several authors and especially for Grassmannians (see []), is that it is *more symmetric* that ordinary cohomology.

In this paper we show that the main tools introduced in [CMP] allow to state, and prove, a general *strange duality* statement for the quantum cohomology of a (co)minuscule homogeneous manifold $X = G/P$. Recall that a $\mathbb{Z}$-basis for the ordinary cohomology ring $H^*(X)$ (or for the Chow ring $A^*(X)$) of X is given by the Schubert classes $\sigma(w)$, where $w \in W_X$ belongs to the set of minimal lengths representatives of W/W_P, the quotient of the Weyl group W of G by the Weyl group W_P of P. To be precise, we let $\sigma(w)$ be the class of the Schubert variety $X(w_X w)$, where w_X denotes the longest element in W_X. So $\sigma(w)$ has degree $\ell(w)$ in the Chow ring (and twice this degree in cohomology). Note that the map $w \mapsto p(w) = w_0 w$, for $w \in W/W_P \simeq W_X$, defines Poincaré duality on X. The map $w \mapsto \iota(w) = w_0^P w$, where w_0^P denotes the longest element of W_P, will play a crucial role in the sequel. For $w = s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}$ a reduced decomposition of $w \in W_X$, we let

$$y(w) = \prod_{i=1}^{\ell(w)} n_{\alpha_i}(\alpha_0)^{\epsilon(\alpha_i)}.$$

Here α_0 denotes the highest root of G, and $n_{\alpha_i}(\alpha_0)$ is the coefficient of α_i when α_0 is written in the basis of simple roots. Moreover, we have let $\epsilon(\alpha) = 1$ if α is a long root, $\epsilon(\alpha) = -1$ if α is short (in the simply-laced case, all roots are considered long). The rational number $y(w)$ is well defined since in the (co)minuscule case, reduced decompositions are uniquely defined up to commutation relations. Finally, we let $\delta(w)$ be the number of occurences, in a reduced decomposition of w, of the simple root β that defines P.

Key words: quantum cohomology, minuscule homogeneous space, quiver, Schubert calculus, strange duality, Gromov-Witten invariant.
Mathematics Subject Classification: 14M15, 14N35

The Schubert classes are still a basis over $\mathbb{Z}[q]$ of the (small) quantum Chow ring $QA^*(X)$, whose associative product is defined in terms of 3-points Gromov-Witten invariants. Denote by $QA^*(X)_{loc}$ its localization at q, that is,

$$QA^*(X)_{loc} = QA^*(X) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q, q^{-1}].$$

Strange duality can be stated in a uniform way as follows.

Theorem 1.1 (Strange Duality) *Let X be a minuscule or cominuscule homogeneous space. The endomorphism ι of $QA^*(X)_{loc}$, defined by*

$$\iota(q) = y(s_{\alpha_0})q^{-1} \quad \textit{and} \quad \iota(\sigma(w)) = q^{-\delta(w)}y(w)\sigma(\iota(w)),$$

is a ring involution.

This result can be stated as a symmetry property of the Gromov-Witten invariants, see Corollary 5.4, which unexpectedly relates certain numbers of small degree rational curves with numbers of high degree rational curves. In fact we will obtain these results by observing that the quantum product with the class of a point maps any Schubert class to another Schubert class, multiplied by some power of q, see Theorem 3.3.

For Grassmannians these results were first proved by Postnikov [P1]; in this case $y(w) = 1$ for any w. For classical Grassmannians our strange duality statement will be deduced from the quantum Pieri formulas of [BKT]. For the two exceptional minuscule spaces, we have used the presentations of the quantum Chow rings obtained in [CMP]. The case of the Cayley plane has been checked by hand, but that of the Freudenthal variety required the help of a computer.

As a consequence of this theorem, we deduce from the formula for the smallest power of q appearing in the quantum product of two Schubert classes, (see [FW] or [CMP]), a formula for the highest power of q in such a product (Corollary 5.5). For u and v in W_X, let us denote, following W. Fulton and C. Woodward [FW], by $\delta(u, v)$ the minimal degree of a rational curve meeting two general translates of $X(u^*)$ and $X(v^*)$ (see also [CMP] page 20 and corollary 4.12 for a combinatorial description).

Theorem 1.2 *For $u, v \in W_X$, the maximal power of q that appears in the quantum product of Schubert classes $\sigma(u) * \sigma(v)$, is*

$$d_{max}(u, v) = \delta(u) - \delta(\iota(u), p(v)) = \delta(v) - \delta(\iota(v), p(u)).$$

2 A partition of the Hasse diagram

Let $X = G/P$ be a (co)minuscule homogeneous variety. In [CMP] we defined the perimeter d_{max} of X. For any non negative integer $d \leq d_{max}$, we introduced certain Schubert subvarieties T_d and Y_d of X, with $T_d \subset Y_d^*$. These varieties allowed us to define a *quantum Poincaré duality* as follows: for any Schubert subvarieties $X(u), X(v) \subset X$, the Gromov-Witten invariant $I_d(Y_d^*, X(u), X(v))$ is non zero if and only if $X(u), X(v)$ are contained in the (smooth) variety T_d, and define Poincaré dual classes in T_d – in which case the invariant equals one. In particular

$$I_d(Y_d^*, \{\text{pt}\}, T_d) = 1.$$

One of the main themes of this paper is to investigate another quantum Poincaré type duality, defined by the non vanishing of the Gromov-Witten invariants $I_d(\{\text{pt}\}, X(u), X(v))$. Let $\delta(u)$ denote the maximal integer d such that $X(u) \subset Y_d^*$. The following facts are taken from [CMP] page 20:

1. There exists a degree d rational curve in X joining the base point $e(1) = P/P$ of $X = G/P$ with the base point $e(u) = uP/P \in X(u)$, if and only if $d \geq \delta(u)$.

2. If β is the simple root defining P, $\delta(u)$ is the number of occurences of s_β in a reduced decomposition of u.

3. $\delta(u) = \delta(w_X, u)$, the minimal length of a Bruhat chain from u to w_X, as defined in [FW].

We add to this list the following essential property.

Lemma 2.1 *We have $T_d \subset X(w) \subset Y_d^*$ if and only if $d = \delta(w_X, w)$.*

Proof. We proved in [CMP, corollary 4.12] a combinatorial characterisation of the smallest power appearing in a quantum product. This gives $\delta(w_X, w) = \min\{d \ / \ T_d \subset X(w)\}$. The lemma will follow from the fact that for all $d \leq d_{max}$, we have $T_{d-1} \not\subset Y_d^*$ and the equivalence

$$T_{d-1} \not\subset X(w) \Leftrightarrow X(w) \subset Y_d^*.$$

These two results come from the following facts on the quivers $Q_{Y_d^*}$ and Q_{T_d} (cf. [CMP] for definitions and results on quivers):

- the quiver $Q_{Y_d^*}$ is obtained from the quiver Q_X by removing all the vertices above the vertex $(\theta(\beta), d)$ where β is the simple root defining X and θ is the Weyl involution
- The vertices of $Q_{T_{d-1}}$ are those under the vertex $(\theta(\beta), d)$.

The vertex $(\theta(\beta), d) \in Q_X$ is in the quiver $Q_{T_{d-1}}$ but not in the quiver $Q_{Y_d^*}$ proving that $T_{d-1} \not\subset Y_d^*$. Furthermore, the condition $T_{d-1} \not\subset X(w)$ is equivalent to the fact that the vertex $(\theta(\beta), d)$ is not in the quiver of $X(w)$ which is also equivalent to the inclusion $X(w) \subset Y_d^*$. $\square$

A nice consequence is that we get a partition of W_X in $d_{max} + 1$ Bruhat intervals,

$$W_X = \bigsqcup_{d=0}^{d_{max}} [T_d, Y_d^*].$$

We will denote by $W_d \subset W_X$ the interval $[T_d, Y_d^*]$. For example, W_1 is the image in $W/W_P \simeq W_X$ of the set of reflections s_α, α a root of G. In particular T_1 is represented by s_{α_0}.

