A simple bijection between permutation tableaux and permutations

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A SIMPLE BIJECTION BETWEEN PERMUTATION TABLEAUX AND PERMUTATIONS

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Permutation tableaux are new objects that come from the enumeration of the totally positive Grassmannian cells [4, 7]. Surprisingly they are also connected to a statistical physics model called the Partially ASymmetric Exclusion Process [1, 2, 3].

Our main interest here is that these tableaux are in bijection with permutations. To our knowledge only one bijection between permutations and permutation tableaux is known and appeared in [5], although several bijections between a subclass called Catalan tableaux and diverse objects counted by Catalan numbers are known [6]. This bijection between permutations and permutation tableaux is quite complicated; but a lot of statistics of the permutation (weak exceedances, crossings [1], alignments [7]...) can be read from the tableau. In particular the number of weak exceedances of the permutation corresponds to the number of rows of the tableau. See [5] for many more details.

Our goal here is therefore to present a simple bijection between permutations and permutation tableaux that maps the descents of the permutation to the columns of the tableau. Our result is

Theorem 1. *There exists a bijection between permutations of $\{1, \ldots, n\}$ with k descents and permutation tableaux of length n with k columns.*

We first start with a few definitions. A descent of a permutation σ of a set S is an entry $i \in S$ such that if $\sigma(j) = i$ then $\sigma(j) > \sigma(j + 1)$. For example, if $\sigma = (2, 4, 8, 5, 1, 6, 3, 7)$ and $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then 5,6 and 8 are the descents of σ.

As in [5], a *permutation tableau* T is a shape (the Ferrers diagram of a partition) together with a filling of the cells with 0's and 1's such that the following properties hold:

(1) Each column contains at least one 1.
(2) There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

An example of a permutation tableau is given in Figure 1. Different statistics on permutation tableaux were defined in [3, 5]. We list a few here. The *length* of a tableau is the half perimeter of its shape. A zero in a permutation tableau is *restricted* if there is a one above it in the same column. A row is *unrestricted* if it does not contain a restricted entry. A restricted zero is a *rightmost* restricted zero if it is restricted and it has no restricted zero to its right in the same row.

We label the boundary of the shape of the tableau from 1 to its length, going from top-right to bottom-left. See Figure 1. This labels the rows and the columns : a South (resp. West) step labelled by i gives the label i to this row (resp. column).

1	0	1
0	0	1
1	1	1
0	1	1
1		

FIGURE 1. Example of a permutation tableau

The cell (i, j) of the tableau corresponds the cell that is in the row labelled by i and the column labelled by j.

In Figure 1, a permutation tableau of shape $(3, 3, 3, 3, 1)$ and length 8 is given. The rows 1,3 and 7 are unrestricted and the rows 2 and 4 are restricted. The rightmost restricted zeros are in cells $(2, 8)$ and $(4, 8)$.

We are now ready to present the bijection. Let σ be a permutation of $\{1, \ldots, n\}$ and let T be its image. We first draw the shape of the tableau T. For i from 1 to n, we draw a West step if i is a descent and a South step otherwise. An example for $\sigma = (2, 4, 8, 5, 1, 6, 3, 7)$ is given in Figure 1, as 5,6 and 8 are the descents of σ.

Now let us fill the cells of the tableau T. Let (i, j) be the Eastmost and Southmost cell of the tableau T that is not filled:

- if i and j are not adjacent in the permutation σ then fill the cell (i, j) with a one.
- otherwise
  - if i is before j, then fill all the empty cells of row i with zeros and delete i from the permutation σ.
  - otherwise fill cell (i, j) with a one and all the empty cells of column j with zeros and delete j from the permutation σ.

At the end of this process, T is filled and σ is the list of the labels of the unrestricted rows of T in increasing order. It is easy to check that T is a permutation tableau and that i is a descent in σ if and only if there is a column labelled by i in T.

Example 1. We start with $\sigma = (2, 4, 8, 5, 1, 6, 3, 7)$, and we draw the shape of T (see Figure 1). We first fill cell $(4, 5)$ and fill it with a 1, as 4 and 5 are not adjacent. We also fill cells $(3, 5)$ and $(2, 5)$ with ones. We fill $(1,5)$ with a one and delete 5 from the permutation. The permutation is now $\sigma = (2, 4, 8, 1, 6, 3, 7)$. We fill cell $(4, 6)$ with a one and then cell $(3, 6)$ with a one and all the cells above in this column with a zero and delete 6 from the permutation. The permutation is now $\sigma = (2, 4, 8, 1, 3, 7)$. Then cell $(7, 8)$ gets a one, cell $(4, 8)$ gets a zero and 4 is deleted. The permutation is now $\sigma = (2, 8, 1, 3, 7)$. Finally cell $(3, 8)$ gets a one, cell $(2, 8)$ gets a zero and 2 is deleted. The permutation is now $(8, 1, 3, 7)$. Cell $(1, 8)$ gets a one and all the cells above get a zero and 8 is deleted. The permutation is finally $(1, 3, 7)$. The result is given in Figure 1.

Example 2. Starting with the permutation $(8, 5, 4, 7, 2, 3, 1, 6)$, we first draw the shape of the tableau T (Figure 2). We first fill cell (2,3) and all the cells to its left with zeros and delete 2 from the permutation. Then we add a one in cell (1,3) and delete 3 from the permutation. Then we fill cell (4,5) with a one and all the cells above with zeros and delete 5 from the permutation. Then we fill cell (6,7) with a one and fill cell (4,6) and all the cells to its left with zeros and delete 4 from the permutation. Then we fill cell (1,7) with a one and delete 7 from the permutation.

Finally we fill cell (6,8) and cell (1,8) with a one and delete 8 from the permutation. See Figure 2.

FIGURE 2. Image of the permutation $(8, 5, 4, 7, 2, 3, 1, 6)$.

The reverse is as easy to define. We start with the tableau T. Then we initialize the permutation σ to the list of the labels of the unrestricted rows in increasing order. Now for each column, starting from the left proceeding to the right, if the column is labelled by j and if (i, j) is the topmost one of the column then we add j to the left of i in the permutation σ. Moreover if column j contains rightmost restricted zeros in rows $i_1, \ldots, i_k$ then we add $i_1, \ldots, i_k$ in increasing order to the left of j in the permutation σ. It is easy to see that this is the reverse mapping.

Example 1. We start with the tableau in Figure 1. The unrestricted rows are rows 1,3 and 7. The rightmost restricted zeros are in cells $(2, 8)$ and $(4, 8)$. We start with the permutation $(1, 3, 7)$, We add 8 to the left of 1 and add 2 and 4 to the left of 8. We get $(2, 4, 8, 1, 3, 7)$. We add 6 to the left of 3 and get $(2, 4, 8, 1, 6, 3, 7)$. Finally we add 5 to the left of 1. The permutation is $(2, 4, 8, 5, 1, 6, 3, 7)$.

Example 2. We start with the tableau in Figure 2. The unrestricted rows are rows 1 and 6. The rightmost restricted zeros are in cells $(4, 7)$ and $(2, 3)$. We start with the permutation $(1, 6)$. We add 8 to the left of 1 and get $(8, 1, 6)$. We add 7 to the left of 1 and 4 to the left of 7 and get $(8, 4, 7, 1, 6)$. We then add 5 to the left of 4 and get $(8, 5, 4, 7, 1, 6)$. Finally we add 3 to the left of 1 and 2 to the left of 3. The result is $(8, 5, 4, 7, 2, 3, 1, 6)$.

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