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Scattering of cosmic strings by black holes: loop formation

Florian Dubath*
Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030, U.S.A.

Mairi Sakellariadou†
Department of Physics, King's College, University of London, Strand, London WC2R 2LS, United Kingdom

Claude Michel Viallet‡
Laboratoire de Physique Théorique et des Hautes Energies, UMR 7589 CNRS - Paris VI - Paris VII, 4 place Jussieu, Boite 126, F-75252 Paris Cedex 05 France

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We study the deformation of a long cosmic string by a nearby rotating black hole. We examine whether the deformation of a cosmic string, induced by the gravitational field of a Kerr black hole, may lead to the formation of a loop of cosmic string. The segment of the string which enters the ergosphere of a rotating black hole gets deformed and, if it is sufficiently twisted, it can self-intersect chopping off a loop of cosmic string. We find that the formation of a loop, via this mechanism, is a rare event. It will only arise in a small region of the collision phase space, which depends on the string velocity, the impact parameter and the black hole angular momentum. We conclude that generically, the cosmic string is simply scattered or captured by the rotating black hole.

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I. INTRODUCTION

Cosmic strings are one-dimensional topological defects [1, 2, 3] which could have been formed at the end of phase transitions associated with spontaneously broken symmetries in the early universe, via the Kibble mechanism [4]. Even though cosmic strings do not play a dominant rôle in structure formation, these objects are expected to be generically formed in the framework of supersymmetric grand unified theories [5]. Cosmic microwave background measurements allow a small, but non negligible, contribution of strings [6], which can be realized in supersymmetric inflationary models [7]. Recently, cosmic strings gained a lot of interest since it was pointed out that fundamental strings, D1-branes and D$(p-2)$-branes wrapping around $(p-3)$-cycles in the compactified dimensions, could also play a cosmic string-like rôle in cosmology [8].

Cosmic string interactions can lead to a network of string loops. Intercommutations of two long strings in two points, or self-intercommutations of a long string in one point, may chop off a loop of cosmic string. In what follows, we explore another mechanism which may lead to the formation of cosmic string loops. More precisely, we examine the possibility that the scattering of a long cosmic string by a rotating black hole can cause a deformation of the long string, which may lead to the formation of a loop of cosmic string. As a segment of a long cosmic string enters the ergosphere of a rotating black hole, with rotation axis not parallel to the axis of the cosmic string, this segment may be deformed and, in some cases, it may get twisted. If the long string self-intersects, as a result of the deformation, a loop of cosmic string may be formed.

The formation of loops of cosmic strings is important for the astrophysical implications of a string network. In particular, cosmic strings can be detected through the gravitational wave background produced by oscillating string loops [9] [1].

The scattering of a long cosmic string by a rotating black hole has been previously studied in the case of a black hole whose axis of rotation is parallel to the axis of the long string, concluding that the long string will be either scattered, or captured by the black hole. Which of the two regimes will take place depends on the impact parameter. The critical value of the impact parameter which separates the two regimes is given in Ref. [11]. The effect of the scattering of a long cosmic string by a rotating black hole was studied in Ref. [12], where it was shown that the string

*Electronic address: dubath@kitp.ucsb.edu
†Electronic address: Mairi.Sakellariadou@kcl.ac.uk
‡Electronic address: viallet@lpthe.jussieu.fr

[1] We note that long cosmic strings also emit gravitational waves, since these are wiggly due to string intercommutations [10].

is displaced in the direction perpendicular to its velocity by an amount which depends on the impact parameter [12]. In this article we generalize the study of Ref. [12] for different configurations of a long cosmic string approaching a rotating black hole.

We conclude that the formation of a loop from the scattering of a long string by a rotating black hole is a rare event. It will only arise in a small region of the collision phase space, which depends on the string velocity, the impact parameter and the black hole angular momentum. More frequently, the cosmic string is either simply scattered or captured by the rotating black hole.

In what follows we assume that the string thickness is negligible as compared to its length. Moreover, the string linear mass density is very small. Then, assuming that the string size is much larger than the Schwarzschild radius of the black hole, while the mass of the string is much smaller than that of the black hole, we can treat the cosmic string as a test object, and we can neglect its gravitational back reaction.

The string is initially sufficiently far from the black hole, so it can be taken as a straight string. As it approaches the black hole, it will move in a Kerr geometry, however since the string is taken to be sufficiently long, the end regions of the infinitely long string will not be influenced by the presence of the gravitational field of the Kerr black hole.

Let us consider the axis of rotation of the Kerr black hole to be along the direction $\mathcal{X}$, in a system of coordinates $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. A cosmic string being initially far away from the black hole is moving towards it. One can investigate the following setups:

- The string is at first parallel to the axis of rotation of the Kerr black hole, and it moves towards the black hole with a velocity perpendicular to the axis of rotation. Thus, the cosmic string is originally along the $\mathcal{X}$-axis and its velocity is along the $\mathcal{Z}$-axis. [We call this case **(i)**.]

- The string is at first perpendicular to the axis of rotation of the Kerr black hole and it moves towards the black hole with velocity parallel to axis of rotation. Thus, the cosmic string is originally along the $\mathcal{Z}$-axis and its velocity is along the $\mathcal{X}$-axis. [We call this case **(ii)**.]