Recall that in [CMP], we proved that Gromov-Witten invariants of degreed d on X can be interpreted as classical intersection numbers on an auxiliary variety F_d. This variety F_d is homogeneous under the same group G as X, and there is an incidence diagram

$$\begin{array}{ccc} I_d & \xrightarrow{p_d} & X \\ \downarrow{\scriptstyle q_d} & & \\ F_d & & \end{array}$$

Then $Z_d \subset F_d$ is the (image by q_d of the) fiber of p_d, and $Y_d \subset X$ is the (image by p_d of the) fiber of q_d. These varieties are given by the following table.

X	d	T_d	Y_d	Z_d
$\mathbb{G}(p,n)$	$\leq \min(p, n-p)$	$\mathbb{G}(p-d, n-2d)$	$\mathbb{G}(d, 2d)$	$\mathbb{G}(p-d, p) \times \mathbb{G}(n-p-d, n-p)$
$\mathbb{G}_\omega(n, 2n)$	$\leq n$	$\mathbb{G}_\omega(n-d, 2n-2d)$	$\mathbb{G}_\omega(d, 2d)$	$\mathbb{G}(n-d, n)$
$\mathbb{G}_Q(n, 2n)$	$\leq [n/2]$	$\mathbb{G}_Q(n-2d, 2n-4d)$	$\mathbb{G}_Q(2d, 4d)$	$\mathbb{G}(n-2d, n)$
$\mathbb{OP}^2$	1	$\mathbb{P}^5$	$\mathbb{P}^1$	$G_Q(5, 10)$
	2	pt	$\mathbb{Q}^8$	$\mathbb{Q}^8$
E_7/P_7	1	$\mathbb{Q}^{10}$	$\mathbb{P}^1$	$\mathbb{OP}^2$
	2	$\mathbb{P}^1$	$\mathbb{Q}^{10}$	$\mathbb{OP}^2$
	3	pt	E_7/P_7	pt

Lemma 2.2 *For any $d \leq d_{max}$, the Bruhat interval $[T_d, Y_d^*]$ in W_X can be identified with the Hasse diagram of a minuscule homogeneous variety Z_d.*

Proof. For any Schubert subvariety $X(w)$ of X, we denoted by $F_d(\widehat{w}) = q_d(p_d^{-1}(X(w))$ the corresponding Schubert subvariety of F_d. The map $w \mapsto \widehat{w}$ identifies the Bruhat interval $[T_d, Y_d^*]$ in W_X with the Bruhat interval $[Z_d^*, F_d]$ in W_{F_d}. By Poincaré duality on F_d and Z_d, this interval is canonically identified with $[pt, Z_d]$, which coincides with the Hasse diagram of Z_d. □

Moreover, each interval W_d, being isomorphic with the Hasse diagram of the smooth homogeneous variety Z_d, is endowed with a natural involution induced by Poincaré duality on the latter. We get a global involution of W_X which we denote by ι. In particular $\iota(T_d) = Y_d^*$, and the fact that $I_d(Y_d^*, \{pt\}, T_d) = 1$ implies that for any $u \in W_X$,

$$\mathrm{codim}(X(u)) + \mathrm{codim}(X(\iota(u))) = \delta(u)c_1(X).$$

3 The quantum product by the class of a point

In this section we want to compare the intersection in F_d of two cells in $[Z_d^*, F_d]$ with an intersection in Z_d. Our first result will hold for arbitrary homogeneous spaces.

Let X be a homogeneous space and $Y \subset X$ a Schubert subvariety. Suppose that Y is smooth, and even homogeneous. The cell decomposition of X defined by the Schubert cells, gives a cell decomposition of Y if we consider only those Schubert cells that are contained in Y. In particular this yields a natural inclusion $W_Y \hookrightarrow W_X$. We denote $p_X : W_X \to W_X$ the Poincaré duality on X and $p_Y : W_Y \to W_Y$ the Poincaré duality on Y.

Consider the submodule $\mathbb{Z}[Y^*, X]$ of $H^*(X)$ generated by the Schubert classes in $[Y^*, X]$. This module has a natural algebra structure: if $[X(u)] \cup [X(v)] = \sum_{w \in W_X} c_{u,v}^w [X(w)]$ in X, set

$$[X(u)] \cdot [X(v)] = \sum_{w \in [Y^*, X]} c_{u,v}^w [X(w)].$$

Let j denote the inclusion $Y \to X$; the morphism of modules $j_* : H_*(Y) \to H_*(X)$ is injective and the dual morphism of algebras $j^* : H^*(X) \to H^*(Y)$ is surjective. The image of j_* is $\mathbb{Z}[pt, Y]$; we denote by j_*^{-1} the inverse map $\mathbb{Z}[pt, Y] \to H_*(Y)$.

Proposition 3.1 *The restriction of j^* to $\mathbb{Z}[Y^*, X]$ is an algebra isomorphism with $H^*(Y)$. Explicitely, it equals $p_Y \circ j_*^{-1} \circ p_X$.*

Proof. The first assertion is an immediate consequence of [CMP, lemma 3.12]. To prove the second one, let $C \in \mathbb{Z}[pt, Y] \subset H_*(X)$, and let $D \in H^*(X)$. Since j_* and j^* are adjoint, we have:

$$\begin{aligned} (j_* p_Y j^* p_X(C), D)_X &= (p_Y j^* p_X(C), j^* D)_Y \\ &= \delta_{j^* p_X(C), j^* D} \\ &= \delta_{p_X(C), D} = (C, D)_X \end{aligned}$$

The second and the third equality follow from the fact that j^* maps bijectively Schubert classes to Schubert classes. We therefore have proved that $C = j_* p_Y j^* p_X(C)$ for all C in $\mathbb{Z}[pt, Y]$ or, equivalently, that $j^* D = p_Y j_*^{-1} p_X(D)$ for all D in $\mathbb{Z}[Y^*, X]$. □

Corollary 3.2 *Let $Y \subset X$ be a homogeneous Schubert subvariety of some rational homogeneous manifold. Let Z be any other Schubert subvariety of X. Then*

$$[Y] \cup [Z] = \begin{cases} p_X p_Y [Z] & \text{if } Z \supset Y^*, \\ 0 & \text{otherwise.} \end{cases}$$

We come back to the (co)minuscule setting. Our main examples of homogeneous Schubert subvarieties are our varieties T_d and Y_d in X and Z_d in F_d. Applying the previous result to the latter varieties, we get the following statement.

Theorem 3.3 *For any $u \in W_X$, we have*

$$\sigma(\mathrm{pt}) * \sigma(u) = q^{\delta(u)}\sigma(p\iota(u)).$$

Proof. Let $u, v \in W_X$. If $I_d(\mathrm{pt}, X(u), X(v)) \neq 0$, then, by corollary 3.21 in [CMP], we have $u \leq w_{Y_d^*}$ and $v \leq w_{Y_d^*}$. Since $I_d(\mathrm{pt}, X(u), X(v)) = I_0(Z_d, F_d(\widehat{u}), F_d(\widehat{v}))$, by [CMP, lemma 3.12], we also have $\widehat{u}, \widehat{v} \geq Z_d^*$, which is equivalent to $u, v \geq T_d$ by the proof of Lemma 2.2. We conclude that $I_d(\mathrm{pt}, X(u), X(v)) \neq 0$ implies that $X(u), X(v) \in [T_d, Y_d^*]$,which is equivalent to $d = \delta(u) = \delta(v)$.

Denote by $j_d : Z_d \hookrightarrow F_d$ the inclusion map. We have

$$I_0(Z_d, F_d(\widehat{v}), F_d(\widehat{u})) = \int_{Z_d} j_d^*[F_d(\widehat{v})] \cup j_d^*[F_d(\widehat{u})].$$

According to proposition 3.1, $\int_{Z_d} j_d^*[F_d(\widehat{v})] \cup j_d^*[F_d(\widehat{u})]$ equals one if $v = \iota(u)$ and equals zero otherwise. The theorem follows. □

4 Strange duality

In this section we prove the strange duality property of the quantum Chow ring stated in Theorem 1.1. For convenience we will in fact prove a slightly different (but equivalent) statement, asserting the existence of a duality mapping q to q^{-1}. With the notable exception of Grassmannians, this is only possible if we extend scalars a little bit.

We will need a case by case analysis to fix the coefficients $\zeta(w)$ in the next statement.

