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Yves Bourgault, Yves Coudière, Charles Pierre

WELL-POSEDNESS OF A PARABOLIC PROBLEM BASED ON A BIDOMAIN MODEL FOR ELECTROPHYSIOLOGICAL WAVE PROPAGATION*

YVES BOURGAULT †, YVES COUDIÈRE ‡, AND CHARLES PIERRE §

Abstract. The purpose of the current paper is to study the well-posedness of the bidomain model. That model is commonly used to simulate electrophysiological wave propagation in the heart. We base our analysis on a formulation of the bidomain model describing two potentials that satisfy a system of coupled parabolic and elliptic PDEs, these being coupled with one or more ODEs representing the ionic activity. The parabolic and elliptic PDEs are first converted into a single parabolic PDE by the introduction of the so-called bidomain operator. We properly define and analyze that bidomain operator. We then present a proof of existence, uniqueness and regularity of a local (in time) solution through a semi-group approach. The bidomain model is next reformulated as a parabolic variational problem, through the introduction of a bidomain bilinear form. A proof of existence and uniqueness of a global solution is obtained using a compactness argument, this time for an ionic model reading as a single ODE but including polynomial nonlinearities. Finally, the hypothesis behind the existence of that global solution are verified for three commonly used ionic models, namely the FitzHugh-Nagumo, Aliev-Panfilov and MacCulloch models.

Key words. reaction-diffusion equation, bidomain model, cardiac electrophysiology

AMS subject classifications. 35A05, 35K57, 35Q80

1. Introduction. A common approach for modeling the propagation of electrophysiological waves in the myocardium consists in defining two electrical potentials, giving rise to the so-called bidomain model [12, 2, 14, 7]. This model is basically a system of reaction-diffusion equations that can be written in at least the two following forms:

1. as two degenerate parabolic PDEs, namely

$$\frac{\partial u}{\partial t} + f(u,w) - \nabla \cdot (\sigma_i \nabla u_i) = s_i, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + f(u,w) + \nabla \cdot (\sigma_e \nabla u_e) = -s_e. \tag{1.2}$$

2. or as a parabolic PDE coupled to an elliptic one, namely

$$\frac{\partial u}{\partial t} + f(u,w) - \nabla \cdot (\sigma_i \nabla u) - \nabla \cdot (\sigma_i \nabla u_e) = s_i, \tag{1.3}$$

$$\nabla \cdot (\sigma_i \nabla u + (\sigma_i + \sigma_e) \nabla u_e) = -(s_i + s_e). \tag{1.4}$$

Here u_i and u_e are the intra and extra-cellular potentials, respectively, $u = u_i - u_e$ is the trans-membrane or action potential, $\sigma_{i,e}$ are conductivity matrices, $f(u,w)$ is the total transmembrane ionic current and $s_{i,e}$ are external applied current sources.

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†Department of Mathematics and Statistics, University of Ottawa, Ottawa (Ont), Canada, K1N 6N5 (ybourg@uottawa.ca).

‡Laboratoire de Mathmatiques Jean Leray, CNRS - UMR 6629, Université de Nantes, 44322 Nantes Cédex 3, France (Yves.Coudiere@univ-nantes.fr).

§Department of Mathematics and Statistics, University of Ottawa, Ottawa (Ont), Canada, K1N 6N5 (cpierre@uottawa.ca).

In each case, the model is closed by a system of ODEs which models the membrane ionic channels:

$$\frac{\partial w}{\partial t} + g(u, w) = 0, \tag{1.5}$$

where w represents the ionic currents or gating variables.

The two versions of the bidomain model can be obtained from one another. For instance, Eq. (1.4) is obtained by substracting Eq. (1.1) from Eq. (1.2) and rewriting the resulting equations in the variables u and u_e instead of u_i and u_e. The *Version 1* of the bidomain model is said to be a degenerate parabolic system as the time-derivatives involve the unknown u instead of the unknowns u_i and u_e occurring in the second-order conduction term.

As far as we know, only one proof of the well-posedness of the bidomain model is available in the literature [5]. This proof is based on a reformulation of (1.1)-(1.2) as an evolution variational inequality in a properly chosen Sobolev space. That approach dealt only with the FitzHugh-Nagumo ionic model. We will use instead more classical formulations of the bidomain model that turn out to be more effective in proving the existence, uniqueness and regularity of the solution for the bidomain model coupled with more complex ionic models.

1.1. Outline of the paper. This paper presents a reformulation of (1.3)-(1.4) as a simple parabolic equation, using the so-called elliptic *bidomain operator*. The special nature of this operator will be covered, both in a strong and a weak (variational) formulation of the equations. A standard compactness technique [9] is used to prove our main result: existence of global (in time) solutions for the weak form of the system (1.3), (1.4), (1.5). The hypotheses on the nonlinear functions f and g, stated precisely in section 3.2, basically allow functions with polynomial growth (with a sign) at infinity. Additionally, w should only be a scalar variable, and the functions f and g have to be both linear with respect to w. Three classical models for numerical electrocardiology are proved to enter our framework, broadening the set of ionic models for which the existence of a solution of the bidomain model is proven as compared to only one model in [5]. We should say that we could prove uniqueness of the solution only for the FitzHugh-Nagumo model.

Our second most important result is a very general result for existence, uniqueness and regularity of local in time solutions. The proof is an application of some general theorems found in [8], and is deduced from a detailed study of the strong bidomain operator. Additionally, it is valid for any kind of locally Lipschitz continuous nonlinear functions f and g. In practice, Lipschitz continuity is not restrictive as solutions tend to remain bounded and non-Lipschitzian ionic currents can be replaced by Lipschitzian functions.

The paper is sketched as follows: some notations are introduced in section 1.2; the reformulation of the bidomain model as a semi-linear parabolic system of equations using the *bidomain operator* and results on local in time strong solutions are given in section 2; the variational approach is covered in section 3, *the variational bidomain model* is stated in sections 3.1 and 3.3, well-posedness is shown in section 3.4 and 3.5, a word on uniqueness is given in section 3.6, and the three examples are considered in section 4.

1.2. Notations. Let Ω be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and its outward normal n. In applications, the domain Ω represents the region occupied by the muscular cells of the whole or a portion of the myocardium. Assume

that the conductivity matrices $\sigma_{i,e}$ are smooth functions of x on $\overline{\Omega}$. The matrix $\sigma_{i,e}$ are also symmetric and satisfy the inequalities

$$m\,|\xi|^2 \leq \xi^t\,\sigma_{i,e}(x)\,\xi \leq M\,|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n \tag{1.6}$$

uniformly over all $x \in \bar{\Omega}$, for some constants $0 < m < M$. We will clarify below the smoothness assumptions on the boundary $\partial\Omega$ and the tensors $\sigma_{i,e}$.

Natural boundary conditions for the electrically isolated heart are

$$(\sigma_i \nabla u_i) \cdot n = 0, \quad (\sigma_e \nabla u_e) \cdot n = 0, \tag{1.7}$$

meaning that no current flows out of the heart. These are the homogeneous Neumann boundary conditions for the system (1.1)-(1.2) in the unknowns u_i and u_e. Linear combinations of the above boundary conditions give:

$$(\sigma_i \nabla(u + u_e)) \cdot n = 0, \quad (\sigma_i \nabla u) \cdot n + ((\sigma_i + \sigma_e)\nabla u_e) \cdot n = 0. \tag{1.8}$$

These are now the homogeneous Neumann boundary conditions for the system (1.3)-(1.4), this time in the unknowns u and u_e. Based on the organization of the myocardium in sheets of aligned fibers, commonly used conductance tensors are of the form:

$$\sigma_{i,e}(x) \;=\; (k_L^{i,e} - k_T^{i,e})\tau \otimes \tau + k_T^{i,e}\, I \tag{1.9}$$

where τ is a unit vector giving the direction of the fibers, $k_L^{i,e}$ are the longitudinal conductances along the fibers and $k_T^{i,e}$ are the conductances in the directions transverse to the fibers, respectively in the intra-cellular (i) or the extra-cellular (e) media [3]. The fibers are usually tangent to the boundary $\partial\Omega$ of the myocardium. It results that $n \cdot \tau = 0$ and any set of boundary conditions given above is equivalent to the standard homogeneous Neumann boundary conditions:

$$\nabla u \cdot n = 0, \quad \nabla u_i \cdot n = 0, \quad \nabla u_e \cdot n = 0. \tag{1.10}$$

Initial conditions are provided on u and w:

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x). \tag{1.11}$$

The functional spaces to properly state the bidomain problem will be specified below, both for strong and weak formulations.

The source terms $s_i(t)$ and $s_e(t)$ are related to the applied stimulating currents. Their regularity will be precised in sections 2 and 3.

We point out that $s_i + s_e$ must have a zero mean value. The physical reason for this is that there is no current flowing out of the heart through its boundary as stated by boundary conditions (1.7) or (1.8), and that the intra- and extra-cellular medias are electrically communicating through the cells membrane. Therefore the current conservation applied to the whole heart, in other words Eq. (1.4) integrated over all Ω, and some integration by parts ensure that $s_i + s_e$ has a zero mean value. That is:

$$\int_\Omega (s_i(x) + s_e(x))\; dx \;= 0\,. \tag{1.12}$$

2. Local Existence and Uniqueness of Solutions based on a Semi-group Approach. We shall here rewrite the bidomain model system of equations into a parabolic semi-linear equation coupled with a $k \times k$ system of ODEs ($w \in \mathbb{R}^k$):

$$\frac{du}{dt} + Au + f(u,w) = s(t) \tag{2.1}$$

$$\frac{dw}{dt} + g(u,w) = 0 \tag{2.2}$$

where the definition of the (strong) bidomain operator A and of the source term $s(t)$ will be precised below in subsection 2.1. Any cell membrane ionic current model might read that form.

We search *strong solutions* (also called *mild solutions*, defined in 2.3) for those equations in the framework of analytical semi-groups and sectorial operators (defined below in 2.1), as presented in [8].

The first unknown u is sought as a twice weakly differentiable function $u \in H^2(\Omega)$. The constraint (1.10) on u is taken into account by choosing the following domain for the unbounded operator $A:\ D(A) \subset L^2(\Omega) \mapsto L^2(\Omega)$:

$$D(A) := \left\{ u \in H^2(\Omega),\ \nabla u \cdot n = 0 \text{ on } \partial\Omega \right\}. \tag{2.3}$$

The second unknown w (a k-dimensional vector here) will be searched in a Banach space B^k: either $B = L^\infty(\Omega)$ or $B = C^\nu(\Omega)$, the space of all (globally) ν-Hölder continuous functions on Ω. This last choice will be needed to establish the regularity of the solutions. In the sequel, ν will represent a real number $0 < \nu < 1$.

In the following we shall always assume Ω to be bounded, its boundary $\partial\Omega$ to have C^2 regularity and the conductivity tensors $\sigma_{i,e}$ to have C^1 regularity over $\overline{\Omega}$.

We shall either consider the two cases where f and g are locally Lipschitzian real functions over $\mathbb{R} \times \mathbb{R}^k$ (which is sufficient to prove the existence and uniqueness of solutions) or the case where f and g are C^2 real functions (which is needed to ensure the regularity of the solutions).