- The string is at first perpendicular to the axis of rotation of the Kerr black hole and it moves towards the black hole with velocity which is also perpendicular to the axis of rotation. Thus, the cosmic string is originally along the $\mathcal{Z}$-axis and it moves towards the Kerr black hole with velocity along the $\mathcal{Y}$-axis. [We call this case **(iii)**.]

Case **(i)** has been studied in Ref. [12]; in what follows we will focus on cases **(ii)** and **(iii)**. Case **(iii)** is the most promising configuration to lead to the formation of a cosmic string loop.

We organize the rest of the paper as follows: In Section II we briefly discuss the string equations of motion in a curved background. We give the string equations of motion for a string in a general background geometry and then we derive them for a straight string far away from a black hole. We proceed with first the Newtonian and then the Lense-Thirring scattering of a straight string in the linearized approximation. In Section III we study the scattering of a long straight string by a Kerr black hole. Our numerical approach is presented in Section IV. We round up with our conclusions in Section V.

We will use units where $G = c = 1$.

II. STRING EQUATIONS OF MOTION IN A CURVED BACKGROUND

A. String equations of motion

The world history of a cosmic string can be expressed by a two-dimensional surface in the four-dimensional space-time, which is called the string worldsheet:

$$x^\mu = x^\mu(\zeta^a) \quad ; \quad a = 0, 1 \ , \tag{II.1}$$

where the worldsheet coordinates ζ^0, ζ^1 are arbitrary parameters chosen so that ζ^0 is time-like ($\equiv \tau$) and ζ^1 is space-like ($\equiv \sigma$).

The string equations of motion, in the limit of a zero thickness string, are derived from the Nambu-Goto effective action which, up to an overall factor, corresponds to the surface area swept out by the string in space-time:

$$S_0[x^\mu] = -\mu \int \sqrt{-\gamma} d^2\zeta \ , \tag{II.2}$$

where γ is the determinant of the two-dimensional worldsheet metric γ_{ab}:

$$\gamma = \det(\gamma_{ab}) = \frac{1}{2}\epsilon^{ac}\epsilon^{bd}\gamma_{ab}\gamma_{cd} \quad , \quad \gamma_{ab} = g_{\mu\nu}x^{\mu}_{,a}x^{\nu}_{,b} \ , \tag{II.3}$$

with $g_{\mu\nu}$ the four-dimensional metric.

One can derive the same string equations of motion by using Polyakov's form of the action [13]

$$S[x^{\mu}, h_{ab}] = -\frac{\mu}{2}\int \sqrt{-h}h^{ab}\gamma_{ab}d^2\zeta \ , \tag{II.4}$$

where h_{ab} is the internal metric with determinant h.

Varying Eq. (II.4) with respect to $x^{\mu}(\zeta^a)$ we obtain the string equations of motion

$$\begin{aligned} \Box x^{\mu} + h^{ab}\Gamma^{\mu}_{\nu\sigma}x^{\nu}_{,a}x^{\sigma}_{,b} &= 0 \ \ \text{(dynamical)} \\ \gamma_{ab} - \frac{1}{2}h_{ab}h^{cd}\gamma_{cd} &= 0 \ \ \text{(constraint)} \ , \end{aligned} \tag{II.5}$$

where

$$\Box = \frac{1}{\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b) \ , \tag{II.6}$$

and $\Gamma^{\mu}_{\nu\sigma}$ is the four-dimensional Christoffel symbol.

We choose the gauge in which h_{ab} is conformal to the flat two dimensional metric $\eta_{ab} = \text{diag}(-1, 1)$. The dynamical and constraint equations, Eqs. (II.5), become

$$\begin{aligned} \Box x^{\mu} + \eta^{ab}\Gamma^{\mu}_{\nu\sigma}x^{\nu}_{,a}x^{\sigma}_{,b} &= 0 \\ \gamma_{01} = g_{\mu\nu}x^{\mu}_{,0}x^{\nu}_{,1} &= 0 \\ \gamma_{00} + \gamma_{11} = g_{\mu\nu}x^{\mu}_{,0}x^{\nu}_{,0} + x^{\mu}_{,1}x^{\nu}_{,1} &= 0 \ , \end{aligned} \tag{II.7}$$

where $\Box \equiv -\partial^2_{\tau} + \partial^2_{\sigma}$ (note that $\partial_A \equiv \partial/\partial A$).

B. Straight string moving in a weak gravitational field

As long as a cosmic string is far away from a black hole, it just feels a weak gravitational field and thus the linearized analysis is sufficient. We present below the linear approach, which can be employed to furnish the initial conditions allowing us to perform a full numerical simulation for the case of a string being close to a rotating black hole, and thus limiting the size of the numerical simulation. This approach has been already used in Ref. [12].