Theorem 4.1 *For any (co)minuscule homogeneous variety X, one can find an algebraic number κ, and a map $\zeta : W_X \to \mathbb{Z}[\kappa]$, such that the correspondence*

$$q \mapsto q^{-1}, \qquad \sigma(w) \mapsto \zeta(w)q^{-\delta(w)}\sigma(\iota(w))$$

defines a ring involution of $QA^*(X)_{loc}[\kappa]$.

Clearly, such an involution changes the degree into its opposite, and the fact that it is involutive is tantamount to the simple relation $\zeta \circ \iota = \zeta^{-1}$. The hard part of the statement is that it is compatible with the quantum product.

To get rid of κ, there remains to compose the previous involution with a degree automorphism (that multiplies a degree d class by t^d for some t). It is then a routine but again case by case check that we obtain Theorem 1.1. That is, we check that

$$\zeta(w) = y(w)y(s_{\alpha_0})^{-\frac{\ell(w)}{c_1(X)}}.$$

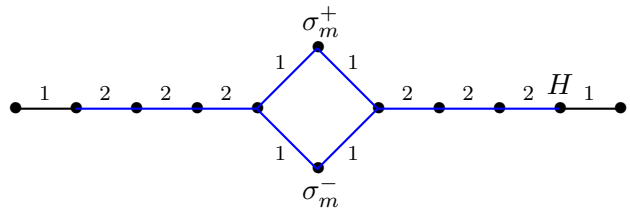
But we have no convincing explanation of why such a formula should hold true.

4.1 Quadrics

We begin with the easy example of quadrics. We will see that already in that case fixing the coefficients involves some subtelties.

Even dimensions

Let $\mathbb{Q}^{2m}$ be a quadric of dimension $2m$, acted on by SO_{2m+2}. The simple root that defines the corresponding maximal parabolic subgroup is α_1. The Hasse diagram is the following:



Indeed, as is well known, there is one Schubert class σ_k in each degre $k \neq m$, and two Schubert classes $\sigma_m^\pm$ in middle degree, defined by the two families of maximal linear spaces in $\mathbb{Q}^{2m}$. Of course $\sigma_1 = H$ is the hyperplane class. For the classical intersection product, we have $H^k = \sigma_k$ for $k < m$, $H^m = \sigma_m^+ + \sigma_m^-$, and $H^k = 2\sigma_k$ for $k > m$. Since $c_1(\mathbb{Q}^{2m}) = 2m$, these formulas remain valid in the quantum Chow ring, except in maximal degree. In fact the quantum Chevalley formula gives the two identities

$$\sigma_{2m-1} * H = \sigma_{2m} + q, \qquad \sigma_{2m} * H = qH.$$

Note that σ_{2m-1} is the class of a line and coincides with $[T_1]$. Also σ_{2m} is the class $[pt]$ of a point.

Proposition 4.2 *The quantum Chow ring $QA^*(\mathbb{Q}^{2m})$ is determined by the formulas*

$$\sigma_m^+ * \sigma_m^- = [pt], \qquad \sigma_m^+ * \sigma_m^+ = \sigma_m^- * \sigma_m^- = q.$$

Note that these relations imply that $H^{2m} = 2[pt] + 2q$ and $H^{2m+1} = 4qH$. Now it is easy to check that $QA^*(\mathbb{Q}^{2m})_{loc}$ has a ring involution given by

$$q \mapsto 1/16q, \qquad H^k \mapsto H^{2m-k}/4q, \qquad [pt] \mapsto [pt]/16q^2,$$

(where the central formula holds for $1 \leq k \leq 2m-1$), and in middle degree by $\sigma_m^\pm \mapsto \sigma_m^\mp/4q$. Multiplying each degree d class by 2^d, we get a slightly different involution given by

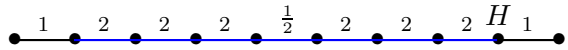
$$\begin{array}{rll} q & \mapsto 2^{2m-4}/q, & \\ \sigma_k & \mapsto 2^{k-1}\sigma_{2m-k}/q & \text{for } 0 < k < m, \\ \sigma_m^\pm & \mapsto 2^{m-2}\sigma_m^\mp, & \\ \sigma_k & \mapsto 2^{k-3}\sigma_{2m-k}/q & \text{for } m < k < 2m, \\ [pt] & \mapsto 2^{2m-4}[pt]/q^2. & \end{array}$$

This is precisely the statement of Theorem 1.1 for $\mathbb{Q}^{2m}$, as one can readily check from the Hasse diagram above. Indeed, we have labeled the edges by the coefficients whose products, taken from the rightmost end of the diagram, give the coefficients $y(w)$. The corresponding coefficients for Theorem 4.1 are the following, where $\zeta(m)$ accounts for the two mid-dimensional classes:

$$\zeta(k) = 4^{\frac{k-m}{m}} \quad \text{for } 0 < k < 2m, \qquad \zeta(0) = \zeta(2m) = 1,$$

Odd dimensions

An odd dimensional quadric $\mathbb{Q}^{2m-1}$ is cominuscule but not minuscule. It has exactly one Schubert class σ_k in each dimension $k < 2m$. Again $\sigma_1 = H$ is the hyperplane class. The Hasse diagram is the simplest possible one; as in the even dimensional case we have labeled the edges by the coefficients whose products, taken from the rightmost end, give the $y(w)$ of Theorem 1.1.



For the classical intersection product, we have $H^k = \sigma_k$ for $k < m$, $H^k = 2\sigma_k$ for $k \geq m$. Since $c_1(\mathbb{Q}^{2m-1}) = 2m-1$, these formulas hold in the quantum Chow ring, except in maximal degree. In fact the quantum Chevalley formula gives the two identities

$$\sigma_{2m-2} * H = \sigma_{2m-1} + q, \qquad \sigma_{2m-1} * H = qH.$$

This is enough to determine the quantum product. The class σ_{2m-2} is that of a line and coincides with $[T_1]$. Also σ_{2m-1} is the class $[pt]$ of a point.

Again it is easy to check that $QA^*(\mathbb{Q}^{2m-1})_{loc}$ has a ring involution given by almost the same formulas that in even dimensions,

$$q \mapsto 1/16q, \qquad H^k \mapsto H^{2m-k-1}/4q, \qquad [pt] \mapsto [pt]/16q^2,$$

(where the central formula holds for $1 \leq k \leq 2m-2$). Multiplying each degree d class by 2^d, we get the slightly different involution given by

$$\begin{array}{rll} q & \mapsto 2^{2m-5}/q, & \\ \sigma_k & \mapsto 2^{k-1}\sigma_{2m-k-1}/q & \text{for } 0 < k < m, \\ \sigma_k & \mapsto 2^{k-3}\sigma_{2m-k-1}/q & \text{for } m \leq k < 2m-1, \\ [pt] & \mapsto 2^{2m-5}[pt]/q^2. & \end{array}$$

This is in perfect agreement with Theorem 1.1.

4.2 Grassmannians

Now suppose $X = \mathbb{G}(p,n)$ is a Grassmannian. Then W_X identifies with the set of partitions inscribed in the rectangle $p \times (n-p)$, and $d_{max} = \min(p, n-p)$. For any $d \subset d_{max}$, Y_d is a Grassmannian $\mathbb{G}(d, 2d)$, and T_d is another Grassmannian $\mathbb{G}(p-d, n-2d)$. In particular the partition $w_{Y_d^*}$ is just a square of size d, while the partition w_{T_d} is the complement of a rectangle of size $(p-d) \times (n-p-d)$.

We deduce that for any partition $\lambda \in W_X$, the degree $d = \delta(\lambda)$ is the size of the biggest square contained in λ. Clearly the interval $[T_d, Y_d^*]$ in W_X identifies with the Hasse diagram of the product $\mathbb{G}(d, p-d) \times \mathbb{G}(d, n-p-d)$. The partition $\iota(\lambda)$ is deduced from λ by taking the complementary partitions in the SW and NE rectangles.

In this case Theorem 4.1 appears in [P1], Theorem 7.5. See also [H] where the involution is interpreted in terms of complex conjugation.

