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Sample path properties of the local time of multifractional Brownian motion

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Abstract : We establish estimates for the local and uniform moduli of continuity of the local time of the multifractional Brownian motion, $B^H = (B^{H(t)}(t), t \in \mathbb{R}^+)$. An analog of the Chung law of the iterated logarithm is studied for B^H and used to obtain the pointwise Hölder exponent of the local time. A kind of local asymptotic self similarity is proved to be satisfied by the local time of B^H .

Key words: Multifractional Brownian motion, Local asymptotic self similarity, LIL of type Chung, Local times.

1 Introduction

The multifractional Brownian motion (mBm), $B^H = (B^{H(t)}(t), t \in \mathbb{R}^+)$, is a Gaussian process which extends the fractional Brownian motion (fBm) by allowing the Hurst function to vary along the time. This provides a tool to model systems whose regularity evolves in time, such as Internet traffic or images. Recently, Boufoussi et al. (2006) have investigated, under mild regularity conditions on $H(\cdot)$, the existence of jointly continuous local times of mBm. Their objective was to explore the link between the pointwise regularity of the local time and the irregularity of the underlying process. This effect of inverse regularity was observed by Berman (1973) for the case of uniform regularity.

The first aim of this paper is to establish estimates for the local and uniform moduli of continuity for the local time of the mBm. Upper bounds for the moduli of continuity of local times have been obtained by Kôno (1977) for Gaussian processes with stationary increments, and more recently Csörgö et al. (1995) have proved upper bounds for the moduli of continuity of the maximum of local times for stationary Gaussian processes and

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Gaussian processes with stationary increments. However, contrary to the classical results, the moduli of continuity obtained in this paper will depend on the point on which the regularity is studied. This is natural, since the mBm has a regularity evolving in time. The approach of the present paper seems to be useful to extend the results of Kôno (1977) and Csörgö et al. (1995) to Gaussian processes without stationary increments.

Chung's form of the law of iterated logarithm for the mBm is established in section 4. This result is used to prove that the pointwise Hölder exponent of its local time with respect to time is equal to $1 - H(t)$ uniformly in x .

The second main objective of the paper concerns the question whether the sample path properties of the mBm can be transferred to its local time. If $H(\cdot)$ is not constant, the mBm will be no more self similar. However, it is proved by Lévy Véhel and Peltier (1995, Proposition 5) that if H is β -Hölder continuous and $\sup_{t \in \mathbb{R}^+} H(t) < \beta$, a local form of self similarity remains, called *local asymptotic self similarity* and defined as follows:

$$\lim_{\rho \rightarrow 0^+} \text{law} \left\{ \frac{B(t + \rho u) - B(t)}{\rho^{H(t)}}, u \in \mathbb{R} \right\} = \text{law} \{ B^{H(t)}(u), u \in \mathbb{R} \}, \quad (1)$$

where $B^{H(t)}$ is a fBm with Hurst parameter $H(t)$. It is the purpose of section 5 to prove that the local time of mBm has a kind of local asymptotic self similarity. Through this result, we obtain some local limit theorems corresponding to the mBm.

We will use C, C_1, \dots to denote unspecified positive finite constants which may not necessarily be the same at each occurrence.

2 Preliminaries

In this section, we present some notations and collect facts about multifractional Brownian motion and its local times.

2.1 Multifractional Brownian motion

The mBm was introduced independently by Lévy Véhel and Peltier (1995) and Benassi et al. (1997). The definition due to Lévy Véhel and Peltier (1995) is based on the moving average representation of fBm, where the constant Hurst parameter H is substituted by a function $H(t)$ as follows :

$$\begin{aligned} B^{H(t)}(t) &= \frac{1}{\Gamma(H(t) + 1/2)} \left(\int_{-\infty}^0 [(t - u)^{H(t)-1/2} - (-u)^{H(t)-1/2}] W(du) \right. \\ &\quad \left. + \int_0^t (t - u)^{H(t)-1/2} W(du) \right); \quad t \geq 0, \end{aligned} \quad (2)$$

where $H(t) : [0, \infty) \rightarrow [\mu, \nu] \subset (0, 1)$ is a Hölder continuous function, W is the standard Brownian motion defined on $(-\infty, +\infty)$. Benassi et al. (1997) defined the mBm by means of the harmonisable representation of fBm as follows :

$$\widehat{B}^{H(t)}(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} d\widehat{W}(d\xi), \quad (3)$$

where $\widehat{W}(\xi)$ is the Fourier transform of the series representation of white noise with respect to an orthonormal basis of $L^2(\mathbb{R})$; we refer to Cohen (1999) for the precise definition and for the fact that \widehat{B}^H is real-valued. Moreover, it is proved in Cohen (1999) that the two representations of mBm are equivalent in law up to a multiplicative deterministic function. The value of this function and the fact that it is bounded away from zero have been obtained by Boufoussi et al. (2006). This equivalence played a crucial role for computing the moments of local times of \widehat{B} .

Various properties of mBm have already been investigated in the literature, related, for instance, to its pointwise and uniform Hölder regularity as well as the local Hausdorff dimension of its sample paths. More precisely, it is known from Lévy Véhel and Peltier (1995, Proposition 10) that, with probability one, for each $t_0 \geq 0$ the Hölder exponent at t_0 of the mBm is equal to $H(t_0)$. Recall that the pointwise Hölder exponent of a stochastic process X at t_0 is defined by

$$\alpha_X(t_0, \omega) = \sup \left\{ \alpha > 0, \lim_{\rho \rightarrow 0} \frac{X(t_0 + \rho, \omega) - X(t_0, \omega)}{\rho^\alpha} = 0 \right\}. \quad (4)$$

In addition, according to the same authors, the local Hausdorff dimension of the graph is $2 - \min_{t \in [a, b]} H(t)$, almost surely for each interval $[a, b] \subset \mathbb{R}^+$.

According to the previous results, the regularity of the mBm depends on the regularity of H . Furthermore, by using Lemma 3.1 in Boufoussi et al. (2006) we can prove easily that the irregularity of H implies the irregularity of the mBm. More precisely, the points of discontinuities of H are also discontinuities of B (see Ayache (2001) Proposition 1).

2.2 Local times

We end this section by briefly recalling some aspects on the theory of local times. For a comprehensive survey on local times of both random and non random vector fields, we refer to Geman and Horowitz (1980), Dozzi (2003) and Xiao (2005).

Let $X = (X(t), t \in \mathbb{R}^+)$ be a real valued separable random process with Borel sample functions. For any Borel set $B \subset \mathbb{R}^+$, the occupation measure of X on B is defined as follows

$$\mu_B(A) = \lambda\{s \in B : X(s) \in A\} \quad \text{for all } A \in \mathcal{B}(\mathbb{R}),$$

where λ is the one dimensional Lebesgue measure on \mathbb{R}^+ . If μ_B is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} , we say that X has a local time on B and define its local time, $L(B, \cdot)$, to be the Radon-Nikodym derivative of μ_B . Here x is the so-called space variable, and B is the time variable.

