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ASYMPTOTIC ESTIMATE FOR PERTURBED SCALAR CURVATURE EQUATION.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the equation \( \Delta u_\epsilon = V_\epsilon u_\epsilon^{(n+2)/(n-2)} + \epsilon W_\epsilon u_\epsilon^\alpha \) with \( \alpha \in \left[ \frac{n}{n-2}, \frac{n+2}{n-2} \right] \) and we give some minimal conditions on \( \nabla V \) and \( \nabla W \) to have an uniform estimate for their solutions when \( \epsilon \to 0 \).

1. INTRODUCTION AND RESULTS.

We denote \( \Delta = -\sum \partial_{ii} \) the geometric Laplacian on \( \mathbb{R}^n, n \geq 3 \).

Let us consider on open set \( \Omega \) of \( \mathbb{R}^n, n \geq 3 \), the following equation:

\[
\Delta u_\epsilon = V_\epsilon u_\epsilon^{(n+2)/(n-2)} + \epsilon W_\epsilon u_\epsilon^\alpha \quad (E_\epsilon)
\]

where \( V_\epsilon \) and \( W_\epsilon \) are two regular functions and \( \alpha \in \left[ \frac{n}{n-2}, \frac{n+2}{n-2} \right] \).

We assume:

\[
0 < a \leq V_\epsilon(x) \leq b, \quad ||\nabla V_\epsilon||_{L^\infty} \leq A \quad (C_1)
\]

\[
0 < c \leq W_\epsilon(x) \leq d, \quad ||\nabla W_\epsilon||_{L^\infty} \leq B \quad (C_2)
\]

Problem: Can we have an \( \sup \times \inf \) estimate with the minimal conditions \( (C_1) \) and \( (C_2) \)?

Note that for \( W \equiv 0 \), the equation \( (E_\epsilon) \) is the wellknown scalar curvature equation on open set of \( \mathbb{R}^n, n \geq 3 \). In this case, there is many results about this equation, see for example [B] and [C-L 1].

When \( \Omega = S_n \) YY. Li, give a flatness condition to have the boundedness of the energy and the existence of the simple blow-up points, see [L1] and [L2].

In [C-L 2], Chen and Lin gave a counterexample of solutions of the scalar curvature equation with unbounded energy. The conditions of Li are minimal in heigh dimension.

Note that, in [C-L 1] and [C-L 3], there is some results concerning Harnack inequalities of type \( \sup \times \inf \) with the “Li-flatness” conditions for the following equation:

\[
\Delta u = V u^{(n+2)/(n-2)} + g(u)
\]

where \( g \) is a regular function ( at least \( C^1 \) ) such that \( g(t)/[t^{(n+2)/(n-2)}] \) is decrasing and tends to \( 0 \) when \( t \to +\infty \). They extend Li result ([L1]) to any open set of the euclidian space.

We can find in [A], some existence results for the presribed scalar curvature equation.

In our work we have no assumption on the energy. We use the blow-up analysis and the moving-plane method, developed by Gidas-Ni-Nirenberg, see [ G-N-N]. This method was used by different authors to have a priori estimates, look for example, [B], [B-L-S] ( in dimension 2), [C-L 1], [C-L 3], [L 1] and [L 2].

We set \( \delta = [(n+2) - \alpha(n-2)]/2, \delta \in [0, 1] \). We have:
Theorem 1. For all \(a, b, c, d, A, B > 0\), for all \(\alpha \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right]\) and all compact set \(K\) of \(\Omega\), there is a positive constant \(c = c(a, b, c, d, A, B, \alpha, K, \Omega, n)\) such that:
\[
\epsilon^{(n-2)/2(1-de)} \left(\sup_K u_\epsilon\right)^{1/3} \times \inf_{\Omega} u_\epsilon \leq c
\]
for all \(u_\epsilon\) solution of \((E_\epsilon)\) with \(V_i\) and \(W_i\) satisfying the conditions \((C_1)\) and \((C_2)\).

Now, we suppose that \(V_i\) satisfies:
\[
0 < a \leq V_i(x) \leq b \quad \text{and} \quad \|\nabla V_i\|_{L^\infty(\Omega)} \leq k\epsilon \quad (C_3)
\]
We have:

Theorem 2. For all \(a, b, c, d, A, B > 0\), for all \(\alpha \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right]\) and all compact set \(K\) of \(\Omega\), there is a positive constant \(c = c(a, b, c, d, A, B, \alpha, K, \Omega, n)\) such that:
\[
\sup_K u_\epsilon \times \inf_{\Omega} u_\epsilon \leq c
\]
for all \(u_\epsilon\) solution of \((E_\epsilon)\) with \(V_i\) and \(W_i\) satisfying the conditions \((C_3)\) and \((C_2)\).