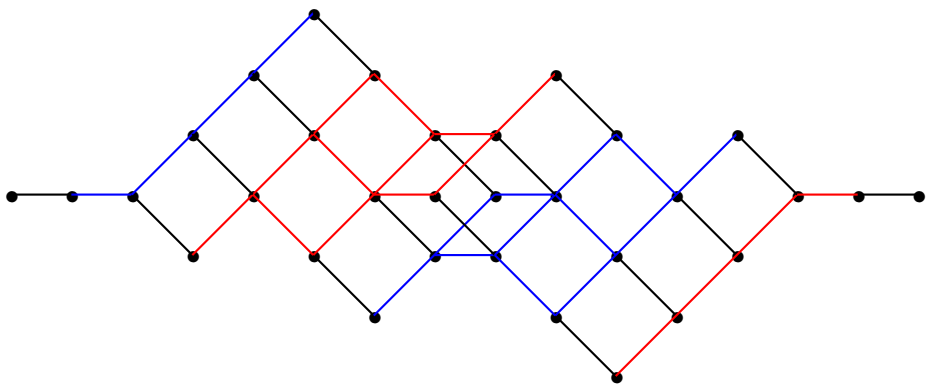
4.3 Lagrangian Grassmannians

Let $X = \mathbb{G}_\omega(n, 2n)$ be a Lagrangian Grassmannian. Recall that W_X identifies with the set of strict partitions $\lambda \subset \rho_n$. In order to simplify notations we let σ_λ denote the class of the Schubert subvariety Poincaré dual to $X(\lambda)$. Its degree is the sum $|\lambda|$ of the parts of λ. Morevoer, $d = \delta(\lambda)$ is simply the number of (non zero) parts of λ, usually called the length and denoted $\ell(\lambda)$. In particular $d_{max} = n$.

Mapping λ to $(\lambda_1 - d, \ldots, \lambda_d - 1)$ we get a partition inscribed in the rectangle $d \times (n-d)$. This identifies the interval W_d of W_X with the Hasse diagram of the Grassmannian $\mathbb{G}(d,n)$. Appying Poincaré duality for that Grassmannian we dedude that the involution ι is given by

$$\iota(\lambda_1, \ldots, \lambda_{\ell(\lambda)}) = (n+1-\lambda_{\ell(\lambda)}, \ldots, n+1-\lambda_1).$$

Example 4.3 *For $n = 5$ we have drawn below the Hasse diagram of $\mathbb{G}_\omega(5, 10)$ and its partition into six disjoint intervals. More precisely we have drawn in black the arrows between different intervals, and in the same color the arrows inside a given interval. Those are the Hasse diagrams of a point, of $\mathbb{P}^4$ and of the Grassmannian $\mathbb{G}(2, 5)$, all appearing twice and symmetrically.*



Proposition 4.4 *For $\lambda \subset \rho_n$ a strict partition, let $z(\lambda) := \ell(\lambda) - \frac{2|\lambda|}{n+1}$. Then $z(\iota(\lambda)) = z(\lambda^*) = -z(\lambda)$, and Theorem 4.1 holds with*

$$\kappa = 2^{-\frac{2}{n+1}} \qquad \text{and} \qquad \zeta(\lambda) = 2^{z(\lambda)}.$$

Proof. We just need to prove that our involution is compatible with the quantum Pieri formula for $\sigma_\lambda * \sigma_k$, for each k. We have $\iota(\sigma_k) = q^{-1}\sigma_{n+1-k}$, so we need to compare $\sigma_\lambda * \sigma_k$ with $\sigma_{\iota(\lambda)} * \sigma_{n+1-k}$.

We use the quantum version of Pieri's rule as stated in [BKT], Theorem 3:

$$\sigma_\lambda * \sigma_k = \sum_\mu 2^{N(\lambda,\mu)}\sigma_\mu + q\sum_\nu 2^{N'(\nu,\lambda)}\sigma_\nu,$$

where the first sum is over all strict partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$, such that the complement μ/λ is a horizontal strip, and the second sum is over all strict partitions $\nu \subset \lambda$ with $|\nu| = |\lambda| - (n + 1 - k)$, such that the λ/ν is a horizontal strip. Moreover $N(\lambda, \mu)$ denotes the number of connected components of μ/λ which do no meet the first column, and $N'(\nu, \lambda)$ is one less than the total number of connected components of λ/ν. (By definition, two boxes are connected if they share an edge or a vertex.) Applying our involution, we get

$$\begin{aligned}\iota(\sigma_\lambda * \sigma_k) = \quad & q^{-\ell(\lambda)}\textstyle\sum_{\ell(\mu)=\ell(\lambda)} 2^{z(\mu)+N(\lambda,\mu)}\sigma_{\iota(\mu)} + q^{-\ell(\lambda)-1}\sum_{\ell(\mu)=\ell(\lambda)+1} 2^{z(\mu)+N(\lambda,\mu)}\sigma_{\iota(\mu)} \\ & +q^{-\ell(\lambda)-1}\textstyle\sum_{\ell(\nu)=\ell(\lambda)} 2^{z(\nu)+N'(\nu,\lambda)}\sigma_{\iota(\nu)} + q^{-\ell(\lambda)}\sum_{\ell(\nu)=\ell(\lambda)-1} 2^{z(\nu)+N'(\nu,\lambda)}\sigma_{\iota(\nu)}. \qquad (1)\end{aligned}$$

Similarly, we deduce again from Pieri's rule that

$$\begin{aligned}\iota(\sigma_\lambda) * \iota(\sigma_k) = \quad & 2^{z(\lambda)+z(k)}\Big(q^{-\ell(\lambda)-1}\textstyle\sum_{\ell(\alpha)=\ell(\lambda)} 2^{N(\iota(\lambda),\alpha)}\sigma_\alpha + q^{-\ell(\lambda)-1}\sum_{\ell(\alpha)=\ell(\lambda)+1} 2^{N(\iota(\lambda),\alpha)}\sigma_\alpha \\ & +q^{-\ell(\lambda)}\textstyle\sum_{\ell(\beta)=\ell(\lambda)} 2^{N'(\beta,\iota(\lambda))}\sigma_\beta + q^{-\ell(\lambda)}\sum_{\ell(\beta)=\ell(\lambda)-1} 2^{N'(\beta,\iota(\lambda))}\sigma_\beta\Big). \qquad (2)\end{aligned}$$

We claim that the four partial sums on the right hand sides of the two identities above correspond pairwise.

Consider the first term on the right hand side of (2). Here α is a strict partition containing $\iota(\lambda)$, of size $|\alpha| = |\iota(\lambda)| + n + 1 - k$, such that $\alpha/\iota(\lambda)$ is a horizontal strip. If we let $\nu = \iota(\alpha)$, we get that ν is contained in λ, $\ell(\nu) = \ell(\alpha)$ and $|\nu| = |\lambda| - (n + 1 - k)$, and $\lambda/\nu \simeq \alpha/\iota(\lambda)$ is a horizontal

strip. Moreover, the fact that $\ell(\alpha) = \ell(\lambda)$ means that $\alpha/\iota(\lambda)$ does not meet the first column, so $N(\iota(\lambda), \alpha) = N'(\nu, \lambda) + 1$. On the other hand,

$$z(\nu) = \ell(\lambda) - \frac{2}{n+1}(|\lambda| - (n+1-k)) = z(\lambda) + z(k) + 1,$$

and therefore $z(\nu) + N'(\nu, \lambda) = z(\lambda) + z(k) + N(\iota(\lambda), \alpha)$. We conclude that the first partial sum of (2) coincides exactly with the third partial sum of (1).

For the second term on the right hand side of (2), the difference with the first term is that $\ell(\alpha) = \ell(\lambda) + 1$, which means that $\alpha/\iota(\lambda)$ does meet the first column. Thus $N(\iota(\lambda), \alpha) = N'(\nu, \lambda)$. On the other hand,

$$z(\nu) = \ell(\lambda) + 1 - \frac{2}{n+1}(|\lambda| + (n+1) - (n+1-k)) = z(\lambda) + z(k),$$

and therefore we get again $z(\nu) + N'(\nu, \lambda) = z(\lambda) + z(k) + N(\iota(\lambda), \alpha)$. We conclude that the second partial sum of (2) coincides exactly with the second partial sum of (1).