By standard monotone class arguments, one can deduce that the local times have a measurable modification that satisfies the following *occupation density formula*: for every Borel set $B \subset \mathbb{R}^+$ and for every measurable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$

$$\int_B f(X(t))dt = \int_{\mathbb{R}} f(x)L(B, x)dx.$$

Recently, Boufoussi et al. (2006) have proved that if the Hurst function is Hölder continuous with exponent β and if $\sup_{t \geq 0} H(t) < \beta$, the mBm has a jointly continuous local time, i.e. the mapping $(t, x) \rightarrow \bar{L}(t, x)$ is continuous. In addition, this local time has the following Hölder continuities : It satisfies for any compact $U \subset \mathbb{R}$

(i)

$$\sup_{x \in U} \frac{|L(t+h, x) - L(t, x)|}{|h|^\gamma} < +\infty \quad a.s., \quad (5)$$

where $\gamma < 1 - H(t)$ and $|h| < \eta$, η being a small random variable almost surely positive and finite,

(ii) for any $I \subset [0, T]$ with small length,

$$\sup_{x, y \in U, x \neq y} \frac{|L(I, x) - L(I, y)|}{|x - y|^\alpha} < +\infty \quad a.s., \quad (6)$$

where $\alpha < \left(\frac{1}{2 \sup_I H(t)} - \frac{1}{2} \right) \wedge 1$.

These results have been used to obtain the local and pointwise Hausdorff dimension of the level sets of the mBm. We refer to Boufoussi et al (2006) for definition and results.

3 Moduli of continuity of the local time

Throughout this section, the Hurst function $H : \mathbb{R}^+ \rightarrow [\mu, \nu] \subset (0, 1)$ is assumed to be any measurable function. The notation $B = (B(t), t \geq 0)$ means that both representations, the moving average and the harmonisable one, can be chosen. Moreover, we say that H satisfies the condition (\mathcal{H}_β) if :

$$H \text{ is } \beta\text{-Hölder continuous with } \sup_{s \geq 0} H(s) < \beta.$$

We give the following improvement of Theorem 3.1 of Boufoussi et al. (2006). Note that the first part reproduces this theorem without assuming (\mathcal{H}_β) .

Theorem 3.1. *Consider a measurable function $H(\cdot) : \mathbb{R}_+ \rightarrow [\mu, \nu] \subset (0, 1)$. The mBm with Hurst function $H(\cdot)$ admits on any interval $[a, b] \subset [0, \infty)$ a square integrable local time. Moreover, if (\mathcal{H}_β) holds, the existence of square integrable local times implies that $H(t) < 1$ for almost all t .*

Proof. We first prove the result for the mBm given by the moving average representation, and the result for \widehat{B} will follow from the equivalence in law proved by Cohen (1999).

Let us write for simplicity

$$B^{H(t)}(t) = \int_{-\infty}^t K_{H(t)}(t, u) dW(u),$$

where

$$K_{H(t)}(t, u) = \frac{1}{\Gamma(H(t) + 1/2)} \left[(t - u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2} \right].$$

For any $t > s$, taking the variance of $B^{H(t)}(t) - B^{H(s)}(s)$, we get

$$\begin{aligned} \text{Var}(B^{H(t)}(t) - B^{H(s)}(s)) &\geq \text{Var}(B^{H(t)}(t) - B^{H(s)}(s)/W(u), u \leq s) \\ &= \text{Var}(B^{H(t)}(t)/W(u), u \leq s), \end{aligned} \quad (7)$$

where the last equality follows from the fact that $B^{H(s)}(s)$ is measurable with respect to $\sigma(W(u), u \leq s)$. Moreover we can write

$$B^{H(t)}(t) = \int_{-\infty}^t K_{H(t)}(t, u) dW(u) = \int_{-\infty}^s K_{H(t)}(t, u) dW(u) + \int_s^t K_{H(t)}(t, u) dW(u).$$

Hence, by using the measurability of $\int_{-\infty}^s K_{H(t)}(t, u) dW(u)$ with respect to $\sigma(W(u)/u \leq s)$, we obtain

$$\begin{aligned} \text{Var}(B^{H(t)}(t)/W(u), u \leq s) &= \text{Var}\left(\int_s^t K_{H(t)}(t, u) dW(u)/W(u), u \leq s\right) \\ &= \text{Var}\left(\int_s^t K_{H(t)}(t, u) dW(u)\right), \end{aligned} \quad (8)$$

where, to obtain the last equality, we have used the fact that $\int_s^t K_{H(t)}(t, u) dW(u)$ is independent of $\sigma(W(u), u \leq s)$ (by the independence of the increments of the Brownian motion). Combining (7) and (8) and denote $C = \sup_{u \in [\mu, \nu]} \Gamma(1/2 + u)$, we obtain

$$\begin{aligned} \text{Var}(B^{H(t)}(t) - B^{H(s)}(s)) &\geq \frac{1}{\Gamma(1/2 + H(t))^2} \int_s^t (t - r)^{2H(t)-1} dr \\ &\geq \frac{1}{2\nu C^2} (t - s)^{2H(t)}. \end{aligned}$$

Therefore

$$\int_{[a,b]} \int_{[a,b]} (\mathbb{E}[B^{H(t)}(t) - B^{H(s)}(s)]^2)^{-1/2} ds dt \leq \sqrt{2\nu} C \int_{[a,b]} \int_{[a,b]} |t - s|^{-\sup_{r \geq 0} H(r)} ds dt. \quad (9)$$

The last integral is finite because $\sup_{r \geq 0} H(r) < 1$. By Theorem 22.1 in Geman and Horowitz (1980), B^H has a local time $L([a, b], \cdot) \in L^2(\mathbb{R})$.

For the mBm $\widehat{B}^{H(t)}$ given by the representation (3), we use the equivalence between $B^{H(t)}$ and $\widehat{B}^{H(t)}$ up to a multiplicative deterministic function $\psi(\cdot)$; i.e. $\widehat{B}^{H(t)}(t) = \psi(t)B^{H(t)}(t)$ in law. First,

$$\begin{aligned} \text{Var}(\widehat{B}^{H(t)}(t) - \widehat{B}^{H(s)}(s)) &= \text{Var}(\psi(t)B^{H(t)}(t) - \psi(s)B^{H(s)}(s)) \\ &\geq \psi^2(t) \text{Var}(B^{H(t)}(t)/W(u), u \leq s) \\ &\geq \frac{\psi^2(t)}{2\nu C^2} (t - s)^{2H(t)}. \end{aligned}$$

Furthermore, according to Boufoussi et al. [(2006), Appendix], the function $\psi(t)$ is bounded away from zero, i.e. there exist $K_1 > 0$ and $K_2 > 0$ such that $K_1 \leq \psi^2(t) \leq K_2$. Hence

$$\text{Var}(\widehat{B}^{H(t)}(t) - \widehat{B}^{H(s)}(s)) \geq \frac{K_1}{2\nu C^2}(t-s)^{2H(t)}.$$

The remainder is the same as (9) and the statements after it.

