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A central limit theorem for a localized version of the SK model

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Abstract

In this note, we consider a SK (Sherrington–Kirkpatrick)-type model on $\mathbb{Z}^d$ for $d \geq 1$, weighted by a function allowing to any single spin to interact with a small proportion of the other ones. In the thermodynamical limit, we investigate the equivalence of this model with the usual SK spin system, through the study of the fluctuations of the free energy.

Key words and phrases: spin glasses, Sherrington-Kirkpatrick, localized mean-field model, cavity method, stochastic calculus.

MSC: primary 60K35; secondary 82D30, 82B44.

1 Introduction

This paper is concerned with a localized version of the Sherrington-Kirkpatrick model with external field, which can be described in the following way: for $N, d \geq 1$, our space of configurations will be $\Sigma = \Sigma_N = \{-1, 1\}^{C_N}$, where C_N is the finite lattice box $C_N = [-N; N]^d$ in $\mathbb{Z}^d$. For a given configuration $\sigma \in \Sigma_N$, we will consider the Hamiltonian

$$-H_N(\sigma) = \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j)\in C_N} q\left(\frac{i-j}{N}\right) g_{(i,j)}\sigma_i\sigma_j + h\sum_{i\in C_N}\sigma_i, \tag{1.1}$$

where β stands for the inverse of the temperature of the system, $\hat{N} = 2N + 1$, (i, j) is the notation for a pair of sites $i, j \in C_N$ (taken only once), $\{g_{(i,j)} : (i, j) \in C_N\}$ is a family of IID standard centered Gaussian random variables, and h represents a constant

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positive external field, under which the spins tend to take the value $+1$. Our localization is represented by the function q, which can be thought of as a smooth frame, and which is only assumed to be defined on $[-1,1]^d$ such that q^2 is of positive type, so that q^2 is non-negative and invariant by symmetry about the origin. We also assume q^2 is a continuous function, including at its periodic boundary $-1 \equiv 1$. The aim of our article is then to study the limit, when $N \to \infty$, of the Gibbs measure $G_N(\sigma)$ defined on Σ_N by

$$G_N(\sigma) = \frac{e^{-H_N(\sigma)}}{Z_N}, \qquad \text{where} \qquad Z_N = Z_N(\beta) = \sum_{\sigma \in \Sigma_N} e^{-H_N(\sigma)},$$

and more specifically, we will concentrate on the so-called free energy of the system, defined by:

$$p(\beta) = \lim_{N\to\infty} \mathbf{E}[p_N(\beta)] = \text{a.s.} - \lim_{N\to\infty} p_N(\beta), \qquad \text{where} \qquad p_N(\beta) = \frac{1}{\hat{N}^d} \log(Z_N(\beta)). \quad (1.2)$$

The model described by (1.1) can be considered as a finite range approximation of the mean field SK model, associated with the Hamiltonian

$$-\hat{H}_N(\sigma) = \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j)\in C_N} g_{(i,j)}\sigma_i\sigma_j + h \sum_{i \in C_N} \sigma_i,$$

for which a large amount of information is now available [6, 9]. It seems then natural to try to approximate the realistic spin glass system, on which we have very little rigorous knowledge (see however [7]), by our localized model (1.1), capturing some of the geometry of the physical spin configuration, but still of a mean-field type in the limit $N \to \infty$. One could then hope to perform an expansion in N in order to quantify the difference between the original SK model and our model (1.1).

In fact, this kind of idea is not new, and goes back at least, in the spin glass context, to [3]. A version of our model with $h = 0$ has been studied then in [10], and more recently, the Kac limit of finite range spin glasses has been considered in [5, 2]. In these latter references, a slightly different point of view is adopted: the finite range model depends on a given parameter $\gamma > 0$, (which would be $1/N$ in our setting), and this localization parameter is sent to 0 *after* the thermodynamical limit in N is taken. It can be shown then, by some nice and soft interpolation arguments, that in the limit $\gamma \to 0$, the free energy of the localized system is the same as the free energy of the SK model, for any value of the parameter $\beta \geq 0$. Notice that the results contained in [5, 2] cannot be applied directly to our model, since in our case the limits $\gamma \to 0$ and $N \to \infty$ are taken at the same time. However, some slight modifications of the computations contained in these papers would also show that our quantity $p_N(\beta)$ defined at (1.2) behaves like the free energy of the SK model for large N.

The goal of our paper is then, in a sense, more modest than [5, 2], since we will only deal with the high temperature region of the model, i.e. small values of β. On the other hand, our scope is to show that the equivalence between the SK model and our localized

model still holds, in the thermodynamical limit, for a second order expansion of the free energy, that is in the central limit theorem regime. More specifically, we will show the following limit result: let γ_0 be the L^2-norm of q in $[-1,1]^d$. For $\beta, h > 0$, let also s be the unique solution to the equation

$$s = \mathbf{E}\left[\tanh^2(\beta z\sqrt{s} + h)\right], \tag{1.3}$$

and $\mathrm{SK}(\beta, h)$ be the function

$$\mathrm{SK}(\beta, h) = \beta^2(1-s)^2/4 + \log 2 + \mathbf{E}\left[\log\left[\cosh\left(\beta z\sqrt{s} + h\right)\right]\right], \tag{1.4}$$

which represents the free energy of the SK model in the high temperature region. Set also $p(\beta, h) = \mathrm{SK}(\gamma_0^{1/2}\beta, h)$. Then, under suitable conditions on q, we have

$$(\mathcal{L}) - \lim_{N\to\infty} \hat{N}^{d/2}\left[p_N(\beta) - p(\beta, h)\right] = Y,$$

where Y is a centered Gaussian random variable with variance $\tau = \tau(\beta, h, \gamma_0)$. The announced equivalence, at the CLT level, between the SK model and the localized one, springs then from the fact that $\tau(\beta, h, 1)$ is also the variance of the Gaussian random variable which shows up in the central limit theorem of the SK case (see [4, 8]).

Let us say a few words about the method we have used in order to get our result: since we are in the high temperature regime, we are allowed to use a cavity type method in order to compute the limit of the overlap of the localized spin system. This yields then the limit of the free energy in a straightforward manner. It has also been shown in [8] that the stochastic calculus tools developed in [1] could be adapted to the case of spin glasses with external field. This induces a powerful method for obtaining central limit theorems for the free energy, and interestingly enough, in this context, a dynamical point of view gives some insight on a static stochastic problem. We will elaborate here on this method in order to treat the localized case, by taking advantage systematically of the Fourier decomposition of q.

Our paper is divided as follows: at Section 2, we compute the simple limit of the overlap function and of $p_N(\beta)$, recovering the results obtained in [2] for the high temperature regime. At Section 3, we derive the announced central limit theorem thanks to stochastic calculus tools.

2 Simple limit of the free energy

Recall that we are dealing with the system induced by the Hamiltonian (1.1), and let us define some additional notations about Gibbs averages: let $f : \Sigma_N^n \to \mathbb{R}$ be a function of n configurations, with $n \geq 1$. Then we set

$$\rho(f) = \frac{1}{Z_N^n} \sum_{\sigma^1,\ldots,\sigma^n} f(\sigma^1, \ldots, \sigma^n) \exp\left(-\sum_{l=1}^n H_N(\sigma^l)\right), \quad \text{and} \quad \nu(f) = \mathbf{E}\left[\rho(f)\right]. \tag{2.1}$$

In the sequel of the paper, we will also write $\partial_u \varphi$ instead of $\partial\varphi/\partial u$ for the derivative of a function φ with respect to a parameter u, and $\bar{p}_N(\beta) = \mathbf{E}[p_N(\beta)]$. With these notations in hand, our strategy in order to get the limit of $p_N(\beta)$ will follow the classical steps of the cavity procedure, namely:

1. Find an expression for $\partial_\beta \bar{p}_N(\beta)$ in terms of an overlap-type function $R^{1,2}$.
2. For a general function $f : \Sigma_N^n \to \mathbb{R}$, find a useful expression for $\partial_v \rho_v(f)$ along a suitable path defined for $v \in [0,1]$, involving a Hamiltonian $H_{N,v}(\sigma)$.
3. Compute $\rho_v(R^{1,2})$ inductively and deduce an expression for $\partial_\beta \bar{p}_N(\beta)$, and then for $\bar{p}_N(\beta)$.