Now consider the third term on the right hand side of (2). Here β is a strict partition contained in $\iota(\lambda)$, with $\ell(\beta) = \ell(\lambda)$ and $|\beta| = |\iota(\lambda)| - k$, such that $\iota(\lambda)/\beta$ is a horizontal strip. If we let $\beta = \iota(\mu)$, we get $\iota(\lambda)/\beta \simeq \mu/\lambda$, so $|\mu| = |\lambda| + k$ and $N'(\beta, \iota(\lambda)) = N'(\lambda, \mu) = N(\lambda, \mu) - 1$. Again we deduce that $z(\lambda) + z(k) + N'(\beta, \iota(\lambda)) = z(\mu) + N(\lambda, \mu)$, so that the third partial sum of (2) coincides exactly with the first partial sum of (1).

Finally, consider the third term on the right hand side of (2). Here β is as before except that $\ell(\beta) = \ell(\lambda) - 1$, so that if we let $\nu = \iota(\beta)$, then $N'(\beta, \iota(\lambda)) = N'(\nu, \lambda)$. Again we conclude that the fourth partial sum of (2) coincides exactly with the fourth partial sum of (1).

This concludes the proof. □

As we have already mentionned, it is possible to get rid of the root of two by composing with a degree automorphism. We get:

Theorem 4.5 *The correspondence*

$$q \mapsto \frac{4}{q}, \qquad \sigma_\lambda \mapsto \left(\frac{2}{q}\right)^{\ell(\lambda)} \sigma_{\iota(\lambda)} \quad for\ \lambda \subset \rho_n,$$

defines a ring involution of $QA^*(G_\omega(n, 2n))_{loc}$.

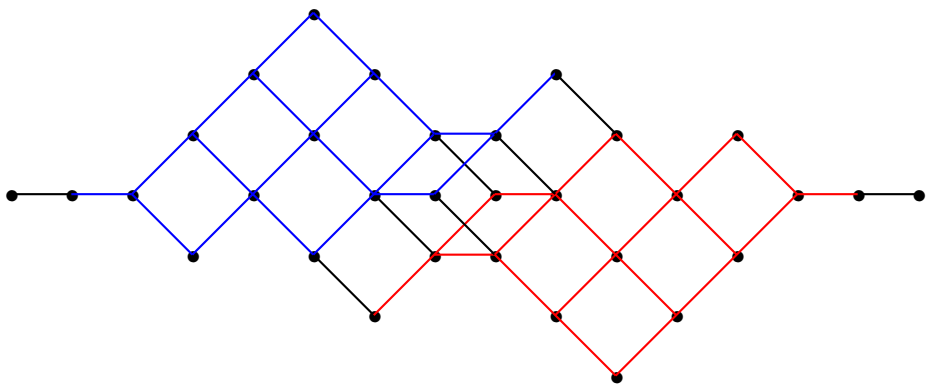
4.4 Orthogonal Grassmannians

Let $X = \mathbb{G}_Q(n+1, 2n+2)$ be a Lagrangian Grassmannian. Recall that W_X identifies again with the set of strict partitions $\lambda \subset \rho_n$. Again we denote by σ_λ the Schubert class Poincaré dual to $[X(\lambda)]$. Its degree is $|\lambda|$, and $\delta(\lambda) = d$ if the number of (non zero) parts of λ is $2d$ or $2d-1$. In particular $d_{max} = [n/2]$.

Mapping λ to $(\lambda_1 - 2d + 1, \ldots, \lambda_{2d})$ (where we let $\lambda_{2d} = 0$ if $\ell(\lambda) = 2d - 1$), we get a partition inscribed in the rectangle $2d \times (n - 2d + 1)$. This identifies the interval W_d of W_X with the Hasse diagram of the Grassmannian $\mathbb{G}(2d, n+1)$. Appying Poincaré duality for that Grassmannian we dedude that the involution ι is given by

$$\iota(\lambda_1, \ldots, \lambda_{2\delta(\lambda)}) = (n - \lambda_{2\delta(\lambda)}, \ldots, n - \lambda_1).$$

Example 4.6 *For $n = 5$ we have drawn below the Hasse diagram of $\mathbb{G}_Q(6, 12)$ and its partition into four disjoint intervals. Note that the diagram is the same as for $\mathbb{G}_\omega(5, 10)$ but the partition is different. Indeed we have only four intervals in that case, isomorphic with the Hasse diagrams of a point and of $G(2, 6)$ appearing twice symmetrically. (Beware that this symmetry is specific to the case where n is odd.)*



Proposition 4.7 *For $\lambda \subset \rho_n$ a strict partition, let $z(\lambda) := \frac{2|\lambda|}{n} - (\ell(\lambda) + \delta_{\lambda_1,n})$. Then $z(\iota(\lambda)) = z(\lambda^*) = -z(\lambda)$, and Theorem 4.1 holds with*

$$\kappa = 2^{-\frac{2}{n}} \qquad \text{and} \qquad \zeta(\lambda) = 2^{z(\lambda)}.$$

Proof. The proof is notably different from that of Theorem 4.5, since we have

$$\iota(\sigma_k) = q^{-1}\sigma_{n,n-k} = q^{-1}\sigma_n * \sigma_{n-k}.$$

Again we check that our formula is compatible with the quantum version of Pieri's rule as stated in [BKT], Theorem 6:

$$\sigma_\lambda * \sigma_k = \sum_\mu 2^{N'(\lambda,\mu)}\sigma_\mu + q\sum_\nu 2^{N'(\lambda,\nu)}\sigma_{\bar\nu},$$

where the first sum is over all strict partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$, such that μ/λ is a horizontal strip, and the second sum is over all partitions $\nu = (n, n, \bar\nu) \supset \lambda$, with $\bar\nu$ strict and $|\nu| = |\lambda| + k$, such that ν/λ is a horizontal strip. Note that the quantum correction is non trivial zero when $\lambda_1 = n$.

We distinguish several cases.

First case: $\lambda_1 < n$. Then

$$\sigma_\lambda * \sigma_k = \sum_{\ell(\mu)=\ell(\lambda)} 2^{N'(\lambda,\mu)}\sigma_\mu + \sum_{\ell(\mu)=\ell(\lambda)+1} 2^{N'(\lambda,\mu)}\sigma_\mu \tag{3}$$

has no quantum correction.

First subcase: $\iota(\lambda)_1 < n$. This means that $\ell(\lambda) = 2\delta(\lambda)$, so that on the right hand side of (3) we have $\delta(\mu) = \delta(\lambda)$ in the first partial sum and $\delta(\mu) = \delta(\lambda) + 1$ in the second one. Hence

$$\iota(\sigma_\lambda * \sigma_k) = q^{-\delta(\lambda)} \sum_{\ell(\mu)=\ell(\lambda)} 2^{z(\mu)+N'(\lambda,\mu)}\sigma_{\iota(\mu)} + q^{-\delta(\lambda)-1} \sum_{\ell(\mu)=\ell(\lambda)+1} 2^{z(\mu)+N'(\lambda,\mu)}\sigma_{\iota(\mu)}. \tag{4}$$

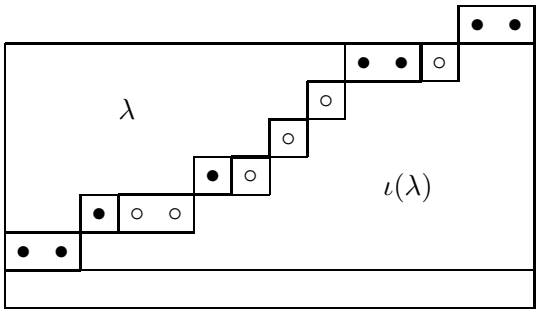
On the other hand $\sigma_{\iota(\lambda)} * \sigma_n = \sigma_{n,\iota(\lambda)}$ has no quantum correction, and the quantum Pieri rule gives

$$\iota(\sigma_\lambda) * \iota(\sigma_k) = 2^{z(\lambda)+z(k)}\Big(q^{-\delta(\lambda)-1}\sum_\alpha 2^{N'(\iota(\lambda),\alpha)}\sigma_{n,\alpha} + q^{-\delta(\lambda)}\sum_\beta 2^{N'((n,\iota(\lambda)),\beta)}\sigma_{\bar\beta}\Big). \tag{5}$$

Consider some μ in the first sum on the right hand side of (4), and let $\beta = \iota(\mu)$. We claim that

$$N'((n, \iota(\lambda)), \beta) = N'(\lambda, \mu) + 1 - \delta_{\mu_1,n}.$$

The following picture should help to see this. We have represented the partition λ in thick lines, so that $\iota(\lambda)$ is its complement (reversed) in the rectangle $\ell(\lambda) \times n$. We have added a line a the bottom of this rectangle to represent $(n, \iota(\lambda))$ (again reversed). The $\bullet$'s represent $\beta/(n, \iota(\lambda))$ (reversed), a horizontal strip inside λ – except possibly if $\ell(\beta) = \ell(\lambda) + 2$, in which case there are some $\bullet$'s on the line above the first line of λ. The $\circ$'s represent μ/λ, again a horizontal strip. On each line they complement the $\bullet$'s of the line above. So we start with a connected component of $\bullet$'s on the SW corner of the picture, and going NE we successively meet the connected components of $\circ$'s and $\bullet$'s. We have thus the same number of components, or one more for the $\bullet$'s if we end by one of these at the NE corner. This is the case if and only if $\mu_1 < n$, so our claim follows.