Let us now prove the second point : According to Boufoussi et al. (2006), we have for any interval $[a, b] \subset [0, \infty)$ with small length,

$$E(B(t) - B(s))^2 \leq C_{\mu, \nu} |t - s|^{2H(t)}, \quad \text{for } s, t \in [a, b]. \quad (10)$$

On the other hand, since the local time exists on the interval $[a, b]$, then according to Geman and Horowitz [(1980), Theorem 22.1 expression (22.3)] the following integral is finite

$$\int_a^b \int_a^b \frac{1}{[E(B(t) - B(s))^2]^{1/2}} ds dt.$$

Then, (10) implies that

$$\int_a^b \int_a^b \frac{1}{(t-s)^{H(t)}} ds dt < \infty.$$

Consequently $H(t) < 1$ for almost all $t \in [a, b]$. Since \mathbb{R}^+ is a countable union of small intervals, the result is proved. \square

In what follows, we are interested in the local and pointwise oscillations (at each t) of the local time of the mBm.

Theorem 3.2. *Let $\{B(t), t \geq 0\}$ be a mBm with Hurst function $H(\cdot)$ satisfying the assumption (\mathcal{H}_β) . Then, for every $t \in \mathbb{R}^+$ and any $x \in \mathbb{R}$, there exist positive and finite constants C_1 and C_2 such that*

$$\limsup_{\delta \rightarrow 0} \frac{L(t + \delta, B(t)) - L(t, B(t))}{\delta^{1-H(t)} (\log \log(\delta^{-1}))^{H(t)}} \leq C_1 \quad a.s. \quad (11)$$

$$\limsup_{\delta \rightarrow 0} \frac{L(t + \delta, x) - L(t, x)}{\delta^{1-H(t)} (\log \log(\delta^{-1}))^{H(t)}} \leq C_2 \quad a.s. \quad (12)$$

Proof. Let $t \geq 0$ be a fixed point and $0 < h < 1$ and denote $H_{t, t+h} = \sup_{s \in [t, t+h]} H(s)$. According to the Fourier analytic approach of Berman (1969) (see also Davies (1976), expression (27)), we have

$$\begin{aligned} & E [L(t+h, B(t)) - L(t, B(t))]^m \\ &= \frac{1}{(2\pi)^m} \int_{[t, t+h]^m} \int_{\mathbb{R}^m} \mathbb{E} \left(e^{i \sum_{j=1}^m u_j (B(s_j) - B(t))} \right) \prod_{j=1}^m du_j \prod_{j=1}^m ds_j. \end{aligned}$$

Let $\tilde{B}(s) = B(s) - B(t)$, $s \geq 0$ and denote by $R(s_1, \dots, s_m)$ the covariance matrix of $(\tilde{B}(s_1), \dots, \tilde{B}(s_m))$ for distinct s_1, \dots, s_m . Let $U = (u_1, \dots, u_m) \in \mathbb{R}^m$ and U^T denote the transpose of U . According to (16) below we have, $\det R(s_1, \dots, s_m) > 0$. Hence, the change of variable $V = R^{1/2}U$, implies

$$\int_{\mathbb{R}^m} \mathbb{E} \left(e^{i \sum_{j=1}^m u_j (B(s_j) - B(t))} \right) du_1 \dots du_m = \frac{(2\pi)^{m/2}}{(\det R(s_1, \dots, s_m))^{1/2}}.$$

Therefore

$$\begin{aligned} E [L(t+h, B(t)) - L(t, B(t))]^m &= \frac{1}{(2\pi)^{m/2}} \int_{[t, t+h]^m} \frac{1}{(\det R(s_1, \dots, s_m))^{1/2}} ds_1 \dots ds_m \\ &= \frac{m!}{(2\pi)^{m/2}} \int_{t < s_1 < \dots < s_m < t+h} \frac{1}{(\det R(s_1, \dots, s_m))^{1/2}} ds_1 \dots ds_m. \end{aligned} \quad (13)$$

In addition

$$\det R(s_1, \dots, s_m) = \text{Var}(\tilde{B}(s_1)) \text{Var}(\tilde{B}(s_2)/\tilde{B}(s_1)) \dots \text{Var}(\tilde{B}(s_m)/\tilde{B}(s_1), \dots, \tilde{B}(s_{m-1})). \quad (14)$$

In order to estimate the previous conditional variances, we distinguish two cases :

Case 1 : The mBm is given by the moving average representation.

Using similar arguments as in the proof of Theorem 2.1, we obtain for any $r, s \in [t, t+h]$, such that $r < s$,

$$\begin{aligned} \text{Var} \left(\tilde{B}(s)/\tilde{B}(u), u \in A, u \leq r \right) &\geq \text{Var} \left(\tilde{B}(s) - \tilde{B}(r) / W(u), u \leq r \right) \\ &\geq \text{Var} \left(B^{H(s)}(s) / W(u), u \leq r \right) \\ &\geq \frac{1}{2\nu C^2} (s-r)^{2H(s)}, \end{aligned} \quad (15)$$

where $C = \sup_{x \in [\mu, \nu]} \Gamma(x + 1/2)$. Combining (14) and (15) we obtain

$$\begin{aligned} \det R(s_1, \dots, s_m) &\geq \frac{1}{(2\nu C^2)^m} \prod_{j=1}^m (s_j - s_{j-1})^{2H(s_j)} \\ &\geq \frac{1}{(2\nu C^2)^m} \prod_{j=1}^m (s_j - s_{j-1})^{2H_{t, t+h}}, \end{aligned} \quad (16)$$

where $s_0 = 0$ and $(s_j - s_{j-1})^{H(s_j)} \geq (s_j - s_{j-1})^{H_{t, t+h}}$, since $(s_j - s_{j-1}) < 1$.

Case 2 : The mBm is given by the harmonizable representation.

