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REGULAR EMBEDDINGS OF MULTIGRAPHS

HUBERT DE FRAYSSEIX AND PATRICE OSSONA DE MENDEZ

ABSTRACT. We prove that the vertex set of any twin-free loopless multigraph G has an embedding into some point set P of some Euclidean space \mathbb{R}^k , such that the automorphism group of G is isomorphic to the isometry group of \mathbb{R}^k globally preserving P .

1. INTRODUCTION

Using spectral analysis, Babai proved in 1978 that the abstract automorphism group of any multigraph G having s distinct eigenvalues with respective multiplicities m_1, m_2, \dots, m_s is a subgroup of $\omega(m_1) \oplus \omega(m_2) \oplus \dots \oplus \omega(m_s)$, where $\omega(m)$ denotes the real orthogonal group of dimension m [1]. As a consequence, if all the eigenvalues of G are simple, the only automorphisms of G are involutions.

Some years before, Mani proved that every triconnected planar graph G can be realized as the 1-skeleton of a convex polytope P in \mathbb{R}^3 such that all automorphisms of G are induced by isometries of P [7]. One non trivial consequence of this result is that the automorphism group $\text{Aut}(G)$ of a triconnected planar graph G has a chain of normal subgroups $\text{Aut}(G) = G_0 \triangleright G_1 \triangleright \dots \triangleright G_m = 1$ where each quotient G_i/G_{i-1} is either cyclic, or isomorphic to a symmetric group or A_5 .

The result of Mani may be expressed in a weaker form: any triconnected planar graph has an embedding f into \mathbb{R}^3 , such that $\text{Aut}(G)$ is the group of isometries of \mathbb{R}^3 globally preserving the point set $P = f(V(G))$, that we shall denote by $\omega(3, P)$.

These two results suggest that a graph G may possibly have some *regular embedding*, that is some embedding $f : V(G) \rightarrow \mathbb{R}^k$ such that $\text{Aut}(G)$ is isomorphic to the group $\omega(k, f(V(G)))$ of isometries of \mathbb{R}^k globally preserving $f(V(G))$, and that this group might be expressed as a subgroup of a group sum relying on spectral considerations. We shall prove that this is indeed the case for any twin-free loopless multigraph. In the remaining of this paper, multigraphs are always assumed to be loopless.

This result will be proved using techniques similar to the one used in a graph symmetry detection heuristic presented in GD'99 [4].

In section 2 we recall several definitions and introduce notations. In section 3 we reduce the study of the automorphism groups of multigraphs to the case of *irreducible* multigraphs, that is to multigraphs having no twin vertices. In section 4 we relate embeddings of a multigraphs to metrics and *distance matrices* on its vertex set. We reduce in section 5 the problem of finding regular embeddings to the one of finding metrics on the vertex set of the multigraph which define *Euclidean*, *reconstructing* and *commuting* distance matrices. We prove in section 6 that Euclidean distance matrices may be built from symmetric real matrices with

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0 on the diagonal, what we call *predistance matrices*. The built distance matrix is commuting/reconstructing if the predistance is. We deduce in section 7 that any commuting reconstructing predistance matrix defines a regular embedding and give some examples of reconstructing commuting predistance matrices. In section 8 we study the special case of regular multigraphs and give a strengthened version of Babai's result [1] in this context. Section 9 is devoted to concluding remarks.

2. DEFINITIONS AND NOTATIONS

Let G be a multigraph. For any $x, y \in V(G)$, the *multiplicity* $\mu(x, y)$ is the number of edges (possibly 0) having x and y as endpoints. An *automorphism* of G is a one-to-one mapping $g : V(G) \rightarrow V(G)$ such that $\mu(g(x), g(y)) = \mu(x, y)$ for any $x, y \in V(G)$. The automorphisms of G define the *automorphism group* $\text{Aut}(G)$ of G .