But $z(\lambda)+z(k)-z(\mu) = \delta_{\mu_1,n}-1$, so $z(\lambda)+z(k)+N'((n,\iota(\lambda)),\beta) = z(\mu)+N'(\lambda,\mu)$. We conclude that the first sum of (4) coincides with the second sum of (5).

Now consider some μ in the second sum on the right hand side of (4). Since $\ell(\mu) = 2\delta(\lambda)+1$ is odd, we get $\iota(\mu) = (n, \alpha)$ for some strict partition α. We claim that

$$N'(\iota(\lambda), \alpha) = N'(\lambda, \mu) - \delta_{\mu_1,n}.$$

Again this implies that the second sum of (4) coincides with the first sum of (5), and we are done.

Second subcase: $\iota(\lambda)_1 = n$. Then $\ell(\lambda) = 2\delta(\lambda) - 1$, and in (3) we always have $\delta(\mu) = \delta(\lambda)$. Hence

$$\iota(\sigma_\lambda * \sigma_k) = q^{-\delta(\lambda)} \sum_\mu 2^{z(\mu)+N'(\lambda,\mu)} \sigma_{\iota(\mu)}. \tag{6}$$

On the other hand, $\sigma_{\iota(\lambda)} * \sigma_n = q\sigma_{\iota(\lambda)/n}$, where the first part of $\iota(\lambda)/n$ is smaller than n. In particular, the quantum Pieri rule for $\sigma_{\iota(\lambda)/n} * \sigma_{n-k}$ has no quantum correction. We get

$$\iota(\sigma_\lambda) * \iota(\sigma_k) = 2^{z(\lambda)+z(k)} q^{-\delta(\lambda)} \sum_\alpha 2^{N'(\iota(\lambda)/n,\alpha)} \sigma_\alpha, \tag{7}$$

the sum being taken over all strict partitions $\alpha \supset \iota(\lambda)/n$, with $|\alpha| = |\iota(\lambda)/n| + n - k = |\iota(\lambda)| - k$, such that $\alpha/(\iota(\lambda)/n)$ is a horizontal strip. Let $\mu = \iota(\alpha)$, and note that $\delta(\mu) = \delta(\alpha) = \delta(\alpha)$ since $\ell(\iota(\lambda)/n) = \ell(\lambda)$ is odd. We claim that

$$N'(\iota(\lambda)/n, \alpha) = N'(\lambda, \mu) + \delta_{\ell(\mu),\ell(\lambda)} - \delta_{\mu_1,n}.$$

Since we have $z(\mu) - z(\lambda) - z(k) = 1 + \ell(\lambda) - \ell(\mu) - \delta_{\mu_1,n} = \delta_{\ell(\mu),\ell(\lambda)} - \delta_{\mu_1,n}$, we conclude that the coefficients of σ_μ in (3) and (6) are equal, which is what we wanted to prove.

Second case: $\lambda_1 = n$. Then we must take care of the quantum correction in Pieri's rule. We write

$$\sigma_\lambda * \sigma_k = \sum_{\ell(\mu)=\ell(\lambda)} 2^{N'(\lambda,\mu)} \sigma_\mu + \sum_{\ell(\mu)=\ell(\lambda)+1} 2^{N'(\lambda,\mu)} \sigma_\mu + q \sum_{\ell(\nu)=\ell(\lambda)} 2^{N'(\lambda,\nu)} \sigma_{\bar{\nu}} + q \sum_{\ell(\nu)=\ell(\lambda)+1} 2^{N'(\lambda,\nu)} \sigma_{\bar{\nu}}. \tag{8}$$

First subcase: $\iota(\lambda)_1 < n$. Then $\delta(\mu) = \delta(\lambda)$ in the first sum of (8) but $\delta(\mu) = \delta(\lambda) + 1$ in the second sum, while $\delta(\bar{\nu}) = \delta(\lambda) - 1$ in the third sum and $\delta(\bar{\nu}) = \delta(\lambda)$ in the last one. Thus

$$\begin{aligned}\iota(\sigma_\lambda * \sigma_k) = \quad & q^{-\delta(\lambda)} \textstyle\sum_{\ell(\mu)=\ell(\lambda)} 2^{z(\mu)+N'(\lambda,\mu)} \sigma_{\iota(\mu)} + q^{-\delta(\lambda)-1} \sum_{\ell(\mu)=\ell(\lambda)+1} 2^{z(\mu)+N'(\lambda,\mu)} \sigma_{\iota(\mu)} \\ & + q^{-\delta(\lambda)} \textstyle\sum_{\ell(\nu)=\ell(\lambda)} 2^{z(\bar{\nu})+N'(\lambda,\nu)} \sigma_{\iota(\bar{\nu})} + q^{-\delta(\lambda)-1} \sum_{\ell(\nu)=\ell(\lambda)+1} 2^{z(\bar{\nu})+N'(\lambda,\nu)} \sigma_{\iota(\bar{\nu})}. \end{aligned} \tag{9}$$

On the other hand $\sigma_{\iota(\lambda)} * \sigma_n = \sigma_{n,\iota(\lambda)}$ and the quantum Pieri rule gives

$$\begin{aligned}\iota(\sigma_\lambda) * \iota(\sigma_k) = \quad & 2^{z(\lambda)+z(k)} \Big(q^{-\delta(\lambda)-1} \textstyle\sum_{\ell(\alpha)=\ell(\lambda)} 2^{N'(\iota(\lambda),\alpha)} \sigma_{n,\alpha} + q^{-\delta(\lambda)-1} \sum_{\ell(\alpha)=\ell(\lambda)+1} 2^{N'(\iota(\lambda),\alpha)} \sigma_{n,\alpha} \\ & + q^{-\delta(\lambda)} \textstyle\sum_{\ell(\beta)=\ell(\lambda)+1} 2^{N'((n,\iota(\lambda)),\beta)} \sigma_{\bar{\beta}} + q^{-\delta(\lambda)} \sum_{\ell(\beta)=\ell(\lambda)+2} 2^{N'((n,\iota(\lambda)),\beta)} \sigma_{\bar{\beta}} \Big). \end{aligned} \tag{10}$$

As above we check that the first, second, third and fourth partial sums in (10) coincide respectively with the second, fourth, third and first partial sum in (9).

Second subcase: $\iota(\lambda)_1 = n$. Same story! □

After renormalizing, we get the following statement:

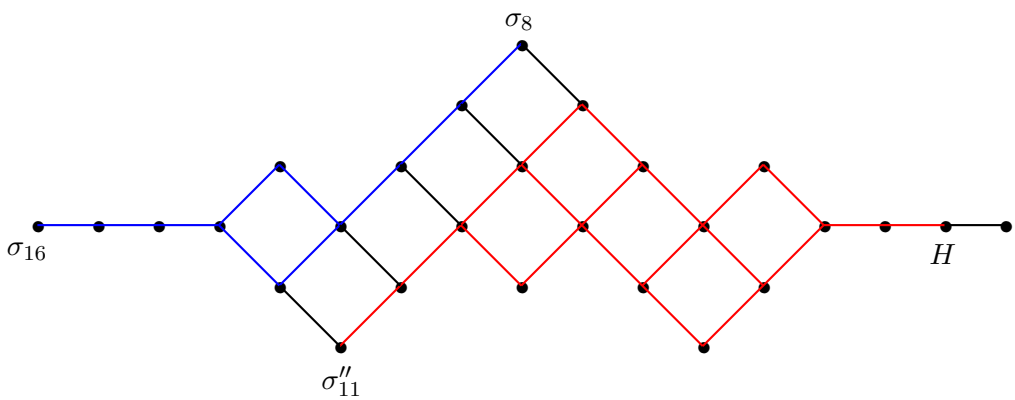
Theorem 4.8 *The correspondence*

$$q \mapsto \frac{4}{q}, \qquad \sigma_\lambda \mapsto \left(\frac{2}{q}\right)^{\delta(\lambda)} 2^{\delta_{\iota(\lambda)_1,n} - \delta_{\lambda_1,n}} \sigma_{\iota(\lambda)} \quad for\ \lambda \subset \rho_n,$$

defines a ring involution of $QA^*(G_Q(n+1, 2n+2))_{loc}$.