Assuming at first that there is no external gravitational field, the 4-dimensional space-time metric $g_{\mu\nu}$ is just the Minkowski metric $\eta_{\mu\nu}$. In coordinates $(\mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and choosing signature $(-, +, +, +)$ one can show that a solution of the equations of motion Eq.(II.5) read

$$\begin{aligned} x^{\mu} &= x^{\mu}_0(\tau, \sigma) = (\tau\cosh(\beta), \tau\sinh(\beta) + \mathcal{X}_0, b, \sigma) \\ \text{or} \ \ x^{\mu} &= x^{\mu}_0(\tau, \sigma) = (\tau\cosh(\beta), b, \tau\sinh(\beta) + \mathcal{Y}_0, \sigma) \ , \\ h_{ab} &= \eta_{ab} = \text{diag}(-1, 1) \ . \end{aligned} \tag{II.8}$$

Equation (II.8a) refers to a long straight string along the $\mathcal{Z}$-axis, being initially at $x^{\mu}(0, \sigma) = (0, \mathcal{X}_0, b, \sigma)$, and moving with velocity $v = \tanh(\beta)$ along the direction $\mathcal{X}$. We consider $b > 0, \mathcal{X}_0 < 0$. This configuration corresponds to case **(ii)**. Similarly, Eq. (II.8b) refers to a long straight string along the $\mathcal{Z}$-axis, being initially at $x^{\mu}(0, \sigma) = (0, b, \mathcal{Y}_0, \sigma)$, and moving with velocity $v = \tanh(\beta)$ along the direction $\mathcal{Y}$. We again consider $b > 0, \mathcal{Y}_0 < 0$. This configuration corresponds to case **(iii)**.

As a second step we consider the case of a weak gravitational field

$$g_{\mu\nu} = \eta_{\mu\nu} + \theta_{\mu\nu} \ , \tag{II.9}$$

in which moves a straight long string

$$x^{\mu}(\xi) = x^{\mu}_0(\xi) + \bar{x}^{\mu}(\xi) \ ; \tag{II.10}$$

$\theta_{\mu\nu}, \bar{x}^\mu(\xi)$ denote a small metric perturbation, and a small perturbation along a straight cosmic string, respectively.

In first order in $\theta_{\mu\nu}$ and $\bar{x}^\mu$, the equations of motion, Eq.(II.5), read

$$\begin{aligned}
\Box x^\mu + \Gamma^\mu_{\alpha\beta}(x_0) x^\alpha_{0,a} x^\beta_{0,b}\eta^{ab} &= 0 \quad \text{(dynamical)} \\
\eta_{\mu\nu}\frac{\partial x_0^\mu}{\partial\tau}\frac{\partial\bar{x}^\nu}{\partial\sigma} + \eta_{\mu\nu}\frac{\partial\bar{x}^\mu}{\partial\tau}\frac{\partial x_0^\nu}{\partial\sigma} + \theta_{\mu\nu}\frac{\partial x_0^\mu}{\partial\tau}\frac{\partial x_0^\nu}{\partial\sigma} &= 0 \quad \text{(constraint)} \\
2\eta_{\mu\nu}\left(\frac{\partial x_0^\mu}{\partial\tau}\frac{\partial\bar{x}^\nu}{\partial\tau} + \frac{\partial x_0^\mu}{\partial\sigma}\frac{\partial\bar{x}^\nu}{\partial\sigma}\right) + \theta_{\mu\nu}\left(\frac{\partial x_0^\mu}{\partial\tau}\frac{\partial x_0^\nu}{\partial\tau} + \frac{\partial x_0^\mu}{\partial\sigma}\frac{\partial x_0^\nu}{\partial\sigma}\right) &= 0 \quad \text{(constraint)} \ . \qquad \text{(II.11)}
\end{aligned}$$

The above equations describe the motion of a long straight cosmic string, being located far from a Kerr black hole.

In the weak field approximation, the gravitational field produced by a black hole of mass M and angular momentum J, rotating around the $\mathcal{X}$-axis, is

$$ds^2 = -\left(1 - \frac{2M}{\mathcal{R}}\right)d\mathcal{T}^2 + \left(1 + \frac{2M}{\mathcal{R}}\right)(d\mathcal{X}^2 + d\mathcal{Y}^2 + d\mathcal{Z}^2) - \frac{4J}{\mathcal{R}^3}(\mathcal{Y}d\mathcal{Z} - \mathcal{Z}d\mathcal{Y})d\mathcal{T}, \qquad \text{(II.12)}$$

where $\mathcal{R}^2 = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$.

One can therefore write the small perturbation $\theta_{\mu\nu}$ on the gravitational field, in terms of the Newtonian and the Lense-Thirring [14, 15] parts, as

$$\begin{aligned}
\theta_{\mu\nu} &= \theta^{\rm N}_{\mu\nu} + \theta^{\rm LT}_{\mu\nu} \\
&= 2\varphi\delta_{\mu\nu} + \frac{4J}{\mathcal{R}^3}\delta^0_{(\mu}\epsilon_{\nu)\alpha 01}\mathcal{X}^\alpha \ , \qquad \text{(II.13)}
\end{aligned}$$

where φ is defined as $\varphi = M/\mathcal{R}$ and $\epsilon_{\alpha\beta\gamma\delta}$ stands for the antisymmetric tensor. The second term of the *r.h.s* of Eq. (II.13b) is due to the rotating black hole around the $\mathcal{X}$-axis.