Using the equivalence in law between the representations (2) and (3), we obtain

$$\tilde{B}(s) = \hat{B}^{H(s)}(s) - \hat{B}^{H(t)}(t) \stackrel{(d)}{=} \psi(s)B^{H(s)}(s) - \psi(t)B^{H(t)}(t),$$

where $\stackrel{(d)}{=}$ stands for the equality in distribution. Therefore, denoting $\bar{B}(s) = \psi(s)B^{H(s)}(s) - \psi(t)B^{H(t)}(t)$, $s \in [t, t+h]$, we obtain

$$\begin{aligned} \det R(s_1, \dots, s_m) &= \det \text{Cov}(\bar{B}(s_1), \dots, \bar{B}(s_m)) \\ &= \text{Var}(\bar{B}(s_1)) \text{Var}(\bar{B}(s_2)/\bar{B}(s_1)) \dots \text{Var}(\bar{B}(s_m)/\bar{B}(s_1), \dots, \bar{B}(s_{m-1})). \end{aligned} \quad (17)$$

On the other hand, for any $r, s \in [t, t+h]$, such that $r < s$,

$$\begin{aligned} \text{Var}(\bar{B}(s)/\bar{B}(u), u \in A, u \leq r) &\geq \text{Var}(\psi(s)B^{H(s)}(s)/W(u), u \leq r) \\ &\geq \frac{K_1}{2\nu C^2}(s-r)^{2H(s)}. \end{aligned} \quad (18)$$

Combining (17), (18) and similar arguments as when proving (16), we obtain

$$\det R(s_1, \dots, s_m) \geq \frac{K_1^m}{(2\nu C^2)^m} \prod_{j=1}^m (s_j - s_{j-1})^{2H_{t,t+h}}, \quad (19)$$

Therefore, we have in both cases

$$\det R(s_1, \dots, s_m) \geq \frac{\tilde{K}^m}{(2\nu C^2)^m} \prod_{j=1}^m (s_j - s_{j-1})^{2H_{t,t+h}}, \quad (20)$$

where $\tilde{K} = \min(K_1, 1)$. Hence, the remainder of the proof does not depend on the representation.

According to (13) and (20), we have

$$\begin{aligned} &E[L(t+h, B(t)) - L(t, B(t))]^m \\ &\leq \left(\frac{\nu C^2}{\tilde{K}\pi}\right)^{m/2} m! \int_{t < s_1 < \dots < s_m < t+h} \prod_{j=1}^m \frac{1}{(s_j - s_{j-1})^{H_{t,t+h}}} ds_1 \dots ds_m. \end{aligned}$$

Now, by an elementary calculation (cf. Ehm (1981)), for all $m \geq 1$, $h > 0$ and $b_j < 1$,

$$\int_{t < s_1 < \dots < s_m < t+h} \prod_{j=1}^m (s_j - s_{j-1})^{-b_j} ds_1 \dots ds_m = h^{m - \sum_{j=1}^m b_j} \frac{\prod_{j=1}^m \Gamma(1 - b_j)}{\Gamma(1 + m - \sum_{j=1}^m b_j)}.$$

Therefore

$$E[L(t+h, B(t)) - L(t, B(t))]^m \leq m! \left(\frac{\nu C^2}{\tilde{K}\pi}\right)^{m/2} h^{m(1-H_{t,t+h})} \frac{(\Gamma(1-H_{t,t+h}))^m}{\Gamma(1+m(1-H_{t,t+h}))}$$

According to Stirling's formula, we have $(m!/\Gamma(1+m(1-H_{t,t+h}))) \leq M^m m!^{H_{t,t+h}}$, $m \geq 2$, for a suitable finite number M . Therefore

$$E \left[\frac{L(t+h, B(t)) - L(t, B(t))}{h^{1-H_{t,t+h}}} \right]^m \leq C^m m!^{H_{t,t+h}}. \quad (21)$$

We shall now prove that for any $K > 0$ there exists a positive and finite constant $A > 0$, depending on t , such that for u small enough,

$$P \left(L(t+h, B(t)) - L(t, B(t)) \geq \frac{Ah^{1-H_{t,t+h}}}{u^{H_{t,t+h}}} \right) \leq \exp(-K/u). \quad (22)$$

Consider u first of the form $u = 1/m$. Combining Tchebychev's inequality and (21) we obtain

$$\begin{aligned} \mathbb{P}\left(\frac{L(t+h, B(t)) - L(t, B(t))}{h^{1-H_{t,t+h}}} \geq Am^{H_{t,t+h}}\right) &\leq \mathbb{E}\left[\frac{L(t+h, B(t)) - L(t, B(t))}{Ah^{(1-H_{t,t+h})}m^{H_{t,t+h}}}\right]^m \\ &\leq \frac{C^m}{A^m} \left(\frac{1}{m}\right)^{mH_{t,t+h}} (m!)^{H_{t,t+h}}. \end{aligned}$$

Using again Stirling's formula, the last expression is at most $\frac{C^m}{A^m} (2\pi m)^{H_{t,t+h}/2} e^{-H_{t,t+h}m}$. This can be written

$$\exp\left(m[\log(C/A) - H_{t,t+h}] + \frac{H_{t,t+h}}{2}[\log(m) + \log(2\pi)]\right), \quad (23)$$

it suffices to take $A > C$ and m_0 large such that for any $m \geq m_0$, (23) is dominated by e^{-2Km} . Moreover for u sufficiently small, there exists $m \geq m_0$ such that $u_{m+1} < u < u_m$ and since $m \geq 1$, $\frac{m}{m+1} \geq \frac{1}{2}$. This proves (22).

In addition, if we take $u(h) = 1/\log \log(1/h)$ and consider first h_m of the form 2^{-m} , then (22) implies

$$P\left(L(t+h_m, B(t)) - L(t, B(t)) \geq Ah_m^{1-H_{t,t+h_m}} \log \log(1/h_m)^{H_{t,t+h_m}}\right) \leq m^{-2},$$

for large m . Consequently, by using Borel Cantelli lemma and monotonicity arguments, we obtain

$$\frac{L(t+h, B(t)) - L(t, B(t))}{h^{1-H_{t,t+h}}} \leq A \log \log(1/h)^{H_{t,t+h}} \quad a.s.$$

In addition the Hölder regularity of H implies that

$$\frac{\log \log(1/h)^{H_{t,t+h}}}{\log \log(1/h)^{H(t)}} = e^{Kh^\beta \log \log \log 1/h} \rightarrow 1 \quad as \quad h \rightarrow 0^+$$

Therefore

$$\log \log(1/h)^{H_{t,t+h}} \leq K \log \log(1/h)^{H(t)}$$

for h small enough. Moreover, using the same arguments (See Remark 3.7 in Boufoussi et al (2006)), we obtain

$$h^{1-H_{t,t+h}} \leq Kh^{1-H(t)}. \quad (24)$$

This completes the proof of (11). Since (12) is proved in a same manner, the proof is omitted here. \square

Remark 3.3. *The result of the previous theorem, with $\sup_{s \in [t, t+h]} H(s)$ instead of $H(t)$, may be proved even when (\mathcal{H}_β) is not satisfied.*

We shall also prove the following uniform result.

