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On Gibbsianness of Random Fields

S. Dachian ^{*} B.S. Nahapetian [†]

Abstract

The problem of characterization of Gibbs random fields is considered. Various Gibbsianness criteria are obtained using the earlier developed one-point framework which in particular allows to describe random fields by means of either one-point conditional or one-point finite-conditional distributions. The main outcome are the criteria in terms of one-point finite-conditional distribution, on the basis of which a simple and comprehensible definition of Gibbs random field is given.

Résumé

Le problème considéré est celui de la caractérisation des champs aléatoires de Gibbs. Divers critères de gibbsianité sont obtenus en utilisant l'approche uniponctuelle développée antérieurement et qui permet en particulier de décrire les champs aléatoires soit par des probabilités conditionnelles uniponctuelles, soit par des probabilités fini-conditionnelles uniponctuelles. Les résultats principaux sont les critères exprimés en termes de probabilités fini-conditionnelles uniponctuelles, à partir desquels une définition simple et compréhensible du champ aléatoire de Gibbs est donnée.

Key words: Gibbsianness, Gibbs random fields, Gibbsian specifications, one-point conditional distribution, one-point finite-conditional distribution.

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Introduction

The classes of processes considered in the random processes theory are usually specified by some properties of their finite-dimensional or conditional distributions. However, as a rule, the study of a particular class goes through some representation theorem expressing processes in terms of simple and convenient objects, such as transition matrices for Markov chains, characteristic functions for processes with independent increments, spectral functions for stationary processes, and so on.

The situation is quite different for the class of Gibbs random fields. Historically, instead of being characterized by the properties of their finite-dimensional or conditional distributions, Gibbs random fields have been defined directly by representation of their conditional distributions in terms of potentials. And only afterwards the problem of characterization of Gibbs random fields was considered.

For the particular case of Gibbs random fields with finite-range potentials, it was shown that they are characterized by strict positivity and Markov properties (see, for example, Averintsev [1, 2, 3] and Sullivan [16]). For the general case of uniformly convergent potentials, Kozlov [13] and Sullivan [17] showed that Gibbs random fields are characterized by strict positivity and quasilocality of their conditional distributions. More precisely, a random field's conditional distribution (which consists of conditional probabilities in finite volumes with infinite boundary conditions and is defined up to a set of probability zero) must have a version which is a strictly positive quasilocal specification. This Gibbsianness criteria is not very handy since one would like a characterization to be in terms of simpler and unambiguously defined objects.

Recently, an approach towards description of random fields was developed by the authors in [4, 5, 6], where Dobrushin's well-known description of random fields by means of conditional distributions was reduced to description of random fields by means of one-point conditional distributions (which consist of conditional probabilities on single sites with infinite boundary conditions). A closely related and in some way complementary description of random fields based on the notion of one-point finite-conditional distribution (the system of conditional probabilities on single sites with finite boundary conditions) was proposed by Dalalyan and Nahapetian in [7]. The one-point framework developed in these papers allows not only to reformulate main results of the random fields theory, but also to carry on its further elaboration. In particular, this framework will be used in the present paper to give a suitable answer to the above mentioned problem of characterization of Gibbs random fields.

The main outcome of the present work are random field Gibbsianness criteria in terms of one-point finite-conditional distribution. These terms meet the above indicated requirements: one-point finite-conditional distribution is a much simpler (than conditional distribution) and unambiguously defined object, since it consists of some ratios of finite-dimensional probabilities of the random field. Moreover, this object uniquely determines (can be identified with) the random field, and so these criteria can serve to define the notion of Gibbs random field in a simple and comprehensible manner. Let us also emphasize that the use of conditional probabilities (with finite boundary conditions) on single sites for the characterization of Gibbs random fields is very natural in light of the one-point framework. Finally, let us mention that the above cited characterization of Gibbs random fields with finite-range potentials can also be considered as being in terms of one-point finite-conditional distribution since in strictly positive case the Markov property can be formulated using these terms.

The random field Gibbsianness criteria in terms of one-point finite-conditional distribution are established in Section 3, which also contains some additional results on characterization of Gibbsian specifications and Gibbs random fields, along with a brief survey of the known ones. Before that, after setting up the notation and giving some necessary prerequisites in Section 1, the authors' approach towards description of random fields (one-point framework) is presented in Section 2.

1 Preliminaries

In this section we briefly recall some necessary notions and facts from the theory of Gibbs random fields.

1.1 Random fields

We consider random fields on the ν -dimensional integer lattice \mathbb{Z}^ν , that is, probability measures \mathbf{P} on $(\mathcal{X}^{\mathbb{Z}^\nu}, \mathcal{F}^{\mathbb{Z}^\nu})$ where $(\mathcal{X}, \mathcal{F})$ is some measurable space of values on single sites (*state space*). Usually the space \mathcal{X} is assumed to be endowed with some topology \mathcal{T} , and \mathcal{F} is assumed to be the Borel σ -algebra for this topology. In this work we concentrate on the case when \mathcal{X} is finite, \mathcal{T} is the discrete topology, and \mathcal{F} is the total σ -algebra, that is, $\mathcal{F} = \mathcal{T} = \text{part}(\mathcal{X})$.

For any $S \subset \mathbb{Z}^\nu$, we denote $\mathcal{E}(S)$ the set of all finite subsets of S , that is, we put $\mathcal{E}(S) = \{\Lambda \subset S : |\Lambda| < \infty\}$ where $|\Lambda|$ is the number of points of the set Λ . For convenience of notation we will omit braces for one-point sets, that is, will write t instead of $\{t\}$. We put also $\mathcal{E}^*(S) = \mathcal{E} \setminus \{\emptyset\}$. For $S = \mathbb{Z}^\nu$ we write $\mathcal{E} = \mathcal{E}(\mathbb{Z}^\nu)$ and $\mathcal{E}^* = \mathcal{E}^*(\mathbb{Z}^\nu)$.

For any $S \subset \mathbb{Z}^\nu$, the space \mathcal{X}^S is the space of all configurations on S . If $S = \emptyset$, we assume that the space $\mathcal{X}^\emptyset = \{\emptyset\}$ where \emptyset is the empty configuration. For any $T, S \subset \mathbb{Z}^\nu$ such that $T \subset S$ and any configuration $\mathbf{x} = \{x_t, t \in S\}$ on S , we denote \mathbf{x}_T the *subconfiguration (restriction)* of \mathbf{x} on T defined by $\mathbf{x}_T = \{x_t, t \in T\}$. For any $T, S \subset \mathbb{Z}^\nu$ such that $T \cap S = \emptyset$ and any configurations \mathbf{x} on T and \mathbf{y} on S , we denote \mathbf{xy} the *concatenation* of \mathbf{x} and \mathbf{y} , that is, the configuration on $T \cup S$ equal to \mathbf{x} on T and to \mathbf{y} on S . For any configuration $\mathbf{x} \in \mathcal{X}^S$, the set $S \subset \mathbb{Z}^\nu$ will be called *support* of \mathbf{x} and we will write $S = \mathfrak{S}(\mathbf{x})$. For any $\Lambda \in \mathcal{E}$, we denote

$$\widetilde{\mathcal{X}}^\Lambda = \bigcup_{\tilde{\Lambda} \in \mathcal{E}^*(\Lambda^c)} \mathcal{X}^{\tilde{\Lambda}}$$

the space of all configurations with non-empty finite support from the outside of Λ .