C. Newtonian scattering of a straight string in the linearized approximation

Let us first consider the Newtonian scattering of a straight cosmic string. We first examine case **(ii)**. The dynamical equations read

$$\begin{aligned}
\left(-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^0 &= -2\sinh(\beta)\cosh(\beta)\varphi_{,1} \\
\left(-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^1 &= 0 \\
\left(-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^2 &= -2\sinh^2(\beta)\varphi_{,2} \\
\left(-\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^3 &= -2\cosh^2(\beta)\varphi_{,3} \ , \qquad \text{(II.14)}
\end{aligned}$$

and the constraint equations are

$$\begin{aligned}
\bar{x}^3_{,\tau} - \cosh(\beta)\bar{x}^0_{,\sigma} + \sinh(\beta)\bar{x}^1_{,\sigma} &= 0 \\
\bar{x}^3_{,\sigma} - \cosh(\beta)\bar{x}^0_{,\tau} + \sinh(\beta)\bar{x}^1_{,\tau} &= -2\varphi\cosh^2(\beta) \ ; \qquad \text{(II.15)}
\end{aligned}$$

we use the notation $\mathcal{W}_{,A} \equiv \partial\mathcal{W}/\partial A$.

The dynamical equations imply that for a given time the cosmic string lies in the $(\mathcal{Y}, \mathcal{Z})$-plane. The string deflection $\mathcal{D}$ along the plane orthogonal to the velocity of the string, given in Ref. [12], is

$$\mathcal{D} = b - 2\pi M\sinh(\beta) \ . \qquad \text{(II.16)}$$

It implies that while at later times segments of the cosmic string which are far away from the rotating black hole keep moving in the $(\mathcal{Y} = b, \mathcal{Z})$-plane, since they are unaffected from the gravitational field of the black hole, the central part of the string being by definition closer to the location of the black hole will be moving in the $(\mathcal{Y} = \mathcal{D}, \mathcal{Z})$-plane instead.

We then study case **(iii)**. The dynamical equations read

$$\begin{aligned}
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^0 &= -2\sinh(\beta)\cosh(\beta)\varphi_{,1}\\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^1 &= -2\sinh^2(\beta)\varphi_{,1}\\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^2 &= 0\\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^3 &= -2\cosh^2(\beta)\varphi_{,3}\ ,
\end{aligned} \tag{II.17}$$

and the constraint equations are

$$\begin{aligned}
\bar{x}^3_{,\tau}-\cosh(\beta)\bar{x}^0_{,\sigma}+\sinh(\beta)\bar{x}^2_{,\sigma} &= 0\\
\bar{x}^3_{,\sigma}-\cosh(\beta)\bar{x}^0_{,\tau}+\sinh(\beta)\bar{x}^2_{,\tau} &= -2\varphi\cosh^2(\beta)\ .
\end{aligned} \tag{II.18}$$

D. Lense-Thirring scattering of a straight string in the linearized approximation

Let us now proceed with the Lense-Thirring scattering of a long straight string, moving in the gravitational field of a black hole of mass M and angular momentum J, in the linearized approximation.

Starting with case **(ii)**, the dynamical equations read

$$\begin{aligned}
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^0 &= \frac{6Jb\sigma}{\mathcal{R}^5}\\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^1 &= 0\\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^2 &= -\frac{6J}{\mathcal{R}^5}(\tau\sinh(\beta)+x_0)\sigma\cosh(\beta)\sinh(\beta)\\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^3 &= \frac{6J}{\mathcal{R}^5}(\tau\sinh(\beta)+x_0)b\cosh(\beta)\sinh(\beta)\ ,
\end{aligned} \tag{II.19}$$

and the constraint equations are

$$\begin{aligned}
\bar{x}^3_{,\sigma}-\cosh(\beta)\bar{x}^0_{,\tau}+\sinh(\beta)\bar{x}^1_{,\tau} &= 0\\
-\cosh(\beta)\bar{x}^0_{,\sigma}+\sinh(\beta)\bar{x}^1_{,\sigma}+\bar{x}^3_{,\tau} &= \frac{2J}{\mathcal{R}^3}b\cosh(\beta)\ ,
\end{aligned} \tag{II.20}$$

where

$$\mathcal{R}=\sqrt{(\tau\sinh(\beta)+\mathcal{X}_0)^2+b^2+\sigma^2}\ . \tag{II.21}$$