4.5 The Cayley plane

For the Cayley plane $\mathbb{OP}^2 = E_6/P_1$, we have $d_{max} = 2$, and the partition of the Hasse diagram is as follows. The two non trivial intervals are isomorphic with the Hasse diagrams of an orthogonal Grassmannian $G_Q(5, 10)$, and a quadric $\mathbb{Q}^8$.



To each Schubert class σ_w of $\mathbb{OP}^2$ we associate a coefficient $\zeta(w)$ as follows, where $y = 2x^2$ and $3y^2 = 1$.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & x & & \frac{1}{x} & & & & & & & \\
 & & & & 1 & & y & & 1 & & \frac{1}{y} & & 1 & & & & \\
1 & \frac{1}{x} & \frac{1}{y} & 2x & & \frac{1}{2x} & & 2x & & \frac{1}{2x} & & 2x & \frac{1}{2x} & & \frac{1}{y} & x & 1 \\
 & & & & 1 & & \frac{1}{y} & & 1 & & y & & 1 & & & & \\
 & & & & & \frac{1}{x} & & & & & & x & & & & &
\end{array}
$$

Observe that $\zeta(\iota(w)) = \zeta(w^*) = \zeta(w)^{-1}$.

Proposition 4.9 *Theorem 4.1 holds for the Cayley plane with $\kappa = 12^{-\frac{1}{4}}$ and $\zeta(w)$ as above.*

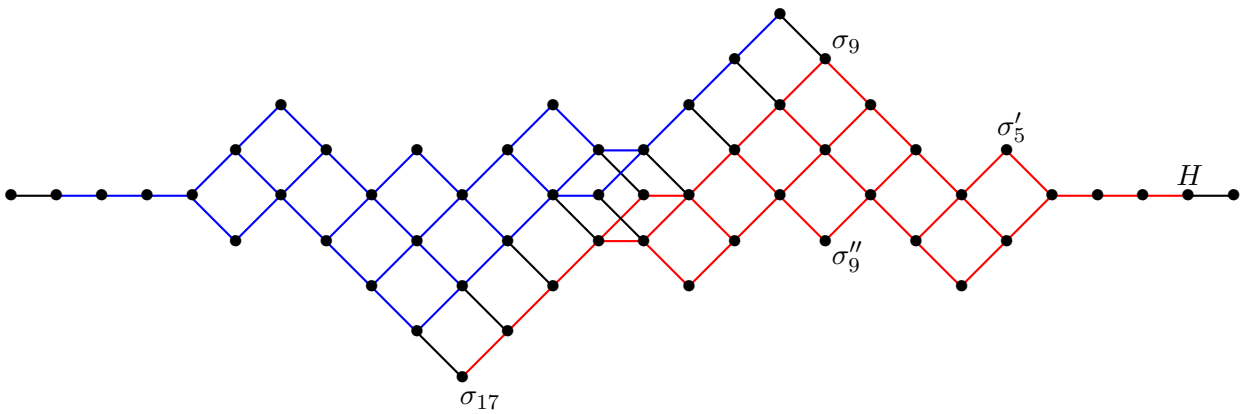
Proof. A presentation for the quantum Chow ring $QA^*(\mathbb{OP}^2)$ has been given in [CMP], Theorem 5.1, with generators H and σ'_4. What we have to check is that the same relations are verified by their images

$$\iota(H) = q^{-1}\sigma''_{11} \qquad \text{and} \qquad \iota(\sigma'_4) = q^{-1}\sigma''_8.$$

This is a lengthy but direct computation. □

4.6 The Freudenthal variety

In this case $d_{max} = 3$. The partition of the Hasse diagram is symmetric, with two intervals that reduce to a point, and two that are isomorphic with the Hasse diagram of the Cayley plane.



In this case we needed a computer to check that Theorem 1.1 does hold. We know from [CMP] that the quantum cohomology ring is generated over $\mathbb{Z}[q]$ by the classes H, σ'_5 and σ_9, and we know the relations explicitely. With the notations of [CMP], our involution is given by

$$q \mapsto 11943936q^{-1}, \quad H \mapsto q^{-1}\sigma_{17}, \quad \sigma'_5 \mapsto 48q^{-1}\sigma'_{13}, \quad \sigma_9 \mapsto 3456q^{-1}\sigma''_9.$$

We first checked that this map preserves the relations and is involutive. Then we computed the image of each Schubert class and checked that it is given by the explicit form of Theorem 1.1.

5 Symmetries of Gromov-Witten invariants

For convenience, denote by p the symmetry of W_X given by Poincaré duality. We deduce from Theorem 4.1 the following identity for the quantum product:

Theorem 5.1 *For any $u, v \in W_X$, we have*

$$\sigma(\iota(u)) * \sigma(p(v)) = q^{\delta(u)+\delta(p(u))-\delta(v)-\delta(p\iota(v))}\sigma(p\iota p(u)) * \sigma(\iota p\iota(v)).$$

Proof. Theorem 4.1 is equivalent to the following identity for Gromov-Witten invariants:

$$I_k(u, v, w) = \zeta(u)\zeta(v)\zeta(w)I_{\delta(u)+\delta(v)-\delta(p(w))-k}(\iota(u), \iota(v), p\iota p(w)). \tag{11}$$

Using the fact that $\zeta(\iota(u)) = \zeta(p(u)) = \zeta(u)^{-1}$, the same identity gives

$$I_\ell(p\iota p(w), \iota(u), \iota(v)) = (\zeta(u)\zeta(v)\zeta(w))^{-1}I_{\delta(p\iota p(w))+\delta(u)-\delta(p\iota(v))-\ell}(\iota p\iota p(w), u, p\iota p\iota(v)).$$

Combining these two relations, we get

$$I_k(u, v, w) = I_{k+\delta(p(w))+\delta(p\iota p(w))-\delta(p\iota(v))-\delta(v)}(u, p\iota p\iota(v), \iota p\iota p(w)),$$

which is equivalent to the identity for the quantum product

$$\sigma(p\iota p\iota(v)) * \sigma(\iota p\iota p(w)) = q^{\delta(p(w))+\delta(p\iota p(w))-\delta(p\iota(v))-\delta(v)}\sigma(v) * \sigma(w).$$

Replacing v by $\iota(u)$ and w by $p(v)$ yields our claim. □

Note that Theorem 5.1 is non trivial only when p and ι don't commute.

Now we observe that Theorem 3.3 can be used to generate more symmetries for the Gromov-Witten invariants. The following statement is a generalization of Proposition 4.10 in [CMP].