Theorem 3.4. Let $\{B(t), t \in [0, 1]\}$ be a mBm with arbitrary Hurst function and denote $H_{0,1} = \sup_{s \in [0,1]} H(s)$. Then, for all $x \in \mathbb{R}$, there exists $C_3 > 0$ such that

$$\limsup_{\substack{|t-s| \searrow 0^+ \\ t,s \in [0,1]}} \frac{|L(t,x) - L(s,x)|}{|t-s|^{1-H_{0,1}} (\log(1/|t-s|))^{H_{0,1}}} \leq C_3 \quad a.s. \quad (25)$$

Proof. By using the same arguments as above, without using the Hölder continuity of H , we show that, with probability one, there exists $m_0(\omega)$ such that for all $m \geq m_0(\omega)$ and $k = 1, \dots, 2^m$

$$L(k2^{-m}, x) - L((k-1)2^{-m}, x) \leq A2^{-(1-H_{0,1})m} (m \log(2))^{H_{0,1}}. \quad a.s.$$

Then by proceeding essentially as in Kôno (1977) we prove (25). \square

Remark 3.5. The results in this section are based on the following property of the mBm:

$$\text{Var}(B(t+h)/B(s) : 0 \leq s \leq t) \geq Ch^{2H_{t,t+h}}, \quad (26)$$

for h small enough. If H is constant, this property is called the one sided strong local nondeterminism; we refer to Monrad and Rootzén (1995 expression (2.2)) and the references therein for definition and applications. The property (26) is satisfied by many other processes with multifractal behavior, such as the Riemann-Liouville mBm $\{X(t), t \geq 0\}$ introduced in Lim (2001) by substituting, in the definition of the Riemann-Liouville fBm, the constant Hurst parameter H by a regular function as follows

$$X(t) = \frac{1}{\Gamma(H(t) + 1/2)} \int_0^t (t-u)^{H(t)-1/2} dW_u.$$

So our results extend those of Kôno (1977) to Gaussian processes without stationary increments.

Remark 3.6. A closer examination of the proof of Theorem 2.1 in Csörgö et al. (1995) shows that the property of stationarity of increments is just used to prove two main ingredients presented in their Lemmas 3.1 and 3.3 and which can in our case be replaced easily by the following properties

1. Let \mathbb{A}_n be the covariance matrix of a Gaussian vector $\{\zeta_i; 1 \leq i \leq n\}$. Then, it is known that the conditional variance can be written as

$$\text{Var}(\zeta_i/\zeta_l, l \neq i, 1 \leq l \leq n) = \frac{|\mathbb{A}_n|}{|\mathbb{A}_n^{(i)}|},$$

where $\mathbb{A}_n^{(i)}$ is the submatrix of \mathbb{A}_n obtained by deleting the i th row and column.

2. If a stochastic process has the property (26), with H constant, then according to (14) its covariance matrix satisfies

$$\det R(t_1, \dots, t_m) \geq C^m \prod_{j=1}^m (t_j - t_{j-1})^{2H}, \quad \text{for } t < t_1 < \dots < t_m < t+h \text{ and } t_0 = 0.$$

Then the arguments of Csörgö et al. (1995, Theorem 2.1) can be adapted to prove that the class of processes satisfying (26) with H constant satisfies

$$\limsup_{T \rightarrow \infty} \sup_{0 \leq t \leq b_T} \frac{L(t + a_T, x) - L(t, x)}{a_T^{1-H} (\log(b_T/a_T) + \log \log(a_T + 1/a_T))^H} < C < \infty \quad a.s. \quad (27)$$

where a_T and b_T are nonnegative functions in $T \geq 0$ such that $\frac{1 + b_T}{a_T} \rightarrow +\infty$ as $T \rightarrow +\infty$.

4 Chung's law of the iterated logarithm

The following theorem establishes an analogue of the Chung law of iterated logarithm for the mBm. The result for the fBm has been obtained by Monrad and Rootzén (1995).

Theorem 4.1. *Let B be a multifractional Brownian motion, assume that H satisfies the condition (\mathcal{H}_β) . Then the following Chung type law of iterated logarithm holds:*

$$\liminf_{\delta \rightarrow 0} \sup_{s \in [t_0, t_0 + \delta]} \frac{|B(s) - B(t_0)|}{(\delta / \log |\log(\delta)|)^{H(t_0)}} = C_{H(t_0)}, \quad a.s. \quad (28)$$

where the constant $C_{H(t_0)}$ is the one appearing in Chung's law for the fBm of Hurst parameter $H(t_0)$.

Proof. We first introduce the process $\{\tilde{B}(t) = B(t) - B^{H(t_0)}(t), t \geq 0\}$, where $B^{H(t_0)}$ denotes a fBm with Hurst parameter $H(t_0)$. According to Theorem 3.3 of Monrad and Rootzén (1995), the fBm with Hurst exponent $H(t_0)$ satisfies (28). Furthermore,

$$\begin{aligned} B(t) - B(t_0) &= (B(t) - B^{H(t_0)}(t)) + (B^{H(t_0)}(t) - B^{H(t_0)}(t_0)), \\ &= \tilde{B}(t) + (B^{H(t_0)}(t) - B^{H(t_0)}(t_0)). \end{aligned}$$

Then (28) will be proved if we show that

$$\lim_{\delta \rightarrow 0} \sup_{t \in [t_0, t_0 + \delta]} \frac{|\tilde{B}(t)|}{(\delta / \log |\log(\delta)|)^{H(t_0)}} = 0. \quad a.s.$$

On the other hand, Lemma 3.1 in Boufoussi et al (2006), implies that there exists a positive constant K such that

$$\sup_{t \in [t_0, t_0 + \delta]} E(\tilde{B}(t))^2 \leq K \delta^{2\beta}.$$

Hence, according to Theorem 2.1 in Adler (1990, p. 43) and a symmetry argument, we have

$$\begin{aligned} P \left(\sup_{t \in [t_0, t_0 + \delta]} |\tilde{B}(t)| \geq u \right) &\leq 2P \left(\sup_{t \in [t_0, t_0 + \delta]} \tilde{B}(t) \geq u \right) \\ &\leq 4 \exp \left(- \frac{\left(u - E \left(\sup_{t \in [t_0, t_0 + \delta]} \tilde{B}(t) \right) \right)^2}{K \delta^{2\beta}} \right). \quad (29) \end{aligned}$$