Choosing initial conditions

$$\begin{aligned}
\bar{x}^0|_{\tau=0} &= 0\\
\bar{x}^2|_{\tau=0} &= \frac{\partial\bar{x}^2}{\partial\tau}|_{\tau=0}=0\\
\bar{x}^3|_{\tau=0} &= \frac{\partial\bar{x}^3}{\partial\tau}|_{\tau=0}=0\ ,
\end{aligned} \tag{II.22}$$

the dynamical equations can be solved analytically. The solution reads

$$
\begin{aligned}
\bar{x}^0 &= 2Jb\Big[\frac{\sinh^2(\beta)\tau+\mathcal{X}_0\sinh(\beta)-\sigma}{\mathcal{A}(\tau+\sigma)\mathcal{R}}-\frac{\sinh^2(\beta)\tau+\mathcal{X}_0\sinh(\beta)+\sigma}{\mathcal{A}(\tau-\sigma)\mathcal{R}} \\
&\quad +\frac{\tau+\sigma-\mathcal{X}_0\sinh(\beta)}{\mathcal{A}(\tau+\sigma)\mathcal{S}(\tau+\sigma)}-\frac{\tau-\sigma-\mathcal{X}_0\sinh(\beta)}{\mathcal{A}(\tau-\sigma)\mathcal{S}(\tau-\sigma)}\Big] \\
\bar{x}^2 &= 2Jb\cosh(\beta)\Big[\frac{\sinh(\beta)[\sinh(\beta)\{b^2+\sigma^2+\sigma\tau\}+\sigma\mathcal{X}_0]}{\mathcal{A}(\tau+\sigma)\mathcal{R}}-\frac{\sinh(\beta)[\sinh(\beta)\{b^2+\sigma^2-\sigma\tau\}-\sigma\mathcal{X}_0]}{\mathcal{A}(\tau-\sigma)\mathcal{R}} \\
&\quad -\frac{\sinh(\beta)[\sinh(\beta)b^2+\sinh(\beta)(\tau+\sigma)^2+\mathcal{X}_0(\tau+\sigma)]}{\mathcal{A}(\tau+\sigma)\mathcal{S}(\tau+\sigma)}+\frac{\sinh(\beta)[\sinh(\beta)b^2+\sinh(\beta)(\tau-\sigma)^2+\mathcal{X}_0(\tau-\sigma)]}{\mathcal{A}(\tau-\sigma)\mathcal{S}(\tau-\sigma)} \\
\bar{x}^3 &= 2Jb\cosh(\beta)\Big[\frac{\sinh^2(\beta)\tau+\mathcal{X}_0\sinh(\beta)+\sigma}{\mathcal{A}(\tau-\sigma)\mathcal{R}}+\frac{\sinh^2(\beta)\tau+\mathcal{X}_0\sinh(\beta)-\sigma}{\mathcal{A}(\tau+\sigma)\mathcal{R}}+\frac{\tau-\sigma-\mathcal{X}_0\sinh(\beta)}{\mathcal{A}(\tau-\sigma)\mathcal{S}(\tau-\sigma)} \\
&\quad +\frac{\tau+\sigma-\mathcal{X}_0\sinh(\beta)}{\mathcal{A}(\tau+\sigma)\mathcal{S}(\tau+\sigma)}-\frac{\tau+\sigma}{(\mathcal{X}_0^2+\mathcal{Y}_0^2)\mathcal{S}(\tau+\sigma)}-\frac{\tau-\sigma}{(\mathcal{X}_0^2+\mathcal{Y}_0^2)\mathcal{S}(\tau-\sigma)}\Big] \ , \qquad \text{(II.23)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R} &= \sqrt{\sinh^2(\beta)\tau^2+2\sinh(\beta)\tau\mathcal{X}_0+\mathcal{X}_0^2+\mathcal{Y}_0^2+\sigma^2} \\
\mathcal{S}(u) &= \sqrt{\mathcal{X}_0^2+\mathcal{Y}_0^2+u^2} \\
\mathcal{A}(u) &= \mathcal{Y}_0^2[1+\sinh(\beta)^2]+[\mathcal{X}_0+\sinh(\beta)u]^2 \ . \qquad \text{(II.24)}
\end{aligned}
$$

Let us proceed with case **(iii)**. The dynamical equations read

$$
\begin{aligned}
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^0 &= -\frac{6J\sigma}{\mathcal{R}^5}(\tau\sinh(\beta)+y_0)\cosh^2(\beta) \\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^1 &= \frac{6Jb\sigma}{\mathcal{R}^5}\cosh(\beta)\sinh(\beta) \\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^2 &= 0 \\
\left(-\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)\bar{x}^3 &= \frac{2J}{\mathcal{R}^5}(\mathcal{R}^2-3b^2) \ , \qquad \text{(II.25)}
\end{aligned}
$$

and the constraint equations are

$$
\begin{aligned}
\bar{x}^3_{,\sigma}-\cosh(\beta)\bar{x}^0_{,\tau}+\sinh(\beta)\bar{x}^2_{,\tau} &= -\frac{2J}{\mathcal{R}^3}\sigma\cosh(\beta)\sinh(\beta) \\
-\cosh(\beta)\bar{x}^0_{,\sigma}+\sinh(\beta)\bar{x}^2_{,\sigma}+\bar{x}^3_{,\tau} &= \frac{2J}{\mathcal{R}^3}b\cosh(\beta) \ , \qquad \text{(II.26)}
\end{aligned}
$$

where

$$
\mathcal{R}=\sqrt{b^2+(\tau\sinh(\beta)+\mathcal{Y}_0)^2+\sigma^2} \ . \qquad \text{(II.27)}
$$

The solution is far more complicated than the one for case **(ii)**[2].