Corollary 5.2 *For any $u \in W_X$, we have*

$$\begin{aligned} [Y_{d_{max}}] * \sigma(u) &= q^{\delta(u)+\delta(p\iota(u))-d_{max}}\sigma(p\iota p\iota(u)), \\ [Y^*_{d_{max}}] * \sigma(u) &= q^{d_{max}-\delta(p(u))}\sigma(\iota p(u)). \end{aligned}$$

Proof. To prove the first identity, multiply the identity of Theorem 3.3 by $\sigma(pt)$ and use the fact that $\sigma(pt) * \sigma(pt) = q^{d_{max}}[Y_{d_{max}}]$. For the second identity, observe that Theorem 3.3 implies that $\sigma(pt)$ is invertible in $QA^*(X)_{loc}$, and its inverse verifies the formula

$$\sigma(pt)^{-1} * \sigma(u) = q^{-\delta(p(u))}\sigma(\iota p(u)).$$

Applying this to the fundamental class $\sigma(1) = 1$ yields $\sigma(pt)^{-1} = q^{-d_{max}}[Y^*_{d_{max}}]$, which we just need to substitute in the previous identity. □

Corollary 5.3 *For any $u, v \in W_X$, we have*

$$q^{\delta(u)}\sigma(p(u)) * \sigma(\iota(v)) = q^{\delta(v)}\sigma(\iota(u)) * \sigma(p(v)).$$

Proof. Multiply the identity of Theorem 3.3 for $\iota(u)$ by the Schubert class $\sigma(\iota(v))$, and use the associativity of the quantum product. □

Together with Theorem 4.1, we get a series of symmetry relations for the Gromov-Witten invariants, which are generated by the following simple ones:

Corollary 5.4 *For any $u, v, w \in W_X$, we have the relation*

$$I_k(u,v,w) = \zeta(u)\zeta(v)\zeta(w) I_{\delta(w)-k}(p(u), p(v), \iota(w)).$$

Proof. In terms of Gromov-Witten invariants, Corollary 5.3 writes

$$I_{k-\delta(u)}(p(u), \iota(v), w) = I_{k-\delta(v)}(\iota(u), p(v), w).$$

Combining with the identity (11), we deduce that

$$\begin{aligned} I_k(u,v,w) &= I_{k+\delta(p(u))-\delta(v)}(\iota p(u), p\iota(v), w) \\ &= I_{k+\delta(p(u))-\delta(v)}(\iota p(u), w, p\iota(v)) \\ &= \zeta(\iota p(u))\zeta(w)\zeta(p\iota(v)) I_{\delta(p(u))+\delta(w)-\delta(v))-(k+\delta(p(u))-\delta(v))}(p(u), \iota(w), p(v)) \\ &= \zeta(u)\zeta(w)\zeta(v) I_{\delta(w)-k}(p(u), p(v), \iota(w)). \end{aligned}$$

This is what we wanted to prove. □

Corollary 5.5 *For $u, v \in W_X$, the maximal power of q that appears in the quantum product of the Schubert classes $\sigma(u) * \sigma(v)$ is*

$$d_{max}(u,v) = \delta(u) - \delta(\iota(u), p(v)) = \delta(v) - \delta(\iota(v), p(u)).$$

Corollary 5.4 suggests to study the group Γ of permutations of W_X^3 generated by (p, p, ι), (p, ι, p), (ι, p, p). Clearly the size of this group is governed by the order η of the permutation $p\iota$ of W_X.

Proposition 5.6 *The order of Γ is $2\eta^2$, and η is given by the following table:*

X	η
$\mathbb{G}(p,n)$	$n/gcd(p, n-p)$
$\mathbb{G}_\omega(n,2n)$	2
$\mathbb{G}_Q(n,2n)$	$4/gcd(2,n)$
$\mathbb{OP}^2$	3
E_7/P_7	2

Note that η always divides the order of the symmetry group of the affine Dynkin diagram of G.

Proof. Let Γ_0 denote the group of permutations of W_X generated by the two involutions p and ι. The order of Γ_0 is 2η. Moreover, the projection on the first factor yields a morphism $\Gamma \to \Gamma_0$ which is obviously surjective. Its kernel consists in the permutations of type $(1, (pi)^k, (ip)^k)$, with $k \in \mathbb{Z}$, so its order is η. Thus the order of Γ is $2\eta^2$.

For the explicit values of η, first consider the case of $X = \mathbb{G}(p,n)$. A partition $\lambda \in W_X$ can be identified with a 01-sequence ω with p ones and $n-p$ zeroes encoding vertical and horizontal steps along the boundary of λ, starting from the SW corner. Then the size of the biggest square contained in λ is the number of zeroes among the first p terms of the sequence. Moreover, reading the sequence backwards we get the sequence ω^* of the Poincaré dual partition $p(\lambda)$. So to get the sequence ω' of $\iota(\lambda)$, we write $\omega = \omega_0\omega_1$ where ω_0 has length p and ω_1 has length $n-p$, and let $\omega' = \omega_0^*\omega_1^*$. To deduce the sequence ω'' of $p\iota(\lambda)$ we simply reverse ω', so $\omega'' = \omega_1\omega_0$. The claim easily follows.

Now suppose $X = \mathbb{G}_Q(n,2n)$. We identify a strict partition $\lambda \in W_X$ with a 01-sequence ω of length n as follows. First we consider it as a usual partition in a square of size $(n+1) \times (n+1)$ and we let ω' be the associated 01-sequence, of length $2n+2$. It begins with a 1 and ends with a 0. We suppress the initial 1...10 sequence. Moreover, since λ is strict every 1 is followed by a 0, which we

suppress. The resulting sequence has length n and is our $\omega = \omega_1 \cdots \omega_n$. Note that the length of λ is the number of 1's.

We check that p and ι are easily expressed in terms of 01-sequences:

$$p(\lambda) \mapsto \bar{\omega}_1 \cdots \bar{\omega}_n, \qquad \iota(\lambda) \mapsto \omega_{n-1} \cdots \omega_1 \omega_0,$$

where $\omega_0 = \omega_1 + \cdots + \omega_n$ $(mod\ 2)$ and $\bar{0} = 1$, $\bar{1} = 0$. So $p\iota(\lambda) \mapsto \bar{\omega}_{n-1} \cdots \bar{\omega}_1 \bar{\omega}_0$, and $p\iota p\iota(\lambda) \mapsto \omega_1 \cdots \omega_{n-1} \omega'_n$, where $\omega'_n = \omega_n$ if n is even and $\omega'_n = \bar{\omega}_n$ if n is odd.

The case of $\mathbb{G}_\omega(n, 2n)$, and also that of E_7/P_7 are trivial, since p commutes with ι. Finally the case of $\mathbb{OP}^2$ follows from an explicit computation. $\square$

For a Grassmannian $\mathbb{G}(p, n)$, we get a $\mathbb{Z}_n$-symmetry only when $(p, n) = 1$, while this symmetry always exists by [P1]. Does the same phenomenon happen for $\mathbb{G}_Q(2m, 4m)$? That is, does our twofold symmetry extend to a fourfold symmetry ?

The cases for which $\eta = 2$ are the most symmetric: the involution ι commutes with Poincaré duality. In particular we get the relation

$$\delta(u) + \delta(p(u)) = d_{max}.$$

The Gromov-Witten invariants are identified by groups of eight according to the following identities:

$$\begin{aligned} I_k(u, v, w) &= \zeta(u)\zeta(v)\zeta(w) I_{\delta(w)-k}(p(u), p(v), \iota(w)) \\ &= I_{k+d_{\max}-\delta(v)-\delta(w)}(u, \iota p(v), \iota p(w)) \\ &= \zeta(u)\zeta(v)\zeta(w) I_{\delta(u)+\delta(v)+\delta(w)-d_{max}-k}(\iota(u), \iota(v), \iota(w)). \end{aligned}$$

In particular all Gromov-Witten invariants can be directly computed from those of degree $k \leq d_{max}/4$.

For the other cases we get even more identities, so that lots of Gromov-Witten invariants can be identified with classical intersection numbers on the same variety.

We close this paper with a "dual quantum Chevalley formula", that we obtain by applying our strange duality Theorem to the quantum Chevalley formula of [CMP]. Remember that α_0 denotes the highest root, and that $\sigma(s_{\alpha_0}) = [T_1]$. The Bruhat interval $[1, T_1]$ has a Poincaré involution p_{T_1}. Finally, recall that we denoted by β the simple root that defines P.

Proposition 5.7 *For any $u \in W_X$, we have*

$$\sigma(u) * \sigma(s_{\alpha_0}) = \delta_{p(u) \leq \alpha_0} \sigma(p_{T_1} p(u)) + q \sum_{\substack{s_\alpha u \to u, \\ \alpha \neq \beta}} n_\alpha(\alpha_0) \sigma(s_\alpha u) + q \delta_{\iota(u) \leq \alpha_0} y(p(s_{\alpha_0})) \sigma(s_\beta u).$$

The classical intersection product $\delta_{p(u) \leq \alpha_0} \sigma(p_{T_1} p(u))$ is in agreement with Proposition 3.2. It would be interesting to extend this formula to more general products.

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