For any $S \subset \mathbb{Z}^\nu$, a probability distribution on \mathcal{X}^S will be denoted by \mathbf{P}_S . Note that if $S = \emptyset$ there exists only one probability distribution $\mathbf{P}_\emptyset(\emptyset) = 1$. For any $T, S \subset \mathbb{Z}^\nu$ such that $T \subset S$ and any \mathbf{P}_S , we denote $(\mathbf{P}_S)_T$ the *marginal distribution (restriction)* of \mathbf{P}_S on T . If $\Lambda \in \mathcal{E}$ and $I \subset \Lambda$, we can write $\mathbf{P}_\Lambda = \{\mathbf{P}_\Lambda(\mathbf{x}), \mathbf{x} \in \mathcal{X}^\Lambda\}$ and

$$(\mathbf{P}_\Lambda)_I(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{X}^{\Lambda \setminus I}} \mathbf{P}_\Lambda(\mathbf{xy}), \quad \mathbf{x} \in \mathcal{X}^I.$$

Any random field \mathbf{P} on \mathbb{Z}^ν can be described in terms of its *finite-dimensional distributions* $\{\mathbf{P}_\Lambda, \Lambda \in \mathcal{E}\}$ which are consistent in the sense that for any $\Lambda \in \mathcal{E}$ and $I \subset \Lambda$ we have $(\mathbf{P}_\Lambda)_I = \mathbf{P}_I$.

Finally, a random field \mathbf{P} will be called *strictly positive* if for any $\Lambda \in \mathcal{E}$ the finite-dimensional distribution \mathbf{P}_Λ is *strictly positive*, that is, $\mathbf{P}_\Lambda(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{X}^\Lambda$.

1.2 Finite-conditional and conditional distributions of random fields

Let \mathbf{P} be some random field. For any $\Lambda \in \mathcal{E}$, we denote $\mathbf{P}_{\mathcal{E}^*(\Lambda^c)}$ the measure on $\widetilde{\mathcal{X}}^\Lambda$ whose projection on \mathcal{X}^Λ is \mathbf{P}_Λ for any $\widetilde{\Lambda} \in \mathcal{E}^*(\Lambda^c)$, that is, $\mathbf{P}_{\mathcal{E}^*(\Lambda^c)}$ is the direct sum of the measures $\mathbf{P}_{\widetilde{\Lambda}}$.

For all $\Lambda \in \mathcal{E}$, the ratios

$$\mathbf{q}_\Lambda^{\widetilde{\mathbf{x}}}(\mathbf{x}) = \frac{\mathbf{P}_{\Lambda \cup \mathfrak{S}(\widetilde{\mathbf{x}})}(\mathbf{x}\widetilde{\mathbf{x}})}{\mathbf{P}_{\mathfrak{S}(\widetilde{\mathbf{x}})}(\widetilde{\mathbf{x}})}, \quad \mathbf{x} \in \mathcal{X}^\Lambda,$$

exist for $\mathbf{P}_{\mathcal{E}^*(\Lambda^c)}$ -almost all $\widetilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^\Lambda$. Any system

$$\widetilde{\mathcal{Q}} = \left\{ \mathbf{Q}_\Lambda^{\widetilde{\mathbf{x}}}, \quad \Lambda \in \mathcal{E} \text{ and } \widetilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^\Lambda \right\}$$

of probability distributions such that for every $\Lambda \in \mathcal{E}$ we have $\mathbf{Q}_\Lambda^{\widetilde{\mathbf{x}}} = \mathbf{q}_\Lambda^{\widetilde{\mathbf{x}}}$ for $\mathbf{P}_{\mathcal{E}^*(\Lambda^c)}$ -almost all $\widetilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^\Lambda$ will be called *finite-conditional distribution* of the random field \mathbf{P} . The subsystem of $\widetilde{\mathcal{Q}}$ consisting of single-site distributions ($|\Lambda| = 1$) will be called *one-point finite-conditional distribution* of \mathbf{P} . Note that in general a random field may have many *versions* both of finite-conditional and one-point finite-conditional distributions. However, for strictly positive random fields these distributions are uniquely determined and consist of strictly positive elements. If this positivity is uniform with respect to $\widetilde{\mathbf{x}}$, the (one-point) finite-conditional distribution will be called *uniformly nonnull*.

Further, for all $\Lambda \in \mathcal{E}$, the limits

$$\mathbf{q}_\Lambda^{\overline{\mathbf{x}}}(\mathbf{x}) = \lim_{\widetilde{\Lambda} \uparrow \mathbb{Z}^\nu \setminus \Lambda} \mathbf{q}_\Lambda^{\overline{\mathbf{x}}_{\widetilde{\Lambda}}}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X}^\Lambda,$$

exist for \mathbf{P}_{Λ^c} -almost all $\overline{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}$. Any system

$$\mathcal{Q} = \left\{ \mathbf{Q}_\Lambda^{\overline{\mathbf{x}}}, \quad \Lambda \in \mathcal{E} \text{ and } \overline{\mathbf{x}} \in \mathcal{X}^{\Lambda^c} \right\}$$

of probability distributions such that for every $\Lambda \in \mathcal{E}$ we have $\mathbf{Q}_\Lambda^{\overline{\mathbf{x}}} = \mathbf{q}_\Lambda^{\overline{\mathbf{x}}}$ for \mathbf{P}_{Λ^c} -almost all $\overline{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}$ will be called *conditional distribution* of the random field \mathbf{P} . The subsystem of \mathcal{Q} consisting of single-site distributions will be called *one-point conditional distribution* of \mathbf{P} . Note that in general a random field \mathbf{P} may have many *versions* both of conditional and one-point conditional distributions (even if \mathbf{P} is strictly positive). Note also that if a random field \mathbf{P} has a strictly positive version of conditional distribution (a version all of whose elements are strictly positive), then \mathbf{P} is necessarily strictly positive itself.

Concluding this section let us also mention that if \mathbf{P} is a Markov random field, the elements of its one-point conditional distribution $\{\mathbf{Q}_t^{\overline{\mathbf{x}}}, \quad t \in \mathbb{Z}^\nu \text{ and } \overline{\mathbf{x}} \in \mathcal{X}^{t^c}\}$ can equally be considered as elements of its one-point finite-conditional distribution $\{\mathbf{Q}_t^{\widetilde{\mathbf{x}}}, \quad t \in \mathbb{Z}^\nu \text{ and } \widetilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t\}$. However the converse is not true since only the distributions $\mathbf{Q}_t^{\widetilde{\mathbf{x}}}$ such that $\mathfrak{S}(\widetilde{\mathbf{x}})$ contains the boundary of the site t correspond to the elements of the one-point conditional distribution.

1.3 Description of random fields by means of conditional distributions

The well-known description of random fields by means of conditional distributions introduced by Dobrushin in [8, 9, 10] is carried out in terms of specifications. A system

$$\mathcal{Q} = \left\{ \mathbf{Q}_\Lambda^{\bar{x}}, \quad \Lambda \in \mathcal{E} \text{ and } \bar{x} \in \mathcal{X}^{\Lambda^c} \right\}$$

of probability distributions is called *specification* if

$$\mathbf{Q}_\Lambda^{\bar{x}}(\mathbf{x}\mathbf{y}) = (\mathbf{Q}_\Lambda^{\bar{x}})_{\Lambda \setminus I}(\mathbf{x}) \mathbf{Q}_I^{\bar{x}\mathbf{x}}(\mathbf{y}) \quad (1)$$

for all $\Lambda \in \mathcal{E}$, $I \subset \Lambda$, $\mathbf{x} \in \mathcal{X}^{\Lambda \setminus I}$, $\mathbf{y} \in \mathcal{X}^I$ and $\bar{x} \in \mathcal{X}^{\Lambda^c}$.

Note that any version of conditional distribution of a random field \mathbf{P} satisfies a somewhat weaker than (1) condition, where \mathbf{P}_{Λ^c} -almost all (and not necessarily all) $\bar{x} \in \mathcal{X}^{\Lambda^c}$ are considered. However, any random field possesses at least one version of conditional distribution being a specification (see Goldstein [12], Preston [14] and Sokal [15]).

One of the main goals of Dobrushin's theory is to study the set of all random fields *compatible* with a given specification, that is, having it as a version of conditional distribution. The best-known sufficient conditions for existence and for uniqueness of random fields compatible with a given specification are quasilocality and Dobrushin's uniqueness conditions respectively. The first one will play an important role in our considerations, so we recall it below.