III. SCATTERING OF A STRAIGHT STRING BY A KERR BLACK HOLE

We now turn to the full non-linear analysis. The Kerr metric, can be presented in the *quasi-Cartesian* coordinate system $(\mathcal{T},\mathcal{X},\mathcal{Y},\mathcal{Z})$ by using the following shift [12] from the Boyer–Lindquist system (t,r,θ,ϕ):

$$
r=\mathcal{R}+M \ , \qquad \text{(III.28)}
$$

[2] Instead of writing down explicitly the quite tedious solution, we give in an appendix the method we follow to solve the system of equations.

and

$$\begin{aligned}\mathcal{T} &= t\\ \mathcal{X} &= \mathcal{R}\cos\theta\\ \mathcal{Y} &= \mathcal{R}\sin\theta\cos\phi\\ \mathcal{Z} &= \mathcal{R}\sin\theta\sin\phi\ ,\end{aligned} \tag{III.29}$$

In these coordinates, the rotation axis of the black hole is the $\mathcal{X}$ axis and the metric takes the form

$$\begin{aligned}ds^2 &= -\left(1-\frac{2M(\mathcal{R}+M)}{A^2}\right)d\mathcal{T}^2+\frac{4aM(\mathcal{R}+M)}{A^2\mathcal{R}^2}\left(\mathcal{Y}d\mathcal{Z}-\mathcal{Z}d\mathcal{Y}\right)d\mathcal{T}\\ &+\left(\frac{\left((\mathcal{R}+M)^2+a^2\right)^2-\frac{a^2\rho^2(\mathcal{Y}^2+\mathcal{Z}^2)}{\mathcal{R}^2}}{(\mathcal{Y}^2+\mathcal{Z}^2)\mathcal{R}^2A^2}\right)\left(\mathcal{Y}d\mathcal{Z}-\mathcal{Z}d\mathcal{Y}\right)^2+\frac{A^2}{\mathcal{R}^2}\left(\frac{1}{\rho^2}+\frac{\mathcal{X}^2}{\mathcal{R}^2(\mathcal{Y}^2+\mathcal{Z}^2)}\right)\left(\mathcal{Y}d\mathcal{Y}+\mathcal{Z}d\mathcal{Z}\right)^2\\ &+\frac{A^2}{\mathcal{R}^2}\left(\frac{\mathcal{X}^2}{\rho^2}+\frac{\mathcal{Y}^2+\mathcal{Z}^2}{\mathcal{R}^2}\right)d\mathcal{X}^2+\frac{2A^2}{\mathcal{R}^2}\left(\frac{1}{\rho^2}-\frac{1}{\mathcal{R}^2}\right)\left(\mathcal{Y}d\mathcal{Y}+\mathcal{Z}d\mathcal{Z}\right)\mathcal{X}d\mathcal{X},\end{aligned} \tag{III.30}$$

where M, J denote the mass and angular momentum, respectively, of the black hole, withe $J = aM$ for $|a| \leq M$; A and ρ are defined as

$$A^2 = (\mathcal{R}+M)^2+\frac{a^2\mathcal{X}^2}{\mathcal{R}^2} \quad \text{and} \quad \rho^2 = (\mathcal{R}+M)^2+a^2-2M\left(\mathcal{R}+M\right)\ . \tag{III.31}$$

Far from the origin of the coordinate system, the metric takes the form given in Eq. (II.12).

IV. NUMERICAL APPROACH

We use the quasi-Cartesian coordinate system given previously. To study the case **(ii)**, or **(iii)**, we let evolve a string initially parallel to $\mathcal{Z}$ axis with initial velocity along the $\mathcal{X}$ axis (case **(ii)**), or the $\mathcal{Y}$ axis (case **(iii)**). We focus on the more promising case of a maximally rotating black hole ($a = M$) and we set the mass equal to unity; we can recover other values by rescaling the coordinates.

The Kerr metric is given analytically in Eq. (III.30) and the Christoffel symbols are obtained by taking the derivatives of the metric. In our simulations we compute the derivatives as a finite difference for a small variation of the coordinates. We discretise a piece of world-sheet with steps $\Delta\sigma$ and $\Delta\tau$. The string position is obtained as the value of the fields $X^\mu(\sigma,\tau)$ on the world-sheet. They are computed using Eq. (II.7) and performing a numerical integration along the τ direction. We need initial and boundary conditions: the initial conditions are the value and τ derivatives of $X^\mu(\sigma,0)$, $V^\mu(\sigma,0) = \partial_\tau X^\mu(\sigma,0)$ on the slice $\tau = 0$. We compute the boundary conditions as the free evolution of the end of the world-sheet[3].