We will start by the first of these steps, for which we will introduce a little more notation: first of all, we assume for the moment the following basic hypothesis on q:

Hypothesis 2.1. *The function q is continuous on $[-1,1]^d$.*

In this case, the function $q^2 : [-1,1]^d \to \mathbb{R}_+$ can be decomposed, as a function of $L^2([-1,1]^d)$, into a Fourier series of the form:

$$q^2(x) = \sum_{k\in\mathbb{Z}^d} \gamma_k e^{\imath\pi k\cdot x}, \quad \text{with} \quad \sum_{k\in\mathbb{Z}^d} \gamma_k^2 < \infty \quad \text{and} \quad \Gamma \equiv \sum_{k\in\mathbb{Z}^d} \gamma_k < \infty. \tag{2.2}$$

Set also $R^{1,2} = \hat{N}^{-d} \sum_{i\in C_N} \sigma_i^1 \sigma_i^2$, and for any $k \in \mathbb{Z}^d$,

$$R_k^{1,2} = \frac{1}{\hat{N}^d} \sum_{i\in C_N} e^{\frac{\imath\pi i\cdot k}{N}} \sigma_i^1 \sigma_i^2, \quad \text{if } k \neq 0, \quad \text{and} \quad R_0^{1,2} = R^{1,2} - r, \tag{2.3}$$

where r is a positive constant, whose exact value will be determined later on. Eventually, we will denote by q_N the function $q(\cdot/N)$ defined on $[-N;N]^d$, and $\mathbb{Z}_*^d = \mathbb{Z}^d \backslash \{0\}$. Then the following relation holds true:

Proposition 2.2. *For all $\beta > 0$, we have*

$$\partial_\beta \bar{p}_N(\beta) = \frac{\beta}{\hat{N}^{2d}} \sum_{(i,j)\in C_N} q_N^2(i-j) - \frac{\beta}{2} \sum_{k\in\mathbb{Z}_*^d} \gamma_k \mathbf{E}\left[\rho(R_k^{1,2})\right] - \frac{\beta}{2}\gamma_0 \mathbf{E}\left[\rho(R^{1,2})\right] + \frac{\Gamma}{2\hat{N}^d}.$$

Proof. Set $B(\sigma) = e^{-H_N(\sigma)}$. Then the previous definitions and an elementary Gaussian integration by parts yield (see [9]):

$$\begin{aligned}
\partial_\beta \bar{p}_N(\beta) &= \frac{\beta}{\hat{N}^{2d}} \mathbf{E}\left[\sum_{(i,j)\in C_N} q_N^2(i-j) - \sum_{(i,j)\in C_N} \sum_{\sigma^1,\sigma^2} \frac{q_N^2(i-j)\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2 B(\sigma^1)B(\sigma^2)}{Z_N^2} \right] \\
&= \frac{\beta}{\hat{N}^{2d}} \left[\sum_{(i,j)\in C_N} q_N^2(i-j) - \sum_{(i,j)\in C_N} q_N^2(i-j) \mathbf{E}\left[\rho(\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2)\right] \right].
\end{aligned}$$

Hence, using the decomposition (2.2) of q^2, we obtain

$$\begin{aligned}\partial_\beta \bar{p}_N(\beta) &= \frac{\beta}{\hat{N}^{2d}}\mathbf{E}\left[\sum_{(i,j)\in C_N} q_N^2(i-j) - \sum_{(i,j)\in C_N}\rho\left(\sum_{k\in\mathbb{Z}^d}\gamma_k e^{\imath\pi\frac{i-j}{N}\cdot k}\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\right)\right]\\ &= \frac{\beta}{\hat{N}^{2d}}\mathbf{E}\Bigg[\sum_{(i,j)\in C_N} q_N^2(i-j) - \sum_{k\in\mathbb{Z}^d}\frac{\gamma_k}{2}\rho\left(\left|\sum_{i\in C_N} e^{\imath\pi\frac{i}{N}\cdot k}\sigma_i^1\sigma_i^2\right|^2\right)\\ &\qquad\qquad + \sum_{k\in\mathbb{Z}^d}\frac{\gamma_k}{2}\rho\left(\sum_{i\in C_N}\left|e^{\imath\pi\frac{i}{N}\cdot k}\sigma_i^1\sigma_i^2\right|^2\right)\Bigg],\end{aligned}$$

from which the desired expression is deduced easily.

□

We are now ready to start the second step of the strategy mentioned above.

2.1 The cavity method

The cavity method will consist here in suppressing in a continuous way the interactions between a certain site $m \in C_N$ and the remaining spins. This will be done in the following way: set $\hat{C}_N^m = \{i \in C_N; i \neq m\}$. We decompose, for all $\sigma \in \Sigma_N$,

$$-H_N(\sigma) = -H_N^m(\rho_m) + \sigma_m\left(h + g_m(\rho_m)\right),$$

where ρ_m denotes the ordered spin values except for the m-th spin, that is, the $(\hat{N}^d - 1)$-tuple $(\sigma_i; i \in \hat{C}_N^m)$ and we denote

$$g_m(\rho_m) = \frac{\beta}{\hat{N}^{d/2}}\sum_{i\in\hat{C}_N^m} q_N(m-i)g_{(m,i)}\sigma_i,$$

and we also use the notation

$$-H_N^m(\rho_m) = \sum_{(i,j)\in\hat{C}_N^m} q_N(i-j)g_{(i,j)}\sigma_i\sigma_j + h\sum_{i\in\hat{C}_N^m}\sigma_i.$$

This new Hamiltonian is similar but not identical to the Hamiltonian $-H_{N-1}$ on Σ_{N-1}.

The path we will build is now of the following form: consider a collection $\{B_{i,m}(v); i \in \hat{C}_N^m, v \in [0,1]\}$ of independent standard Brownian motions, and $\{X(v); v \in [0,1]\}$ an independent reversed time Brownian motion, all defined on the probability space $(\Omega, \mathcal{F}, P)$. Notice that $g_{(i,m)}$ can be seen as the final value $B_{i,m}(1)$ of the Brownian motion $B_{i,m}$, and that X is the solution to a stochastic differential equation of the form

$$X(v) = \eta - \int_0^v \frac{X(s)}{1-s}ds + W(v), \quad v \in [0,1], \tag{2.4}$$

where η is a standard Gaussian random variable and W another Brownian motion, independent of the remainder of the randomness. Notice that, for notational sake, we have written $B_{i,m}$ instead of $B_{(i,m)}$. Set then $\hat{r} = \gamma_0^{1/2} r$ and for $v \in [0,1]$, define

$$-H_{N,v}(\sigma) = -H_N^m(\rho_m) + \frac{\beta}{\hat{N}^{d/2}} \sum_{i \in \hat{C}_N^m} q_N(m-i)\sigma_i\sigma_m B_{i,m}(v) + \beta\hat{r}^{1/2} \sum_{i \in \hat{C}_N^m} \sigma_i X(t) + h \sum_{i \in \hat{C}_N^m} \sigma_i.$$

For $f : \Sigma_N^n \to \mathbb{R}$, denote also by $\rho_v(f)$ the associated Gibbs average, defined in a similar way to (2.1), and $\nu_{m,v}(f) = \mathbf{E}[\rho_v(f)]$.