In order to ease the matrix presentation, we shall assume that the vertex set of a multigraph G of order n is $\{1, \dots, n\}$. Then the *adjacency matrix* of G is the symmetric matrix with entries $\mathbf{A}_{i,j} = \mu(i, j)$ and any automorphism $g \in \text{Aut}(G)$ may be written as a permutation matrix \mathbf{g} , where $\mathbf{g}_{i,j} = \delta_{j,g(i)}$ (δ is the usual Kronecker symbol). By permissive abridgment, $\text{Aut}(G)$ would as well denote the group of these permutation matrices. Notice that $\text{Aut}(G)$ may be then described as the group of the permutation matrices commuting with \mathbf{A} . Also, we will denote by \mathbf{E}_i the n column matrix whose j th entry is $\delta_{i,j}$, $\mathbf{1}$ the n column matrix filled with 1s and by \mathbf{J} the $n \times n$ matrix $\mathbf{1}\mathbf{1}^t$.

The symmetric group acting on a finite set P will be denoted by $\mathfrak{S}(P)$. The real orthogonal group of dimension n will be denoted by $\omega(n)$. If P is a set of points of \mathbb{R}^n , the subgroup of $\omega(n)$ globally preserving P will be denoted by $\omega(n, P)$. Notice that $\omega(n, P) < \mathfrak{S}(P)$.

3. REDUCING GRAPHS

Definition 3.1. Let G be a multigraph. G is *reducible* if there exists two vertices x_1, x_2 such that $\mu(x_1, y) = \mu(x_2, y)$ for any $y \in V(G) \setminus \{x_1, x_2\}$. The vertices x_1 and x_2 are said to be *twins*. If a multigraph is twin-free, it is *irreducible*.

Lemma 3.1. Let X be a subset of vertices, any two elements of which are twins. Then, $\mu(x, y)$ is constant for $x \neq y \in X$.

Proof. Let $X = \{x_1, x_2, \dots, x_k\}$. Then, for $1 < i < j \leq k$: $\mu(x_i, x_j) = \mu(x_1, x_j)$ as x_1 and x_i are twins and $\mu(x_1, x_j) = \mu(x_1, x_2)$ as x_2 and x_j are twins. \square

Lemma 3.2. Let G be a multigraph. For any distinct vertices x, y, z the following holds:

$$\begin{aligned} \text{If } x \text{ and } y \text{ are twins, that is: } & \forall v \in V(G) \setminus \{x, y\}, \quad \mu(x, v) = \mu(y, v) \\ \text{and } y \text{ and } z \text{ are twins, that is: } & \forall v \in V(G) \setminus \{y, z\}, \quad \mu(y, v) = \mu(z, v) \\ \text{then } x \text{ and } z \text{ are twins, that is: } & \forall v \in V(G) \setminus \{x, z\}, \quad \mu(x, v) = \mu(z, v). \end{aligned}$$

Proof. Let $v \in V(G) \setminus \{x, z\}$.

If $v \neq y$: $\mu(x, v) = \mu(y, v)$ (as x and y are twins) and $\mu(y, v) = \mu(z, v)$ (as y and z are twins) hence $\mu(x, v) = \mu(z, v)$.

Otherwise $\mu(x, z) = \mu(x, y)$ (as y and z are twins) and $\mu(x, z) = \mu(y, z)$ (as x and y are twins) hence $\mu(x, v) = \mu(z, v)$. \square

Corollary 3.3. *The vertex set of any multigraph G has a unique partition $\mathcal{P} = \{V_1, \dots, V_k\}$, such that any V_i is a maximal subset of twin vertices of G . This partition is the twin-decomposition of G .*

Corollary 3.4. *Let G be a multigraph, let \mathcal{P} be its twin decomposition and let G/\mathcal{P} denote the quotient multigraph. Then*

$$(1) \quad \text{Aut}(G) = \bigoplus_{X \in \mathcal{P}} \mathfrak{S}(X) \oplus \text{Aut}(G/\mathcal{P})$$

Moreover, no subset $Y \subseteq V(G/\mathcal{P})$ of cardinality at least two is such that $\mathfrak{S}(Y)$ is a subgroup of $\text{Aut}(G/\mathcal{P})$.