Let $S \subset \mathbb{Z}^\nu$. A real-valued function g on \mathcal{X}^S is called *quasilocal* if

$$\lim_{\Lambda \uparrow S} \sup_{\mathbf{x}, \mathbf{y} \in \mathcal{X}^S : \mathbf{x}_\Lambda = \mathbf{y}_\Lambda} |g(\mathbf{x}) - g(\mathbf{y})| = 0,$$

or equivalently if g is a uniform limit of functions depending on values of configuration on finite sets of sites only (*local* functions). Note also that the quasilocality is nothing but continuity with respect to the topology \mathcal{T}^S and, taking into account that \mathcal{X}^S is compact, the strict positivity and uniform nonnullness conditions are equivalent for quasilocal functions.

A specification $\mathcal{Q} = \left\{ \mathbf{Q}_\Lambda^{\bar{x}}, \quad \Lambda \in \mathcal{E} \text{ and } \bar{x} \in \mathcal{X}^{\Lambda^c} \right\}$ is called (*quasi*)*local* if for any $\Lambda \in \mathcal{E}$ and $\mathbf{x} \in \mathcal{X}^\Lambda$ the function $\bar{x} \mapsto \mathbf{Q}_\Lambda^{\bar{x}}(\mathbf{x})$ on \mathcal{X}^{Λ^c} is (quasi)local.

Finally, a specification will be called *strictly positive* if all its elements are strictly positive.

1.4 Gibbs random fields and Gibbsian specifications

The main object of consideration of the present paper are Gibbs random fields. The latter are defined in terms of Gibbsian specifications, which in turn are defined in terms of potentials.

Any function Φ on $\widetilde{\mathcal{X}}^\phi$ taking values in $\mathbb{R} \cup \{+\infty\}$ is called (*interaction*) *potential*. A potential Φ is called *convergent* if it is real-valued and the series

$$\sum_{\tilde{J} \in \mathcal{E}(t^c)} \Phi(x_{\tilde{x}_{\tilde{J}}}) \quad (2)$$

converge for all $t \in \mathbb{Z}^\nu$, $x \in \mathcal{X}^t$ and $\bar{x} \in \mathcal{X}^{t^c}$.

For an arbitrary convergent potential Φ one can construct the specification $\mathcal{Q} = \{\mathbf{Q}_\Lambda^{\bar{\mathbf{x}}}, \Lambda \in \mathcal{E} \text{ and } \bar{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}\}$ given by Gibbs formulae

$$\mathbf{Q}_\Lambda^{\bar{\mathbf{x}}}(\mathbf{x}) = \frac{\exp(-U_\Lambda^{\bar{\mathbf{x}}}(\mathbf{x}))}{\sum_{\mathbf{y} \in \mathcal{X}^\Lambda} \exp(-U_\Lambda^{\bar{\mathbf{x}}}(\mathbf{y}))}, \quad \Lambda \in \mathcal{E}, \mathbf{x} \in \mathcal{X}^\Lambda, \bar{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}, \quad (3)$$

where

$$U_\Lambda^{\bar{\mathbf{x}}}(\mathbf{x}) = \sum_{J: \emptyset \neq J \subset \Lambda} \sum_{\tilde{\mathbf{j}} \in \mathcal{E}(\Lambda^c)} \Phi(\mathbf{x}_J \bar{\mathbf{x}}_{\tilde{\mathbf{j}}}), \quad \Lambda \in \mathcal{E}, \mathbf{x} \in \mathcal{X}^\Lambda, \bar{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}. \quad (4)$$

The specification \mathcal{Q} is called *Gibbsian with potential* Φ . Any random field compatible with \mathcal{Q} is called *Gibbs random field with potential* Φ .

A convergent potential Φ is called *finite-range potential* if in (2) the summation can be taken over some finite (independent of $\bar{\mathbf{x}}$) subset of $\mathcal{E}(t^c)$.

A convergent potential Φ is called *uniformly convergent* if the convergence in (2) (and hence in (4)) is uniform with respect to $\bar{\mathbf{x}}$. In this paper we consider uniformly convergent potentials only, so Gibbsian specifications and Gibbs random fields with uniformly convergent potentials will be called shortly *Gibbsian specifications* and *Gibbs random fields* correspondingly.

2 One-point framework

The idea that it is possible to describe and study random fields using one-point conditional probabilities only goes back to Dobrushin [8]. However it was realized much later in the authors works [4, 5, 6] using one-point conditional distributions, and in Dalalyan and Nahapetian [7] using one-point finite-conditional distributions.

2.1 Description of specifications and random fields by means of one-point conditional distributions

In this section we briefly recall the main results of the authors works [4, 5, 6].

In these papers, under wide positivity assumptions (*very weak positivity*) a necessary and sufficient conditions for a system $\{\mathbf{Q}_t^{\bar{\mathbf{x}}}, t \in \mathbb{Z}^\nu \text{ and } \bar{\mathbf{x}} \in \mathcal{X}^{t^c}\}$ of probability distributions to be contained in some specification were established. A system satisfying these conditions was called 1-specification. It was equally shown that the specification containing the given 1-specification is uniquely determined by some explicit formulae involving the elements of this 1-specification only. Moreover, since these formulae make use of finite number of elementary operations, the whole specification is quasilocal if and only if the 1-specification is, and the set of random fields compatible with the 1-specification coincides with the set of random fields compatible with the whole specification. So, whole Dobrushin's theory can be reformulated in terms of 1-specifications and one can speak about description of random fields by means of one-point conditional distributions.

Let us now give some more details in the particular strictly positive case.

The definition of strictly positive 1-specification can be formulated in the following way: a system

$$\mathcal{Q} = \left\{ \mathbf{Q}_t^{\bar{\mathbf{x}}}, \quad t \in \mathbb{Z}^\nu \text{ and } \bar{\mathbf{x}} \in \mathcal{X}^{t^c} \right\}$$

of strictly positive probability distributions will be called *1-specification* if

$$\begin{aligned} \mathbf{Q}_t^{\bar{x}v}(x) \mathbf{Q}_s^{\bar{x}x}(y) \mathbf{Q}_t^{\bar{x}y}(u) \mathbf{Q}_s^{\bar{x}u}(v) &= \mathbf{Q}_s^{\bar{x}u}(y) \mathbf{Q}_t^{\bar{x}y}(x) \mathbf{Q}_s^{\bar{x}x}(v) \mathbf{Q}_t^{\bar{x}v}(u) \\ \text{for all } t, s \in \mathbb{Z}^\nu, x, u \in \mathcal{X}^t, y, v \in \mathcal{X}^s \text{ and } \bar{x} \in \mathcal{X}^{\{t,s\}^c}. \end{aligned} \quad (5)$$

Further, a 1-specification $\mathcal{Q} = \{\mathbf{Q}_t^{\bar{x}}, t \in \mathbb{Z}^\nu \text{ and } \bar{x} \in \mathcal{X}^{t^c}\}$ is called *(quasi)local* if for any $t \in \mathbb{Z}^\nu$ and $x \in \mathcal{X}^t$ the function $\bar{x} \mapsto \mathbf{Q}_t^{\bar{x}}(x)$ on \mathcal{X}^{t^c} is (quasi)local. Finally, a random fields \mathbf{P} is called *compatible* with a 1-specification if the latter is a version of one-point conditional distribution of \mathbf{P} .