Denote for simplicity $\Lambda = \sup_{t \in [t_0, t_0 + \delta]} \tilde{B}(t)$. By (29) we obtain

$$\begin{aligned}
E(\Lambda) &\leq \int_0^{+\infty} P \left(\sup_{t \in [t_0, t_0 + \delta]} |\tilde{B}(t)| > x \right) dx \\
&\leq 4 \int_0^{+\infty} \exp \left(-\frac{[x - E\Lambda]^2}{K\delta^{2\beta}} \right) dx \\
&= \frac{4\sqrt{K}\delta^\beta}{\sqrt{2}} \int_{-\sqrt{2/K}\frac{E\Lambda}{\delta^\beta}}^{\infty} e^{-y^2/2} dy \\
&\leq 4\sqrt{K}\pi\delta^\beta
\end{aligned}$$

It follows that

$$(u - E\Lambda)^2 \geq \frac{1}{2}u^2 - (E\Lambda)^2 \geq \frac{1}{2}u^2 - 16K\pi\delta^{2\beta}$$

Consequently (29) becomes

$$P \left(\sup_{t \in [t_0, t_0 + \delta]} |\tilde{B}(t)| \geq u \right) \leq C \exp \left(-\frac{u^2}{K\delta^{2\beta}} \right). \quad (30)$$

Since $H(t_0) < \beta$, there exists $0 < \xi < \beta - H(t_0)$. Consider $\delta_n = n^{1/(2(\xi + H(t_0) - \beta))}$ and $u_n = \delta_n^{H(t_0) + \xi}$. Therefore, according to (30), we have

$$\sum_{n=1}^{\infty} P \left(\sup_{t \in [t_0, t_0 + \delta_n]} |\tilde{B}(t)| \geq u_n \right) \leq \sum_{n=1}^{\infty} \exp \left(-\frac{1}{K}n \right) < \infty$$

It follows from the Borel Cantelli lemma that there exists $n_0 = n_0(\omega)$ such that for all $n \geq n_0$, $\sup_{s \in [t_0, t_0 + \delta_n]} |\tilde{B}(s)| \leq \delta_n^{H(t_0) + \xi}$ almost surely. Furthermore, for $\delta_{n+1} \leq \delta \leq \delta_n$, we have almost surely

$$\begin{aligned}
\sup_{s \in [t_0, t_0 + \delta]} |\tilde{B}(s)| &\leq \sup_{s \in [t_0, t_0 + \delta_n]} |\tilde{B}(s)| \\
&\leq \delta_n^{H(t_0) + \xi} \leq \delta^{H(t_0) + \xi} \left(\frac{\delta_n}{\delta_{n+1}} \right)^{H(t_0) + \xi} \\
&\leq 2^\theta \delta^{H(t_0) + \xi}
\end{aligned}$$

where $\theta = \frac{H(t_0) + \xi}{2(\beta - H(t_0) - \xi)}$. Hence

$$\lim_{\delta \rightarrow 0} \sup_{t \in [t_0, t_0 + \delta]} \frac{|\tilde{B}(t)|}{(\delta / \log |\log(\delta)|)^{H(t_0)}} \leq \lim_{\delta \rightarrow 0} \delta^\xi (\log |\log(\delta)|)^{H(t_0)} = 0. \quad a.s.$$

This completes the proof of the theorem. \square

Remark 4.2. Observe that by the ideas used to prove the previous theorem many laws of iterated logarithm (LIL) proved for the fBm can now be obtained for the mBm and with the same constants. For example, we have the following LIL (c.f. Li and Shao (2001), equation (7.5) for the fBm)

$$\limsup_{\delta \rightarrow 0} \sup_{s \in [t_0, t_0 + \delta]} \frac{|B(s) - B(t_0)|}{\delta^{H(t_0)} (\log |\log(\delta)|)^{1/2}} = \sqrt{2} V_{H(t_0)}(B), \quad a.s. \quad (31)$$

where

$$V_{H(t_0)}(\widehat{B}) = \sqrt{\frac{\pi}{H(t_0) \Gamma(2H(t_0)) \sin(\pi H(t_0))}}$$

and

$$V_{H(t_0)}(B^H) = \frac{\left(\int_{-\infty}^0 \left[(1-s)^{H(t_0)-\frac{1}{2}} - (-s)^{H(t_0)-\frac{1}{2}} \right]^2 ds \right)^{1/2} + \frac{1}{2H(t_0)}}{\Gamma(H(t_0) + 1/2)}.$$

Other laws of iterated logarithm and uniform moduli of continuity are obtained for \widehat{B} by Benassi et al. (1997, Theorem 1.7) via wavelet technics.

We are now able to prove the following

Proposition 4.3. Let B be a multifractional Brownian motion, assume that H satisfies the condition (\mathcal{H}_β) . The pointwise Hölder exponent α_L of $\sup_{x \in \mathbb{R}} L(\cdot, x)$ at t satisfies

$$\alpha_L(t) = 1 - H(t) \quad a.s.$$

Proof. The lower bound was already given in Boufoussi et al. (2006, Corollary 3.6). The upper bound is a consequence of the Chung's law of the mBm. Indeed, since the local time vanishes outside the range of B , we obtain

$$\begin{aligned} \delta &= \int_{\mathbb{R}} L([t_0, t_0 + \delta], x) dx \\ &\leq \sup_{x \in \mathbb{R}} L([t_0, t_0 + \delta], x) \sup_{s, t \in [t_0, t_0 + \delta]} |B(s) - B(t)| \\ &\leq 2 \sup_{x \in \mathbb{R}} L([t_0, t_0 + \delta], x) \sup_{s \in [t_0, t_0 + \delta]} |B(s) - B(t_0)|. \end{aligned} \quad (32)$$

Combining (28) and (32) we obtain that there exists a positive constant C such that

$$\limsup_{\delta \rightarrow 0} \sup_{x \in \mathbb{R}} \frac{L(t_0 + \delta, x) - L(t_0, x)}{\delta^{1-H(t_0)} (\log \log(\delta^{-1}))^{H(t_0)}} \geq C \quad a.s.$$

Which with the definition of the pointwise Hölder exponent proves the result. \square

5 Asymptotic results

It is well known that technics for proving limit theorems related to self similar processes use the self similarity of their local times. It is natural to expect the same when dealing with locally asymptotically self similar processes (lass for brevity). Thus it will be of some interest to know if the local times satisfy a kind of lass property.