Proof. Assume there exists a subset $Y \subseteq V(G/\mathcal{P})$ of cardinality at least two is such that $\mathfrak{S}(Y)$ is a subgroup of $\text{Aut}(G/\mathcal{P})$. Let τ be a transposition in $\mathfrak{S}(Y)$ exchanging two vertices a and b of G/\mathcal{P} . As τ is an automorphism of G/\mathcal{P} it follows that a and b are twins in G/\mathcal{P} . Identifying a and b with the classes of twins of vertices in G , it follows that any $x \in a$ is a twin of any $y \in b$ in G hence $a \cup b$ is a class of twins, what contradicts the maximality of the classes in \mathcal{P} . \square

4. EMBEDDINGS AND DISTANCES

Recall that a *metric* on a set X is a mapping $d : X^2 \rightarrow \mathbb{R}^+$ satisfying the usual axioms of a metric, that is:

$$\begin{aligned} \forall (x, y) \in X^2, \quad d(x, y) &= d(y, x) \\ \forall (x, y) \in X^2, \quad d(x, y) &= 0 \iff x = y \\ \forall (x, y, z) \in X^3, \quad d(x, z) &\leq d(x, y) + d(y, z) \end{aligned}$$

An *Euclidean metric* on a set X is a metric d on X such that there exist some Euclidean space \mathbb{R}^k and an embedding $f : X \rightarrow \mathbb{R}^k$ so that, for any $x, y \in X$:

$$\text{dist}(f(x), f(y)) = d(x, y)$$

where dist is the Euclidean metric of \mathbb{R}^k . Any embedding of a multigraph G into some Euclidean space \mathbb{R}^k defines an Euclidean metric d_f on $V(G)$ by $d_f(x, y) = \text{dist}(f(x), f(y))$.

Definition 4.1. The *distance matrix* of a metric d on the set $\{1, \dots, n\}$ is the $n \times n$ real symmetric matrix \mathbf{D} defined by:

$$\forall i, j \in \{1, \dots, n\}, \quad \mathbf{D}_{i,j} = d(i, j)^2$$

Notice that the entries of \mathbf{D} are the **squares** of the distances and not the distances themselves.

Definition 4.2. Let G be a multigraph and let \mathbf{D} be a distance matrix of a metric defined on $V(G) = \{1, \dots, n\}$.

The distance matrix \mathbf{D} is

- *Euclidean* if the metric from which \mathbf{D} comes from is Euclidean. A compatible embedding of G into an Euclidean space \mathbb{R}^k is then called a **\mathbf{D} -embedding** of G ;
- *reconstructing* if G may be reconstructed from \mathbf{D} , that is:

$$\exists \Xi : \mathbb{R} \rightarrow \mathbb{N}, \quad \forall i \neq j \in \{1, \dots, n\}, \quad \mu(i, j) = \Xi(\mathbf{D}_{i,j});$$

- *commuting* if any automorphism of G commutes with \mathbf{D} , that is:

$$\forall g \in \text{Aut}(G), \quad g\mathbf{D} = \mathbf{D}g$$

or, equivalently:

$$\forall g \in \text{Aut}(G), \forall i, j \in \{1, \dots, n\} \quad \mathbf{D}_{g(i),g(j)} = \mathbf{D}_{i,j}.$$

5. DISTANCE MATRICES AND REGULAR EMBEDDINGS

Definition 5.1. Let G be a multigraph and let k be an integer. A *regular embedding* of G into \mathbb{R}^k is a mapping $f : V(G) \rightarrow \mathbb{R}^k$ such that $\text{Aut}(G)$ is isomorphic to $\omega(k, f(V(G)))$.

We recall the classical following theorem (which proof may be found in any undergraduate textbook):

Theorem 5.1 (Isometry Extension Theorem). *Let $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ be families of points of an Euclidean affine space \mathcal{E} (with distance dist) such that:*

$$\forall (i, j) \in I^2 \quad \text{dist}(a_i, a_j) = \text{dist}(b_i, b_j)$$

Then there exists an isometry f of \mathcal{E} such that:

$$\forall i \in I \quad f(a_i) = b_i$$

Moreover, if $(b_i)_{i \in I}$ spans \mathcal{E} then f is uniquely determined.

Theorem 5.2. *Let f be a one-to-one embedding of a multigraph G into the Euclidean space \mathbb{R}^k (with metric dist) and let \mathbf{D} be the Euclidean distance matrix of the metric defined by f on $V(G)$.*

Assume \mathbf{D} is reconstructing. Then f is regular if and only if $f(V(G))$ spans \mathbb{R}^k and \mathbf{D} is commuting.