Recall now the following elementary lemma from [8]:

Lemma 2.3. *For $k \geq 1$, let $\{B_l;\, l \leq k\}$ be a collection of independent standard Brownian motions. Let also X be the solution to (2.4), and $\varphi : \mathbb{R}^{k+1} \to \mathbb{R}$ be a C^2 function having at most exponential growth together with its first two derivatives. Then, for any $v \in [0,1]$,*

$$\begin{aligned}
&\mathbf{E}\left[\varphi(B_1(v), \ldots, B_k(v), X(v))\right] \\
&= \mathbf{E}\left[\varphi(\eta)\right] + \frac{1}{2}\int_0^v \sum_{l=1}^k \mathbf{E}\left[\partial^2_{x_l x_l}\varphi(B_1(s), \ldots, B_k(s), X(s))\right] ds \\
&\qquad - \frac{1}{2}\int_0^v \mathbf{E}\left[\partial^2_{x_{k+1}x_{k+1}}\varphi(B_1(s), \ldots, B_k(s), X(s))\right] ds.
\end{aligned}$$

These preliminary tools yield the following differentiation rule:

Proposition 2.4. *For any $f : \Sigma_N^n \to \mathbb{R}$ and $v \in [0,1]$, the derivative of $\nu_{m,v}(f)$ is given by*

$$\begin{aligned}
\partial_v \nu_{m,v}(f) = \beta^2 \sum_{k \in \mathbb{Z}^d} \gamma_k e^{\imath\pi m\cdot k/N} \Bigg(&\sum_{1 \leq l < l' \leq n} \nu_{m,v}\left(f \bar{R}_k^{l,l'} \sigma_m^{l'} \sigma_m^l\right) \\
&- n\sum_{l=1}^n \nu_{m,v}\left(f \bar{R}_k^{l,n+1} \sigma_m^l \sigma_m^{n+1}\right) + \frac{n(n+1)}{2} \nu_{m,v}\left(f \bar{R}_k^{n+1,n+2} \sigma_m^{n+1} \sigma_m^{n+2}\right) \Bigg). \quad (2.5)
\end{aligned}$$

Proof. This result stems from an easy application of Lemma 2.3 to the function

$$\varphi\left(B_{i,m}(v), i \in C_N^m;\, X_m(v)\right) \equiv \rho_{m,v}(f).$$

The computations of the second derivatives of φ are a matter of easy (though cumbersome) calculations, and are left to the reader for sake of conciseness.

□

Now that the variations of $\nu_{m,v}(f)$ have been computed, we can proceed to get some bounds on the overlaps $R_k^{1,2}$.

2.2 Bounds on the overlap

Let us start with three lemmas whose proofs follow essentially that of the corresponding result in the cavity method for the standard Sherrington-Kirkpatrick model, and which will be combined with the explicit expression for $\nu_{m,v}$ in Proposition 2.4 and a separate calculation of $\nu_{m,v}$ for $v = 0$ to obtain information on the actual expected overlaps, under $\nu = \nu_{m,1}$, i.e. for $v = 1$.

Lemma 2.5. *There exist two positive constants $c_{n,q,\beta}$ and $c'_{n,q,\beta}$ that depend only on n, q and β, and are uniformly bounded in β for $\beta \in [0,1]$, such that if f is a positive function on Σ_N^n then for all $m \in C_N$,*

$$\nu_{m,v}(f) \leq c_{n,q,\beta}\, \nu(f), \tag{2.6}$$

and

$$|\nu_{m,v}(f) - \nu_{m,0}(f)| \leq c'_{n,q,\beta} \beta^2 \nu^{1/2}\left(|f|^2\right) \left[\sum_{k\in\mathbb{Z}^d} \gamma_k \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right]. \tag{2.7}$$

Proof. See [9, Propositions 2.4.6 and 2.4.7]. □

For $v = 0$, the expression of $\nu_v(f)$ can be simplified for a large class of functions f on Σ_N.

Lemma 2.6. *For fixed $m \in \Sigma_N$, let f be a function on Σ_N^n that does not depend on the values $\sigma_m^1, \sigma_m^2, \ldots, \sigma_m^n$. Then for any subset I of $\{1, \ldots, n\}$ we have*

$$\nu_{m,0}\left(f \prod_{l\in I} \sigma_m^l\right) = \mathbf{E}\left[\tanh(Y)^{|I|}\right] \nu_{m,0}(f),$$

where Y is the Gaussian random variable defined as:

$$Y = \beta z \sqrt{\gamma_0 r} + h,$$

with a standard normal variable z.

Proof. See [9, Lemma 2.4.4]. □

On the other hand, some symmetry properties for $v = 1$ yield the following kind of estimate:

Lemma 2.7. *Set*

$$\delta_v \equiv \sum_{k\in\mathbb{Z}^d} \gamma_k \nu_v\left(|R_k^{1,2}|^2\right), \quad \text{for} \quad v \in [0,1]. \tag{2.8}$$

Then

$$\delta_1 = \gamma_0 \nu\left(\left(R^{1,2} - r\right)\left(\sigma_m^1 \sigma_m^2 - r\right)\right) + \frac{1}{\hat{N}^d} \sum_{k\in\mathbb{Z}_*^d} \sum_{i\in C_N} \gamma_k e^{\imath\pi i\cdot k/N} \nu\left(\bar{R}_k^{1,2} \sigma_i^1 \sigma_i^2\right).$$

Now, in order to exploit Lemma 2.6, we must modify the above expression for δ_1 by completing the following two tasks:

(i) estimate the error made by replacing the arguments of $\nu_{i,0}$ by functions that are of the same form as those in Lemma 2.6;

(ii) estimate the error made by replacing ν by $\nu_{m,0}$ (or $\nu_{i,0}$ as appropriate).

2.2.1 Task (i). Separation of cavity variable from others

It is sufficient to replace $R^{1,2}$ by the same quantity with the m-th term omitted: define

$$R^{1,2}_{-,m} := \frac{1}{\hat{N}^d} \sum_{i \in \hat{C}^m_N} \sigma^1_i \sigma^2_i = R^{1,2} - \frac{1}{\hat{N}^d} \sigma^1_m \sigma^2_m = R^{1,2} + O\left(\frac{1}{\hat{N}^d}\right),$$

and similarly let

$$\begin{aligned} R^{1,2}_{k,-,m} &:= \frac{1}{\hat{N}^d} \sum_{i \in \hat{C}^m_N} \sigma^1_i \sigma^2_i e^{\imath \pi i \cdot k/N} \\ &= R^{1,2}_k - \frac{1}{\hat{N}^d} \sigma^1_m \sigma^2_m e^{\imath \pi m \cdot k/N} = R^{1,2}_k + O\left(\frac{1}{\hat{N}^d}\right). \end{aligned}$$

Then we have the following relation, whose elementary proof is omitted:

Lemma 2.8. *For any $v \in [0,1]$ and δ_v defined at (2.8), it holds that:*

$$\begin{aligned} \delta_0 = \gamma_0 \nu_{m,0} \left(\left(R^{1,2}_{-,m} - r \right) \left(\sigma^1_m \sigma^2_m - r \right) \right) \\ + \sum_{k \in \mathbb{Z}^d_*} \frac{\gamma_k}{\hat{N}^d} \sum_{i \in C_N} \nu_{i,0} \left(\bar{R}^{1,2}_{k,-,i} \sigma^1_i \sigma^2_i \right) e^{\imath \pi i \cdot k/N} + O\left(\hat{N}^{-d} \right). \end{aligned}$$

Thanks to a Lemma by Lattala and Guerra, we can now choose r in order to eliminate one of the terms in our overlap calculation δ_0 (at $v = 0$ with separated spins), as an immediate consequence of Lemma 2.6.

Lemma 2.9. *For any choice of the parameters $\beta, \gamma_0, h > 0$, the equation*

$$r = \mathbf{E}\left[\tanh^2 \left(\beta z \sqrt{\gamma_0 r} + h \right) \right] \tag{2.9}$$

has a unique solution $r \in [0,1]$, and we have

$$\nu_{m,0} \left(\left(R^{1,2}_{-,m} - r \right) \left(\sigma^1_m \sigma^2_m - r \right) \right) = 0.$$

Let us take advantage of this relation, and try to write δ_0 in terms of r: going back to the expression in Lemma 2.8, we get

$$\begin{aligned} \delta_0 &= \sum_{k \in \mathbb{Z}^d_*} \frac{\gamma_k}{\hat{N}^d} \sum_{i \in C_N} e^{\imath \pi i \cdot k/N} \nu_{i,0} \left(\bar{R}^{1,2}_{k,-,i} \sigma^1_i \sigma^2_i \right) + O\left(\frac{1}{N^d} \right) \\ &= r \sum_{k \in \mathbb{Z}^d_*} \gamma_k \left(A^1_k + A^2_k + A^3_k \right) + O\left(\frac{1}{N^d} \right), \end{aligned} \tag{2.10}$$

with

$$A_k^1 := S_k\, \nu\left(\bar{R}_k^{1,2}\right), \qquad A_k^2 := \frac{1}{\hat{N}^d} \sum_{i \in C_N} \nu_{i,0}\left(\bar{R}_{k,-,i}^{1,2} - \bar{R}_k^{1,2}\right) e^{\imath\pi i\cdot k/N}$$

$$A_k^3 := \frac{1}{\hat{N}^d} \sum_{i \in C_N} \left[\nu_{i,0}\left(\bar{R}_k^{1,2}\right) - \nu\left(\bar{R}_k^{1,2}\right)\right] e^{\imath\pi i\cdot k/N},$$

where we have set

$$S_k = \frac{1}{\hat{N}^d} \sum_{i \in C_N} e^{\imath\pi i\cdot k/N}. \tag{2.11}$$