The two stimulating currents s_i and s_e in (1.1)-(1.2) will throughout that section be considered as functions of the time variable t into $L^2(\Omega)$, *i.e.*: $s_{i,e} : t \mapsto s_{i,e}(t) \in L^2(\Omega)$. We will moreover assume here these functions to be locally Hölder continuous in time, $s_{i,e} \in C^\gamma_{loc}([0,T), L^2(\Omega))$. Namely, there exists a constant $\gamma > 0$ such that for any time t there exists a neighborhood ω of t and a constant C such that:

$$\forall \delta \in \omega, \quad \|s_{i,e}(t) - s_{i,e}(\delta)\|_{L_2} \leq C|t-\delta|^\gamma. \tag{2.4}$$

In the case $\gamma = 1$, that assumption merely means that $s_{i.e}$ are locally Lipschitzian in time.

The system (2.1)-(2.2) is rewritten as a single equation

$$\frac{dz}{dt} + \mathcal{A}z + F(z) = S(t) \tag{2.5}$$

where $z = (u,w)$, $F(z) = (f(u,w), g(u,w))$, $S(t) = (s(t), 0)$ and $\mathcal{A}$ is the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset Z \mapsto Z$ on the Banach space $Z = L^2(\Omega) \times B^k$ defined as $\mathcal{A} = A \times 0$ and so with domain $D(\mathcal{A}) = D(A) \times B^k$.

We shall study equation (2.5) in the framework of analytical semi-group theory and sectorial operators as presented in [8]. Let us recall the definition of a sectorial operator:

DEFINITION 2.1. *$\mathcal{A} : D(\mathcal{A}) \subset Z \mapsto Z$ on the Banach space Z is sectorial if $\mathcal{A}$ is dense, closed and if there exists a sector $S_{\omega,\theta} \subset \mathbb{C}$ in the complex plane $\mathbb{C}$,*

$$S_{\omega,\theta} = \{\lambda \in \mathbb{C},\ |\arg(\lambda - \omega)| < \theta\} \cup \{\omega\}, \quad \omega \in \mathbb{R},\ 0 \leq \theta < \frac{\pi}{2},$$

such that:

$$\mathbb{C} - S_{\omega,\theta} \subset \rho(\mathcal{A})$$
$$\forall \lambda \in \mathbb{C} - S_{\omega,\theta}, \quad \| (\lambda I - A)^{-1} \|_Z \leq \frac{M}{|\lambda - \omega|},$$

where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$ and $\| \cdot \|_Z$ the norm on the space $\mathcal{L}(Z)$ of the bounded operators on Z.

However we will only use the following three characterizations of sectorial operators:

1. a self-adjoint operator on a Hilbert space which is dense and whose spectrum is bounded below is sectorial,
2. a bounded operator is always sectorial,
3. if $A_{1,2} : D(A_{1,2}) \subset Z_{1,2} \mapsto Z_{1,2}$ are two sectorial operators on the spaces Z_1 and Z_2 then $A_1 \times A_2$ is sectorial on $Z_1 \times Z_2$ with domain $D(A_1) \times D(A_2)$.

2.1. Definition of the bidomain operator A: spectral decomposition. For equation (2.1) to be equivalent with (1.3)-(1.4), the *strong bidomain* operator A and the source term $s(t)$ have to read

$$Au - s(t) = -\nabla \cdot (\sigma_i \nabla u) - \nabla \cdot (\sigma_i \nabla u_e) - s_i, \tag{2.6}$$

where the function u_e is implicitly given by

$$\nabla \cdot ((\sigma_e + \sigma_i)\nabla u_e) = -\nabla \cdot (\sigma_i \nabla u) - (s_i + s_e). \tag{2.7}$$

Remark that the solution of (2.7) is defined up to an additive constant. Hence, for a Banach functional space X, we shall consider also the Banach space $X/\mathbb{R} \simeq \{u \in X,\ \int_\Omega u(x)dx = 0\}$, defining $L^2(\Omega)/\mathbb{R}$, $D(A)/\mathbb{R} \subset L^2(\Omega)/\mathbb{R}$, and $H^1(\Omega)/\mathbb{R}$. For any $u \in L^2(\Omega)$, we denote by $[u] = u - \int_\Omega u(x)dx \in L^2(\Omega)/\mathbb{R}$.

Now, for any $u \in D(A)$, the solution of (2.7) is divided into $\tilde{u}_e$ and $\bar{u}_e$, respectively solutions of

$$-\nabla \cdot ((\sigma_e + \sigma_i)\nabla \tilde{u}_e) = \nabla \cdot (\sigma_i \nabla u),$$
$$-\nabla \cdot ((\sigma_e + \sigma_i)\nabla \bar{u}_e) = (s_i + s_e).$$

The functions $-\nabla \cdot (\sigma_i \nabla u)$ and $-(s_i + s_e)$ are in $L^2(\Omega)/\mathbb{R}$, due to the definition of $D(A)$ and to relation (1.12) on $s_i + s_e$.Hence, these two problems have unique weak (variational) solutions in $H^1(\Omega)/\mathbb{R}$ from a simple application of the Lax-Milgram theorem.

From the additional regularity assumptions on $\partial\Omega$ and $\sigma_{i,e}$, we also have that $\tilde{u}_e \in D(A)/\mathbb{R}$ and $\bar{u}_e \in D(A)/\mathbb{R}$, see [6]. Specifically, let

$$A_{i,e} : v \in D(A)/\mathbb{R} \mapsto -\nabla \cdot (\sigma_{i,e} \nabla v) \in L^2(\Omega)/\mathbb{R}$$

denote the two strong elliptic operators associated to σ_i and σ_e. We can write that

$$\tilde{u}_e = (A_i + A_e)^{-1} (\nabla \cdot (\sigma_i \nabla u)) = -(A_i + A_e)^{-1} A_i[u],$$
$$\bar{u}_e = (A_i + A_e)^{-1}(s_i + s_e),$$

and subsequently define A over $D(A)$ and s by

$$Au = -\nabla\cdot(\sigma_i\nabla u) - \nabla\cdot(\sigma_i\nabla\tilde{u}_e) = A_i[u] - A_i(A_i+A_e)^{-1}A_i[u], \tag{2.8}$$

$$s(t) = -\nabla\cdot(\sigma_i\nabla\bar{u}_e) - s_i = A_i(A_i+A_e)^{-1}(s_i+s_e) - s_i. \tag{2.9}$$

so that (2.6) holds. An easy computation shows that

$$\begin{aligned} A_i - A_i(A_i+A_e)^{-1}A_i &= A_i\left(\mathrm{Id} - (A_i+A_e)^{-1}A_i\right) \\ &= A_i(A_i+A_e)^{-1}((A_i+A_e)-A_i) = A_i(A_i+A_e)^{-1}A_e = \left(A_e^{-1}+A_i^{-1}\right)^{-1}, \end{aligned}$$

and thus the restriction $\overline{A}$ to $D(A)/\mathbb{R}$ of A is given by:

$$\overline{A} = \left(A_e^{-1}+A_i^{-1}\right)^{-1}$$

PROPOSITION 2.2. *The strong bidomain operator A defined by* (2.8) *for $u\in D(A)$ is self-adjoint and non negative, therefore it is sectorial. Moreover there exists a Hilbert base $(\psi_n)_{n\geq 0}$ of $L^2(\Omega)$, where ψ_0 is a constant, and a sequence of eigenvalues $0=\lambda_0<\lambda_1<\cdots<\lambda_n<\cdots\to\infty$ such that:*

$$D(A) = \left\{ u\in L^2(\Omega),\ \sum_{n\geq 0}\lambda_n^2(u,\psi_n)^2<\infty \right\},\quad Au = \sum_{n\geq 0}\lambda_n(u,\psi_n)\psi_n, \tag{2.10}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$.
Moreover, the source term defined by (2.9) *is locally Hölder continuous in time.*

Proof. The operator $\overline{A} = (A_e^{-1}+A_i^{-1})^{-1}$ is defined from $D(A)/\mathbb{R} := D(\overline{A})$ onto $L^2(\Omega)/\mathbb{R}$. The operator $B = A_e^{-1}+A_i^{-1}$ is defined on $L^2(\Omega)/\mathbb{R}$, with range $D(\overline{A})$ by definition.

From regularity assumptions, $A_{i,e}$ are self-adjoint, positive and have compact inverses from $L^2(\Omega)/\mathbb{R}$ into $L^2(\Omega)/\mathbb{R}$. As a consequence, B is self-adjoint, positive and compact. Thus, there exists an Hilbert basis $(\psi_n)_{n\geq 1}$ of $L^2(\Omega)/\mathbb{R}$ and a sequence $0<\lambda_1\leq\cdots\leq\lambda_n\to\infty$ such that $Bu = \sum_{n\geq 1}\dfrac{1}{\lambda_n}(u,\psi_n)\psi_n$, and the spectral theorem ensures that the operator B^{-1} is defined by

$$D(B^{-1}) = \left\{ u\in L^2(\Omega)/\mathbb{R},\ \sum_{n\geq 1}\lambda_n^2(u,\psi_n)^2<\infty \right\},\quad B^{-1}u = -\sum_{n\geq 1}\lambda_n(u,\psi_n)\psi_n,$$

and is self-adjoint on $L^2(\Omega)/\mathbb{R}$.

We easily have $D(\overline{A}) = B(L^2(\Omega)/\mathbb{R}) = D(B^{-1})$, and then $\overline{A}=B$ is self-adjoint, positive and has the spectral decomposition shown. This implies that A is also self-adjoint and positive, and (2.10) holds (with $\lambda_0=0$ and ψ_0 a constant function having norm 1), since $Au = \overline{A}[u]$ and $(Au,v) = (\overline{A}[u],[v])$.

Now, the source term $s(t)$ is given by (2.9). With the regularity assumptions on $\partial\Omega$ and $\sigma_{i,e}$, we have that (see [6]),

$$(A_i+A_e)^{-1} : L^2(\Omega)/\mathbb{R}\mapsto H^2(\Omega)/\mathbb{R}$$

is well-defined and bounded. Since $A_i : H^2(\Omega)/\mathbb{R}\mapsto L^2(\Omega)/\mathbb{R}$ is also bounded then $A_i(A_i+A_e)^{-1} : L^2(\Omega)/\mathbb{R}\mapsto L^2(\Omega)/\mathbb{R}$ is a bounded operator. Therefore, if $s_{i,e} : t\mapsto L^2(\Omega)$ are locally Hölder continuous or locally Lipschitzian, then the same result holds for $s : t\mapsto L^2(\Omega)$. ∎

2.2. Local existence of solutions and uniqueness. Let us first define what we mean by strong (or *mild*) solutions for (2.5):

DEFINITION 2.3. *A function $z : [0,T) \mapsto Z$ is a strong solution of equation* (2.5) *with initial data z_0 if:*

1. *$z : [0,T) \mapsto Z$ is continuous and $z(0) = z_0$,*
2. *$z : (0,T) \mapsto Z$ is Fréchet differentiable,*
3. *$t \in [0,T) \mapsto F(z(t)) \in Z$ is well defined, locally Hölder continuous on $(0,T)$ and is continuous at $t = 0$,*
4. *$\forall\ t \in (0,T)$, $z(t) \in D(\mathcal{A})$ and* (2.5) *is verified.*

One problem is that the function F (considered as a real function on $\mathbb{R}\times\mathbb{R}^k$) cannot be extended to a function on the whole Banach space Z, since a locally Lipschitz real function do not map $L^2(\Omega)$ into $L^2(\Omega)$.