The above mentioned explicit formulae determining the elements of the specification $\mathcal{Q} = \{\mathbf{Q}_\Lambda^{\bar{x}}, \Lambda \in \mathcal{E} \text{ and } \bar{x} \in \mathcal{X}^{\Lambda^c}\}$ containing the given strictly positive 1-specification have the following form: for all $\Lambda \in \mathcal{E}$ and $\bar{x} \in \mathcal{X}^\Lambda$ one has

$$\mathbf{Q}_\Lambda^{\bar{x}}(\mathbf{x}) = \frac{\mathbf{Q}_{t_1}^{\bar{x}\mathbf{u}_{\{t_2, \dots, t_n\}}}(x_{t_1}) \mathbf{Q}_{t_2}^{\bar{x}\mathbf{x}_{\{t_1\}}^{\mathbf{u}_{\{t_3, \dots, t_n\}}}}(x_{t_2}) \cdots \mathbf{Q}_{t_n}^{\bar{x}\mathbf{x}_{\{t_1, \dots, t_{n-1}\}}}(x_{t_n})}{\mathbf{Q}_{t_1}^{\bar{x}\mathbf{u}_{\{t_2, \dots, t_n\}}}(u_{t_1}) \mathbf{Q}_{t_2}^{\bar{x}\mathbf{x}_{\{t_1\}}^{\mathbf{u}_{\{t_3, \dots, t_n\}}}}(u_{t_2}) \cdots \mathbf{Q}_{t_n}^{\bar{x}\mathbf{x}_{\{t_1, \dots, t_{n-1}\}}}(u_{t_n})} \times C, \quad \mathbf{x} \in \mathcal{X}^\Lambda,$$

where C is the normalizing factor. Here some fixed configuration $\mathbf{u} \in \mathcal{X}^\Lambda$ and some enumeration t_1, \dots, t_n of elements of Λ are chosen arbitrary. Note that the right hand side of these formulae does not depend on this choice (correctness of the formulae) thanks to consistency condition (5). Note also, that these formulae imply that the specification containing a strictly positive 1-specification is necessarily strictly positive itself.

2.2 Description of random fields by means of one-point finite-conditional distributions

Now we turn to the problem of description of random fields by means of one-point finite-conditional distributions considered in Dalalyan and Nahapetian [7]. This description is closely related (and in some way complementary) to the one presented in the previous section.

First, let us note that the description of specifications by means of 1-specifications is algebraic in nature. Therefore, the necessary and sufficient conditions for a system $\tilde{\mathcal{q}} = \{\mathbf{Q}_t^{\tilde{x}}, t \in \mathbb{Z}^\nu \text{ and } \tilde{x} \in \widetilde{\mathcal{X}}^t\}$ of strictly positive probability distributions to be contained in some system $\tilde{\mathcal{Q}} = \{\mathbf{Q}_\Lambda^{\tilde{x}}, \Lambda \in \mathcal{E} \text{ and } \tilde{x} \in \widetilde{\mathcal{X}}^\Lambda\}$ of probability distributions satisfying

$$\begin{aligned} \mathbf{Q}_\Lambda^{\tilde{x}}(\mathbf{x}\mathbf{y}) &= (\mathbf{Q}_\Lambda^{\tilde{x}})_{\Lambda \setminus I}(\mathbf{x}) \mathbf{Q}_I^{\tilde{x}\mathbf{x}}(\mathbf{y}) \\ \text{for all } \Lambda \in \mathcal{E}, I \subset \Lambda, \mathbf{x} \in \mathcal{X}^{\Lambda \setminus I}, \mathbf{y} \in \mathcal{X}^I \text{ and } \tilde{x} \in \widetilde{\mathcal{X}}^\Lambda \end{aligned} \quad (6)$$

are the following:

$$\begin{aligned} \mathbf{Q}_t^{\tilde{x}v}(x) \mathbf{Q}_s^{\tilde{x}x}(y) \mathbf{Q}_t^{\tilde{x}y}(u) \mathbf{Q}_s^{\tilde{x}u}(v) &= \mathbf{Q}_s^{\tilde{x}u}(y) \mathbf{Q}_t^{\tilde{x}y}(x) \mathbf{Q}_s^{\tilde{x}x}(v) \mathbf{Q}_t^{\tilde{x}v}(u) \\ \text{for all } t, s \in \mathbb{Z}^\nu, x, u \in \mathcal{X}^t, y, v \in \mathcal{X}^s \text{ and } \tilde{x} \in \widetilde{\mathcal{X}}^{\{t,s\}}. \end{aligned} \quad (7)$$

Note also that if $\tilde{\mathcal{q}}$ is the one-point finite-conditional ($\tilde{\mathcal{Q}}$ is the finite-conditional) distribution of some strictly positive random field, then it necessarily satisfies the condition (7) (the condition (6)). However, in order for a system $\tilde{\mathcal{q}}$ satisfying (7)

(a system $\tilde{\mathcal{Q}}$ satisfying (6)) to be the one-point finite-conditional (the finite-conditional) distribution of some random field one needs some additional conditions. It turns out that such a condition is the following:

$$\begin{aligned} \mathbf{Q}_t^v(x) \mathbf{Q}_s^x(y) \mathbf{Q}_t^y(u) \mathbf{Q}_s^u(v) &= \mathbf{Q}_s^u(y) \mathbf{Q}_t^y(x) \mathbf{Q}_s^x(v) \mathbf{Q}_t^v(u) \\ \text{for all } t, s \in \mathbb{Z}^\nu, x, u \in \mathcal{X}^t \text{ and } y, v \in \mathcal{X}^s. \end{aligned} \quad (8)$$

More precisely, in [7] it was shown that the strict positivity of elements and the fulfillment of the conditions (7) and (8) are necessary and sufficient for a system $\{\mathbf{Q}_t^{\tilde{\mathbf{x}}}, t \in \mathbb{Z}^\nu \text{ and } \tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t\}$ of probability distributions to be the one-point finite-conditional distribution of some strictly positive random field. It was equally shown that this random field is uniquely determined. In particular, a strictly positive random field is uniquely determined by its one-point finite-conditional distribution. So, one can speak about description of random fields by means of one-point finite-conditional distributions.

3 Gibbsianness criteria

In this section we turn to the main subject of the present work: the problem of characterization of Gibbs random fields. But first, let us mention several results concerning Gibbsianness of specifications.

3.1 Specification Gibbsianness criteria

The problem of characterization of the class of Gibbsian specifications with potentials satisfying some given conditions was a subject of consideration of many authors. For example one can refer to Averintsev [1, 2, 3] and Sullivan [16] for finite-range potentials, Kozlov [13] and Sullivan [17] for uniformly convergent potentials, our works [4, 5] for more general potentials (which in particular can assume the value $+\infty$).

Concerning Gibbsian specifications (with uniformly convergent potentials) considered in this paper, the best-known criterion is the following one (see, for example, Georgii [11]).

CRITERION 1. — *A specification is Gibbsian if and only if it is quasilocal and strictly positive.*

Combining the results of Section 2.1 with Criterion 1, one clearly gets the following simpler criterion already obtained by the authors in [4, 5].

CRITERION 2. — *A specification is Gibbsian if and only if the 1-specification contained in it is quasilocal and strictly positive.*

Since the uniform convergence of potential assures the quasilocality of the 1-specification expressed by Gibbs formulae, one can also obtain the following corollary of Criterion 2.

CRITERION 3. — *A specification is Gibbsian if and only if the 1-specification contained in it can be expressed by Gibbs formulae (3) and (4) with some uniformly convergent potential.*

Concerning Gibbsian specifications with finite-range potentials, let us recall that they are characterized by strict positivity and locality properties of their single-site

parts, while corresponding Gibbs random fields are characterized by strict positivity and Markov properties. The last characterization can be considered as being in terms of one-point finite-conditional distribution, since the latter is sufficient for the formulation of the Markov property for strictly positive random fields. Establishment of a similar characterization of Gibbs random fields with uniformly convergent potentials is much less straightforward. We will establish such characterizations in Sections 3.4 and 3.5. Before that, in the next two sections, translating and simplifying Criteria 1–3 we obtain random field Gibbsianness criteria in terms of conditional and one-point conditional distributions correspondingly.

3.2 Random field Gibbsianness criteria in terms of conditional distribution

Specification Gibbsianness criteria allow one to obtain random field Gibbsianness criteria in terms of conditional distribution. For example, combining the definition of Gibbs random field with Criterion 1, one directly gets the following characterization: a random field is a Gibbs random field if and only if it has a version of conditional distribution which is a strictly positive quasilocal specification. This criterion can be simplified in the following way.