We can bound now δ_0 in the following way:

Lemma 2.10. *Recall that Γ has been defined at (2.2). Then there exists a constant κ that depends on β and q but is bounded for β bounded such that*

$$|\delta_0| \leq \kappa \left(\frac{\Gamma}{N} + \beta^2 \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right) \left[\sum_{k \in \mathbb{Z}^d} \gamma_k \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right]\right). \tag{2.12}$$

Proof. Go back to relation (2.10), and let us bound the terms A_1^k, A_2^k, A_3^k for $k \in \mathbb{Z}^d$. First of all, the estimation of A_1^k is controlled by S_k. However,

$$S_k = \frac{1}{\hat{N}^d} \sum_{j_1=-N}^{N} \cdots \sum_{j_d=-N}^{N} \prod_{l=1}^{d} e^{\imath\pi k_l j_l/N} = \frac{1}{\hat{N}^d} \prod_{l=1}^{d} \left[(-1)^{k_l} \mathbf{1}_{(k_l \neq 0)} + \hat{N} \mathbf{1}_{(k_l=0)}\right],$$

by an elementary argument on sums of geometric sequences. Hence,

$$|S_k| \leq \frac{\kappa}{N^{Z(k)}}, \quad \text{where} \quad Z(k) = \text{number of components of } k \text{ that are non-zero}, \tag{2.13}$$

and thus,

$$\Big| \sum_{k \in \mathbb{Z}_*^d} \gamma_k A_k^1 \Big| = \Big| \nu\left(\bar{R}_k^{1,2}\right) \sum_{k \in \mathbb{Z}_*^d} \gamma_k S_k \Big| \leq \frac{\kappa\Gamma}{N}. \tag{2.14}$$

Furthermore, it is easily checked that

$$|A_k^2| \leq \frac{\kappa}{N} \quad \text{and} \quad |A_k^3| \leq \kappa\beta^2 \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right) \left[\sum_{k \in \mathbb{Z}^d} \gamma_k \nu^{1/2}\left(\left|R_k^{1,2}\right|^2\right)\right], \tag{2.15}$$

by applying inequality (2.7) in Lemma 2.5. Our claim is then proved easily by putting together (2.14) and (2.15).

□

Remark 2.11. *In the remainder of the article, κ will stand for a positive constant depending on β and q and that is uniformly bounded in the range of our parameter β; we will allow κ to change from line to line*

Notice that, for our result on the fluctuations of Z_N, we will need an improved bound on A_k^1. This can be achieved under the following additional condition:

Hypothesis 2.12. *Going back to the decomposition (2.2), we assume that there exists an integer $\hat{d} > d/2$ such that, for every $k \in \mathbb{Z}_*^d$ such that $\gamma_k \neq 0$, the number $Z(k)$ of components of k that are non-zero satisfies $Z(k) \geq \hat{d}$.*

Then Lemma 2.10 can be enhanced in the following way:

Corollary 2.13. *Assume q satisfies Hypothesis 2.1 and 2.12. Then*

$$|\delta_0| \leq \kappa \left(\frac{\Gamma}{N^{\hat{d}}} + \beta^2 \nu^{1/2} \left(\left| R_k^{1,2} \right|^2 \right) \left[\sum_{k \in \mathbb{Z}^d} \gamma_k \nu^{1/2} \left(\left| R_k^{1,2} \right|^2 \right) \right] \right).$$

Proof. This is trivially checked by going through the computations of Lemma 2.10 again, and taking into account (2.13).

□

2.2.2 Task (ii). Difference between the overlaps at $v = 0$ and $v = 1$

We are ready to state and prove the result which completes task (ii).

Lemma 2.14. *There exists a constant κ that depends on β and q but is bounded for β bounded, such that*

$$|\delta_1 - \delta_0| \leq +\kappa \beta^2 \Gamma \sum_{k \in \mathbb{Z}^d} \gamma_k \nu \left(\left| R_k^{1,2} \right|^2 \right). \tag{2.16}$$

Proof. We can first write, using Lemma 2.5,

$$\begin{aligned} &\sum_{k \in \mathbb{Z}_*^d} \gamma_k \left| \frac{1}{\hat{N}^d} \sum_{i \in C_N} \left[\nu \left(\bar{R}_k^{1,2} \sigma_i^1 \sigma_i^2 \right) - \nu_{i,0} \left(\bar{R}_k^{1,2} \sigma_i^1 \sigma_i^2 \right) \right] e^{\imath \pi i \cdot k/N} \right| \\ &\leq \kappa \beta^2 \sum_{k \in \mathbb{Z}_*^d} \gamma_k \nu^{1/2} \left(\left| R_k^{1,2} \right|^2 \right) \left[\sum_{k \in \mathbb{Z}^d} \gamma_k \nu^{1/2} \left(\left| R_k^{1,2} \right|^2 \right) \right]. \end{aligned} \tag{2.17}$$

Similarly we can obtain

$$\begin{aligned} &\gamma_0 \left| \nu \left(\left(R^{1,2} - r \right) \left(\sigma_m^1 \sigma_m^2 - r \right) \right) - \nu_{i,0} \left(\left(R^{1,2} - r \right) \left(\sigma_m^1 \sigma_m^2 - r \right) \right) \right| \\ &\leq \gamma_0 \kappa \beta^2 \nu^{1/2} \left(\left| R_0^{1,2} \right|^2 \right) \left[\sum_{k \in \mathbb{Z}^d} \gamma_k \nu^{1/2} \left(\left| R_k^{1,2} \right|^2 \right) \right]. \end{aligned} \tag{2.18}$$

We now get, by putting together (2.17) and (2.18) and applying Jensen's inequality, that

$$|\delta_1 - \delta_0| = \kappa \beta^2 \left[\sum_{k \in \mathbb{Z}^d} \gamma_k \nu^{1/2} \left(\left| R_k^{1,2} \right|^2 \right) \right]^2 \leq \kappa \beta^2 \Gamma \sum_{k \in \mathbb{Z}^d} \gamma_k \nu \left(\left| R_k^{1,2} \right|^2 \right),$$

which proves the lemma.

□

2.2.3 Self-averaging overlap limit

We are now in a position to estimate δ_1. We show that for small β, this expected total overlap, which is recentered using the value r, converges to 0 at the speed $1/N$ as long as q is a continuous function.

Proposition 2.15. *Let $\beta > 0$ and let $r = r(\beta)$ be the solution of (2.9). Let κ be the constant defined in Lemma 2.8 and Lemma 2.14, i.e. κ is a constant that depends on β and q but is bounded for β bounded. Assume that q satisfies Hypothesis 2.1, and that β is so small that $2\kappa\beta^2\Gamma < 1$. In that case, we have, for N large enough, with $R_k^{1,2}$ defined by relation (2.3),*

$$0 \leq \nu\left(\sum_{k\in\mathbb{Z}^d} \gamma_k \left|R_k^{1,2}\right|^2\right) \leq \frac{r\Gamma}{(1-2\kappa\beta^2\Gamma)N}.$$

Proof. We have, using (2.12) and (2.16),

$$\begin{aligned} 0 \leq \delta_1 = \nu\left(\sum_{k\in\mathbb{Z}^d} \gamma_k \left|R_k^{1,2}\right|^2\right) &\leq |\delta_0 - \delta_1| + r|\delta_0| \\ &\leq 2\kappa\beta^2\Gamma\nu\left(\sum_{k\in\mathbb{Z}^d} \gamma_k \left|R_k^{1,2}\right|^2\right) + O\left(\frac{1}{N}\right), \end{aligned} \tag{2.19}$$

where we recall that $O(N^{-1})$ is a function that tends to zero as fast as N^{-1} and that this convergence holds uniformly in all parameters. Moreover, since κ is bounded for β bounded, for β sufficiently small we can make $2\kappa\beta^2\Gamma$ smaller than 1. The result of the proposition follows.