However, using the Sobolev continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$ in dimension $d \leq 3$, the reaction term F in (2.5) maps $Z^1 := D(\mathcal{A})$ into Z and the subsequent function $F :\ Z^1 \mapsto Z$ is also locally Lipschitzian.

Let us define for $\alpha \geq 0$ the unbounded operator $A^\alpha : D(A^\alpha) \subset L^2(\Omega) \mapsto L^2(\Omega)$ as:

$$D(A^\alpha) = \left\{ u \in L^2(\Omega),\ \sum_{n\geq 0} \lambda_n^{2\alpha}(u,\psi_n)^2 < \infty \right\}, \quad A^\alpha u = \sum_{n\geq 0} \lambda_n^\alpha (u,\psi_n)\psi_n.$$

The fractional spaces $D(A^\alpha)$, equipped with the norm $\|u\|_\alpha = \|u + A^\alpha u\|_{L^2}$, are Banach spaces. Moreover (see [8]), for any $0 \leq \alpha \leq \beta$, we have the continuous and dense embedding $D(A^\beta) \subset D(A^\alpha)$. These spaces form a sequence of decreasing functional spaces composed of functions whose regularity increases from $L^2(\Omega)$ ($\alpha = 0$) to $D(A) \subset H^2(\Omega)$ ($\alpha = 1$).

With the regularity we assumed for $\partial\Omega$, we have the following embedding lemma (see [8]).

LEMMA 2.4 (Dan Henry). *Let A be a sectorial operator on $L^2(\Omega)$ such that $D(A) \subset H^2(\Omega)$. We then have the following continuous embeddings for $0 \leq \alpha < 1$:*

$$D(A^\alpha) \subset L^\infty(\Omega) \quad \textit{for } \frac{d}{4} < \alpha, \tag{2.11}$$

$$D(A^\alpha) \subset C^\nu(\Omega) \quad \textit{for } 0 < \nu < 2\alpha - \frac{d}{2}. \tag{2.12}$$

Let us now introduce the fractional powers of $\mathcal{A}$ and the associated fractional spaces Z^α:

$$\mathcal{A}^\alpha = A^\alpha \times 0, \quad Z^\alpha := D(\mathcal{A}^\alpha) = D(A^\alpha) \times B^k.$$

Using the embedding (2.11), one has for $d/4 < \alpha < 1$, that $Z^\alpha \subset L^\infty(\Omega) \times B^k$. Thus for $B = L^\infty(\Omega)$ and with f and g assumed to be locally Lipschitzian real functions, $F : Z^\alpha \mapsto Z$ is well defined and also is locally Lipschitzian.

Similarly, using the embedding (2.12), for $0 < \nu < 2\alpha - d/2$, $Z^\alpha \subset C^\nu(\Omega) \times B^k$. Thus for $B = C^\nu(\Omega)$, with f and g assumed to be C^2 regular real functions, $F :\ Z^\alpha \mapsto Z$ is well defined and is locally Lipschitzian (because a C^2 real function maps $C^\nu(\Omega)$ into $C^\nu(\Omega)$ and is locally Lipschitzian when $0 < \nu < 1$).

THEOREM 2.5 (Local existence and uniqueness of the solution). *Let us assume that the dimension $d = 1, 2$ or 3. We consider $d/4 < \alpha < 1$ such that $F : Z^\alpha \mapsto Z$ is*

locally Lipschitzian. Then for any $z_0 \in Z^\alpha$, there exists $T > 0$ and a unique strong solution $z(t)$ defined on $[0, T)$ for the Cauchy problem

$$\frac{dz}{dt} + \mathcal{A}z + F(z) = -S(t), \quad z(0) = z_0. \tag{2.13}$$

We point out that choosing α such that $F : Z^\alpha \mapsto Z$ is locally Lipschitzian imposes a strong constraint on the initial data $z_0 = (u_0, w_0) \in D(A^\alpha) \times B^k$. In dimension 3 for instance, one must have $\alpha > 3/4$: that is u_0 *"is almost in $H^2(\Omega)$"*.

Proof. That theorem is a direct application of the local existence and uniqueness theorem in [8] since:

- there always exists $0 \leq \alpha < 1$ such that F extend to a function $F : Z^\alpha \mapsto Z$ locally Lipschitzian.
- $\mathcal{A} = A \times 0$ is sectorial because A is sectorial and because the zero operator on B^k is bounded and therefore sectorial,
- $t \mapsto S(t)$ is locally Hölder continuous for some $0 < \nu < 1$ or locally Lipschitzian by assumption on the stimulating currents s_i and s_e.

□

2.3. Regularity of the solutions. We will assume throughout that subsection that $B = C^\nu(\Omega)$ and that the reaction terms f and g have C^2 regularity. We will moreover assume that the boundary $\partial\Omega$ of the domain has $C^{2+\nu}$ regularity.

We point out that ν represents here any real number $0 < \nu < 1$.

The operator $\mathcal{A}$ has a smoothing effect of the solutions of (2.5): for an initial value $u_0 \in D(A^\alpha)$, the solution satisfy $u(t) \in D(A)$ for $t > 0$. This is due to the following elliptic regularity result (see [6]):

LEMMA 2.6. *Let σ be a uniformly elliptic tensor on Ω whose components belong to $C^{1+\nu}(\Omega)$ for some $\nu > 0$. We also assume the boundary $\partial\Omega$ to have $C^{2+\nu}$ regularity. If $u \in D(A)$ satisfies $\nabla \cdot (\sigma \nabla u) \in C^\nu(\Omega)$, then $u \in C^{2+\nu}(\Omega)$.*

Beyond this space smoothing effect on the first unknown u, a time smoothing also occurs as shown by the following (see [8]).

LEMMA 2.7. *With the assumption of theorem 2.5, let $t \in (0, T) \mapsto z(t) \in D(\mathcal{A}) = Z^1$ be the solution of the Cauchy problem (2.13). We have $Z^1 \subset Z^\gamma$ for any $\gamma < 1$, and the solution moreover satisfies: $t \in (0, T) \mapsto z(t) \in Z^\gamma$ is continuously (Fréchet) differentiable, for any $\gamma < 1$.*

Together with the previous elliptic regularity argument, these two results imply that the solutions for (2.5) actually are classical solutions provided that the initial data w_0 for the second variable w is smooth enough.

THEOREM 2.8. *Consider $d/4 < \alpha < 1$ and $0 < \nu < 2\alpha - d/2$ such that the embedding $D(A^\alpha) \subset C^\nu(\Omega)$ holds. The stimulating currents $s_{i,e} : t \mapsto L^2(\Omega)$ are assumed locally Hölder continuous (2.4), and such that $s_{i,e}(t) \in C^\nu(\Omega)$ for all $t \geq 0$. For $z_0 = (u_0, w_0) \in D(A^\alpha) \times [C^\nu(\Omega)]^k$, the associated solution on $(0, T)$, considered as a real function of the space variable x and the time variable t, satisfies*

- *$(x, t) \in \overline{\Omega} \times (0, T) \mapsto z(x, t) = (u(x, t), w(x, t))$ is continuously differentiable in t,*
- *$(x, t) \in \overline{\Omega} \times (0, T) \mapsto u(x, t)$ is twice continuously differentiable in x, i.e. $u(\cdot, t) \in C^2(\overline{\Omega})$.*

Proof. Using the embedding (2.12) ensures that the solution $t \in (0, T) \mapsto z(t) \in C^\nu(\Omega) \times [C^\nu(\Omega)]^k$ is continuously (Fréchet) differentiable. This actually implies that $(x, t) \in \overline{\Omega} \times (0, T) \mapsto z(x, t) = (u(x, t), w(x, t))$ is continuously differentiable in t.

Let us now prove that $Au \in C^\nu(\Omega)$ for $t \in (0,T)$. One has $Au = -du/dt - f(u,w) - s(t)$. Easily, $f(u,w) \in C^\nu(\Omega)$ and also $du/dt \in C^\nu(\Omega)$ thanks to lemma 2.7. Now $s(t) = -A_e(A_i + A_e)^{-1}(s_i + s_e) + s_e$ and $(s_i + s_e)(t) \in C^\nu(\Omega)$ by assumption. By lemma 2.6, the function $(A_i + A_e)^{-1}(s_i + s_e)$ belongs to $C^{\nu+2}(\Omega)$ and then $s(t) \in C^\nu(\Omega)$.

Consequently, $Au \in C^\nu(\Omega)$. Remark that $-A_e(A_i + A_e)^{-1}A_i[u] = Au$ (with the previous notations). Lemma 2.6 ensures that $A_e^{-1}Au \in C^{\nu+2}(\Omega)$, and therefore $(A_i + A_e)A_e^{-1}Au \in C^\nu(\Omega)$ and at last $[u] = -A_i^{-1}(A_i + A_e)A_e^{-1}Au \in C^{\nu+2}(\Omega)$. This implies that $x \mapsto u(x,t) \in C^2(\overline{\Omega})$ for $t \in (0,T)$. □

3. Global Solution based on a Variational Formulation.

3.1. The bidomain bilinear form. This section is motivated by the idea that (1.3)-(1.4) has a variational formulation. It is obtained by multiplying Eq. (1.3) by a test function $v \in H^1(\Omega)$, multiplying Eq. (1.4) by a test function $v_e \in H^1(\Omega)$, integrating by parts the second order terms and adding the two resulting variational equations. When first setting the nonlinearity $f(u,w) = 0$ it reads:

$$\frac{d}{dt}\int_\Omega uv\,dx + \int_\Omega (\sigma_i\nabla u)\cdot\nabla v\,dx + \int_\Omega (\sigma_i\nabla u_e)\cdot\nabla v\,dx + \int_\Omega (\sigma_i\nabla u)\cdot\nabla v_e\,dx + \int_\Omega ((\sigma_e+\sigma_i)\nabla u_e)\cdot\nabla v_e = \langle s_i, v\rangle + \langle s_i + s_e, v_e\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $(H^1(\Omega))'$ and $H^1(\Omega)$.

Without the nonlinear term, the equation above has solutions (u, u_e) defined only up to an additive constant. The nonlinear term determines u in $H^1(\Omega)$ but u_e remains defined up to an additive constant. As a consequence, we shall seek a solution u_e in the functional space $U = H^1(\Omega)/\mathbb{R}$. The space U is a Hilbert space with the usual inner product on $H^1(\Omega)$. The inner product $(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx$ can also be used, while the control on the $L^2(\Omega)$ semi-norm can be done through the Poincaré-Wirtinger inequality [1]. The topological dual of U is denoted by U'.