CRITERION 4. — *A random field is a Gibbs random field if and only if it has a version of conditional distribution which is quasilocal and strictly positive.*

Since the strict positivity of a version of conditional distribution implies the strict positivity of the random field, the criterion is immediately deduced from the following proposition which is of general interest.

PROPOSITION 1. — *If a strictly positive random field has a quasilocal version of conditional distribution, the latter is unique and is necessarily a specification.*

Proof. Let \mathbf{P} be a strictly positive random field. First, note that the measure \mathbf{P} is everywhere dense, that is, $\mathbf{P}(A) > 0$ for any non-empty open set $A \in \mathcal{J}^{\mathbb{Z}^\nu} \setminus \{\emptyset\}$. Indeed, since such a set A necessarily contains a non-empty cylinder subset A' , which in turn contains a subset $\{\bar{\mathbf{x}} \in \mathcal{X}^{\mathbb{Z}^\nu} : \bar{\mathbf{x}}_\Lambda = \mathbf{x}^\circ\}$ where $\Lambda \in \mathcal{E}^*$ and $\mathbf{x}^\circ \in \mathcal{X}^\Lambda$, we have $\mathbf{P}(A) \geq \mathbf{P}(A') \geq \mathbf{P}_\Lambda(\mathbf{x}^\circ) > 0$. An important evident property of everywhere dense measures is the following: if a continuous function is equal to zero almost everywhere (with respect to such a measure), then it is equal to zero everywhere.

Now, suppose $\{\mathbf{Q}_\Lambda^{\bar{\mathbf{x}}}, \Lambda \in \mathcal{E} \text{ and } \bar{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}\}$ and $\{\mathbf{q}_\Lambda^{\bar{\mathbf{x}}}, \Lambda \in \mathcal{E} \text{ and } \bar{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}\}$ are two quasilocal versions of conditional distribution of \mathbf{P} . Hence, for any $\Lambda \in \mathcal{E}$ and $\mathbf{x} \in \mathcal{X}^\Lambda$, the function $\bar{\mathbf{x}} \mapsto \mathbf{Q}_\Lambda^{\bar{\mathbf{x}}}(\mathbf{x}) - \mathbf{q}_\Lambda^{\bar{\mathbf{x}}}(\mathbf{x})$ on \mathcal{X}^{Λ^c} is quasilocal and equal to zero \mathbf{P}_{Λ^c} -almost everywhere. Since quasilocality is nothing but continuity and the measure \mathbf{P}_{Λ^c} is everywhere dense, this function is equal to zero everywhere. So, the uniqueness is proved.

Finally, suppose $\mathcal{Q} = \{\mathbf{Q}_\Lambda^{\bar{\mathbf{x}}}, \Lambda \in \mathcal{E} \text{ and } \bar{\mathbf{x}} \in \mathcal{X}^{\Lambda^c}\}$ is (the unique) quasilocal version of conditional distribution of \mathbf{P} . For any $\Lambda \in \mathcal{E}$, $I \subset \Lambda$, $\mathbf{x} \in \mathcal{X}^{\Lambda \setminus I}$ and $\mathbf{y} \in \mathcal{X}^I$ consider the function $\bar{\mathbf{x}} \mapsto \mathbf{Q}_\Lambda^{\bar{\mathbf{x}}}(\mathbf{x}\mathbf{y}) - (\mathbf{Q}_\Lambda^{\bar{\mathbf{x}}})_{\Lambda \setminus I}(\mathbf{x}) \mathbf{Q}_I^{\bar{\mathbf{x}}\mathbf{x}}(\mathbf{y})$ on \mathcal{X}^{Λ^c} . This function is clearly quasilocal and, as it follows from the properties of conditional probabilities, is equal to zero \mathbf{P}_{Λ^c} -almost everywhere. Hence it is equal to zero everywhere, and so \mathcal{Q} is a specification. \square

Let us note that Criterion 4 was as a matter of fact obtained in Sullivan [17] using a different approach.

3.3 Random field Gibbsianness criteria in terms of one-point conditional distribution

Considering Criterion 4, one can notice that it is formulated in terms of whole conditional distribution of random field. However, in view of Section 2.1, one-point conditional distribution should be sufficient.

Indeed, combining the definition of Gibbs random field with Criterion 2 and taking into account the results of Section 2.1, one directly gets the following characterization: a random field is a Gibbs random field if and only if it has a version of one-point conditional distribution which is a strictly positive quasilocal 1-specification. As in the preceding section we can simplify this criterion in the following way.

CRITERION 5. — *A random field is a Gibbs random field if and only if it has a version of one-point conditional distribution which is quasilocal and strictly positive.*

The criterion is immediately deduced from the following two propositions which are of general interest.

PROPOSITION 2. — *If a random field \mathbf{P} has a strictly positive version of one-point conditional distribution, then \mathbf{P} is strictly positive itself.*

Proof. Let us suppose that the random field \mathbf{P} is not strictly positive. In this case we can find some $\Lambda \in \mathcal{E}^*$, $t \in \Lambda$ and $\mathbf{z} \in \mathcal{X}^\Lambda$ such that $\mathbf{P}_\Lambda(\mathbf{z}) = 0$ and $\mathbf{P}_{\Lambda \setminus t}(\mathbf{z}_{\Lambda \setminus t}) > 0$ (recall that $\mathbf{P}_\emptyset(\emptyset) = 1$). Now denote

$$A = \left\{ \bar{\mathbf{x}} \in \mathcal{X}^{t^c} : \bar{\mathbf{x}}_{\Lambda \setminus t} = \mathbf{z}_{\Lambda \setminus t} \right\}.$$

Obviously $\mathbf{P}_{t^c}(A) = \mathbf{P}_{\Lambda \setminus t}(\mathbf{z}_{\Lambda \setminus t}) > 0$. Introduce also

$$B = \bigcap_{\tilde{\Lambda} \in \mathcal{E}(t^c)} \left\{ \bar{\mathbf{x}} \in \mathcal{X}^{t^c} : \mathbf{P}_{\tilde{\Lambda}}(\bar{\mathbf{x}}_{\tilde{\Lambda}}) > 0 \right\}.$$

Since B is a countable intersection of sets of probability 1, we have $\mathbf{P}_{t^c}(B) = 1$. So, it comes $\mathbf{P}_{t^c}(A \cap B) > 0$.

For all $\bar{\mathbf{x}} \in A \cap B$ and all $\tilde{\Lambda} \in \mathcal{E}(t^c)$ such that $\tilde{\Lambda} \supset \Lambda \setminus t$, we have

$$\mathbf{q}_t^{\bar{\mathbf{x}}_{\tilde{\Lambda}}}(\mathbf{z}_t) \triangleq \frac{\mathbf{P}_{t \cup \tilde{\Lambda}}(\mathbf{z}_t \bar{\mathbf{x}}_{\tilde{\Lambda}})}{\mathbf{P}_{\tilde{\Lambda}}(\bar{\mathbf{x}}_{\tilde{\Lambda}})} = 0.$$

Hence, for all $\bar{\mathbf{x}} \in A \cap B$ we get

$$\lim_{\tilde{\Lambda} \uparrow \mathcal{Z}^{\nu \setminus t}} \mathbf{q}_t^{\bar{\mathbf{x}}_{\tilde{\Lambda}}}(\mathbf{z}_t) = 0$$

which contradicts the existence of a strictly positive version of one-point conditional distribution of \mathbf{P} . \square

PROPOSITION 3. — *If a strictly positive random field has a quasilocal version of one-point conditional distribution, the latter is unique and is necessarily a 1-specification.*

Proof. The uniqueness is proved following exactly the same argument as in the proof of Proposition 1.