□

Corollary 2.16. *Under the same assumptions as in Proposition 2.15, but assuming additionally that condition 2.12 holds true, then the conclusion of Proposition 2.15 holds with N replaced by $N^{\hat{d}}$.*

Proof. This follows trivially from the proof of Proposition 2.15 if we modify the argument in order to take into account Corollary 2.13.

□

2.3 Consequence for the partition function

We can now apply the previous computations in order to get the simple limit of $\bar{p}_N(\beta)$, which recovers, in the high temperature region, the results contained in [2].

Theorem 2.17. *Under the hypotheses of Proposition 2.15, we have*

$$\left|\bar{p}_N(\beta) - SK\left(\gamma_0^{1/2}\beta, h\right)\right| \leq \frac{C(\beta)}{N}, \tag{2.20}$$

where the constant C depends on h, q, and β, and is bounded for $\beta \in [0, \beta_0]$.

Proof. Recall from Proposition 2.2 that

$$\partial_\beta \bar{p}_N = \frac{\beta}{\hat{N}^{2d}} \sum_{(i,j)\in C_N} q_N^2(i-j) - \frac{\beta}{2} \sum_{k\in\mathbb{Z}_*^d} \gamma_k \nu\left(\left|R_k^{1,2}\right|^2\right) - \frac{\beta}{2}\gamma_0 \nu\left(\left|R_0^{1,2}+r\right|^2\right) + \frac{\Gamma}{2\hat{N}^d}. \quad (2.21)$$

The first term on the right-hand side can be handled easily: indeed, we have

$$\begin{aligned}
\frac{1}{\hat{N}^{2d}} \sum_{(i,j)\in C_N} q_N^2(i-j) &= \frac{1}{\hat{N}^{2d}} \sum_{(i,j)\in C_N} \sum_{k\in\mathbb{Z}^d} \gamma_k e^{\imath\pi k\cdot i/N} e^{-\imath\pi k\cdot j/N} \\
&= \frac{1}{2\hat{N}^{2d}} \sum_{i\neq j} \sum_{k\in\mathbb{Z}^d} \gamma_k e^{\imath\pi k\cdot i/N} e^{-\imath\pi k\cdot j/N} \\
&= \frac{1}{2}\left(\gamma_0 + \sum_{k\in\mathbb{Z}_*^d} \gamma_k |S_k|^2 - \frac{\Gamma}{\hat{N}^d}\right),
\end{aligned}$$

where S_k has been defined at (2.11). Thus,

$$\left|\frac{\beta}{\hat{N}^{2d}} \sum_{(i,j)\in C_N} q_N^2(i-j) - \frac{\beta\gamma_0}{2}\right| \leq \frac{\kappa}{N},$$

and Proposition 2.15 then easily yields, for $\beta < \beta_0$,

$$\left|\partial_\beta \bar{p}_N - \frac{\beta\gamma_0}{2} - \frac{r^2\gamma_0\beta}{2}\right| \leq \frac{\kappa}{N} + \gamma_0 r\beta \left|\nu\left(R_0^{1,2}\right)\right|. \quad (2.22)$$

Furthermore, it can be shown, along the same lines as in [9], that for $\beta < \beta_0$, we have

$$\left|\nu\left(R_0^{1,2}\right)\right| \leq K(\beta, q)/N,$$

where $K(\beta, q)$ is bounded for β bounded. This inequality and (2.22) now imply

$$\left|\partial_\beta \bar{p}_N - \frac{\gamma_0\beta}{2}\left(1-r^2\right)\right| \leq \frac{\kappa}{N}$$

where κ depends only on β, h, q and is bounded for $\beta \in [0, \beta_0]$. The theorem follows by integrating $\partial_\beta \bar{p}_N$, and using trivial calculations and known facts about the function SK. □

The final result we present in this section shows that while the complete structure of q does not seem to effect the limiting behavior of the partition function beyond the average value γ_0 of q^2, the speed of convergence towards this value may depend heavily on the behavior of q. We show that the speed can be increased to the order $N^{-\hat{d}}$ as long as the hypotheses of Corollary 2.16 hold.

Corollary 2.18. *Under the hypotheses of Corollary 2.16, we have*

$$\left|\bar{p}_N(\beta) - SK\left(\gamma_0^{1/2}\beta, h\right)\right| \leq \frac{\beta C(\beta)}{N^{\hat{d}}}$$

where the constant C depends on h, q, and β, and is bounded for $\beta \in [0, \beta_0]$.

Proof. In the proof of Theorem 2.17, some estimates are already of order N^{-d}. For the others, we may use Corollary 2.16 instead of Proposition 2.15 in all its occurrences. This improves all estimates that were originally of order N^{-1} to the order $N^{-\hat{d}}$, with the exception of the estimation of the first term on the right-hand side of (2.21). But here again, one can easily check that this term is of order $N^{-\hat{d}}$ under the conditions of Corollary 2.16.

□

3 Fluctuations of the free energy

In this section, we will turn to our main aim, that is the central limit theorem governing the fluctuations of Z_N. More specifically, we will get the following:

Theorem 3.1. *For β small enough, $t \in [0,1]$, if q satisfies Hypothesis 2.1 and 2.12, then*

$$(\mathcal{L}) - \lim_{N\to\infty} \hat{N}^{d/2} \left[p_N(\beta) - p(\beta,h)\right] = Y,$$

where Y is a centered Gaussian random variable with variance τ, and τ is given by

$$\tau = \hat{\tau} - \frac{\beta^2\gamma_0 r^2}{2}, \qquad \text{where} \qquad \hat{\tau} = \mathrm{Var}\left[\log(\cosh(\beta\sqrt{\gamma_0 r}z + h))\right] \tag{3.1}$$

with r defined by (2.9), and a standard Gaussian random variable z.

This kind of result is usually obtained by letting all the interactions between spins tend to 0 at once, and this procedure can be somewhat simplified by considering the computations from a stochastic calculus point of view. This leads us to consider a new path $t \mapsto H_{N,t}(\sigma)$ defined for $t \in [0,1]$ by

$$-H_{N,t}(\sigma) = \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j)\in C_N} B_{i,j}(t) q_N(i-j)\sigma_i\sigma_j + \beta\hat{r}^{1/2} \sum_{i\in C_N} X_i(t)\sigma_i + \sum_{i\in C_N} h\sigma_i, \tag{3.2}$$

where $\hat{r} = \gamma_0 r$ and $\{B_{i,j}; (i,j) \in C_N\}$ is again a collection of independent standard Brownian motions, and $\{X_i; i \in C_N\}$ is a family of independent reversed time Brownian motion, which can be seen as the solution to some stochastic differential equations of the form (2.4), for a family $\{\eta_i; i \in C_N\}$ (resp. $\{W_i; i \in C_N\}$) of standard Gaussian random variables (resp. of independent Brownian motions). Here again, we will assume that all these objects are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Notice that, in order to spare notations, we have called this modified Hamiltonian $H_{N,t}$ again, like in Section 2, hoping that this won't lead to any confusion. We will also denote by $Z_N(t)$ and $\rho_t(f)$ the partition function and the Gibbs average associated with $H_{N,t}$. Eventually, for $t \in [0,1]$, we define

$$p_{\beta,h,t} = \frac{\beta^2\gamma_0 t}{4}(1-r)^2 + \log(2) + \mathbf{E}\left[\log(\cosh(\beta\sqrt{\gamma_0 r}z + h))\right],$$

and we observe that $p_{\beta,h,1} = p(\beta,h)$. Once this path has been defined, the strategy which leads to Theorem 3.1 can be summarized as follows: apply Itô's formula in order to:

1. Compute the variations of $t \mapsto e^{-H_{N,t}(\sigma)}$.
2. Get an equation for the evolution of $Y_N(t) \equiv \hat{N}^{d/2}[p_N(\beta) - p_{\beta,h,t}]$.
3. Find a limit for $\mathbf{E}[e^{iuY_N(t)}]$ for any $u \in \mathbb{R}$.