Note that in [B], we have some results as the previous but for prescribed scalar curvature equation with subcritical exponent tending to the critical. Here, we have a \(\sup \times \inf\) inequality for the scalar curvature equation, with critical exponent, perturbed by a nonlinear term. We can see the influence of this non-linear term.

2. PROOFS OF THE THEOREMS.

Proof of the theorem 1.

Without loss of generality, we suppose \(\Omega = B_1\) the unit ball of \(\mathbb{R}^n\). We want to prove an a priori estimate around 0. We can also suppose \(\epsilon \to 0\), the case \(\epsilon \not\to 0\) is solved in [B].

Let \((u_i)\) and \((V_i)\) be a sequences of functions on \(\Omega\) such that:
\[
\Delta u_i = V_i u_i^{(n+2)/(n-2)} + \epsilon_i W_i u_i^\alpha, \quad u_i > 0,
\]
with \(0 < a \leq V_i(x) \leq b\), \(0 < a \leq W_i(x) \leq d\), \(\|V_i\|_{L^\infty} \leq A\) and \(\|W_i\|_{L^\infty} \leq B\).

We argue by contradiction and we suppose that the \(\sup \times \inf\) is not bounded.

We have:
\[
\forall c, R > 0 \exists u_{c,R} \text{ solution of } (E_1) \text{ such that:}
\]
\[
\epsilon^{(n-2)/2(1-de)} R^{n-2} \left(\sup_{B(0, R)} u_{c,R}\right)^{1/3} \times \inf_{\Omega} u_{c,R} \geq c, \quad (H)
\]

Proof of the proposition (blow-up analysis)

There is a sequence of points \((y_i)_i, y_i \to 0\) and two sequences of positive real numbers \((l_i)_i, (L_i)_i, l_i \to 0, L_i \to +\infty\), such that if we set \(v_i(y) = \frac{u_i(y + y_i)}{u_i(y_i)}\), we have:
\[
0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2}, \quad \beta_i \to 1.
\]
\[
v_i(y) \to \left(\frac{1}{1 + |y|^2}\right)^{(n-2)/2} \epsilon_i^{(n-2)/2(1-de)} \left[\inf_{B_1} u_i\right], \quad \text{uniformly on all compact set of } \mathbb{R}^n.
\]

Proof of the proposition:

We use the hypothesis \((H)\), we take two sequences \(R_i > 0, R_i \to 0\) and \(c_i \to +\infty\), such that,
Let $x_i \in B(x_0, R_i)$ be a point such that $\sup_{B(0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = (R_i - |x - x_i|)^{(n-2)/2} u_i(x), x \in B(x_i, R_i)$. Then, $x_i \to 0$.

We have:

$$\max_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i^{(n-2)/2} u_i(x_i) \geq \sqrt{c_i} \to +\infty.$$ 

We set:

$$l_i = R_i - |y_i - x_i|, \quad \bar{u}_i(y) = u_i(y_i + y), \quad v_i(z) = \frac{u_i[y_i + (z/[u_i(y_i)])^{2/(n-2)}]}{u_i(y_i)}.$$ 

Clearly we have, $y_i \to x_0$. We also obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2(n-2)} [u_i(y_i)]^{2/(n-2)}} \geq \frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}} = c_i^{1/2(n-2)} \to +\infty.$$ 

If $|z| \leq L_i$, then $y = [y_i + z/[u_i(y_i)]^{2/(n-2)}] \in B(y_i, \delta_i l_i)$ with $\delta_i = \frac{1}{(c_i)^{1/2(n-2)}}$ and $|y - y_i| < R_i - |y_i - x_i|$, thus, $|y - x_i| < R_i$ and, $s_i(y) \leq s_i(y_i)$. We can write:

$$u_i(y)(R_i - |y - y_i|)^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2}.$$ 

But, $|y - y_i| \leq \delta_i l_i$, $R_i > l_i$ and $R_i - |y - y_i| \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$. We obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1 - \delta_i)}\right]^{(n-2)/2} \leq 2^{(n-2)/2}.$$ 

We set, $\beta_i = \left(\frac{1}{1 - \delta_i}\right)^{(n-2)/2}$, clearly, we have, $\beta_i \to 1$.