In our simulations the initial conditions are the values of the $X^\mu(\sigma,0), V^\mu(\sigma,0)$ at the point σ_i, $i \in \{0;...;4000\}$ given by

$$\begin{aligned}X^\mu(\sigma_i,0) &= (0,-\mathcal{X}_0,b,(i-2000)\delta\mathcal{Z})\ ,\\ V^\mu(\sigma_i,0) &= (\cosh(\beta),\sinh(\beta),0,0)\ ,\end{aligned} \tag{IV.32}$$

for the case **(ii)**, and

$$\begin{aligned}X^\mu(\sigma_i,0) &= (0,b,-\mathcal{Y}_0,(i-2000)\delta\mathcal{Z})\ ,\\ V^\mu(\sigma_i,0) &= (\cosh(\beta),0,\sinh(\beta),0)\ ,\end{aligned} \tag{IV.33}$$

for the case **(iii)**. In the above, b is the impact parameter, $\delta\mathcal{Z} = 0.1/M$, β denotes the velocity parameter and $\mathcal{X}_0$ ($\mathcal{Y}_0$) is the distance between the cosmic string and the black hole at the beginning of the simulation. In order to avoid numerical instability we set the *proper time step* $\Delta\tau = 10^{-3}$.

For a cosmic string which is deformed by a Kerr black hole but with insufficient twist, so that a loop is not formed when the string comes close to the black hole, we stop the simulation when the string starts drifting away from the

[3] In a more accurate simulation both initial and boundary conditions should be computed using the weak field evolution.

black hole. The reason for this is that the deformation of the central segments of the cosmic string will be propagated towards the end parts of the string, leaving behind a wiggly, but on the large-scale straight, cosmic string. We also stop the simulation if we encounter an infinity, denoting that a point along the string crosses the horizon of the black hole.

To display the string position we produce equal-time slices. Such slices are obtained by searching all the world-sheet points (σ, τ) such that $X^0(\sigma, \tau) = \mathcal{T}$ for fixed $\mathcal{T}$ and plotting the $(\mathcal{X}, Y, Z)$.

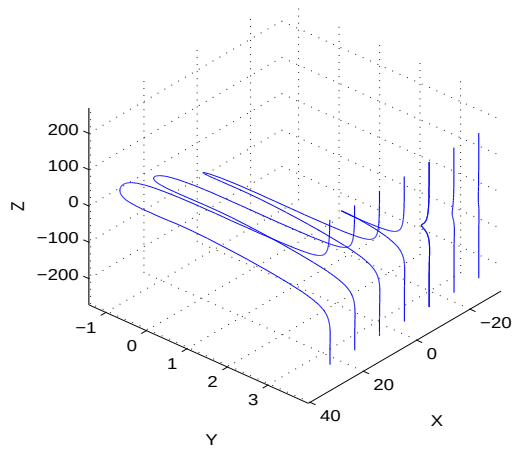

FIG. 1: Timeslices for the case **(ii)** with $\beta = 0.5$ and $b = 5$. We show the slices for time $\mathcal{T} = 25, 37.5, \ldots, 100$.

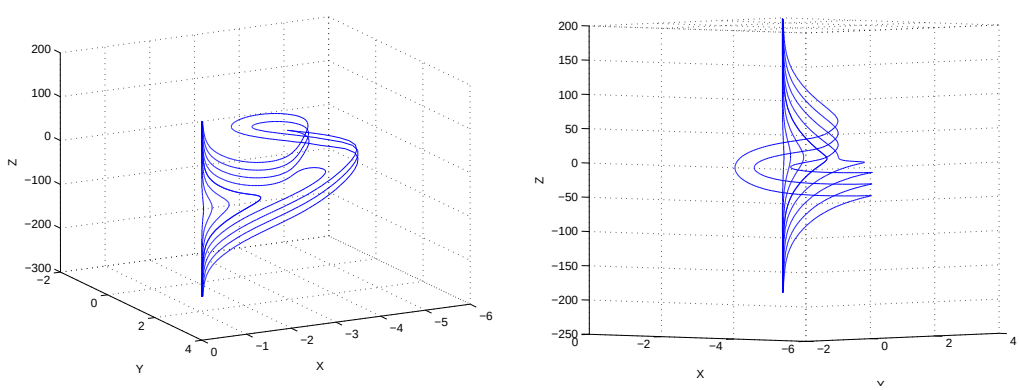

FIG. 2: Same as Fig. 1 but with the $\mathcal{X}$ motion subtracted in order to show the string deformation.

V. CONCLUSIONS

We have studied the deformation of a long cosmic string entering the ergosphere of a Kerr black hole. The aim of this study is to investigate how often such a deformation can be accompanied by a sufficient twist, such that there is a string self-intersection leading to a loop formation.

Among the configurations we have studied, the case of a long cosmic string being at first perpendicular to the axis of rotation of the Kerr black hole and moving towards it with velocity perpendicular to the rotation axis, is the most prominent to lead to loop formation. [It is our case **(iii)**.]

A numerical investigation has shown us that loop formation is a rather rare event, while in general the cosmic string will be just scattered from the black hole or captured by it.

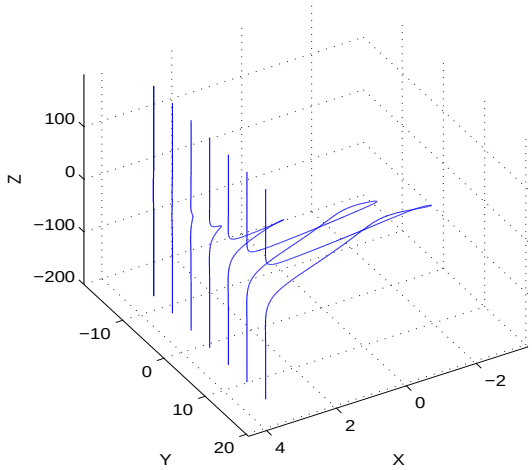

FIG. 3: Timeslices for the case **(iii)** with $\beta = 0.5$ and $b = 3.3$. We show the slices for time $\mathcal{T} = 44, 55, \ldots, 110$.