Proof. Assume $f(V(G))$ spans \mathbb{R}^k and that \mathbf{D} is commuting.

Let dist be the usual Euclidean metric of \mathbb{R}^k . Let $g \in \text{Aut}(G)$. Then, for any vertices x, y of G , $\text{dist}(f(g(x)), f(g(y))) = \text{dist}(f(x), f(y))$. According to Theorem 5.1, g may be extended to a unique isometry of \mathbb{R}^k preserving $f(V(G))$, that is: to a unique element of $\omega(k, f(V(G)))$.

Let \tilde{g} denote the isometry defined by the automorphism g . Obviously, if $g_1, g_2 \in \text{Aut}(G)$, $\widetilde{g_1 g_2} = \tilde{g}_1 \tilde{g}_2$. Thus we have defined a group morphism from $\text{Aut}(G)$ to $\omega(k, f(V(G)))$.

Now assume $\phi \in \omega(k, f(V(G)))$. Define $g : V(G) \rightarrow V(G)$ by $f(g(x)) = \phi(f(x))$. Notice that g is well defined as ϕ is obviously one-to-one. As \mathbf{D} is reconstructing, there exists a mapping $\Xi : \mathbb{R} \rightarrow \mathbb{N}$ such that $\Xi(\mathbf{D}_{i,j}) = \mu(i, j)$. Let $i \neq j \in \{1, \dots, n\}$. Then, for any $i \neq j \in \{1, \dots, n\}$, we have: $\mu(i, j) = \Xi(\mathbf{D}_{i,j})$. As ϕ is an isometry, the distance between $f(i)$ and $f(j)$ is the same as the distance as the distance between $\phi(f(i))$ and $\phi(f(j))$, that is: between $f(g(i))$ and $f(g(j))$ (by the definition of g). It follows that $\mathbf{D}_{g(i),g(j)} = \mathbf{D}_{i,j}$ hence $\mu(g(i), g(j)) = \mu(i, j)$. It follows that g is an automorphism of G . Moreover, ϕ extends $f^{-1} \circ g \circ f$, thus $\phi = \tilde{g}$. It follows that $g \mapsto \tilde{g}$ is actually a group isomorphism from $\text{Aut}(G)$ to $\omega(k, f(V(G)))$.

Conversely, assume f is a regular embedding. Then $f(V(G))$ spans \mathbb{R}^k for otherwise $\omega(k, f(V(G)))$ would not be a finite group although $\text{Aut}(G)$ is. Let

$\phi \in \omega(k, f(V(G)))$. Define $\tilde{\phi} : V(G) \rightarrow V(G)$ by $f(\tilde{\phi}(x)) = \phi(f(x))$. As previously, we get $\mu(\tilde{\phi}(x), \tilde{\phi}(y)) = m$, hence $\tilde{\phi}$ is an automorphism of G such that $d(f(\tilde{\phi}(x)), f(\tilde{\phi}(y))) = d(f(x), f(y))$. Moreover, $\widetilde{\phi_1\phi_2} = \tilde{\phi}_1\tilde{\phi}_2$. It follows that $\phi \mapsto \tilde{\phi}$ is a group-morphism. As it is clearly one-to-one and as $\text{Aut}(G)$ and $\omega(k, f(V(G)))$ are isomorphic by assumption, $\phi \mapsto \tilde{\phi}$ is onto. It follows that $d(f(g(x)), f(g(y))) = d(f(x), f(y))$ for any $g \in \text{Aut}(G)$, that is: \mathbf{D} is commuting. \square

We shall consider now this result from another point of view:

Corollary 5.3. *Let G be a multigraph and let \mathbf{D} be an Euclidean reconstructing and commuting distance matrix on G . Then \mathbf{D} defines a regular embedding of G .*

Proof. Consider any \mathbf{D} -embedding f of G and the subspace spanned by $f(V(G))$ and apply Theorem 5.2. \square

Our main problem is now to build such a distance matrix.