The function $v_i$ satisfies:

$$\Delta v_i = \tilde{V}_i v_i^{(n+2)/(n-2)} + \epsilon_i \tilde{W}_i \left[\frac{v_i^{n/(n-2)}}{u_i(y_i)^{n+2/(n-2)}}\right]^{\alpha}.$$ 

where, $\tilde{V}_i(y) = V_i \left[ y + y/[u_i(y_i)]^{2/(n-2)}\right]$ and $\tilde{W}_i(y) = W_i \left[ y + y/[u_i(y_i)]^{2/(n-2)}\right]$.

Without loss of generality, we can suppose that $\tilde{V}_i \to V(0) = n(n-2)$.

We use the elliptic estimates, Ascoli and Ladyzenskaya theorems to have the uniform convergence of $(v_i)$ to $v$ on compact set of $\mathbb{R}^n$. The function $v$ satisfies:

$$\Delta v = n(n-2)v^{N-1}, \quad v(0) = 1, \quad 0 \leq v \leq 2^{(n-2)/2},$$ 

By the maximum principle, we have $v > 0$ on $\mathbb{R}^n$. If we use Caffarelli-Gidas-Spruck result, (see [C-G-S]), we obtain, $v(y) = \left(\frac{1}{1 + |y|^2}\right)^{(n-2)/2}$. We have the same properties that in [B].

**Polar Coordinates** (Moving-Plane method)

Now, we must use the same method than in the Theorem 1 of [B]. We will use the moving-plane method.

We must prove the lemma 2 of [B].

We set $t \in ]-\infty, -\log 2]$ and $\theta \in S_{n-1}$:
\( w_i(t, \theta) = e^{(n-2)t/2} u_i(y_i + e^t \theta), \quad \bar{V}_i(t, \theta) = V_i(y_i + e^t \theta) \) and \( \bar{W}_i(t, \theta) = W_i(y_i + e^t \theta) \).

We consider the following operator \( L = \partial_t - \Delta_\sigma - \frac{(n-2)^2}{4}, \) with \( \Delta_\sigma \) the Laplace-Baltrami operator on \( S_{n-1} \).

The function \( w_i \) is solution of:

\[
-L w_i = \bar{V}_i w_i^{N-1} + \epsilon_i e^{[(n+2)-(n-2)\alpha)t/2} \bar{W}_i w_i^\alpha.
\]

For \( \lambda \leq 0 \) we set:

\[
t^\lambda = 2\lambda - t \ w_i^\lambda(t, \theta) = w_i(t^\lambda, \theta), \quad \bar{V}_i^\lambda(t, \theta) = \bar{V}_i(t^\lambda, \theta) \text{ et } \bar{W}_i^\lambda(t, \theta) = \bar{W}_i(t^\lambda, \theta).
\]

**Remark:** Here we work on \([\lambda, t_i] \times S_{n-1}\), with \( \lambda \leq - \frac{2}{n-2} \log u_i(y_i) + 2 \) and \( t_i \leq \log \sqrt{t_i} \), where \( t_i \) is chosen as in the proposition.

First, like in [B], we have the following lemma:

**Lemma 1:**

Let \( A_\lambda \) be the following property:

\[ A_\lambda = \{ \lambda \leq 0, \exists (t_\lambda, \theta_\lambda) \in [\lambda, t_i] \times S_{n-1}, \ w_i^\lambda(t_\lambda, \theta_\lambda) = w_i(t_\lambda, \theta_\lambda) \geq 0 \}. \]

Then, there is \( \nu \leq 0 \), such that for \( \lambda \leq \nu \), \( A_\lambda \) is not true.

Like in the proof of the Theorem 1 of [B], we want to prove the following lemma:

**Lemma 2:**

For \( \lambda \leq 0 \) we have:

\[
w_i^\lambda - w_i \leq 0 \Rightarrow -L(w_i^\lambda - w_i) \leq 0,
\]

on \([\lambda, t_i] \times S_{n-1}\).

Like in [B], we have:

**A useful point:**

\( \xi_i = \sup \{ \lambda \leq \lambda_i = 2 + \log \eta_i, \ w_i^\lambda - w_i < 0, \text{ on } [\lambda, t_i] \times S_{n-1} \}. \) The real \( \xi_i \) exists.