We will first study and reformulate all the bilinear forms associated to the second order spatial derivatives in Eq. (1.3)-(1.4). We will look for a solution u in the quotient space U and later on extend the bilinear form so that u can be taken in all $H^1(\Omega)$. The control of the constant or the average value of the action potential u will then be done through the nonlinear ionic current $f(u,w)$.

In this section, the source terms $s_i(t)$ and $s_e(t)$ are taken in U' for any $t \geq 0$. Moreover, to extend these linear forms over $H^1(\Omega)$ (so that $s_i,\ s_e \in (H^1(\Omega))'$), we impose the following condition related with the physical relation (1.12):

$$\langle s_i + s_e\,,\,1\rangle \;=\; 0 \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality product on $(H^1(\Omega))'\times H^1(\Omega)$.

In order to study the bidomain spatial operator, we only have to focus on the steady equation

$$\int_\Omega (\sigma_i\nabla u)\cdot\nabla v\,dx + \int_\Omega (\sigma_i\nabla u_e)\cdot\nabla v\,dx + \int_\Omega (\sigma_i\nabla u)\cdot\nabla v_e\,dx + \int_\Omega ((\sigma_e+\sigma_i)\nabla u_e)\cdot\nabla v_e = \langle s_i, v\rangle + \langle s_i + s_e, v_e\rangle, \tag{3.2}$$

Like in the previous case (strong bidomain operator), we have to solve the equation for u_e first, and then use the solution back in Eq. (3.2).

Let us define the following bilinear forms $a_{i,e}(\cdot,\cdot)$ on $U \times U$:

$$a_{i,e}(u,v) \equiv \int_\Omega \sigma_{i,e} \nabla u \cdot \nabla v \, dx, \quad \forall (u,v) \in U \times U. \tag{3.3}$$

Under the hypothesis (1.6), it can easily be shown that the bilinear forms $a_{i,e}(\cdot,\cdot)$ are symmetric coercive and continuous on the space U.

We next define the bilinear form $b : (U \times U) \times (U \times U) \to \mathbb{R}$ as

$$\begin{aligned} b((u,u_e),(v,v_e)) = &\int_\Omega (\sigma_i \nabla u) \cdot \nabla v \, dx + \int_\Omega (\sigma_i \nabla u_e) \cdot \nabla v \, dx \\ &+ \int_\Omega (\sigma_i \nabla u) \cdot \nabla v_e \, dx + \int_\Omega ((\sigma_e + \sigma_i) \nabla u_e) \cdot \nabla v_e \, dx. \end{aligned} \tag{3.4}$$

The bilinear form $b(\cdot,\cdot)$ appears in the variational formulation (3.2) of a simplified form of the bidomain model (1.3)-(1.4). It has nice properties:

LEMMA 3.1. *The bilinear form $b(\cdot,\cdot)$ is symmetric, continuous and coercive on $(U \times U) \times (U \times U)$ for the usual product norm $\|(v,v_e)\|_{U\times U} = (\|v\|^2 + \|v_e\|^2)^{1/2}$.*

Proof. The bilinear form $b(\cdot,\cdot)$ can be rewritten as

$$\begin{aligned} b((u,u_e),(v,v_e)) &= \int_\Omega (\sigma_i \nabla (u+u_e)) \cdot \nabla (v+v_e) \, dx + \int_\Omega (\sigma_e \nabla u_e) \cdot \nabla v_e \, dx \\ &= a_i(u+u_e, v+v_e) + a_e(u_e, v_e) \end{aligned}$$

Clearly the bilinear form is symmetric. Using the inequalities (1.6) and the fact that $(\|v_e\|^2 + \|v+v_e\|^2)^{1/2}$ defines a norm equivalent to the norm $\|\cdot\|_{U\times U}$ on $U \times U$, it can easily be seen that the bilinear form is continuous and coercive. ∎

A consequence of the above lemma and the Lax-Milgram theorem is that for any s_i, $s_e \in U'$, there exists unique u, $u_e \in U$ such that

$$b((u,u_e),(v,0)) = \langle s_i, v \rangle, \qquad \forall v \in U, \tag{3.5}$$

$$b((u,u_e),(0,v_e)) = \langle s_i + s_e, v_e \rangle, \qquad \forall v_e \in U, \tag{3.6}$$

or equivalently,

$$b((u,u_e),(v,v_e)) = \langle s_i, v \rangle + \langle s_i + s_e, v_e \rangle, \quad \forall (v,v_e) \in U \times U.$$

Moreover, the solution (u,u_e) can be bounded by (s_i,s_e), namely

$$\|(u,u_e)\|_{U\times U} \le \mathcal{C} \|(s_i,s_e)\|_{U'\times U'}.$$

The variational system (3.5)-(3.6) is nothing but a *steady pure conduction problem* providing the two potentials u and u_e in the myocardium subject to steady source currents s_i and s_e, *assuming that no ions flow through the cellular membranes*. This is comforting to see that this simpler problem (that occurs in a dead heart!) is well-posed.

As a matter of fact, Eq (3.6) is the weak form of (1.4). Its solution is of the general form $u_e = \tilde{u}_e + \bar{u}_e$ where (using the definition of b and $a_{i,e}$):

$$(a_i + a_e)(\bar{u}_e, v_e) = \langle s_i + s_e, v_e \rangle, \quad \forall v_e \in U, \tag{3.7}$$

$$(a_i + a_e)(\tilde{u}_e, v_e) + a_i(u, v_e) = 0, \quad \forall v_e \in U. \tag{3.8}$$

One has to note here that the variational relation (3.7) holds in U only, which do not contain the space $C_c^\infty(\Omega)$. A solution $\bar{u}_e$ of Eq. (3.7) will however satisfy $\nabla \cdot ((\sigma_i + \sigma_e)\nabla \bar{u}_e) = s_i + s_e$ in the sense of distributions thanks to relation (3.1) on the source terms. For the same reason a solution $\tilde{u}_e$ of Eq. (3.8) will satisfy $\nabla \cdot ((\sigma_i + \sigma_e)\nabla \tilde{u}_e) = -\nabla \cdot (\sigma_i \nabla u)$ in the sense of distributions because one has $\langle \nabla \cdot (\sigma_i \nabla u)\,,\, 1\rangle = 0$.

Given $u \in U$, the homogeneous problem (3.8) has a unique solution $\tilde{u}_e \in U$, given by the Lax-Milgram theorem. Additionally, this solution verifies

$$\|u\|_U \leq \|(u, \tilde{u}_e)\|_{U\times U} \leq \mathcal{C}\|u\|_U \tag{3.9}$$

for $\mathcal{C} = \left(1 + \frac{M^2}{4m^2}\right)^{1/2}$. As a consequence, we can define the following linear subspace of the product space $U \times U$:

$$Y = \{(u, \tilde{u}_e) \in U \times U,\ b((u, \tilde{u}_e), (0, v_e)) = 0, \quad \forall v_e \in U\}, \tag{3.10}$$

that is the space of all the weak solutions to the homogeneous equation associated to (1.4). That subspace is well-defined, non-empty and closed (continuity of b). Hence it is a Hilbert space with the inner product inherited from $U \times U$.

The bidomain bilinear form is the restriction of the bilinear form b to the subspace Y. More precisely, the mapping

$$\Phi : u \in U \mapsto \Phi u = (u, \tilde{u}_e) \in Y,$$

where $\tilde{u}_e$ is the solution of Eq. (3.8), is a one-to-one, linear and continuous (due to (3.9)) mapping between U and Y, so that U and Y can be identified (with their respective topologies).

We define the bidomain bilinear form $\bar{a} : U \times U \to \mathbb{R}$ using the bilinear form b and the mapping Φ as:

$$\bar{a}(u, v) = b(\Phi u, \Phi v), \quad \forall (u, v) \in U \times U. \tag{3.11}$$

After simplification, we also have

$$\bar{a}(u, v) = b((u, u_e), (v, 0)), \quad \forall (u, v) \in U \times U,\ (u, u_e) = \Phi u.$$

The bilinear form $\bar{a}(\cdot, \cdot)$ inherits nice properties from $b(\cdot, \cdot)$:

LEMMA 3.2. *The bidomain bilinear form $\bar{a}(\cdot, \cdot)$ defined by* (3.11) *is symmetric, continuous and coercive on U.*

Proof. The result easily follows from (3.11) and because Φ is a one-to-one, linear and continuous mapping, and b is symmetric, bilinear, continuous and coercive on $U \times U$. □

Now, in the homogeneous case where $s_i + s_e = 0$, using the one-to-one mapping Φ shows that solving Eq. (3.2) is equivalent to look for $u \in U$ such that

$$\bar{a}(u, v) = \langle s_i, v\rangle, \quad \forall v \in U. \tag{3.12}$$

This problem is a reformulation of the bidomain pure conduction problem as a mono-domain pure conduction problem, but no longer having a bilinear form derived from usual elliptic second order partial differential operators.

The non homogeneous case where $s_i + s_e \neq 0$ can also be handled in a variational setting by modifying the right-hand-side functional s_i in Eq. (3.12). Indeed, the

equation (3.7) has a unique solution $\bar{u}_e$. Substituting $u_e = \bar{u}_e + \tilde{u}_e$ with $(u, \tilde{u}_e) = \Phi u$ in Eq. (3.2) (or equivalently in Eq. (3.5)) gives

$$b((u, \tilde{u}_e), (v, 0)) + b((0, \bar{u}_e), (v, 0)) = \langle s_i, v \rangle, \quad \forall v \in U.$$

But $b((u, \tilde{u}_e), (v, 0)) = b(\Phi u, (v, 0)) = \bar{a}(u, v)$ and $b((0, \bar{u}_e), (v, 0)) = a_i(\bar{u}_e, v)$, so that Eq. (3.2) is equivalent to

$$\bar{a}(u, v) = \langle s_i, v \rangle - a_i(\bar{u}_e, v), \quad \forall v \in U,$$

that is, the source terms in Eq. (3.12) is replaced by

$$s = s_i + \nabla \cdot (\sigma_i \nabla \bar{u}_e). \tag{3.13}$$

At last, the operator $\bar{a}$ is extended to $H^1(\Omega) \times H^1(\Omega)$, in order to state the full bidomain problem:

DEFINITION 3.3. *The bidomain bilinear form a is defined on $H^1(\Omega) \times H^1(\Omega)$ by*

$$a(u, v) = \bar{a}([u], [v]), \quad \forall (u, v) \in H^1(\Omega) \times H^1(\Omega),$$

where $\bar{a}$ is defined on $U \times U$ by (3.11).