6. EUCLIDEAN DISTANCE MATRICES FROM PREDISTANCE MATRICES

Given a distance dist on the set $\{1, \dots, n\}$, it is classical to define the corresponding bilinear form \mathbf{B} by:

$$(2) \quad \mathbf{B} = -\frac{1}{2}(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{D}(\mathbf{I} - \frac{1}{n}\mathbf{J})$$

where $\mathbf{D}_{i,j} = \text{dist}(i, j)^2$.

It is well known that the distance dist is Euclidean (i.e. allows an isometric embedding into some Euclidean space) if and only if \mathbf{B} is positive semi-definite [2, 6]. Such a result extends to a characterization of those symmetric real matrices which are Euclidean distance matrices.

Definition 6.1. A *predistance matrix* on G is an $n \times n$ symmetric real matrix \mathbf{P} with 0 on the diagonal (notice that negative entries are allowed).

The *bilinear form* $\Lambda(\mathbf{P})$ of \mathbf{P} is defined by:

$$\Lambda(\mathbf{P}) = -\frac{1}{2}(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{P}(\mathbf{I} - \frac{1}{n}\mathbf{J})$$

Lemma 6.1. *Let \mathbf{D} be a predistance matrix and let k be an integer. Then, the following statements are equivalent:*

- (1) \mathbf{D} is an Euclidean distance matrix on G , and there exists a \mathbf{D} -embedding of G in \mathbb{R}^k ;
- (2) the matrix $\Lambda(\mathbf{D})$ is positive semi-definite and $\text{rank}(\Lambda(\mathbf{D})) \leq k$.

Proof. Assume $\Lambda(\mathbf{D})$ is positive semi-definite. Then the square of the distance defined by $\Lambda(\mathbf{D})$ between the basis points e_i and e_j is $(\mathbf{E}_i - \mathbf{E}_j)^t \Lambda(\mathbf{D})(\mathbf{E}_i - \mathbf{E}_j)$. As $(\mathbf{I} - \frac{\mathbf{J}}{n})(\mathbf{E}_i - \mathbf{E}_j) = (\mathbf{E}_i - \mathbf{E}_j)$, we get that the square of this distance equals $(\mathbf{D}_{i,j} + \mathbf{D}_{j,i} - \mathbf{D}_{i,i} - \mathbf{D}_{j,j})/2 = \mathbf{D}_{i,j}$ as \mathbf{D} is symmetric with zero on the diagonal. We define $f(i)$ as the projection of e_i orthogonal to the isotropic space of \mathbf{B} .

Conversely, assume \mathbf{D} is an Euclidean distance matrix and let $f : [1; n] \rightarrow \mathbb{R}^k$ be a \mathbf{D} -embedding of G . Then $\Lambda(\mathbf{D})$ is the Gram matrix of the vectors $f(i)$, i.e. $\Lambda(\mathbf{D})_{i,j} = \langle f(i), f(j) \rangle$. It is well known that this Gram matrix determines the vectors $f(i)$ up to isometry. \square

Let \mathbf{P} be a predistance matrix. The condition that $\Lambda(\mathbf{P})$ has no negative eigenvalue is quite difficult to handle. However, we know the following about $\Lambda(\mathbf{P})$:

- $\Lambda(\mathbf{P})$ is symmetric real and thus diagonalizable,
- $\mathbf{1} \in \ker(\Lambda(\mathbf{P}))$ as $(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{1} = \mathbf{0}$,
- $\Lambda(\mathbf{P})$ has an orthogonal basis $\mathcal{B}(\Lambda(\mathbf{P}))$ of eigenvectors including $\mathbf{1}$. The eigenvectors of $\mathcal{B}(\Lambda(\mathbf{P})) \setminus \{\mathbf{1}\}$ have eigenvalues $\lambda_1 > \dots > \lambda_r$ with respective multiplicities m_1, \dots, m_r .

Now consider the *reduced distance matrix* $\mathbf{P}^\star = \mathbf{P} - 2\lambda_r(\mathbf{J} - \mathbf{I})$.