First, we have:

\[
w_i(2\xi_i - t, \theta) = w_i([\xi_i - t + \xi_i - \log \eta_i - 2] + (\log \eta_i + 2)) \]

the definition of \( w_i \) and the fact that, \( \xi_i \leq t \), we obtain:

\[
w_i(2\xi_i - t, \theta) = e^{[(n-2)(\xi_i - t + \xi_i - \log \eta_i - 2)]/2 \log 2} e^{(\xi_i - t)(\xi_i - \log \eta_i - 2)} \leq 2(\xi_i - t)(\xi_i - \log \eta_i - 2) = c.
\]

**Proof of the Lemma 2:**

We know that:

\[
-L(w_i^{\xi_i} - w_i) = [\bar{V}_i^{\xi_i}(w_i^{\xi_i})^{N-1} - \bar{V}_i w_i^{N-1}] + \epsilon_i e^{\delta t \xi_i} \bar{W}_i^{\xi_i}(w_i^{\xi_i})^\alpha - e^{\delta t} \bar{W}_i w_i^\alpha,
\]

with \( \delta = [(n+2)-(n-2)\alpha]/2. \)

We denote by \( Z_1 \) and \( Z_2 \) the following terms:

\[
Z_1 = (\bar{V}_i^{\xi_i} - \bar{V}_i)(w_i^{\xi_i})^{N-1} + \bar{V}_i([w_i^{\xi_i}]^{N-1} - w_i^{N-1}),
\]

and
$Z_2 = \epsilon_i (W^t_{i_1} - W_{i_1}) (u^t_{i_1})^\alpha e^{\delta t_{i_1}} + \epsilon_i e^{\delta t_{i_1}} W_{i_1} [(u^t_{i_1})^\alpha - w_{i_1}^\alpha] + \epsilon_i W_{i_1}^\alpha (e^{\delta t_{i_1}} - e^{\delta t_{i'}})
$.

But, using the same method as in the proof of the theorem 1 of [B], we have:

$$w_{i_1}^{t_{i_1}} \leq w_i \text{ et } w_{i_1}^{t_{i_1}}(t, \theta) \leq \bar{c} \text{ pour tout } (t, \theta) \in [\xi_i, \log 2] \times S_{n-1},$$

where \(\bar{c}\) is a positive constant not depending on \(t\) for \(\xi_i \leq \log n_i + 2\):

$$|\bar{V}_{i_1}^{t_{i_1}} - \bar{V}_i| \leq A(e^t - e^{t_{i_1}}) \text{ et } |\bar{V}_{i_1}^{t_{i_1}} - W_i| \leq B(e^t - e^{t_{i_1}}),$$

Then,

$$Z_1 \leq A (w_{i_1}^{t_{i_1}})^{N-1} (e^t - e^{t_{i_1}}) \text{ et } Z_2 \leq \epsilon_i B (w_{i_1}^{t_{i_1}})^\alpha (e^t - e^{t_{i_1}}) + \epsilon_i e (e^{\delta t_{i_1}} - e^{\delta t_{i'}}),$$

and,

$$-L(w_{i_1}^{t_{i_1}} - w_i) \leq (w_{i_1}^{t_{i_1}})^\alpha [(A w_{i_1}^{t_{i_1}})^{N-1} - A w_{i_1}^{t_{i_1}} + \epsilon_i B (e^t - e^{t_{i_1}}) + \epsilon_i e (e^{\delta t_{i_1}} - e^{\delta t_{i'}})].$$

But, \(w_{i_1}^{t_{i_1}} \leq \bar{c}\), we obtain:

$$-L(w_{i_1}^{t_{i_1}} - w_i) \leq (w_{i_1}^{t_{i_1}})^\alpha [(A w_{i_1}^{t_{i_1}})^{N-1} - \epsilon_i B (e^t - e^{t_{i_1}}) + \epsilon_i e (e^{\delta t_{i_1}} - e^{\delta t_{i'}})]. \quad (1)$$

We must see the sign of:

$$\tilde{Z} = [(A w_{i_1}^{t_{i_1}})^{N-1} - \epsilon_i B (e^t - e^{t_{i_1}}) + \epsilon_i e (e^{\delta t_{i_1}} - e^{\delta t_{i'}})].$$

But \(\alpha \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right], \delta = \frac{n+2 - (n-2)\alpha}{2} \in \left[0, 1\right]\)

For \(t \leq t_i < 0\), we have:

$$e^t \leq e^{(1-\delta)t_i} e^{\delta t}, \text{ for all } t \leq t_i,$$

and the fact that \(t_{i_1} \leq t (\xi_i \leq t)\), by integration of the previous two members, we obtain:

$$e^t - e^{t_{i_1}} \leq \frac{e^{(1-\delta)t_i}}{\delta} (e^{\delta t} - e^{\delta t_{i_1}}), \text{ for all } t \leq t_i,$$

We can write:

$$(e^{\delta t_{i_1}} - e^{\delta t}) \leq \frac{\delta}{e^{(1-\delta)t_i}} (e^{t_{i_1}} - e^t).$$