LEMMA 3.4. *The bilinear form $a(\cdot, \cdot)$ defined by definition 3.3 is symmetric, continuous and positive on $H^1(\Omega)$.*

Proof. It obviously has exactly the properties of $\bar{a}$, except coercivity; but still $a(u, u) \geq 0$. As a matter of fact, the kernel of a is the space of functions constant on Ω. ∎

3.2. Formulation of the non-linear problem and hypothesis. We will be working in the framework of the Sobolev spaces and will note $H := L^2(\Omega)$, and $V := H^1(\Omega)$ with their usual norms, and simply note L^p for $L^p(\Omega)$ whenever their is no ambiguity. The topological dual space of H is identified to H and we need to have

$$V \subset L^p \subset H \equiv H' \subset L^{p'} \subset V', \tag{3.14}$$

for $p \geq 2$, where the embeddings are continuous and the injection $V \mapsto H$ is compact. It is possible if

(H1) the Sobolev embedding $H^1 \subset L^p$ holds: $2 \leq p$ if $d = 2$; or $2 \leq p \leq 6$ if $d = 3$.

The problem (1.3)-(1.4)+(1.5) can now be formulated in a variational way. As for the steady conduction problem, we can look for a solution such that $u_e(t) = \bar{u}_e(t) + \tilde{u}_e(t)$ in U with $\bar{u}_e(t)$ solution of (3.7) and $([u(t)], \tilde{u}_e(t)) \in Y$ and rewrite the problem as: Find u and w verifying the following variational equations

$$\langle \partial_t u(t), \phi \rangle + a(u(t), \phi) + \int_\Omega f(u(t), w(t)) \phi \, dx = \langle s, \phi \rangle, \tag{3.15}$$

$$\langle \partial_t w(t), \psi \rangle + \int_\Omega g(u(t), w(t)) \psi \, dx = 0, \tag{3.16}$$

for any test functions ϕ and ψ in V. Here s is defined by Eq. (3.13).

To ensure the well-posedness of the variational formulation (3.15)-(3.16), hypothesis (H2)-(H3) below on the nonlinear terms are formulated so that the functions f and g taken on $L^p \times L^2$ have their range in $L^{p'}$ $(1/p + 1/p' = 1)$ and L^2, respectively, as will be shown in lemma 3.5. Hence, the integral terms in Eq. (3.15) and (3.16) are duality products respectively in $L^{p'} \times L^p$ and $L^2 \times L^2$.

A fourth hypothesis (H4) will also be needed to get the *a priori* estimates in section 3.4.2.

At last, we only consider the case of real functions w, $w : (t,x) \mapsto w(t,x) \in \mathbb{R}$. Restricting w to be a scalar function as opposed to a vector function as in Section 2 will allow the proof of existence of a global (in time) solution instead of a local solution.

The non-linear functions have to verify the following hypothesis:

(H2) The functions f and g are linear with respect to w:

$$f(u,w) = f_1(u) + f_2(u)w, \quad g(u,w) = g_1(u) + g_2 w, \tag{3.17}$$

where f_1, f_2, g_1 are continuous real functions defined on $\mathbb{R}$ and $g_2 \in \mathbb{R}$.

(H3) There exists constants $c_i \geq 0$ $(i = 1 \dots 6)$ such that for any $u \in \mathbb{R}$,

$$|f_1(u)| \leq c_1 + c_2|u|^{p-1}, \tag{3.18}$$

$$|f_2(u)| \leq c_3 + c_4|u|^{p/2-1}, \tag{3.19}$$

$$|g_1(u)| \leq c_5 + c_6|u|^{p/2}, \tag{3.20}$$

where $p \geq 2$.

(H4) There exists constants $a, \lambda > 0$, $b, c \geq 0$ such that for any $(u,w) \in \mathbb{R}^2$,

$$\lambda u f(u,w) + w g(u,w) \geq a|u|^p - b\left(\lambda|u|^2 + |w|^2\right) - c. \tag{3.21}$$

Lemma 3.5. *Under hypotheses (H2) and (H3), the nonlinear terms f and g map $L^p \times L^2$ to $L^{p'}$ and L^2. Specifically, for any $(u,w) \in \mathbb{R}^2$, we have*

$$|f(u,w)| \leq A_1 + A_2|u|^{p-1} + A_3|w|^{2/p'}, \quad |g(u,w)| \leq B_1 + B_2|u|^{p/2} + B_3|w|, \tag{3.22}$$

where the A_i $(i = 1 \dots 3)$ and B_i $(i = 1 \dots 3)$ are numerical constant that depend only on the c_i $(i = 1 \dots 6)$ and on p.

Proof. For $(u,w) \in \mathbb{R}^2$, we have from (H2) and (H3),

$$|f(u,w)| \leq c_1 + c_2|u|^{p-1} + c_3|w| + c_4|w||u|^{p/2-1}, \quad |g(u,w)| \leq c_5 + c_6|u|^{p/2} + |g_2||w|.$$

The second inequality above is exactly the second inequality in (3.22). With $u, w \in L^2(\Omega)$, the right-hand side is clearly in L^2, hence $g(u,w) \in L^2$.

If $p = 2$ then the first inequality in (3.22) is deduced immediately from the first inequality above. Otherwise, it is proved by Young's inequality for $\beta = 2/p' > 1$:

$$|w||u|^{p/2-1} \leq \frac{|w|^\beta}{\beta} + \frac{|u|^{(p/2-1)\beta'}}{\beta'}.$$

Since $\left(\frac{p}{2} - 1\right)\beta' = \left(\frac{p}{2} - 1\right) 2\frac{p-1}{p-2} = p - 1$, with $\frac{1}{\beta} + \frac{1}{\beta'} = 1$, we have

$$|f(u,w)| \leq c_1 + \left(c_2 + \frac{c_4}{\beta'}\right)|u|^{p-1} + c_3|w| + \frac{c_4}{\beta}|w|^\beta.$$

But $\beta > 1$ and then we also have $|w| \leq \frac{|w|^\beta}{\beta} + \frac{1}{\beta'}$, so that it can be found positive constants A_1, A_2 and A_3 such that

$$|f(u,w)| \leq A_1 + A_2|u|^{p-1} + A_3|w|^\beta,$$

which is exactly the first inequality in (3.22).

Now, we have from Jensen inequality and (3.22),

$$\int_\Omega |f(u,w)|^{p'} \leq (A_1+A_2+A_3)^{p'/p}\left(A_1|\Omega| + A_2\int_\Omega |u|^p + A_3\int_\Omega |w|^2\right), \tag{3.23}$$

$$\int_\Omega |g(u,w)|^2 \leq (B_1+B_2+B_3)\left(B_1|\Omega| + B_2\int_\Omega |u|^p + B_3\int_\Omega |w|^2\right), \tag{3.24}$$

because $(p-1)p' = p$, $\beta p' = 2$. □

3.3. Weak operator. Of course, the whole variational process can be handled through operators. These are weak operators, defined from U onto U' and are extensions of the strong operators defined in section 2. Namely we are able to define $A^w_{i,e}$ and $\overline{A}^w$ by duality by setting

$$\langle A^w_{i,e}u, v\rangle = a_{i,e}(u,v), \quad \langle \overline{A}^w u, v\rangle = \bar{a}(u,v), \quad \forall (u,v)\in U\times U.$$

They are all one-to-one continuous mappings from U onto U', and we can also write

$$\bar{u}_e = (A^w_i + A^w_e)^{-1}(s_i+s_e), \quad \tilde{u}_e = -(A^w_i+A^w_e)^{-1}A_i u,$$

and then

$$\begin{aligned}\overline{A}^w &= A^w_i(A^w_i+A^w_e)^{-1}A^w_e = ((A^w_i)^{-1}+(A^w_e)^{-1})^{-1},\\ s(t) &= s_i(t) - A^w_i(A^w_i+A^w_e)^{-1}(s_i(t)+s_e(t)).\end{aligned}$$

At last, the bidomain weak operator A^w is defined by duality from V onto V' by

$$\langle A^w u, v\rangle := \bar{a}([u],[v]) = \langle \overline{A}^w[u],[v]\rangle = \langle \overline{A}^w[u], v\rangle, \quad \forall (u,v)\in V\times V.$$

Again, we have that $A^w u = \overline{A}^w[u]$ for any $u\in V$. The weak solution of the bidomain problem is also the solution of

$$\begin{aligned}\frac{du}{dt} + f(u,w) + A^w u &= s,\\ \frac{dw}{dt} + g(u,w) &= 0,\end{aligned}$$

with the initial condition (1.11). Note that, although it is a positive operator, A^w is not in general a differential operator, being the harmonic average of elliptic operators.

3.4. Existence for the Initial value Problem. We shall now prove the existence of weak solutions to the bidomain equations (1.3), (1.4), (1.5) with initial data (1.11) and boundary conditions (1.7). The nonlinearity functions f and g verify hypotheses (H2), (H3), (H4). Three examples of models in electrocardiology satisfying these assumptions will be given in section 4.

A weak solution is a couple of functions (u,w) that verifies the variational equations (3.15) and (3.16) for any test functions ϕ and ψ in V. The solution will be found with $u(t)$ and $w(t)$ respectively in the Sobolev spaces V and H for almost every $t>0$. The functions u and w will also be continuous from $[0,T]$ respectively to V' and H, for the initial data to make sense.

Approximate solutions are constructed using the method of Faedo-Galerkin. A priori estimates will be established thanks to the hypothesis (H1) on p and (H2)-(H4) on the nonlinear terms.

An *a priori* estimate on u in $H^1 \cap L^p$ is obtained in a classical way [9], and from that estimate on u, we find a subsequence that converges strongly[1] in H. Such strong convergence in H is not possible for w from the sole L^2 *a priori* estimate on w. In any case, we will be able to pass to the limit in the variational formulation because f and g are both linear with respect to w, resulting in the following theorem:

THEOREM 3.6. *Suppose that hypotheses (H1) to (H4) on p and the functions f and g are satisfied. If* $u_0 \in L^2(\Omega)$ *and* $w_0 \in L^2(\Omega)$ *and* $s \in C^0(\mathbb{R}; (H^1(\Omega))')$*, there exists two functions* $u : \mathbb{R}^+ \to H^1(\Omega) \cap L^p(\Omega)$ *and* $w : \mathbb{R}^+ \to L^2(\Omega)$ *such that, for any* $T > 0$*,*

$$u \in C^0([0,T]; (H^1)') \cap L^2(0,T; H^1) \cap L^p(]0,T[\times\Omega), \quad \frac{du}{dt} \in L^{p'}(0,T; (H^1)'),$$
$$w \in C^0([0,T]; L^2), \quad \frac{dw}{dt} \in L^2(]0,T[\times\Omega), \quad 1/p + 1/p' = 1,$$

that verify (3.15), (3.16) *and* (1.11) *for any test functions* ϕ *and* ψ *in* V*.*

We use the classical spaces $L^p(0,T;X)$ $(1 \le p \le \infty)$ of vector valued functions $f : t \in]0,T[\mapsto f(t) \in X$ where X is a Banach space[2]. The derivative $\partial_t f$ (or $\frac{df}{dt}$) of this function is taken in the space of vector valued distributions from $]0,T[$ onto X [9]. A distribution f and a function $\underline{f} \in L^p(0,T;X)$ are identified if

$$\langle f, \phi \rangle = \int_0^T \underline{f}(t)\phi(t)dt \quad \forall \phi \in \mathcal{D}(0,T),$$

where $\mathcal{D}(0,T)$ is the space of real functions C^∞ on $]0,T[$ with compact support in $]0,T[$.