5.1 Lass for local times

The answer to the preceding question is affirmative in the case of the mBm and the result is given by the following

Theorem 5.1. *Let B be a multifractional Brownian motion, assume that H satisfies the condition (\mathcal{H}_β) . Then, for any fixed t_0 , the local time of the mBm is locally asymptotically self similar with parameter $1 - H(t_0)$, in the sense that for every $x \in \mathbb{R}$, the processes $\{Y_\rho(t, x), t \in [0, 1]\}_{\rho > 0}$ defined by*

$$Y_\rho(t, x) = \frac{L(t_0 + \rho t, \rho^{H(t_0)}x + B(t_0)) - L(t_0, \rho^{H(t_0)}x + B(t_0))}{\rho^{1-H(t_0)}},$$

converge in law to the local time, $\{\ell(t, x), t \in [0, 1]\}$, of a fBm Z with Hurst parameter $H(t_0)$, i.e.

$$\lim_{\rho \rightarrow 0} \text{law}\{Y_\rho(t, x), t \in [0, 1]\} = \text{law}\{\ell(t, x), t \in [0, 1]\}, \quad (33)$$

where the convergence is in the space of continuous functions endowed with the norm of the uniform convergence.

The motivation to consider the processes Y_ρ follows from the occupation density formula and the lass property (1) of the mBm. Indeed, according to the occupation density formula, the local time has the following representation

$$L(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iux} \left(\int_0^t e^{iuB(s)} ds \right) du.$$

Consequently,

$$Y_\rho(t, x) = \frac{1}{2\pi\rho^{1-H(t_0)}} \int_{\mathbb{R}} e^{-iy(\rho^{H(t_0)}x + B(t_0))} \int_{t_0}^{t_0 + \rho t} e^{iyB(s)} ds dy.$$

Using the change of variables $r = \frac{s - t_0}{\rho}$ and $\rho^{H(t_0)}y = v$, the right hand side of the previous expression becomes

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ivx} \int_0^t \exp\left(iv \frac{B(t_0 + \rho r) - B(t_0)}{\rho^{H(t_0)}} \right) dr dv,$$

which is the local time of the Gaussian process $\left\{ B^\rho(r) = \frac{B(t_0 + \rho r) - B(t_0)}{\rho^{H(t_0)}}, r \in [0, 1] \right\}$.

We need the following lemma for the proof of the finite dimensional convergence.

Lemma 5.1. *Let $\{X(t), t \in [0, 1]\}$ be a stochastic process in the Skorohod space $D([0, 1])$ and define for a fixed interval $I = [a, b] \subset \mathbb{R}$ the map*

$$\phi_I(X)(t) = \int_0^t 1_{\{X(s) \in I\}} ds.$$

If $\{X_n(t), t \in [0, 1]\}_{n \geq 1}$ is a family of processes which converges in law in $D([0, 1])$ to X , then $\phi_I(X_n)$ converges in law to $\phi_I(X)$.

Proof. The lemma is a consequence of the continuity of the map $X(\cdot) \rightarrow \phi_\varepsilon(X(\cdot))$ in J_1 topology on $D([0, 1])$ at almost all sample points of the process X , which is proved on page 11 in Kesten and Spitzer (1979). \square

Proof of Theorem 5.1. To prove the convergence in law, we proceed in two steps. First we prove the tightness of the family $\{Y_\rho(t, x), t \in [0, 1]\}_{\rho > 0}$ in the space of continuous functions. By using (21) and (24), for ρ small enough, we obtain

$$\begin{aligned} \mathbb{E} |Y_\rho(t, x) - Y_\rho(s, x)|^m &= \frac{\mathbb{E} [L(t_0 + \rho t, \rho^{H(t_0)}x + B(t_0)) - L(t_0 + \rho s, \rho^{H(t_0)}x + B(t_0))]^m}{\rho^{(1-H(t_0))m}}, \\ &\leq C_m |t - s|^{(1-H(t_0))m}. \end{aligned}$$

We can take $m > \frac{1}{1-H(t_0)}$ to get the tightness.

Now, we prove the convergence of the finite dimensional distributions of Y_ρ , as ρ tends to 0, to those of the local time ℓ of a fBm, Z , with Hurst parameter $H(t_0)$. We need to show that for any $d \geq 1$, $a_1, \dots, a_d \in \mathbb{R}$ and $t_1, \dots, t_d \in [0, 1]$, the following convergence holds,

$$\sum_{j=1}^d a_j Y_\rho(t_j, x) \xrightarrow{\mathcal{W}} \sum_{j=1}^d a_j \ell(t_j, x) \quad \text{as } \rho \rightarrow 0.$$

We will show the convergence of the corresponding characteristic function. More precisely we prove that

$$\left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j Y_\rho(t_j, x) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \ell(t_j, x) \right] \right| \rightarrow 0, \quad \text{as } \rho \rightarrow 0.$$

Let's introduce the following notations

$$\begin{aligned} I_1^{\varepsilon, \rho} &= \left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j Y_\rho(t_j, x) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(B^\rho(t_j)) \right] \right|, \\ I_2^{\varepsilon, \rho} &= \left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(Z(t_j)) ds \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(B^\rho(t_j)) ds \right] \right|, \\ \text{and } I_3^\varepsilon &= \left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \phi_{\varepsilon, x}(Z(t_j)) ds \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \ell(t_j, x) \right] \right|. \end{aligned}$$

Therefore

$$\left| \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j Y_\rho(t_j, x) \right] - \mathbb{E} \exp \left[i\lambda \sum_{j=1}^d a_j \ell(t_j, x) \right] \right| \leq I_1^{\varepsilon, \rho} + I_2^{\varepsilon, \rho} + I_3^\varepsilon. \quad (34)$$

On the other hand, Y_ρ is the local time of B^ρ and by using the mean value theorem and the occupation density formula we obtain

$$\begin{aligned} I_1^{\varepsilon,\rho} &\leq C \max_{1 \leq j \leq d} E |Y_\rho(t_j, x) - \phi_{\varepsilon,x}(B^\rho(t_j))| \\ &= C \max_{1 \leq j \leq d} E \left| \frac{1}{\varepsilon} \int_x^{x+\varepsilon} Y_\rho(t_j, y) dy - Y_\rho(t_j, x) \right| \end{aligned} \quad (35)$$

Moreover, since the stochastic process $\{Y_\rho(t, y), y \in \mathbb{R}\}$, is almost surely continuous in y for every t , according to the dominated convergence theorem, (35) converges to zero as ε tends to zero independently of ρ .

We deal now with $I_2^{\varepsilon,\rho}$. Since the family of processes $\{B^\rho(t), t \in [0, 1]\}_{\rho>0}$ converges in distribution to the fBm $\{Z(t), t \in [0, 1]\}$ with Hurst parameter $H(t_0)$, the second term converges to zero as ρ tends to 0 by Lemma 5.1 .

The last term in (34) is treated in a similar way as the first and the proof of the finite dimensional convergence is complete . \square

5.2 Limit theorems

The following result is an immediate consequence of the lass of the mBm.