Then,

$$-L(w_{i_1}^{t_{i_1}} - w_i) \leq (w_{i_1}^{t_{i_1}})^\alpha [-\frac{\epsilon_i \delta e}{e^{(1-\delta)t_i}} + A \bar{c}^{N-1} - \epsilon_i B (e^t - e^{t_{i_1}})].$$

The term \(\frac{\epsilon_i \delta e}{e^{(1-\delta)t_i}} - A \bar{c}^{N-1} - \epsilon_i B\) is positive if:

$$\epsilon_i e^{-[(1-\delta)t_i]} \to +\infty,$$

then,

$$\epsilon_i^{(n-2)/2(1-\delta)} e^{-(n-2)/2t_i} \to +\infty.$$ 

If we take, \(t_i = -\frac{2}{3(n-2)} \log u_i(y_i)\), we have:

$$\epsilon_i^{(n-2)/2(1-\delta)} [u_i(y_i)]^{1/3} \to +\infty.$$ 

It is given by our Hypothesis in the proposition.

But the Hopf Maximum principle gives:

$$\min_{\theta \in S_{n-1}} w_i(t_i, \theta) \leq \max_{\theta \in S_{n-1}} w_i (2\xi_i - t_i, \theta),$$

then,
\[ e^{(n-2)t_i} u_i(y_i) \min_{B_2(0)} u_i \leq c, \]

and,

\[ [u_i(y_i)]^{1/3} \min_{B_2(0)} u_i \leq c, \]

Contradiction.

**Proof of the Theorem 2.**

The proof is similar than the proof of the theorem 1. Only the end of the proof is different.

**Step 1:** The blow-up analysis give:

There is a sequence of points \( y_i \), \( y_i \to 0 \) and two sequences of positive real numbers \((l_i)\), \((L_i)\), \( l_i \to 0 \), \( L_i \to +\infty \), such that if we set \( v_i(y) = \frac{u_i(y + y_i)}{u_i(y_i)} \), we have:

\[ 0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2}, \quad \beta_i \to 1. \]

\[ v_i(y) \to \left( \frac{1}{1 + |y|^2} \right)^{(n-2)/2}, \quad \text{uniformly on all compact set of } \mathbb{R}^n. \]

\[ l_i^{(n-2)/2} u_i(y_i) \times \inf_{B_2} u_i \to +\infty, \]

**Step 2:** Application of the Hopf maximum principle.

We have the same notation that in the proof of the theorem 1. First, we take \( t_i = \sqrt{l_i} \) as in the Step 1 and we look to the end of the proof of the theorem 1. We replace \( A \) by \( k \epsilon_i \). We want to proof that:

\[ w_i - w_i \leq 0 \Rightarrow -L(w_i - w_i) \leq 0, \]

on \([\xi, t_i] \times S_{n-1}\). We have the same definition for \( \xi \) (as in the proof of the theorem 1).

For \( t \leq t_i < 0 \), we have:

\[ e^t \leq e^{(1-\delta)t_i} e^{\delta t}, \quad \text{for all } t \leq t_i, \]

and the fact that \( t^\xi \leq t \) (\( \xi \leq t \)), by integration of the previous two members, we obtain:

\[ e^t - e^{e^{\xi t}} \leq \frac{e^{(1-\delta)t_i}}{\delta}(e^{\delta t} - e^{\delta t_i}), \quad \text{for all } t \leq t_i, \]

We can write:

\[ (e^{e^{\xi t}} - e^{\delta t}) \leq \frac{\delta}{e^{(1-\delta)t_i}} (e^{\xi t} - e^t). \]

Then,

\[ -L(w_i^{\xi t} - w_i) \leq (w_i^{\xi t})^\alpha \left[ \frac{\epsilon_i \delta c}{e^{(1-\delta)t_i}} + k \epsilon_i c^{N-1-\alpha} + \epsilon_i B (e^t - e^{\xi t}) \right]. \]

The term \( \frac{\delta c}{e^{(1-\delta)t_i}} - k c^{N-1-\alpha} - B \) is positive because \( t_i \to -\infty \) and \( \delta \in [0, 1] \).

But the Hopf Maximum principle, gives:

\[ \min_{\theta \in S_{n-1}} w_i(t_i, \theta) \leq \max_{\theta \in S_{n-1}} w_i(2 \xi - t_i, \theta), \]

then,

\[ e^{(n-2)t_i} u_i(y_i) \min_{B_2(0)} u_i \leq c, \]

and,
\[ t_i^{(n-2)/2} u_i(y_i) \min_{B_2(0)} u_i \leq c, \]

Contradiction with the step 1.

**Références:**


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