To prove the second assertion, suppose $\{\mathbf{Q}_t^{\bar{\mathbf{x}}}, t \in \mathbb{Z}^\nu \text{ and } \bar{\mathbf{x}} \in \mathcal{X}^{t^c}\}$ is (the unique) quasilocal version of one-point conditional distribution of a strictly positive random field \mathbf{P} . For any $t, s \in \mathbb{Z}^\nu$, $x, u \in \mathcal{X}^t$ and $y, v \in \mathcal{X}^s$ consider the function $\bar{\mathbf{x}} \mapsto \mathbf{Q}_t^{\bar{\mathbf{x}}v}(x) \mathbf{Q}_s^{\bar{\mathbf{x}}x}(y) \mathbf{Q}_t^{\bar{\mathbf{x}}y}(u) \mathbf{Q}_s^{\bar{\mathbf{x}}u}(v) - \mathbf{Q}_s^{\bar{\mathbf{x}}u}(y) \mathbf{Q}_t^{\bar{\mathbf{x}}y}(x) \mathbf{Q}_s^{\bar{\mathbf{x}}x}(v) \mathbf{Q}_t^{\bar{\mathbf{x}}v}(u)$ on $\mathcal{X}^{\{t,s\}^c}$. Applying the reasoning used in the proof of Proposition 1, it clearly comes that this function is equal to zero everywhere. \square

Concluding this section, let us note that combining the definition of Gibbs random field with Criterion 3 and taking into account the results of Section 2.1, one directly gets the following criterion which gives a simplified definition of Gibbs random field.

CRITERION 6. — *A random field is a Gibbs random field if and only if it has a version of one-point conditional distribution which can be expressed by Gibbs formulae (3) and (4) with some uniformly convergent potential.*

3.4 Random field Gibbsianness criteria in terms of one-point finite-conditional distribution I

In order to apply the random field Gibbsianness criteria presented above, one needs to verify the existence of a version of (one-point) conditional distribution possessing some properties. But in general, such type condition seems quite difficult to check. Below we give random field Gibbsianness criteria in terms of simpler and unambiguously defined objects, more precisely in terms of one-point finite-conditional distribution.

CRITERION 7. — *A random field is a Gibbs random field if and only if it is strictly positive and its one-point finite-conditional distribution $\{\mathbf{q}_t^{\tilde{\mathbf{x}}}, t \in \mathbb{Z}^\nu \text{ and } \tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t\}$ satisfy one of the following equivalent conditions:*

(A) *the limits*

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \mathbf{q}_t^{\bar{\mathbf{x}}^\Lambda}(x), \quad t \in \mathbb{Z}^\nu, x \in \mathcal{X}^t, \bar{\mathbf{x}} \in \mathcal{X}^{t^c},$$

exist, are nonnull uniformly with respect to $\bar{\mathbf{x}}$, and the convergence is uniform with respect to $\bar{\mathbf{x}}$,

(B) *the limits*

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \mathbf{q}_t^{\bar{\mathbf{x}}^\Lambda}(x), \quad t \in \mathbb{Z}^\nu, x \in \mathcal{X}^t, \bar{\mathbf{x}} \in \mathcal{X}^{t^c},$$

exist, are strictly positive, and the convergence is uniform with respect to $\bar{\mathbf{x}}$.

Proof. The sufficiency is quite evident. Indeed, the strictly positive limits supposed to exist form a strictly positive version of one-point conditional distribution of the random field. The uniformity of convergence guarantees that this version is quasilocal and so, the sufficiency follows from Criterion 5. Let us also note that at the same time this quasilocality clearly yields the equivalence of the conditions (A) and (B).

Now let us turn to the proof of the necessity. Let \mathbf{P} be a Gibbs random field. According to Criterion 5 it has a quasilocal and strictly positive version

$\mathcal{Q} = \{\mathbf{Q}_t^{\bar{x}}, t \in \mathbb{Z}^\nu \text{ and } \bar{x} \in \mathcal{X}^{t^c}\}$ of one-point conditional distribution. So, to conclude the proof it is sufficient to show that

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \sup_{\bar{x} \in \mathcal{X}^{t^c}} |\mathbf{q}_t^{\bar{x}^\Lambda}(x) - \mathbf{Q}_t^{\bar{x}}(x)| = 0$$

for all $t \in \mathbb{Z}^\nu$ and $x \in \mathcal{X}^t$.

For this we need the following inequality due to Sullivan:

$$\inf_{\bar{y} \in \mathcal{X}^{t^c} : \bar{y}_\Lambda = z} \mathbf{Q}_t^{\bar{y}}(x) \leq \mathbf{q}_t^z(x) \leq \sup_{\bar{y} \in \mathcal{X}^{t^c} : \bar{y}_\Lambda = z} \mathbf{Q}_t^{\bar{y}}(x) \quad (9)$$

for all $t \in \mathbb{Z}^\nu$, $\Lambda \in \mathcal{E}^*(t^c)$, $x \in \mathcal{X}^t$ and $z \in \mathcal{X}^\Lambda$. This inequality is clearly valid since

$$\mathbf{q}_t^z(x) = \frac{\mathbf{P}_{t \cup \Lambda}(xz)}{\mathbf{P}_\Lambda(z)} = \frac{1}{\mathbf{P}_\Lambda(z)} \int_{\{\bar{y} \in \mathcal{X}^{t^c} : \bar{y}_\Lambda = z\}} \mathbf{Q}_t^{\bar{y}}(x) \mathbf{P}_{t^c}(d\bar{y}).$$

Taking this inequality into account, it remains to verify that

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \sup_{\bar{x} \in \mathcal{X}^{t^c}} \left| \inf_{\bar{y} \in \mathcal{X}^{t^c} : \bar{y}_\Lambda = \bar{x}_\Lambda} \mathbf{Q}_t^{\bar{y}}(x) - \mathbf{Q}_t^{\bar{x}}(x) \right| = 0$$

and

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \sup_{\bar{x} \in \mathcal{X}^{t^c}} \left| \sup_{\bar{y} \in \mathcal{X}^{t^c} : \bar{y}_\Lambda = \bar{x}_\Lambda} \mathbf{Q}_t^{\bar{y}}(x) - \mathbf{Q}_t^{\bar{x}}(x) \right| = 0$$

for all $t \in \mathbb{Z}^\nu$ and $x \in \mathcal{X}^t$. To show the first one we write

$$\sup_{\bar{x} \in \mathcal{X}^{t^c}} \left| \inf_{\bar{y} \in \mathcal{X}^{t^c} : \bar{y}_\Lambda = \bar{x}_\Lambda} \mathbf{Q}_t^{\bar{y}}(x) - \mathbf{Q}_t^{\bar{x}}(x) \right| \leq \sup_{\bar{x} \in \mathcal{X}^{t^c}} \sup_{\bar{y} \in \mathcal{X}^{t^c} : \bar{y}_\Lambda = \bar{x}_\Lambda} \left| \mathbf{Q}_t^{\bar{y}}(x) - \mathbf{Q}_t^{\bar{x}}(x) \right|$$

and use the quasilocality of \mathcal{Q} . The second one is proved similarly. \square

Considering Criterion 7, let us note that in our opinion it gives the simplest and the most comprehensible characterization of the class of Gibbs random fields. Having this criteria in mind, one can define the notion of Gibbs random field in the following way:

DEFINITION. — A random field \mathbf{P} is called Gibbs random field if

- 1) for any $\Lambda \in \mathcal{E}$ and $\mathbf{x} \in \mathcal{X}^\Lambda$ one has $\mathbf{P}_\Lambda(\mathbf{x}) > 0$,
- 2) the limits

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \frac{\mathbf{P}_{t \cup \Lambda}(x\bar{x}_\Lambda)}{\mathbf{P}_\Lambda(\bar{x}_\Lambda)}, \quad t \in \mathbb{Z}^\nu, x \in \mathcal{X}^t, \bar{x} \in \mathcal{X}^{t^c},$$

exist, are strictly positive, and the convergence is uniform with respect to \bar{x} .