We will detail now this global strategy.

3.1 Preliminary computations

Let us first study the dynamics of $t \mapsto e^{-H_{N,t}(\sigma)}$:

Proposition 3.2. *For $t \in [0,1]$ and $\sigma \in \Sigma_N$, we have*

$$e^{-H_{N,t}(\sigma)} = \exp\left(\sum_{i \in C_N} \sigma_i(\beta\hat{r}^{1/2}\eta_i + h)\right) + F_N^1(t) + F_N^2(t),$$

with

$$\begin{aligned} F_N^1(t) &= \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j) \in C_N} q_N(i-j)\sigma_i\sigma_j \int_0^t e^{-H_{N,s}(\sigma)} dB_{i,j}(s) \\ &\quad + \beta\hat{r}^{1/2} \sum_{i \in C_N} \sigma_i \int_0^t e^{-H_{N,s}(\sigma)} dW_i(s), \end{aligned}$$

and

$$\begin{aligned} F_N^2(t) &= -\beta\hat{r}^{1/2} \sum_{i \in C_N} \sigma_i \int_0^t e^{-H_{N,s}(\sigma)} \frac{X_i}{1-s} ds \\ &\quad + \frac{\beta^2}{2}\left(\frac{1}{\hat{N}^d} \sum_{(i,j) \in C_N} q_N^2(i-j) + \hat{r}\hat{N}^d\right) \int_0^t e^{-H_{N,s}(\sigma)} ds. \end{aligned}$$

Proof. The exponential function being a C^2 function with a nicely controlled growth, we can apply Ito's formula and we obtain

$$e^{-H_{N,t}(\sigma)} = e^{-H_{N,0}(\sigma)} - \int_0^t e^{-H_{N,s}(\sigma)} dH_{N,s}(\sigma) + \frac{1}{2}\int_0^t e^{-H_{N,s}(\sigma)} d\langle H_{N,\cdot}\rangle_s,$$

where $\langle M \rangle_t$ denotes the quadratic variation process of a semi-martingale M. Now, we can evaluate the quantity $\langle H_{N,\cdot}\rangle_s$, since all of the $B_{i,j}$, W_i are independent and using the fact that any finite variation process have a null quadratic variation. We get

$$\begin{aligned} \langle H_{N,\cdot}\rangle_t &= \frac{\beta^2}{\hat{N}^d} \sum_{(i,j) \in C_N} q_N^2(i-j)(\sigma_i\sigma_j)^2 t + \beta^2\hat{r} \sum_{i \in C_N} \sigma_i^2 t \\ &= \frac{\beta^2}{\hat{N}^d} \sum_{(i,j) \in C_N} q_N^2(i-j)t + \beta^2\hat{r}\hat{N}^d t, \end{aligned}$$

from which our claim is easily shown.

□

We will turn now to the fluctuations of $p_N(\beta)$. Namely set, for $t \in [0,1]$,

$$Y_N(t) = \hat{N}^{d/2}\left(\frac{1}{\hat{N}^d}\log(Z_N(t)) - p_{\beta,h,t}\right), \tag{3.3}$$

and define also the function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(x) = \log(\cosh(\beta\hat{r}^{1/2}x + h)).$$

Then the semi-martingale Y_N can be decomposed in the following way:

Proposition 3.3. *Recall that $\Gamma = \sum_{k\in\mathbb{Z}^d}\gamma_k$. Then, for $t \in [0,1]$, $Y_N(t)$ satisfies:*

$$Y_N(t) = U_N + \sum_{l\leq 2} M_{l,N}(t) - (V_{1,N}(t) - V_{2,N}(t)) + V_{3,N}(t),$$

where the random variable U_N and the processes $M_{l,N}$ and $V_{k,N}$ are defined by:

$$\begin{aligned}
U_N &= \hat{N}^{d/2}\left(\frac{1}{\hat{N}^d}\sum_{i\in C_N}\Phi(\eta_i) - \mathbf{E}\Phi(z)\right)\\
M_{1,N}(t) &= \frac{\beta}{\hat{N}^d}\sum_{(i,j)\in C_N} q_N(i-j)\int_0^t \rho_s(\sigma_i\sigma_j)dB_{i,j}(s)\\
M_{2,N}(t) &= \frac{\beta\hat{r}^{1/2}}{\hat{N}^{d/2}}\sum_{i\in C_N}\int_0^t \rho_s(\sigma_i)dW_i(s)\\
V_{1,N}(t) &= \frac{\beta\hat{r}^{1/2}}{\hat{N}^{d/2}}\sum_{i\in C_N}\int_0^t \rho_s(\sigma_i)\frac{X_i(s)}{1-s}ds\\
V_{2,N}(t) &= \frac{\beta^2\hat{r}}{\hat{N}^{d/2}}\sum_{i\in C_N}\int_0^t \rho_s(1-\sigma_i^1\sigma_i^2)ds\\
V_{3,N}(t) &= \left[\frac{\Gamma}{\hat{N}^{d/2}} + \frac{2}{\hat{N}^{3d/2}}\sum_{(i,j)\in C_N} q_N^2(i-j) - \gamma_0\hat{N}^{d/2}\right]\frac{\beta^2 t}{4}\\
&\quad - \frac{\hat{N}^{d/2}\beta^2}{4}\sum_{k\in\mathbb{Z}^d}\gamma_k\int_0^t \rho_s\left(\left|R_k^{1,2}\right|^2\right)ds.
\end{aligned}$$

Proof. Notice that $Z_N(t)$ is almost surely a strictly positive random variable. Thus, Itô's formula can be applied to $\log(Z_N(t))$, and we obtain

$$\log(Z_N(t)) = \log(Z_N(0)) + \int_0^t \frac{dZ_N(s)}{Z_N(s)} - \frac{1}{2}\int_0^t \frac{d\langle Z_N\rangle_s}{Z_N^2(s)}. \tag{3.4}$$

The first two terms in the right hand side of (3.4) are easily computed. Indeed, it is easily checked that

$$\log(Z_N(0)) = \hat{N}^d\log 2 + \sum_{i\in C_N}\log[\cosh(\beta\hat{r}^{1/2}\eta_i + h)]. \tag{3.5}$$

Furthermore, invoking the fact that $Z_N(t) = \sum_{\sigma\in\Sigma_N} e^{-H_{N,t}(\sigma)}$ and Proposition 3.2, we have

$$\int_0^t \frac{dZ_N(s)}{Z_N(s)} = \frac{\beta}{\hat{N}^{d/2}} \sum_{(i,j)\in C_N} q_N(i-j) \int_0^t \rho_s(\sigma_i\sigma_j) dB_{i,j}(s) + \beta\hat{r}^{1/2} \sum_{i\in C_N} \int_0^t \rho_s(\sigma_i) dW_i(s)$$
$$- \beta\hat{r}^{1/2} \sum_{i\in C_N} \int_0^t \rho_s(\sigma_i) \frac{X_i(s)}{1-s} ds + \left(\frac{\beta^2}{2\hat{N}^d} \sum_{(i,j)\in C_N} q_N^2(i-j) + \frac{\beta^2\hat{r}\hat{N}^d}{2} \right) t. \quad (3.6)$$

Thus, putting together (3.4), (3.5) and (3.6), we have obtained that

$$Y_N(t) = U_N + M_{1,N}(t) + M_{2,N}(t) - V_{1,N}(t) + \tilde{V}_{2,N}(t) - \frac{1}{2\hat{N}^{d/2}} \int_0^t \frac{d\langle Z_N\rangle_s}{Z_N^2(s)}, \quad (3.7)$$

where

$$\tilde{V}_{2,N}(t) = \left(\frac{\beta^2}{2\hat{N}^{3d/2}} \sum_{(i,j)\in C_N} q_N^2(i-j) + \frac{\beta^2\hat{r}\hat{N}^{d/2}}{2} \right) t - \frac{\beta^2\gamma_0\hat{N}^{d/2}t}{4} (1-r)^2 .$$