3.4.1. Construction of an approximate solution. We use the special basis $(\psi_i)_{i\in\mathbb{N}}$ defined in Prop. 2.2. The ψ_i are also solution of the eigenproblem

$$\forall v \in V, \quad a(\psi_i, v) = \lambda_i (\psi_i, v)_H.$$

The functions $\psi_i \in V$ for all $i \ge 0$ form an orthonormal Hilbert basis in H and $0 \le \lambda_0 < \lambda_1 \ldots$. For $m \ge 1$, we note $V_m = \text{span}(\psi_0, \ldots, \psi_m) \subset V \subset L^2$. We are looking for a couple of functions $t \mapsto (u_m(t), w_m(t))$ with

$$u_m(t) = \sum_{i=0}^m u_{im}(t)\psi_i \in V_m, \quad w_m(t) = \sum_{i=0}^m w_{im}(t)\psi_i \in V_m$$

where $(u_{im}(t), w_{im}(t))_{i=0\ldots m}$ are real valued functions solutions of

$$\frac{d}{dt}u_{im}(t) + \lambda_i u_{im}(t) + \int_\Omega f(u_m(t), w_m(t))\psi_i = s_i(t), \tag{3.25}$$
$$\frac{d}{dt}w_{im}(t) + \int_\Omega g(u_m(t), w_m(t))\psi_i = 0 \tag{3.26}$$

for $i = 0 \ldots m$, where $s_i(t) = \langle s(t), \psi_i \rangle_{V' \times V}$, and with initial data

$$u_m(0) = u_{m0}, \quad w_m(0) = w_{m0}. \tag{3.27}$$

[1] Here, we also need the injection $H^1 \subset L^p$ to estimate $\frac{du}{dt}$.
[2] X alternatively is V, V' or H here.

Since u_0 and w_0 are in H, we can take u_{m0} and w_{m0} to be the H orthogonal projections of u_0 and w_0 on V_m:

$$\|u_{m0} - u_0\|_H \to 0, \quad \|w_{m0} - w_0\|_H \to 0 \quad \text{as } m \to \infty. \tag{3.28}$$

Equations (3.25) and (3.26) make sense because $u_m \in V \subset L^p$, $w_m \in L^2$ so that $f(u_m, w_m) \in L^{p'}$ and $g(u_m, w_m) \in L^2$ (from hypothesis (H2)+(H3) and lemma 3.5) and $\psi_i \in V$ so that $\psi_i \in L^p$ and $\psi_i \in L^2$. Moreover, it can easily be seen that the last three terms in Eq. (3.25) and the last term in Eq. (3.26) are continuous functions of u_{im} and w_{im}.

The initial value problem composed of the $2m+2$ differential equations (3.25)-(3.26) with initial data (3.27) has a maximal solution defined for $t \in [0, t_m[$ with u_{im} and w_{im} in C^1 (theorem of Cauchy-Peano from [4] – page 59). If (u_m, w_m) is not a global solution (i.e. $t_m < +\infty$) then it is unbounded in $[0, t_m[$. It will be shown in the next section, using *a priori* estimates, that (u_m, w_m) remains bounded for all time, namely $t_m = +\infty$.

3.4.2. A priori estimates. We prove the following lemma in this section:

LEMMA 3.7. *Using the λ of Hypothesis (H4), for any $T > 0$, the solutions u_m, w_m exist on $[0, T]$ and there are constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ such that*

$$\lambda\|u_m(t)\|_H^2 + \|w_m(t)\|_H^2 \le \mathcal{C}_1, \qquad \forall t \in [0, T], \tag{3.29}$$

$$\|u_m\|_{L^p(]0,T[\times\Omega)} \le \mathcal{C}_2, \tag{3.30}$$

$$\|u_m\|_{L^2(]0,T[;V)} \le \mathcal{C}_3, \tag{3.31}$$

$$\int_0^T \|u_m'(t)\|_{V'}^{p'} dt \le \mathcal{C}_4. \tag{3.32}$$

Multiplying (3.25) by λu_{im} (λ defined in hypothesis (H4)), multiplying (3.26) by w_{im}, and summing over $i = 0 \dots m$ yields

$$\frac{1}{2}\lambda\frac{d}{dt}\|u_m\|_H^2 + \frac{1}{2}\frac{d}{dt}\|w_m\|_H^2 + \lambda a(u_m, u_m) + \int_\Omega (\lambda f(u_m, w_m)u_m + g(u_m, w_m)w_m) = \langle s(t), u_m\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between V' and V. With the properties of a and hypothesis (H4) we have for any $t \in [0, t_m)$,

$$\begin{aligned}
&\frac{1}{2}\lambda\frac{d}{dt}\|u_m\|_H^2 + \frac{1}{2}\frac{d}{dt}\|w_m\|_H^2 + \alpha\lambda\int_\Omega |\nabla u_m|^2 + a\int_\Omega |u_m|^p \\
\le& b\left(\lambda\|u_m\|_H^2 + \|w_m\|_H^2\right) + c|\Omega| + \|s(t)\|_{V'}\|u_m(t)\|_V \\
\le& b\left(\lambda\|u_m\|_H^2 + \|w_m\|_H^2\right) + c|\Omega| + \frac{1}{2\xi}\|s(t)\|_{V'}^2 + \frac{\xi}{2}\|u_m(t)\|_V^2,
\end{aligned}$$

for any $\xi > 0$. And then, for $\xi = \alpha\lambda$, we have (because $\|u_m(t)\|_V^2 = \|u_m(t)\|_H^2 + \int_\Omega |\nabla u_m|^2$),

$$\frac{1}{2}\frac{d}{dt}\left(\lambda\|u_m\|_H^2 + \|w_m\|_H^2\right) + \frac{\alpha\lambda}{2}\int_\Omega |\nabla u_m|^2 + a\int_\Omega |u_m|^p \le \tilde{b}\left(\lambda\|u_m\|_H^2 + \|w_m\|_H^2\right) + c|\Omega| + \frac{1}{2\alpha\lambda}\|s(t)\|_{V'}^2. \tag{3.33}$$

where $\tilde{b} = b + \alpha/2$.

We know that $\|u_m(0)\|_H \leq \|u_0\|_H$, $\|w_m(0)\|_H \leq \|w_0\|_H$, Ω is bounded, and $\int_0^t \|s(\tau)\|^2_{V'} d\tau \leq t \sup_{[0,t]} \|s(t)\|^2_{V'}$. From the Gronwall inequality, there exists a constant $\mathcal{C} > 0$ that depends only on $\sigma_{i,e}$, f, g, u_0, w_0, Ω, s and t_m, such that

$$0 \leq t < t_m \;\Rightarrow\; \lambda \|u_m(t)\|^2_H + \|w_m(t)\|^2_H \leq \mathcal{C}.$$

As an immediate consequence, the solution u_m, w_m cannot explode in finite time, it therefore exists over any finite interval $[0, T]$ with the estimate (3.29) of lemma 3.7 being valid.

Coming back to (3.33), we also have the estimate (3.30) of lemma 3.7 with

$$(\mathcal{C}_2)^p = \frac{\tilde{b}}{a}\mathcal{C}T + \frac{c}{a}|\Omega|T + \frac{1}{2a\alpha\lambda}\mathcal{S} + \frac{1}{2a}\left(\lambda\|u_0\|^2_H + \|w_0\|^2_H\right),$$

where $\mathcal{S} = \int_0^T \|s(t)\|^2_{V'}$.

Integrating (3.33) on $]0, T[$, one obtain

$$\int_0^T \int_\Omega |\nabla u_m|^2 \leq \frac{2\tilde{b}}{\alpha\lambda}\mathcal{C}T + \frac{2c}{\alpha\lambda}|\Omega|T + \frac{1}{(\alpha\lambda)^2}\mathcal{S} + \frac{1}{\alpha\lambda}\left(\lambda\|u_0\|^2_H + \|w_0\|^2_H\right) := (\mathcal{D}_1)^2. \quad (3.34)$$

This last estimate combined with (3.29) easily gives the bound (3.31).

The remaining estimate on u'_m is the most difficult. Consider now the projection $P_m : V' \to V'$ defined for $u \in V'$ by

$$P_m u = \sum_{i=0}^{m} \langle u, \psi_i \rangle \psi_i,$$

where $\langle \cdot, \cdot \rangle$ still denotes the duality product between V' and V. It is equivalently defined as the unique element of V_m such that $\langle u, v \rangle = \langle P_m u, v \rangle$ for all $v \in V_m$. Since $u_m \in V_m \subset V \subset L^p$ and $w_m \in L^2$, we have $f(u_m, w_m) \in L^{p'} \subset V'$, and also $u'_m \in V_m$; and then for any $v \in V$,

$$\frac{d}{dt}(u_m, v)_H = (u'_m, v)_H = \langle u'_m, v \rangle,$$
$$\int_\Omega f(u_m, w_m) v = \langle f(u_m, w_m), v \rangle.$$

And then equation (3.25) reads as

$$u'_m = -P_m(A_w u_m) - P_m(f(u_m, w_m)), \quad (3.35)$$

where A_w is the operator associated to the bilinear operator $a(\cdot, \cdot)$ on $V \times V$. For $u, v \in V$, we recall that

$$\langle A_w u, v \rangle = a(u, v) = \sum_{i=0}^{+\infty} \lambda_i (u, \psi_i)_H (v, \psi_i)_H,$$
$$C_0^2 \int_\Omega |\nabla u|^2 dx \leq a(u, u) \leq C_1^2 \int_\Omega |\nabla u|^2 dx,$$
$$\|A_w u\|_{V'} \leq c_7 \|u\|_V.$$

Now, we have

$$\int_0^T \|A_w u_m(t)\|_{V'}^2 dt \le c_7^2 \int_0^T \|u_m\|_V^2 \le c_7^2 \int_0^T \left(\|u_m(t)\|_H^2 + \int_\Omega |\nabla u_m(t)|^2 \right) dt$$
$$\le c_7^2 \left(\frac{\mathcal{C}}{\lambda} T + (\mathcal{D}_1)^2 \right) := (\mathcal{D}_2)^2,$$

and then

$$\|A_w u_m\|_{L^{p'}(0,T;V')}^{p'} = \int_0^T \|A_w u_m(t)\|_{V'}^{p'} dt$$
$$\le T^{1-p'/2} \left(\int_0^T \|A_w u_m\|_{V'}^2 \right)^{p'/2} \le (\mathcal{D}_2)^{p'} T^{1-p'/2}.$$

On the other hand, from (3.23) (and because $L^{p'} \subset V'$ – Sobolev embedding), we have

$$\|f(u_m, w_m)\|_{L^{p'}(0,T;V')}^{p'} \le c_8 \|f(u_m, w_m)\|_{L^{p'}(0,T;L^{p'})}^{p'} = c_8 \int_0^T \int_\Omega |f(u_m, w_m)|^{p'}$$
$$\le c_8 (A_1 + A_2 + A_3)^{p'/p} \int_0^T \left(A_1 |\Omega| + A_2 \int_\Omega |u_m|^p + \int_\Omega |w_m|^2 \right)$$
$$\le c_8 (A_1 + A_2 + A_3)^{p'/p} \left(A_1 |\Omega| T + A_2 (\mathcal{C}_2)^p + \mathcal{C} T \right) := (\mathcal{D}_3)^{p'}.$$