Lemma 6.2. $\mathcal{B}(\Lambda(\mathbf{P}))$ is an orthogonal basis of eigenvectors of $\Lambda(\mathbf{P}^\star)$, and the eigenvalue (for $\Lambda(\mathbf{P}^\star)$) of $v \in \mathcal{B}(\Lambda(\mathbf{P}))$ is

$$\begin{cases} 0 & \text{if } v = \mathbf{1} \\ \lambda - \lambda_r & \text{if } v \neq \mathbf{1} \text{ and } \Lambda(\mathbf{P})v = \lambda v \end{cases}$$

Thus, $\Lambda(\mathbf{P}^\star)$ is positive semi-definite and has corank $m_r + 1$.

Proof. If $v \neq \mathbf{1}$, then $v \in \mathbf{1}^\perp$, and thus $(\mathbf{I} - \frac{\mathbf{J}}{n})v = v$. Hence, if $\Lambda(\mathbf{P})v = \lambda v$, then

$$\begin{aligned} \Lambda(\mathbf{P}^\star)v &= -\frac{1}{2}(\mathbf{I} - \frac{\mathbf{J}}{n})\mathbf{P}^\star(\mathbf{I} - \frac{\mathbf{J}}{n})v \\ &= -\frac{1}{2}(\mathbf{I} - \frac{\mathbf{J}}{n})(\mathbf{P} - 2\lambda_r(\mathbf{J} - \mathbf{I}))(\mathbf{I} - \frac{\mathbf{J}}{n})v \\ &= -\frac{1}{2}(\mathbf{I} - \frac{\mathbf{J}}{n})\mathbf{P}(\mathbf{I} - \frac{\mathbf{J}}{n})v + \lambda_r(\mathbf{I} - \frac{\mathbf{J}}{n})(\mathbf{J} - \mathbf{I})(\mathbf{I} - \frac{\mathbf{J}}{n})v \\ &= \Lambda(\mathbf{P})v - \lambda_r v && \text{(as } \mathbf{J}v = \mathbf{0}) \\ &= (\lambda - \lambda_r)v \end{aligned}$$

□

7. RECONSTRUCTING AND COMMUTING PREDISTANCE MATRICES

Definition 7.1. A predistance matrix \mathbf{P} is

- *commuting* if \mathbf{P} commutes with any automorphism of G , that is:

$$\forall g \in \text{Aut}(G), \quad g\mathbf{P} = \mathbf{P}g$$

or, equivalently:

$$\forall g \in \text{Aut}(G), \forall i, j \in \{1, \dots, n\}, \quad P_{g(i), g(j)} = P_{i, j};$$

- *reconstructing* if there exists a mapping $\Xi : \mathbb{R} \rightarrow \mathbb{N}$ such that $\mu(i, j) = \Xi(P_{i, j})$, or equivalently:

$$\forall i, j, i', j' \in \{1, \dots, n\} \quad P_{i, j} = P_{i', j'} \Rightarrow \mu(i, j) = \mu(i', j')$$

Notation 7.2. Let \mathbf{P} be a predistance matrix. As $\Lambda(\mathbf{P})$ has 0 as an eigenvalue, we may define $\zeta(\mathbf{P})$ as follows:

- if the smallest eigenvalue λ of $\Lambda(\mathbf{P})$ is negative and has multiplicity m then $\zeta(\mathbf{P}) = n - m - 1$,
- if the smallest eigenvalue of $\Lambda(\mathbf{P})$ is 0 with multiplicity $m > 1$ then $\zeta(\mathbf{P}) = n - m$,
- if the smallest eigenvalue of $\Lambda(\mathbf{P})$ is 0 with multiplicity 1 and if the second smallest eigenvalue of $\Lambda(\mathbf{P})$ has multiplicity m then $\zeta(\mathbf{P}) = n - m - 1$.