Let us compute now the term $d\langle Z_N\rangle_s/Z_N^2(s)$: according to Proposition 3.2, $Z_N(t)$ is a continuous semi-martingale, whose martingale part is

$$\begin{aligned} \hat{M}_N(t) &= \frac{\beta}{\hat{N}^{d/2}} \sum_{\sigma\in\Sigma_N} \sum_{(i,j)\in C_N} q_N(i-j)\sigma_i\sigma_j \int_0^t e^{-H_N(s)} dB_{i,j}(s) \\ &\quad + \beta\hat{r}^{1/2} \sum_{\sigma\in\Sigma_N} \sum_{i\in C_N} \sigma_i \int_0^t e^{-H_N(s)} dW_i(s). \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^t \frac{1}{Z_N^2(s)} d\langle Z_N\rangle_s &= \int_0^t \frac{1}{Z_N^2(s)} d\langle \hat{M}_N\rangle_s \\ &= \beta^2\hat{r} \sum_{i\in C_N} \int_0^t \rho_s(\sigma_i^1\sigma_i^2) ds + \frac{\beta^2}{\hat{N}^d} \sum_{(i,j)\in C_N} q_N^2(i-j) \int_0^t \rho_s\left(\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\right) ds. \end{aligned}$$

Recall now that $q^2(x)$ can be decomposed into $q^2(x) = \sum_{k\in\mathbb{Z}^d} \gamma_k e^{\imath\pi k\cdot x}$. Thus,

$$\int_0^t \frac{d\langle Z_N\rangle_s}{Z_N^2(s)} = \frac{\beta^2}{\hat{N}^d} \sum_{(i,j)\in C_N} \sum_{k\in\mathbb{Z}^d} \gamma_k e^{\imath\pi k\cdot(i-j)/N} \int_0^t \rho_s\left(\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\right) ds$$
$$+ \beta^2\hat{r} \sum_{i\in C_N} \int_0^t \rho_s(\sigma_i^1\sigma_i^2) ds.$$

In order to simplify this last expression, we will use the elementary identity

$$\left| \sum_{i\in C_N} z_i \right|^2 = \sum_{i\in C_N} |z_i|^2 + 2 \sum_{(i,j)\in C_N} z_i\overline{z}_j, \quad (3.8)$$

valid for any family of complex numbers, and applied here to $z_i = e^{\imath\pi k\cdot i/N}\sigma_i^1\sigma_i^2$. Recalling furthermore the definition (2.3) of $R_k^{1,2}$, we end up with

$$\int_0^t \frac{d\langle Z_N\rangle_s}{Z_N^2(s)} = \frac{\hat{N}^d\beta^2}{2}\sum_{k\in\mathbb{Z}_*^d}\gamma_k\int_0^t \rho_s\left(\left|R_k^{1,2}\right|^2\right)ds - \frac{\beta^2\Gamma t}{2} + \frac{\beta^2\gamma_0\hat{N}^d}{2}\int_0^t \rho_s\left(\left(R^{1,2}\right)^2\right)ds + \beta^2\hat{N}^d\hat{r}\int_0^t \rho_s(R^{1,2})ds.$$

Let us introduce now artificially the quantity $R_0^{1,2}$, by writing $(R^{1,2})^2 = (R_0^{1,2})^2 + 2rR_0^{1,2} + r^2$. This gives

$$\int_0^t \frac{d\langle Z_N\rangle_s}{Z_N^2(s)} = \frac{\hat{N}^d\beta^2}{2}\sum_{k\in\mathbb{Z}^d}\gamma_k\int_0^t \rho_s\left(\left|R_k^{1,2}\right|^2\right)ds - \frac{\beta^2\Gamma t}{2} - \frac{\beta^2\hat{N}^d\gamma_0 r^2 t}{2} + 2\beta^2\hat{N}^d\hat{r}\int_0^t \rho_s(R^{1,2})ds. \quad (3.9)$$

Similarly to [8], let us notice that, since $X_i(s) \sim \mathcal{N}(0, 1-s)$, a simple Gaussian integration by parts yields

$$\mathbf{E}\left[\rho_s(\sigma_i)\frac{X_i(s)}{1-s}\right] = \mathbf{E}\left[\partial_{X_i(s)}\rho_s(\sigma_i)\right] = \beta\hat{r}^{1/2}\mathbf{E}\left[\rho_s(1-\sigma_i^1\sigma_i^2)\right]. \quad (3.10)$$

This elementary consideration will induce us to add and subtract $\frac{\beta^2\hat{r}}{\hat{N}^{d/2}}\sum_i \rho_s(1-\sigma_i^1\sigma_i^2)$ to the expression (3.7). By plugging moreover (3.9), we get

$$Y_N(t) = U_N + M_{1,N}(t) + M_{2,N}(t) - (V_{1,N}(t) - V_{2,N}(t)) + \hat{V}_{3,N}(t), \quad (3.11)$$

where

$$\begin{aligned}\hat{V}_{3,N}(t) = &-\frac{\beta^2\hat{r}}{\hat{N}^{d/2}}\sum_{i\in C_N}\int_0^t \rho_s(1-\sigma_i^1\sigma_i^2)ds + \frac{\beta^2}{2\hat{N}^{3d/2}}\sum_{(i,j)\in C_N} q_N^2(i-j)t \\ &+ \frac{\beta^2\hat{r}}{2}\hat{N}^{d/2}t - \frac{\hat{N}^{d/2}\beta^2}{4}\sum_{k\in\mathbb{Z}^d}\gamma_k\int_0^t \rho_s\left(\left|R_k^{1,2}\right|^2\right)ds + \frac{\beta^2\Gamma t}{4\hat{N}^{d/2}} \\ &+ \frac{\beta^2\hat{N}^{d/2}\gamma_0 r^2 t}{4} - \beta^2\hat{N}^{d/2}\hat{r}\int_0^t \rho_s(R^{1,2})ds - \frac{\beta^2\gamma_0\hat{N}^{d/2}t}{4}(1-r)^2.\end{aligned}$$

This last expression can be simplified a little to give

$$\hat{V}_{3,N}(t) = \left[\frac{\Gamma}{\hat{N}^{d/2}} + \frac{2}{\hat{N}^{3d/2}}\sum_{(i,j)\in C_N} q_N^2(i-j) - \gamma_0\hat{N}^{d/2}\right]\frac{\beta^2 t}{4} - \frac{\hat{N}^{d/2}\beta^2}{4}\sum_{k\in\mathbb{Z}^d}\gamma_k\int_0^t \rho_s\left(\left|R_k^{1,2}\right|^2\right)ds,$$

that is $\hat{V}_{3,N}(t) = V_{3,N}(t)$. By reporting this equality in (3.11), the proof is now complete. □

The last preliminary result we will need in order to establish our CLT is a self-averaging result for $R_k^{1,2}$ under the measure ρ_t at a fixed value of the parameter $t \in [0,1]$.

Proposition 3.4. *Under the hypotheses of Corollary 2.16, let $t \in [0,1]$. Then*

$$\sum_{k\in\mathbb{Z}^d} \gamma_k \mathbf{E}\left[\rho_t(|R_k^{1,2}|^2)\right] = O\left(\frac{1}{N^{\hat{d}}}\right), \tag{3.12}$$

for $\hat{d} > d/2$, uniformly in t.

Proof. This is an easy elaboration of the computations leading to Proposition 2.15, by considerering the path $v \in [0,t] \mapsto H_{N,t,v}(\sigma)$ defined by

$$-H_{N,t,v}(\sigma) = -H_{N,t}^m(\sigma) + \frac{\beta}{\hat{N}^{d/2}} \sum_{i\in C_N^m} B_{i,m}(v) q_N(i-m)\sigma_i\sigma_m + \beta\hat{r}^{1/2} X_m(v)\sigma_m + h\sigma_m,$$

with obvious notations.