As a consequence, we get from (3.35),

$$\|u_m'\|_{L^{p'}(0,T;V')}^{p'} \le \int_0^T \|P_m\|_{\mathcal{L}(V',V')}^{p'} \left(\|A_w u_m\|_{V'} + \|f(u_m, w_m)\|_{V'} \right)^{p'}$$
$$\le \|P_m\|_{\mathcal{L}(V',V')}^{p'} 2^{p'/p} \left((\mathcal{D}_2)^{p'} T^{1-p'/2} + (\mathcal{D}_3)^{p'} \right) := (\mathcal{C}_4)^{p'}. \quad (3.36)$$

To obtain the estimate (3.32) of lemma 3.7, it remains only to bound the norm of P_m as an operator from V' to V'. First, remark that the restriction of P_m to V can be viewed as an operator from V onto V (since $P_m(V') \subset V_m \subset V$), given by

$$\forall u \in V, \quad P_m u = \sum_{i=0}^{m} (u, \psi_i)_H \psi_i.$$

Its transpose P_m^T identifies with $P_m : V' \to V'$, and then we have $\|P_m\|_{\mathcal{L}(\mathcal{V}',\mathcal{V}')} = \|P_m\|_{\mathcal{L}(\mathcal{V},\mathcal{V})}$. For $u \in V$ we can compute

$$a(P_m u, P_m u) = \sum_{i=0}^{+\infty} \lambda_i (P_m u, \psi_i)_H (P_m u, \psi_i)_H$$
$$= \sum_{i=0}^{m} \lambda_i (u, \psi_i)_H (u, \psi_i)_H \le \sum_{i=0}^{+\infty} \lambda_i |(u, \psi_i)_H|^2 = a(u, u).$$

Additionally, it is easy to see that $\|P_m u\|_H \le \|u\|_H$ (in fact P_m coincides with the L^2 perpendicular projection on V_m). As a consequence,

$$\|P_m u\|_V = \left(\int_\Omega |\nabla P_m u|^2 dx + \|P_m u\|_H^2 \right)^{1/2} \le \left(\frac{C_1^2}{C_0^2} \int_\Omega |\nabla u|^2 dx + \|u\|_H^2 \right)^{1/2} \le \frac{C_1}{C_0} \|u\|_V$$

(because $C_1 \ge C_0$). It shows that P_m is uniformly bounded in V': $\|P_m\|_{\mathcal{L}(\mathcal{V}',\mathcal{V}')} \le C_1/C_0$.

3.4.3. Convergence towards a solution. As a consequence of lemma 3.7, the sequence (u_m) remains in a compact set of $L^2(0,T;H)$. Indeed, it follows from a classical compactness result, see for instance [9], theorem 5.1 page 58.

As a consequence, we can construct subsequences of u_m and w_m, still denoted by u_m and w_m, such that

- $u_m \to u$ weak in $L^p(]0,T[\times\Omega)$,
- $u_m \to u$ weak-$\star$ in $L^\infty(0,T;H)$ and weak in $L^2(0,T;V)$,
- $u_m \to u$ strong in $L^2(0,T;H)$, and then almost everywhere in $]0,T[\times\Omega$,
- $w_m \to w$ weak-$\star$ in $L^\infty(0,T;H)$, and then weak in $L^2(0,T;H)$.

We are left proving that the limits u and w of these subsequences verify the equations (3.15) and (3.16). in the sense of distributions on $\mathcal{D}(0,T)$.

For $i \geq 1$ fixed, we naturally have

$$-\int_0^T \int_\Omega u'_m \psi_i \phi = \int_0^T \int_\Omega u_m \psi_i \phi' \to \int_0^T \int_\Omega u \psi_i \phi',$$
$$-\int_0^T \int_\Omega w'_m \psi_i \phi = \int_0^T \int_\Omega w_m \psi_i \phi' \to \int_0^T \int_\Omega w \psi_i \phi',$$

for any $\phi \in \mathcal{D}(0,T)$, because $\psi_i \phi' \in L^2(0,T;H)$. Similarly, $a(u,\psi)$ is bilinear and continuous on $V \times V$, and then

$$\int_0^T \int_\Omega a(u_m, \psi_i \phi) \to \int_0^T \int_\Omega a(u, \psi_i \phi),$$

for any $\phi \in \mathcal{D}(0,T)$, from the weak convergence of u_m in $L^2(0,T;V)$.

Concerning the nonlinear terms, we use hypothesis (H2) to write

$$f(u_m, w_m) = f_1(u_m) + f_2(u_m) w_m = f_1(u_m) + (f_2(u_m) - f_2(u))\, w_m + f_2(u) w_m,$$
$$g(u_m, w_m) = g_1(u_m) + g_2 w_m.$$

Now, $u_m \to u$ a.e. in $]0,T[\times\Omega$ and f_1 is continuous, so that $f_1(u_m)\psi_i\phi \to f_1(u)\psi_i\phi$ a.e.in $]0,T[\times\Omega$. Additionally, for any $\phi \in \mathcal{D}(0,T)$,

$$\int_0^T \int_\Omega |f_1(u_m)\psi_i\phi| \leq \|f(u_m,0)\|_{L^{p'}(0,T;L^{p'})} \|\psi_i\phi\|_{L^p(0,T;L^p)} \leq M,$$

where M does not depend on m, so that we can apply the dominated convergence theorem of Lebesgue:

$$\int_0^T \int_\Omega f_1(u_m)\psi_i\phi \to \int_0^T \int_\Omega f_1(u)\psi_i\phi.$$

Similarly, g_1 is continuous and then $g_1(u_m)\psi_i\phi \to g_1(u)\psi_i\phi$ a.e.in $]0,T[\times\Omega$; and $g_1(u_m)\psi_i\phi$ is bounded in $L^1(]0,T[\times\Omega)$,

$$\int_0^T \int_\Omega |g_1(u_m)\psi_i\phi| \leq \|g(u_m,0)\|_{L^2(0,T;L^2)} \|\psi_i\phi\|_{L^2(0,T;L^2)} \leq M$$

where M does not depend on m. From the theorem of Lebesgue we have:

$$\int_0^T \int_\Omega g_1(u_m)\psi_i\phi \to \int_0^T \int_\Omega g_1(u)\psi_i\phi.$$

Since $w_m \to w$ weak in $L^2(0,T;H)$ we naturally have

$$\int_0^T \int_\Omega g_2 w_m \psi_i \phi \to \int_0^T \int_\Omega g_2 w \psi_i \phi.$$

The second term in f is such that, for $\beta = \frac{2p}{p-2}$ (verifying $\frac{1}{\beta} + \frac{1}{2} + \frac{1}{p} = 1$), we have

$$\left| \int_0^T \int_\Omega \left(f_2(u_m) - f_2(u)\right) w_m \psi_i \phi \right| \\ \leq \|f_2(u_m) - f_2(u)\|_{L^\beta(0,T;L^\beta)} \|w_m\|_{L^2(0,T;H)} \|\psi_i \phi\|_{L^p(0,T;L^p)},$$

where $\|w_m\|_{L^2(0,T;H)}$ is bounded. Again, $f_2(u_m) \to f_2(u)$ a.e. in $]0,T[\times\Omega$ (continuity of f_2), and

$$\int_0^T \int_\Omega |f_2(u_m) - f_2(u)|^\beta \leq \int_0^T \int_\Omega 2^{\beta-1} \left(|f_2(u_m)|^\beta + |f_2(u)|^\beta\right).$$

But from (3.19) (hypothesis (H3)), we have

$$\forall u \in \mathbb{R}, \quad |f_2(u)|^\beta \leq (c_3 + c_4)^{\beta-1} \left(c_3 + c_4 |u|^{\beta(p/2-1)}\right) = (c_3 + c_4)^{\beta-1} \left(c_3 + c_4 |u|^p\right),$$

so that $f_2(u_m)$ and $f_2(u)$ are bounded in $L^\beta(0,T;L^\beta)$. As a consequence, the theorem of Lebesgue applies and $\|f_2(u_m) - f_2(u)\|_{L^\beta(0,T;L^\beta)} \to 0$, and then

$$\int_0^T \int_\Omega \left(f_2(u_m) - f_2(u)\right) w_m \psi_i \phi \to 0.$$

At last, we also have $f_2(u)\psi_i \in L^2(0,T;H)$ because $\frac{2}{\beta} + \frac{2}{p} = 1$ and then

$$|f_2(u)\psi_i|^2 \leq \frac{2}{p} |\psi_i|^p + \left(1 - \frac{2}{p}\right) |f_2(u)|^\beta.$$

As a consequence,

$$\int_0^T \int_\Omega f_2(u) w_m \psi_i \phi \to \int_0^T \int_\Omega f_2(u) w \psi_i \phi.$$

Gathering all these results, we have

$$-\int_0^T (u, \psi_i)_H \phi' + \int_0^T a(u, \psi_i)\phi + \int_0^T \int_\Omega f(u,w)\psi_i \phi = \int_0^T \langle s(t), \psi_i \rangle \phi,$$
$$-\int_0^T (w, \psi_i)_H \phi' + \int_0^T \int_\Omega g(u,w)\psi_i \phi = 0,$$

for any $i \geq 1$. This is exactly (3.15) and (3.16) in the sense of distribution on $\mathcal{D}(0,T)$. The conclusion holds because $(\psi_i)_{i\geq 0}$ is dense in V.

3.5. Continuity. From (3.15) and (3.16), we can see that the weak derivatives of u and w are such that $\frac{du}{dt} \in L^{p'}(0,T;V')$ and $\frac{dw}{dt} \in L^2(0,T;H)$. From a classical theorem (e.g. see Lemma 1.1 of Chapter 3 in [13]), we have $u \in C^0([0,T];V')$ and $w \in C^0([0,T];H)$, and $u(0) = u_0$ in V', $w(0) = w_0$ in H.