Taking this definition as the definition of Gibbs random field, the following theorem about representation of conditional distribution of Gibbs random fields is clearly valid.

REPRESENTATION THEOREM. — If \mathbf{P} is a Gibbs random field, then \mathbf{P} has a version of conditional distribution which can be expressed by Gibbs formulae (3) and (4) with some uniformly convergent potential.

Conversely, if a random field \mathbf{P} has a version of conditional distribution which can be expressed by Gibbs formulae (3) and (4) with some uniformly convergent potential, then \mathbf{P} is a Gibbs random field.

3.5 Random field Gibbsianness criteria in terms of one-point finite-conditional distribution II

At first sight, the above presented Criterion 7 deals with one-point finite-conditional distribution only. However, in fact it imposes conditions equally on its limit, that is, on one-point conditional distribution. Besides, the existence and positivity of uniform limits might be difficult to check.

The following and last criterion is perhaps more applicable and really deals with one-point finite-conditional distribution only. Before formulating it, let us agree that in the sequel when we use the notation \mathbf{x}_T we presume that only configurations \mathbf{x} such that $\mathfrak{S}(\mathbf{x}) \supset T$ are considered.

CRITERION 8. — *A random field is a Gibbs random field if and only if it is strictly positive, its one-point finite-conditional distribution $\{\mathbf{q}_t^{\tilde{\mathbf{x}}}, t \in \mathbb{Z}^\nu \text{ and } \tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t\}$ is uniformly nonnull and one of the following equivalent conditions holds:*

(C) for any $t \in \mathbb{Z}^\nu$ and $x \in \mathcal{X}^t$ one has

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \sup_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \widetilde{\mathcal{X}}^t : \tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right| = 0,$$

(D) for any $t \in \mathbb{Z}^\nu$ and $x \in \mathcal{X}^t$ one has

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \sup_{J \in \mathcal{L}^*(t^c)} \sup_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathcal{X}^J : \tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right| = 0,$$

(E) for any $t \in \mathbb{Z}^\nu$ and $x \in \mathcal{X}^t$ one has

$$\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \sup_{\tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{x}}_\Lambda}(x) \right| = 0.$$

Proof. First we concentrate on the condition (E). Clearly

$$\sup_{\tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{x}}_\Lambda}(x) \right| = \sup_{I \in \mathcal{L}(t^c) : I \supset \Lambda} \sup_{\tilde{\mathbf{x}} \in \mathcal{X}^{t^c}} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}^I}(x) - \mathbf{q}_t^{\tilde{\mathbf{x}}_\Lambda}(x) \right|,$$

and so the condition (E) is nothing but the Cauchy condition for the existence of the uniform limits considered in Criterion 7. The sufficiency now clearly follows from Criterion 7 since the Cauchy principle yields the existence of the uniform limits, and the uniform nonnullness of one-point finite-conditional distribution guarantees their strict positivity. The necessity also follows from Criterion 7 since the condition (E) is ensured by the Cauchy principle, and the uniform nonnullness of one-point finite-conditional distribution can be easily obtained from (9) and the condition (A) (use the first inequality of (9) and the uniform nonnullness of limits considered in the condition (A)).

It remains to check the equivalence of the conditions (C), (D) and (E). The implications (C) \Rightarrow (D) and (C) \Rightarrow (E) are trivial since

$$\sup_{J \in \mathcal{L}^*(t^c)} \sup_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathcal{X}^J : \tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right| \leq \sup_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \widetilde{\mathcal{X}}^t : \tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right|$$

and

$$\sup_{\tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{x}}_\Lambda}(x) \right| \leq \sup_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \widetilde{\mathcal{X}}^t : \tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right|.$$

Similarly, the inequality

$$\sup_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \widetilde{\mathcal{X}}^t : \tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right| \leq 2 \sup_{\tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^t} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{x}}_\Lambda}(x) \right|$$

yields the implications (E) \Rightarrow (C). To prove the last implication (D) \Rightarrow (C), we need the following lemma.

LEMMA. — Let $\{\mathbf{q}_I^{\tilde{\mathbf{x}}}, I \in \mathcal{E} \text{ and } \tilde{\mathbf{x}} \in \widetilde{\mathcal{X}}^I\}$ be the finite-conditional distribution of some strictly positive random field. Then the set

$$A = \left\{ \bar{\mathbf{x}} \in \mathcal{X}^{\mathbb{Z}^\nu} : \lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus I} \mathbf{q}_I^{\bar{\mathbf{x}}_\Lambda}(x) \text{ exists for every } I \in \mathcal{E} \text{ and } x \in \mathcal{X}^I \right\}$$

is of probability 1 and possesses the following property: if $\bar{\mathbf{x}} \in A$ then $z\bar{\mathbf{x}}_{J^c} \in A$ for all $J \in \mathcal{E}$ and $z \in \mathcal{X}^J$.

Proof. Since the set A is a countable intersection of sets of probability 1, it is also of probability 1. It remains to show that if $\bar{\mathbf{x}} \in A$ then $\bar{\mathbf{y}} = z\bar{\mathbf{x}}_{t^c} \in A$ for all $t \in \mathbb{Z}^\nu$ and $z \in \mathcal{X}^t$, that is, $\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus I} \mathbf{q}_I^{\bar{\mathbf{y}}_\Lambda}(\mathbf{x})$ exists for every $I \in \mathcal{E}$ and $\mathbf{x} \in \mathcal{X}^I$. This is trivial if $t \in I$ (since in this case $\bar{\mathbf{y}}_\Lambda = \bar{\mathbf{x}}_\Lambda$) and clearly follows from the relation

$$\mathbf{q}_I^{\bar{\mathbf{y}}_\Lambda}(\mathbf{x}) = \mathbf{q}_I^{z\bar{\mathbf{x}}_{\Lambda \setminus t}}(\mathbf{x}) = \frac{\mathbf{q}_{t \cup I}^{\bar{\mathbf{x}}_{\Lambda \setminus t}}(z\mathbf{x})}{(\mathbf{q}_{t \cup I}^{\bar{\mathbf{x}}_{\Lambda \setminus t}})_t(z)}, \quad \Lambda \ni t,$$

otherwise. \triangleleft

Returning to the proof of the implication (D) \Rightarrow (C), let us fix some $t \in \mathbb{Z}^\nu$ and $x \in \mathcal{X}^t$, denote

$$f(\Lambda) = \sup_{J \in \mathcal{E}^*(t^c)} \sup_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathcal{X}^J : \tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right|,$$

and for any $\varepsilon > 0$ choose (according to the condition (D)) some $\Lambda_\varepsilon \in \mathcal{E}$ such that $|f(\Lambda)| < \varepsilon$ for all $\Lambda \in \mathcal{E}$, $\Lambda \supset \Lambda_\varepsilon$.