□

3.2 Proof of Theorem 3.1

For an arbitrary $u \in \mathbb{R}$, we will try to control $\Phi_{N,u}(t) \equiv \mathbf{E}[e^{\imath u Y_N(t)}]$. To this purpose, we will apply Itô's formula to the complex valued C_b^2 function $x \mapsto e^{\imath u x}$. We obtain, for any $t \in [0,1]$,

$$e^{\imath u Y_N(t)} = D_{1,N} + \sum_{m=2}^{8} D_{m,N}(t),$$

where

$$\begin{aligned}
D_{1,N} &= e^{\imath u U_N} \\
D_{2,N}(t) &= \frac{\imath u\beta}{\hat{N}^d} \sum_{(i,j)\in C_N} q_N(i-j) \int_0^t e^{\imath u Y_N(s)} \rho_s(\sigma_i\sigma_j) dB_{i,j} \\
D_{3,N}(t) &= \frac{\imath u\beta\hat{r}^{1/2}}{\hat{N}^{d/2}} \sum_{i\in C_N} \int_0^t e^{\imath u Y_N(s)} \rho_s(\sigma_i) dW_i \\
D_{4,N}(t) &= -\frac{\imath u\beta\hat{r}^{1/2}}{\hat{N}^{d/2}} \sum_{i\in C_N} \int_0^t e^{\imath u Y_N(s)} \rho_s(\sigma_i) \frac{X_i(s)}{1-s} ds \\
D_{5,N}(t) &= \frac{\imath u\beta^2\hat{r}}{\hat{N}^{d/2}} \sum_{i\in C_N} \int_0^t e^{\imath u Y_N(s)} \rho_s(1-\sigma_i^1\sigma_i^2) ds,
\end{aligned}$$

and

$$\begin{aligned}
D_{6,N}(t) = \frac{\imath u\beta^2}{4} \left[\frac{\Gamma}{\hat{N}^{d/2}} + \frac{2}{\hat{N}^{3d/2}} \sum_{(i,j)\in C_N} q_N^2(i-j) - \gamma_0 \hat{N}^{d/2}\right] \int_0^t e^{\imath u Y_N(s)} ds \\
- \frac{\imath u \hat{N}^{d/2}\beta^2}{4} \sum_{k\in\mathbb{Z}^d} \gamma_k \int_0^t e^{\imath u Y_N(s)} \rho_s\left(\left|R_k^{1,2}\right|^2\right) ds.
\end{aligned}$$

The terms $D_{7,N}(t)$ and $D_{8,N}(t)$are the ones associated with the quadratic variation process of $Y_N(t)$, that is:

$$\begin{aligned} D_{7,N}(t) &= -\frac{u^2\beta^2}{2\hat{N}^{2d}} \sum_{(i,j)\in C_N} q_N^2(i-j) \int_0^t e^{\imath u Y_N(s)} \rho_s(\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2) ds \\ &= -\frac{u^2\beta^2}{2\hat{N}^{2d}} \sum_{(i,j)\in C_N} \sum_{k\in\mathbb{Z}^d} \gamma_k e^{\imath\pi k\cdot(i-j)/N} \int_0^t e^{\imath u Y_N(s)} \rho_s(\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2) ds \\ D_{8,N}(t) &= -\frac{u^2\beta^2\hat{r}}{2\hat{N}^{d}} \sum_{i\in C_N} \int_0^t e^{\imath u Y_N(s)} \rho_s(\sigma_i^1\sigma_i^2) ds = -\frac{u^2\beta^2\hat{r}}{2} \int_0^t e^{\imath u Y_N(s)} \rho_s(R^{1,2}) ds. \end{aligned}$$

Let us find some estimates for the expected value of all the terms we have obtained in this decomposition. First of all, the usual central limit theorem for IID random variables yields

$$\mathbf{E}[D_{1,N}(t)] = e^{-\hat{\tau}^2 u^2/2} + O\left(\frac{1}{N^{d/2}}\right), \tag{3.13}$$

where $\hat{\tau}$ is defined at (3.1). Furthermore, the terms $D_{2,N}(t)$ and $D_{3,N}(t)$ are obviously of zero mean.

In order to control $D_{4,N}(t)$, let us perform again the Gausssian integration by parts (3.10), which can be read here as

$$\begin{aligned} \mathbf{E}[D_{4,N}(t)] &= -\frac{\imath u\beta\hat{r}^{1/2}}{\hat{N}^{d/2}} \sum_{i\in C_N} \int_0^t \mathbf{E}\left[\partial_{X_i(s)}(e^{\imath u Y_N(s)}\rho_s(\sigma_i))\right] ds \\ &= \frac{u^2\beta\hat{r}^{1/2}}{\hat{N}^{d/2}} \sum_{i\in C_N} \int_0^t \mathbf{E}\left[\partial_{X_i(s)}(Y_N(s)) e^{\imath u Y_N(s)}\rho_s(\sigma_i)\right] ds - \mathbf{E}[D_{5,N}(t)]. \end{aligned}$$

Furthermore,

$$\partial_{X_i(s)}(Y_N(s)) = \frac{\partial_{X_i(s)}(Z_N(s))}{N^{d/2} Z_N(s)} = \frac{\beta\hat{r}^{1/2}}{N^{d/2}} \rho_s(\sigma_i),$$

and therefore, we obtain

$$\begin{aligned} \mathbf{E}[D_{4,N}(t) + D_{5,N}(t)] &= u^2\beta^2\hat{r} \int_0^t \mathbf{E}\left[e^{\imath u Y_N(s)}\rho_s(R^{1,2})\right] ds \\ &= u^2\beta^2\gamma_0 r^2 \int_0^t \Phi_{N,u}(s) ds + O\left(\frac{1}{N^{\hat{d}}}\right), \end{aligned} \tag{3.14}$$

owing to Proposition 3.4.

Let us turn now to the estimation of $D_{6,N}(t)$: notice that, under Hypothesis 2.1 and 2.12, it is easily checked that

$$\left| \frac{1}{\hat{N}^{2d}} \sum_{(i,j)\in C_N} q_N^2(i-j) - \frac{\gamma_0}{2} \right| = O\left(\frac{1}{\hat{N}^{\hat{d}}}\right),$$

which together with a direct application of Proposition 3.4, shows that

$$\mathbf{E}[D_{6,N}(t)] = O\left(\frac{1}{\hat{N}^{\epsilon}}\right),$$

where $\epsilon = \hat{d} - d/2$.

As far as $D_{7,N}(t)$ is concerned, notice that by relation (3.8), we have

$$\begin{aligned} D_{7,N}(t) = &-\frac{u^2\beta^2}{4}\sum_{k\in\mathbb{Z}_*^d}\gamma_k\int_0^t e^{\imath uY_N(s)}\rho_s\left(|R_k^{1,2}|^2\right)ds \\ &-\frac{u^2\beta^2\gamma_0}{4}\int_0^t e^{\imath uY_N(s)}\rho_s\left(|R^{1,2}|^2\right)ds + \frac{u^2\beta^2\Gamma}{4\hat{N}^d}\int_0^t e^{\imath uY_N(s)}ds, \end{aligned} \tag{3.15}$$

and thanks to proposition 3.4, we get

$$D_{7,N}(t) = -\frac{u^2\beta^2\gamma_0 r^2}{4}\int_0^t e^{\imath uY_N(s)}ds + O\left(\frac{1}{N^{\hat{d}}}\right), \tag{3.16}$$

and thus

$$\mathbf{E}[D_{7,N}(t)] = -\frac{u^2\beta^2\gamma_0 r^2}{4}\int_0^t \Phi_{N,u}(s)ds + O\left(\frac{1}{N^{\hat{d}}}\right).$$

Eventually, in a similar way we have

$$\mathbf{E}\left[D_{8,N}(t)\right] = -\frac{u^2\beta^2\gamma_0 r^2}{2}\int_0^t \Phi_{N,u}(s)ds + O\left(\frac{1}{\hat{N}^{\hat{d}}}\right).$$

Putting together the previous estimates on $\mathbf{E}[D_{1,N}(t)], \ldots, \mathbf{E}[D_{8,N}(t)]$, we have finally:

$$\psi_{N,u}(t) = e^{-\frac{\nu^2u^2}{2}} + \frac{(u\beta\sqrt{\gamma_0}r)^2}{4}\int_0^t \psi_{N,u}(s)ds + \hat{R}_{N,u}(t),$$

with

$$\left|\hat{R}_{N,u}(t)\right| \leq \frac{\kappa}{N^{\epsilon}},$$

which ends the proof by a Gronwall type argument.

□

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