3.6. Uniqueness. Assume that (u_1, w_1) and (u_2, w_2) are two weak solutions of (3.15), (3.16) with the same initial data (u_0, w_0). Using the equalities (in $\mathcal{D}'(0,T)$)

$$\left\langle \frac{du}{dt}, u \right\rangle_{V' \times V} = \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}, \quad \left(\frac{dw}{dt}, w \right)_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)},$$

for functions $u \in L^2(0,T;L^2)$ such that $\frac{du}{dt} \in L^{p'}(0,T,V')$ and $w \in L^2(0,T;L^2)$ such that $\frac{dw}{dt} \in L^2(0,T;L^2)$, we naturally have

$$\frac{1}{2}\frac{d}{dt}\|u_1 - u_2\|^2_{L^2} + a(u_1 - u_2, u_1 - u_2) + \int_\Omega \left(f(u_1, w_1) - f(u_2, w_2)\right)(u_1 - u_2) = 0,$$
$$\frac{1}{2}\frac{d}{dt}\|w_1 - w_2\|^2_{L^2} + \int_\Omega \left(g(u_1, w_1) - g(u_2, w_2)\right)(w_1 - w_2) = 0.$$

With a linear combination of these two equations, we will be able to conclude using a Gronwall inequality if we can bound below

$$\Phi(u_1, w_1, u_2, w_2) = \int_\Omega \mu \left(f(u_1, w_1) - f(u_2, w_2)\right)(u_1 - u_2) + \left(g(u_1, w_1) - g(u_2, w_2)\right)(w_1 - w_2) dx$$

for some $\mu > 0$. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(u, w) = \begin{bmatrix} \mu f(u, w) \\ g(u, w)) \end{bmatrix}$$

and denote by $z \in \mathbb{R}^2$ the vector $z = (u, w)^T$. Then we have

$$\Phi(u_1, w_1, u_2, w_2) = \Phi(z_1, z_2) = \int_\Omega (F(z_1) - F(z_2)) \cdot (z_1 - z_2) dx,$$

where $\cdot$ denotes the inner product in $\mathbb{R}^2$. Here F is continuously differentiable, so that Taylor expansion with an integral remainder implies that $\forall z_1, z_2 \in \mathbb{R}^2$

$$F(z_1) - F(z_2) = \int_0^1 [\nabla F(z_\theta)](z_1 - z_2)\, d\theta$$

where $z_\theta = \theta z_1 + (1 - \theta) z_2$ and $\nabla F = \begin{pmatrix} \mu \partial_u f & \mu \partial_w f \\ \partial_u g & \partial_w g \end{pmatrix}$.

Now, let $Q(z) = \frac{1}{2}(\nabla F(z)^T + \nabla F(z))$ be the symmetric part of ∇F for $z \in \mathbb{R}^2$, and denote by $\lambda_1(z) \leq \lambda_2(z)$ its eigenvalues. We can complete the proof under the hypothesis that

$$\exists C \in \mathbb{R},\ \forall z \in \mathbb{R}^2, \quad \lambda_2(z) \geq \lambda_1(z) \geq C. \tag{3.37}$$

Indeed, in that case, we have

$$\Phi(z_1, z_2) = \int_\Omega \int_0^1 (z_1 - z_2)^T [\nabla F(z_\theta)](z_1 - z_2)\, d\theta dx \geq C \int_\Omega \int_0^1 |z_1 - z_2|^2\, d\theta dx$$
$$\geq C \min(1, \mu^{-1}) \left(\mu \|u_1 - u_2\|^2_{L^2} + \|w_1 - w_2\|^2_{L^2}\right),$$

and consequently,

$$\frac{1}{2}\frac{d}{dt}Y \leq -C\min(1,\mu^{-1})Y, \tag{3.38}$$

where $Y = \left(\mu\|u_1 - u_2\|_{L^2}^2 + \|w_1 - w_2\|_{L^2}^2\right)$.

Using the lemma of Gronwall, we have proved the following result:

THEOREM 3.8. *If the condition (3.37) is satisfied, then the solution obtained in Theorem 3.6 is unique.*

Remark that Eq. (3.38) also provide a stability estimate with respect to the initial condition.

We will apply that theorem to the first example presented below, though it is not clear how to obtain uniqueness for the last two examples.

4. Examples.

4.1. FitzHugh-Nagumo.

The FitzHugh-Nagumo model reads as

$$f(u,w) = u(u-a)(u-1) + w, \quad g(u,w) = -\epsilon(ku - w),$$

with $0 < a < 1$, $k, \epsilon > 0$. The functions f and g are obviously of the form (3.17) with f_1, f_2, g_1 continuous and $g_2 = \epsilon$. Using Young's inequality, we have

$$|u|^2 \leq \frac{2|u|^3}{3} + \frac{1}{3}, \quad |u| \leq \frac{|u|^3}{3} + \frac{2}{3}, \quad |u| \leq \frac{|u|^2}{2} + \frac{1}{2}, \tag{4.1}$$

and then (H3) holds with $p = 4$ (and $c_4 = 0$):

$$\begin{aligned}
&|f_1(u)| = |u(u-a)(u-1)| \leq \frac{2}{3}a + \frac{1}{3}(1+a) + \left(\frac{1}{3}a + \frac{2}{3}(1+a) + 1\right)|u|^3,\\
&|f_2(u)| = 1,\\
&|g_1(u)| = \epsilon k|u| \leq \frac{1}{2}\epsilon k + \frac{1}{2}\epsilon k|u|^2.
\end{aligned}$$

Consider the function $E(u,w) = \epsilon k u f(u,w) + w g(u,w)$ defined in $\mathbb{R}^2$. We have

$$E(u,w) = \epsilon k u^4 - \epsilon k(1+a)u^3 + \epsilon k a u^2 + \epsilon w^2 \geq \epsilon k\left(|u|^4 - (1+a)|u|^3\right).$$

With Young's inequality, we can find a constant $\gamma > 0$ such that

$$(1+a)|u|^3 \leq \frac{|u|^4}{2} + \gamma.$$

Consequently,

$$E(u,w) + \epsilon k\gamma \geq \frac{\epsilon k}{2}|u|^4,$$

which is exactly (H4) with $\lambda = k\epsilon$, $a = k\epsilon/2$, $b = 0$ and $c = k\epsilon\gamma$.

As regards the uniqueness of the solution, we verify the condition (3.37) to apply Theorem 3.8. One easily calculates

$$\nabla F(z) = \begin{bmatrix} \mu(3u^2 - 2(1+a)u + a) & \mu \\ -\epsilon k & \epsilon \end{bmatrix}.$$

Taking $\mu = \epsilon k$, we get rid of the antisymmetric part in the quadratic form and easily bound below the eigenvalues by $C = \epsilon\min(k(a-(1+a)^2/3), 1)$.

4.2. Aliev-Panfilov. The Aliev-Panfilov model [10] is

$$f(u,w) = ku(u-a)(u-1) + uw, \quad g(u,w) = \epsilon\left(ku(u-1-a) + w\right),$$

with $0 < a < 1$, $k, \epsilon > 0$. The functions f and g are obviously of the form (3.17) with f_1, f_2, g_1 continuous and $g_2 = \epsilon$. Using the inequalities (4.1), we get (H3) with $p = 4$ (and $c_4 = 1$, $c_3 = 0$):

$$\begin{aligned}
&|f_1(u)| = k|u(u-a)(u-1)| \le \frac{2}{3}ka + \frac{1}{3}k(1+a) + \left(\frac{1}{3}a + \frac{2}{3}(1+a) + 1\right)k|u|^3,\\
&|f_2(u)| = |u|,\\
&|g_1(u)| = \epsilon k|u(u-1-a)| \le \frac{1}{2}\epsilon k(1+a) + \left(\frac{1}{2}(1+a) + 1\right)\epsilon k|u|^2.
\end{aligned}$$

Now, we compute the function $E(u,w) = \lambda u f(u,w) + w g(u,w)$. It is

$$E(u,w) = \lambda k u^4 - \lambda k(1+a)u^3 + \lambda k a u^2 + (\lambda + \epsilon k)u^2 w - \epsilon k(1+a)uw + \epsilon w^2. \quad (4.2)$$

Here, we will prove (3.21) because it allows negative $|u|^2$ and $|w|^2$ bounds below, so that the terms in u^3 and $u^2 w$ can be manipulated to enter the main bound $\lambda k u^4$. For instance with $\lambda = \epsilon k$, we write

$$|(1+a)u^3| \le \frac{3}{4}\left(\alpha|u|^3\right)^{4/3} + \frac{1}{4}\left(\frac{1+a}{\alpha}\right)^4, \quad (4.3)$$

$$|u^2 w| \le \frac{1}{2}(\beta|u|^2)^2 + \frac{1}{2}\left(\frac{|w|}{\beta}\right)^2, \quad (4.4)$$

$$|uw| \le \frac{1}{2}|u|^2 + \frac{1}{2}|w|^2, \quad (4.5)$$

for any $\alpha > 0$ and $\beta > 0$, and then

$$\begin{aligned}
E(u,w) \ge{}& \left(\epsilon k^2 - \epsilon k^2 \frac{3}{4}\alpha^{4/3} - \epsilon k\beta^2\right)|u|^4\\
&- \frac{1}{4}\epsilon k^2\left(\frac{1+a}{\alpha}\right)^4 - \epsilon k\frac{1}{\beta^2}|w|^2 - \epsilon k\frac{1+a}{2}|u|^2 - \epsilon k\frac{1+a}{2}|w|^2 + \epsilon|w|^2 + \epsilon k^2 a|u|^2.
\end{aligned}$$

Now, we can simply take α and β such that

$$\frac{3}{4}\alpha^{4/3} = \frac{1}{2}, \quad \text{and } \frac{1}{4}\epsilon k^2 = \epsilon k\beta^2,$$

and we get (3.21) with

$$\begin{aligned}
a &= \frac{1}{4}\epsilon k^2,\\
b &= \max\left(\epsilon k\left(\frac{1}{\beta^2} + \frac{1+a}{2}\right) - \epsilon,\ \frac{1+a}{2} - ak\right),\\
c &= \frac{1}{4}\epsilon k^2\left(\frac{1+a}{\alpha}\right)^4.
\end{aligned}$$

4.3. MacCulloch. The model introduced by McCulloch [11] is

$$f(u,w) = bu(u-a)(u-1) + uw, \quad g(u,w) = \epsilon\left(-ku + w\right),$$

with $0 < a < 1$, and $b, k, \epsilon > 0$. The functions f and g are obviously of the form (3.17) with f_1, f_2, g_1 continuous and $g_2 = \epsilon$. Using the inequalities (4.1), we get (H3) with $p = 4$ (and $c_4 = 1$, $c_3 = 0$):

$$\begin{aligned}
&|f_1(u)| = b|u(u-a)(u-1)| \leq \frac{2}{3}ba + \frac{1}{3}b(1+a) + \left(\frac{1}{3}a + \frac{2}{3}(1+a) + 1\right)b|u|^3,\\
&|f_2(u)| = |u|,\\
&|g_1(u)| = \epsilon k|u| \leq \frac{1}{2}\epsilon k + \frac{1}{2}\epsilon k|u|^2.
\end{aligned}$$

Using again (4.3)-(4.5), we have this time

$$\begin{aligned}
E(u,w) &= \lambda b u^4 - \lambda b(1+a)u^3 + \lambda b a u^2 + \lambda u^2 w - \epsilon k u w + \epsilon w^2\\
&\geq \lambda\left(b - \frac{3}{4}\alpha^{4/3}b - \frac{\beta^2}{2}\right)u^4 - \frac{1}{4}\left(\frac{1+a}{\alpha}\right)^4\lambda b\\
&\qquad - \frac{1}{2\beta^2}\lambda|w|^2 - \frac{\epsilon k}{2}|u|^2 - \frac{\epsilon k}{2}|w|^2 + \epsilon|w|^2 + \lambda b a|u|^2,
\end{aligned}$$

and (3.21) holds if we take

$$\frac{3}{4}\alpha^{4/3} = \frac{1}{2}, \quad \text{and } \frac{1}{4}b = \frac{\beta^2}{2}.$$

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