First, we will show that $\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \mathbf{q}_t^{\bar{\mathbf{x}}_\Lambda}(x)$ exists for every $\bar{\mathbf{x}} \in \mathcal{X}^{t^c}$. Let us take some $\bar{\mathbf{x}}^\circ \in A$ (according to the lemma, the set A is of probability 1 and so is not empty) and consider $\bar{\mathbf{y}} = \bar{\mathbf{x}}_{\Lambda_\varepsilon} \bar{\mathbf{x}}_{\Lambda_\varepsilon^c}^\circ \in A$. So, we can find some $\Lambda'_\varepsilon \in \mathcal{E}$ such that $|\mathbf{q}_t^{\bar{\mathbf{y}}_I}(x) - \mathbf{q}_t^{\bar{\mathbf{y}}_J}(x)| < \varepsilon$ for all $I, J \in \mathcal{E}$, $I \supset \Lambda'_\varepsilon$, $J \supset \Lambda'_\varepsilon$. Thus, for all $I, J \in \mathcal{E}$ such that $I \supset \Lambda_\varepsilon \cup \Lambda'_\varepsilon$ and $J \supset \Lambda_\varepsilon \cup \Lambda'_\varepsilon$ we can write

$$\begin{aligned} \left| \mathbf{q}_t^{\bar{\mathbf{x}}_I}(x) - \mathbf{q}_t^{\bar{\mathbf{x}}_J}(x) \right| &\leq \left| \mathbf{q}_t^{\bar{\mathbf{x}}_I}(x) - \mathbf{q}_t^{\bar{\mathbf{y}}_I}(x) \right| + \left| \mathbf{q}_t^{\bar{\mathbf{y}}_I}(x) - \mathbf{q}_t^{\bar{\mathbf{y}}_J}(x) \right| + \left| \mathbf{q}_t^{\bar{\mathbf{y}}_J}(x) - \mathbf{q}_t^{\bar{\mathbf{x}}_J}(x) \right| \\ &< f(\Lambda_\varepsilon) + \varepsilon + f(\Lambda_\varepsilon) < 3\varepsilon, \end{aligned}$$

and hence $\lim_{\Lambda \uparrow \mathbb{Z}^\nu \setminus t} \mathbf{q}_t^{\bar{\mathbf{x}}_\Lambda}(x)$ exists according to Cauchy principle.

Further, for every $\bar{\mathbf{x}} \in \mathcal{X}^{t^c}$ consider the set $V(\bar{\mathbf{x}}) = \{\bar{\mathbf{y}} \in \mathcal{X}^{t^c} : \bar{\mathbf{y}}_{\Lambda_\varepsilon} = \bar{\mathbf{x}}_{\Lambda_\varepsilon}\}$. Clearly these sets are either mutually disjoint or coinciding, and there is only a finite number k (more precisely $k = |\mathcal{X}^{\Lambda_\varepsilon}|$) of different sets among them. Hence there exists a finite collection $\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^k \in \mathcal{X}^{t^c}$ such that

$$\mathcal{X}^{t^c} = \bigcup_{i=1}^k V(\bar{\mathbf{x}}^i).$$

(This fact equally follows from the compactness of \mathcal{X}^{t^c} .) So, using Cauchy principle we can find some $\Lambda_\varepsilon'' \in \mathcal{E}$ such that $|\mathbf{q}_t^{\bar{\mathbf{x}}^i}(x) - \mathbf{q}_t^{\bar{\mathbf{x}}^j}(x)| < \varepsilon$ for all $i = 1, \dots, k$ and all $I, J \in \mathcal{E}$, $I \supset \Lambda_\varepsilon''$, $J \supset \Lambda_\varepsilon''$.

Now, let the set $\Lambda \in \mathcal{E}$ be such that $\Lambda \supset \Lambda_\varepsilon \cup \Lambda_\varepsilon''$, the sets $I, J \in \mathcal{E}$ be such that $I \supset \Lambda$ and $J \supset \Lambda$, and the configurations $\tilde{\mathbf{x}} \in \mathcal{X}^I$ and $\tilde{\mathbf{y}} \in \mathcal{X}^J$ be such that $\tilde{\mathbf{x}}_\Lambda = \tilde{\mathbf{y}}_\Lambda$. Clearly, we can find some $i \in \{1, \dots, k\}$ such that $\bar{\mathbf{x}}_{\Lambda_\varepsilon}^i = \tilde{\mathbf{x}}_{\Lambda_\varepsilon} = \tilde{\mathbf{y}}_{\Lambda_\varepsilon}$, and thus we may write

$$\begin{aligned} \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right| &\leq \left| \mathbf{q}_t^{\tilde{\mathbf{x}}}(x) - \mathbf{q}_t^{\bar{\mathbf{x}}^i}(x) \right| + \left| \mathbf{q}_t^{\bar{\mathbf{x}}^i}(x) - \mathbf{q}_t^{\bar{\mathbf{x}}^j}(x) \right| + \left| \mathbf{q}_t^{\bar{\mathbf{x}}^j}(x) - \mathbf{q}_t^{\tilde{\mathbf{y}}}(x) \right| \\ &< f(\Lambda_\varepsilon) + \varepsilon + f(\Lambda_\varepsilon) < 3\varepsilon \end{aligned}$$

which shows that the condition (C) holds. \square

In conclusion let us note that the analogues of Criteria 7 and 8 formulated in terms of whole finite-conditional distribution are of course valid. Concerning the first one, we would like to mention that its necessity statement was as a matter of fact contained in the proof of Lemma 1 of Sullivan [17], whose argument we follow while proving Criterion 7. As to the second one, let us mention that the part utilizing the analogue of the condition (D) was stated in Theorems 1 and 2 of Kozlov [13]. It should be pointed out that the author does not provide the proof of the sufficiency statement of Theorem 2 (leaving it, as he says, to the reader). However, our considerations show that the proof of this statement is neither intuitive, nor technically simple. Moreover, the validity of the statement seems dubitable in the settings of Kozlov [13] where the state space is not supposed to be finite or even compact.

References

- [1] M.B. AVERINTSEV, “A way of describing random fields with a discrete argument”, *Probl. Inf. Transm.* **6**, pp. 169–175, 1970.
- [2] M.B. AVERINTSEV, “Description of Markovian random fields by Gibbsian conditional probabilities”, *Theory Probab. Appl.* **17**, pp. 20–33, 1972.
- [3] M.B. AVERINTSEV, “Gibbsian distribution of random fields whose conditional probabilities may vanish”, *Probl. Inf. Transm.* **11**, pp. 326–334, 1975.
- [4] S. DACHIAN, B.S. NAHAPETIAN, “An approach towards description of random fields”, *Université du Maine*, preprint no. 98–5, Mai 1998.

- [5] S. DACHIAN, B.S. NAHAPETIAN, “*Description of random fields by means of one-point conditional distributions and some applications*”, Markov Processes and Related Fields **7**, pp. 193–214, 2001.
- [6] S. DACHIAN, B.S. NAHAPETIAN, “*Description of specifications by means of probability distributions in small volumes under condition of very weak positivity*”, J. Stat. Phys. **117**, no. 1–2, pp. 281–300, 2004.
- [7] A. DALALYAN, B.S. NAHAPETIAN, “*Description of random fields by means of one-point finite-conditional distributions*”, work in progress.
- [8] R.L. DOBRUSHIN, “*The description of a random field by means of conditional probabilities and conditions of its regularity*”, Theory of Probability and Applications **13**, pp. 197–224, 1968.
- [9] R.L. DOBRUSHIN, “*Gibbsian random fields for lattice systems with pairwise interaction*”, Funct. Anal. Appl. **2**, pp. 292–301, 1968.
- [10] R.L. DOBRUSHIN, “*The problem of uniqueness of a Gibbs random field and the problem of phase transitions*”, Funct. Anal. Appl. **2**, pp. 302–312, 1968.
- [11] H.-O. GEORGII, “*Gibbs Measures and Phase Transitions*”, de Gruyter, Berlin, 1988.
- [12] S. GOLDSTEIN, “*A note on specifications*”, Z. Wahrsch. verw. Geb. **46**, pp. 45–51, 1978.
- [13] O.K. KOZLOV, “*Gibbs Description of a System of Random Variables*”, Probl. Inf. Transm. **10**, pp. 258–265, 1974.
- [14] C. PRESTON, “*Construction of specifications*”, in Quantum Fields – Algebras, Processes (Bielefeld symposium 1978), pp. 269–292, ed. L. Streit. Wien-New York : Springer, 1980.
- [15] A.D. SOKAL, “*Existence of compatible families of proper regular conditional probabilities*”, Z. Wahrsch. verw. Geb. **56**, pp. 537–548, 1981.
- [16] W.G. SULLIVAN, “*Finite range random fields and energy fields*”, J. Math. Anal. Appl. **44**, pp. 710–724, 1973.
- [17] W.G. SULLIVAN, “*Potentials for almost Markovian random fields*”, Commun. Math. Phys. **33**, pp. 61–74, 1973.