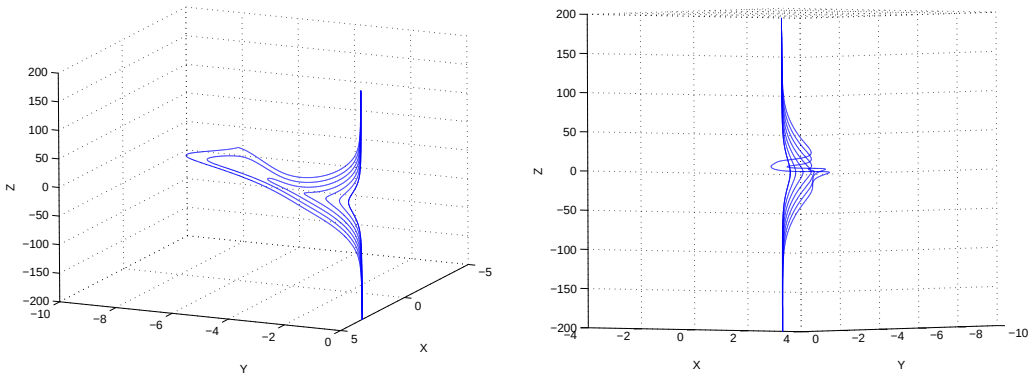

FIG. 4: Same as Fig. 3 but with the $\mathcal{Y}$ motion subtracted in order to show the string deformation. Note that the center of the string has been twisted of about 90 degrees.

Acknowledgments

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Appendix

Rather than giving explicitely the lengthy formulae for the solutions of the dynamical equations, Eq. (II.25), of case **(iii)** with constraint equations Eq. (II.26), we describe here the resolution method of such equations.

The form of the equations is

$$\partial_\sigma^2 \varphi(\sigma,\tau) - \partial_\tau^2 \varphi(\sigma,\tau) = f(\sigma,\tau) \ . \tag{V.34}$$

The corresponding homogeneous equation is

$$\partial_\sigma^2 \varphi_{\rm h} - \partial_\tau^2 \varphi_{\rm h} = 0 \ . \tag{V.35}$$

The general solution of Eq. (V.34) is the sum of the general solution φ_h of the homogeneous equation Eq. (V.35) and a particular solution φ_{part} of Eq. (V.34).

The general solution φ_h is just

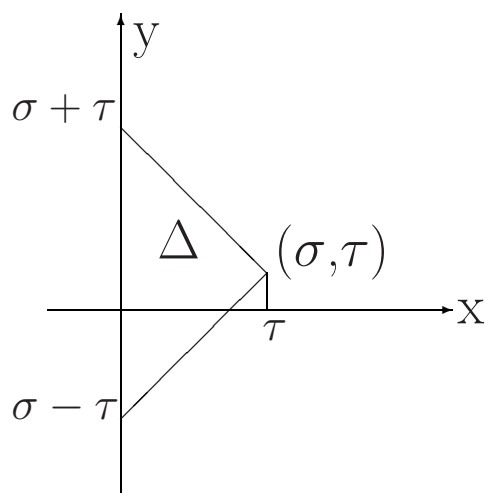
$$\varphi_{\rm h}(\sigma,\tau) = a\,(\sigma+\tau) + b\,(\sigma-\tau) \ , \tag{V.36}$$

wih arbitrary functions of one variable a and b.

It is possible to write a particular solution $\varphi_{\rm part}$ as a double integral of the right hand side of (V.34).

$$\varphi_{\rm part} = \int_\Delta f(x,y)\, dx\, dy \ , \tag{V.37}$$

where the domain of integration Δ is a triangle shown in the figure below.



The particular solution $\varphi_{\rm part}$ may be written in more than one way. Two are useful:

$$\varphi_{\rm part}(\sigma,\tau) = \frac{1}{2}\int_{\sigma-\tau}^{\sigma} dy \int_0^{\tau-\sigma+y} dx\ f(x,y) + \frac{1}{2}\int_{\sigma}^{\sigma+\tau} dy \int_0^{\tau+\sigma-y} f(x,y) \tag{V.38}$$

and

$$\varphi_{\rm part}(\sigma,\tau) = \frac{1}{2}\int_0^\tau dx \int_{\sigma-\tau+x}^{\sigma+\tau-x} dy\ f(x,y) \tag{V.39}$$

Expression (V.38) [resp. (V.39)] is useful when $f(x,y)$ may be written simply as a derivative $\partial_x\, g(x,y)$, [resp. $\partial_y\, g(x,y)$]. Only one integration is then to be performed.

Of course equation (V.34) is complemented with boundary conditions which will determine the unknown functions a and b.