Theorem 7.1. *Let G be an irreducible multigraph. Any commuting reconstructing predistance matrix \mathbf{P} defines a regular embedding f into $\mathbb{R}^{\zeta(\mathbf{P})}$.*

Proof. Let \mathbf{P} be a commuting reconstructing predistance matrix \mathbf{P} . The corresponding reduced distance matrix \mathbf{P}^* , according to Lemma 6.1 and Lemma 6.2 is Euclidean and there exists a \mathbf{P}^* embedding f on G into $\mathbb{R}^{\zeta(\mathbf{P})}$. As \mathbf{P} is reconstructing and commuting, so is \mathbf{P}^* . Assume $f(x) = f(y)$. Since f is reconstructing, x and y are twins of G , contradicting its irreducibility. Hence f is one-to-one and, according to Theorem 5.2, f is a regular embedding. \square

Corollary 7.2. *Let G be an irreducible multigraph and let \mathbf{P} be a reconstructing predistance matrix which commutes with the automorphisms of G .*

Denote by $\mathbf{1}\mathbb{R}$ the line spanned by $\mathbf{1}$ and by \bigoplus^\perp the orthogonal direct sum of vector spaces. Let $\mathbf{1}\mathbb{R} \bigoplus^\perp E_1 \bigoplus^\perp E_2 \bigoplus^\perp \dots \bigoplus^\perp E_r$ be a decomposition into eigenspaces of $\Lambda(\mathbf{P})$, the eigenvalues associated with E_1, \dots, E_r being $\lambda_1 > \lambda_2 > \dots > \lambda_r$ (notice that some λ_i may be 0). Then the abstract automorphism group of G is a subgroup of

$$\omega(\dim E_1, P_1) \oplus \omega(\dim E_2, P_2) \oplus \dots \oplus \omega(\dim E_{r-1}, P_{r-1})$$

where P_i is the orthogonal projection of the image of G under the regular embedding defined by \mathbf{P} into the subspace E_i .

Here some examples of simple commuting reconstructing predistance matrices of a simple connected graph G (the two last examples only apply if G has order at least 3):

$$\begin{array}{ll} \text{Adjacency} & \mathbf{A}_{i,j} = \begin{cases} 0, & \text{if } i = j \text{ or } i \text{ and } j \text{ are adjacent} \\ 1, & \text{otherwise} \end{cases} \\ \text{Graph distance} & \mathbf{S}_{i,j} = \begin{cases} 0, & \text{if } i = j \\ l, & \text{if } l \text{ is the graph distance from } i \text{ to } j \end{cases} \\ \text{Bisected Czekanovski-Dice} & \mathbf{C}_{i,j} = \begin{cases} 0, & \text{if } i = j \\ 1 - \frac{2}{d(i)+d(j)}, & \text{if } i \text{ and } j \text{ are adjacent} \\ 1, & \text{otherwise} \end{cases} \\ \text{Q-distance} & \mathbf{Q}_{i,j} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \text{ and } j \text{ are non adjacent} \\ 1 - \frac{1}{\sqrt{d(i)d(j)}}, & \text{otherwise} \end{cases} \end{array}$$

Theorem 7.3. *Any irreducible multigraph has a regular embedding into some Euclidean space.*

Proof. The adjacency matrix \mathbf{A} of G defined by $\mathbf{A}_{i,j} = \mu(i, j)$ is commuting and reconstructing. Hence the result follows from Theorem 7.1. \square

8. REGULAR MULTIGRAPHS

As noted in section 7, the adjacency matrix \mathbf{A} defines a commuting reconstructing predistance matrix. In the particular case where G is d -regular, we have $\mathbf{A}\mathbf{J} = d\mathbf{J}$ thus $\Lambda(\mathbf{A}) = (1 - d/n)^2\mathbf{A}$. Moreover (see [3]):

- d is an eigenvalue of G with eigenvector $\mathbf{1}$.
- If G is connected, then the multiplicity of d is one.
- For any eigenvalue λ of G , we have $|\lambda| \leq d$.

In this case we deduce a strengthening of Babai's result [1] mentioned in section 1:

Corollary 8.1. *Let G be a connected irreducible regular multigraph having s distinct eigenvalues $\lambda_1 > \lambda_2 > \dots > \lambda_s$ with respective multiplicities m_1, m_2, \dots, m_s . Then, the abstract automorphism group γ of G is a subgroup of*

$$\omega(m_2, P_2) \oplus \omega(m_3, P_3) \oplus \dots \oplus \omega(m_{s-1}, P_{s-1}).$$