Proposition 5.2. *Let B be a multifractional Brownian motion, assume that H satisfies the condition (\mathcal{H}_β) and denote by $\ell(t, x)$ the local time of a fBm with Hurst parameter $H(t_0)$. Then for every $f \in L^1(\mathbb{R})$, locally Riemann integrable and with compact support such that $\int_{\mathbb{R}} f(x)dx \neq 0$, the following convergence in law holds*

$$\lim_{\lambda \rightarrow \infty} \lim_{\rho \rightarrow 0^+} \frac{1}{\lambda^{1-H(t_0)}} \int_0^{\lambda t} f \left(\frac{B(\rho s + t_0) - B(t_0)}{\rho^{H(t_0)}} \right) ds = \int_{\mathbb{R}} f(x)dx \cdot \ell(t, 0).$$

If H is constant, B is a fBm. Being self similar with stationary increments, the term on the left side of the previous formula is, for H constant, equal in law to $\frac{1}{\lambda^{1-H}} \int_0^{\lambda t} f(B(s))ds$. We retrieve the result by Kasahara and Kosugi (1997) for the fBm.

Proof. The proof of the proposition relies on the lass property and standard limit theorem for the fBm. We sketch it for the sake of completeness.

Combining the fact that f is locally Riemann integrable, the lass property of the mBm and a result in Jegannathan [(2004), Theorem 6 and the comment after it], we obtain

$$\int_0^{\lambda t} f \left(\frac{B(\rho s + t_0) - B(t_0)}{\rho^{H(t_0)}} \right) ds \xrightarrow{(d)} \int_0^{\lambda t} f(B^{H(t_0)}(s))ds \text{ as } \rho \rightarrow 0^+.$$

Moreover, using the occupation density formula and the self similarity of the fBm we obtain

$$\frac{1}{\lambda^{1-H(t_0)}} \int_0^{\lambda t} f(B^{H(t_0)}(s))ds \stackrel{(d)}{=} \int_{\mathbb{R}} f(x)\ell(t, \lambda^{-H(t_0)}x)dx,$$

where $\stackrel{(d)}{=}$ denotes the equality in distribution. Then the result of the theorem follows from the continuity of $\ell(t, x, \omega)$ with respect to the space variable. \square

However the presence of a double limit may be not convenient, and the result may be better if λ and ρ are dependent. We prove the following local limit theorem

Theorem 5.3. *Let B be a multifractional Brownian motion, assume that H satisfies the condition (\mathcal{H}_β) . Denote by $\ell(t, x)$ the local time of a fBm with Hurst parameter $H(t_0)$ and consider $f \in L^1(\mathbb{R})$ such that :*

$$\int_{\mathbb{R}} |f(x)| |x|^\xi dx < \infty \text{ for some } 0 < \xi < \frac{1}{2 \sup_{t \geq 0} H(t)} - \frac{1}{2}, \text{ and } \int_{\mathbb{R}} f(x) dx \neq 0 \quad (36)$$

Then, the following convergence in law holds

$$\frac{1}{\psi(\rho)} \int_{t_0}^{\rho t + t_0} f\left(\frac{B(s) - B(t_0) - \rho^{H(t_0)} y}{\theta(\rho)}\right) ds \xrightarrow{(d)} \int_{\mathbb{R}} f(x) dx \cdot \ell(t, y), \text{ as } \rho \rightarrow 0^+,$$

where $\theta(\cdot)$ and $\psi(\cdot)$ satisfy $\frac{\psi(\rho)}{\theta(\rho)} = \rho^{1-H(t_0)}$, and $\frac{\theta(\rho)}{\rho^{H(t_0)}} = o(1)$.

Proof. Using the occupation density formula we obtain

$$\begin{aligned} & \frac{1}{\psi(\rho)} \int_{t_0}^{\rho t + t_0} f\left(\frac{B(s) - B(t_0) - \rho^{H(t_0)} y}{\theta(\rho)}\right) ds \\ &= \int_{\mathbb{R}} f(x) \frac{L(\rho t + t_0, \theta(\rho)x + \rho^{H(t_0)} y + B(t_0)) - L(t_0, \theta(\rho)x + \rho^{H(t_0)} y + B(t_0))}{\rho^{1-H(t_0)}} dx \\ &= \frac{L(\rho t + t_0, \rho^{H(t_0)} y + B(t_0)) - L(t_0, \rho^{H(t_0)} y + B(t_0))}{\rho^{1-H(t_0)}} \times \int_{\mathbb{R}} f(x) dx \end{aligned} \quad (37)$$

$$+ \int_{\mathbb{R}} f(x) \frac{L(I_0, \theta(\rho)x + \rho^{H(t_0)} y + B(t_0)) - L(I_0, \rho^{H(t_0)} y + B(t_0))}{\rho^{1-H(t_0)}} dx, \quad (38)$$

where we denote in the last expression $I_0 = [t_0, t_0 + \rho t]$ for simplicity. According to Theorem 5.1, the expression (37) converges in distribution to $\ell(t, y) \times \int_{\mathbb{R}} f(x) dx$. It suffices now to prove that (38) converges to 0 in some strong sense. We have

$$\begin{aligned} & E \left| \int_{\mathbb{R}} f(x) \frac{L(I_0, \theta(\rho)x + \rho^{H(t_0)} y + B(t_0)) - L(I_0, \rho^{H(t_0)} y + B(t_0))}{\rho^{1-H(t_0)}} dx \right| \\ & \leq \int_{\mathbb{R}} |f(x)| \left\| \frac{L(I_0, \theta(\rho)x + \rho^{H(t_0)} y + B(t_0)) - L(I_0, \rho^{H(t_0)} y + B(t_0))}{\rho^{1-H(t_0)}} \right\|_{L^2(\Omega)} dx. \end{aligned} \quad (39)$$

Using essentially the same arguments as when proving expression (22) and Remark 3.7 in Boufoussi et al (2006), but for the process $X(t) = B(t) - B(t_0)$ instead of the mBm, we obtain

$$\left\| \frac{L(I_0, \theta(\rho)x + \rho^{H(t_0)} y + B(t_0)) - L(I_0, \rho^{H(t_0)} y + B(t_0))}{\rho^{1-H(t_0)}} \right\|_{L^2(\Omega)} \leq C t^{1-H(t_0)(1+\xi)} |x|^\xi \left(\frac{\theta(\rho)}{\rho^{H(t_0)}} \right)^\xi,$$

for ρ small enough and all $0 < \xi < \frac{1}{2 \sup_{t \geq 0} H(t)} - \frac{1}{2}$. Hence, (39) is dominated by

$$Ct^{1-H(t_0)(1+\xi)} \int_{\mathbb{R}} |f(x)| |x|^\xi dx \times \left(\frac{\theta(\rho)}{\rho^{H(t_0)}} \right)^\xi. \quad (40)$$

This last integral is finite by assumption (36), and then (40) tends to zero as ρ tends to zero. This completes the proof. \square

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