Proof. Assume G is connected. Let $\mathcal{B}(\mathbf{A})$ be an orthogonal basis of eigenvectors of \mathbf{A} . As $\mathbf{J}\mathbf{v} = 0$ for any vector orthogonal to $\mathbf{1}$ and as $\mathbf{J}\mathbf{1} = n\mathbf{1}$, $\mathcal{B}(\mathbf{A})$ is also a basis of eigenvectors of $\Lambda(\mathbf{A})$. The eigenvalue (for $\Lambda(\mathbf{A})$) of an eigenvector \mathbf{v} with eigenvalue λ (for \mathbf{A}) is clearly λ if $\lambda \neq d$ and 0 if $\lambda = d$. As the eigenvalue d is maximal and simple for \mathbf{A} , we deduce that $\zeta(\mathbf{A}) = n - m - 1$, where m is the multiplicity of the smallest eigenvalue of \mathbf{A} . Moreover, the greatest eigenvalue is simple and corresponds to eigenvector $\mathbf{1}$, according to Perron-Frobenius theorem [5, 8]. \square

As a matter of fact, instead of $\omega(m_2, P_2) \oplus \omega(m_3, P_3) \oplus \dots \oplus \omega(m_{s-1}, P_{s-1})$, we may prove a similar result with $\omega(m_3, P_3) \oplus \dots \oplus \omega(m_{s-1}, P_{s-1}) \oplus \omega(m_s, P_s)$ by considering $2\lambda_2(\mathbf{J} - \mathbf{I}) - \mathbf{A}$ instead of $\mathbf{A} - 2\lambda_s(\mathbf{J} - \mathbf{I})$.

A *strongly regular graph* is a regular graph such that there exist constants λ, μ so that any two adjacent vertices have exactly λ common neighbors, and every two non-adjacent vertices have exactly μ common neighbors [9]. As a strongly regular graph has exactly three distinct eigenvalues [3], we may choose to keep among E_2 and E_3 the one having the smallest dimension. Hence every connected irreducible strong regular graph of order n has a regular embedding into an Euclidean space of dimension at most $\lfloor \frac{n-1}{2} \rfloor$. As an example, the Petersen graph has a regular embedding into \mathbb{R}^4 , which is optimal (the automorphism group of the Petersen graph cannot be realized as the isometry group of a set of points in \mathbb{R}^3).

9. CONCLUSION

The techniques presented here allow to construct regular embeddings from weakly constrained matrices: we only ask that they should be symmetric, have 0 on the diagonal, are reconstructing and commute with every automorphism.

Recall that a *cellular algebra*, or *coherent algebra*, is an algebra of $n \times n$ complex matrices which has a basis $\{\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_t\}$ consisting of matrices with entries 0 and 1 satisfying the following conditions:

- (1) $\mathbf{B}_0 + \mathbf{B}_1 + \dots + \mathbf{B}_t = \mathbf{J}$;
- (2) $\exists 0 \leq r \leq t : \mathbf{B}_0 + \mathbf{B}_1 + \dots + \mathbf{B}_r = \mathbf{I}$;
- (3) the set $\{\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_t\}$ is closed under transposition.

The unique minimal cellular algebra which contains \mathbf{A} as an element is the *cellular algebra generated by \mathbf{A}* and is denoted by $\langle\langle \mathbf{A} \rangle\rangle$. Notice that a basis of this cellular algebra may be constructed in polynomial time. In the case where \mathbf{A} is the adjacency matrix of a simple graph G , the cellular algebra $\langle\langle \mathbf{A} \rangle\rangle$ is called the cellular algebra *generated by G* , or the cellular algebra *of G* . This algebra may contain non symmetric basis elements. However, any matrix in $\langle\langle \mathbf{A} \rangle\rangle$ commutes with any automorphism of G . It follows that any commuting predistance matrix \mathbf{P}

in $\langle\langle A \rangle\rangle$ may be written as $\sum_{i=r+1}^t \lambda_i (\mathbf{B}_i + \mathbf{B}_i^t)$, with $\lambda_i \in \mathbb{R}$ for $r < i \leq t$ and thus form an affine space \mathcal{E} . By perturbing each of them (by adding $\epsilon(\mathbf{P})\mathbf{A}$, for instance) we obtain reconstructing and commuting predistance matrices forming a dense subset of \mathcal{E} .

We hope that the suggested approach will be fruitful and more practical than the usual techniques arising from algebra and spectral analysis.

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