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# Unbiased minimum variance estimation for systems with unknown exogenous inputs

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**Key words** - Minimum variance; Kalman filter; Unknown disturbance; Stability; Convergence

**Abstract** - A new method is developed for the state estimation of linear discrete-time stochastic system in the presence of unknown disturbance. The obtained filter is optimal in the unbiased minimum variance sense. The necessary and sufficient conditions for the existence and the stability of the filter are given.

## 1. Introduction

The problem of the state estimation in the presence of unknown inputs has received considerable attention in the last two decades. In particular, for deterministic systems the observer design has been treated and the existence and stability conditions are well established (see Darouach et al., 1994, and references therein). However for stochastic systems few approaches have been developed. The most common approach is to treat the unknown input  $d(t)$  as a stochastic process with known wide-sense description (known mean and covariance for example) or as a constant bias, this approach was introduced by Friedland (1969) and several extensions have appeared in the literature (see Ignani, 1990; Zhou et al., 1993).

Generally, there may be no any knowledge concerning the model of these inputs. As shown by Kitanidis (1987), the problem is of great importance in geophysical and environmental applications, where one can not make any assumption on the evolution of the unknown inputs and the filtering is too dependent on poorly justified assumptions about how  $d(t)$  varies in time and about its statistical characteristics. Others applications of the state estimation in presence of unknown inputs are in the fault detection and isolation (FDI) problems (see Patton et al., 1989; Basseville, 1994).

In this paper, we derive the unbiased minimum-variance linear state estimation in presence of unknown deterministic inputs for discrete-time stochastic systems. The results from the approach developed in Kitanidis (1987) are extended. In particular a new method for the design of the filter is given and the stability and convergence conditions are established from the standard Kalman filtering case.

## 2. Filter design

Let us consider the linear discrete-time stochastic system

$$x_{k+1} = \Phi_k x_k + G_k d_k + w_k \quad (1-a)$$

$$y_k = H_k x_k + v_k \quad (1-b)$$

where  $x_k \in \mathbf{R}^n$  is the state,  $d_k \in \mathbf{R}^p$  is the unknown input, and  $y_k \in \mathbf{R}^m$  is the output.  $v_k$  and  $w_k$  are zero mean white sequences uncorrelated with each other and with the initial state of the system with covariances

$$E(w_k w_j^T) = Q_k \delta_{kj}, E(v_k v_j^T) = R_k \delta_{kj}, Q_k \geq 0, \text{ and } R_k \geq 0.$$

where  $\delta_{kj}$  is the Kronecker delta and  $E$  denotes the expectation operator.  $\Phi_k$ ,  $G_k$  and  $H_k$  are known matrices of appropriate dimensions. We assume that  $m \geq p$ , and without loss of generality  $\text{rank } H_k = m$  and  $\text{rank } G_k = p$ .

The problem is to design an unbiased minimum variance linear estimator of the state  $x_k$  given measurements up to time instant  $k$  without any information concerning the disturbance  $d_k$ .

Let  $\hat{x}_k$  denotes the estimate of the state  $x_k$  at time instant  $k$  using measurement up till time  $k$ , this estimator will be unbiased if

$$E(\hat{x}_k - x_k) = 0$$

The estimation error covariance matrix is given by

$$P_k = E[(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T]$$

Consider the linear filter proposed by Kitanidis (1987)

$$\hat{x}_{k+1} = \Phi_k \hat{x}_k + L_{k+1} (y_{k+1} - H_{k+1} \Phi_k \hat{x}_k) \quad (2)$$

where  $L_{k+1}$  is an  $n \times m$  matrix which must be determined such that

$$E(\hat{x}_{k+1} - x_{k+1}) = 0$$

This condition is realized if (see Kitanidis, 1987)

$$L_{k+1} H_{k+1} G_k - G_k = 0 \quad (3)$$

The estimation error covariance matrix of the estimator (2) is

$$P_{k+1} = (I - L_{k+1} H_{k+1}) (\Phi_k P_k \Phi_k^T + Q_k) (I - L_{k+1} H_{k+1})^T + L_{k+1} R_{k+1} L_{k+1}^T \quad (4)$$

In Kitanidis (1987) the matrix  $L_{k+1}$  is obtained by minimizing the trace of the estimation error covariance matrix (4) under the constraint (3).

Here we present a new approach which consists of parameterizing the solution of (3) by an arbitrary matrix which can be obtained to lead to a minimum variance estimator. This formulation permits to obtain the optimal filter under less restrictive conditions.

Now equation (3) has a solution if  $\text{rank } H_{k+1} G_k = \text{rank } G_k = p$ . Under this condition, the solution is given by

$$L_{k+1} = G_k \Pi_k + K_{k+1} T_k \quad (5)$$

with  $\Pi_k = (H_{k+1}G_k)^+ \in \mathbf{R}^{p \times m}$  and  $T_k = \alpha_k (I - (H_{k+1}G_k) (H_{k+1}G_k)^+) \in \mathbf{R}^{(m-p) \times m}$ , where  $\alpha_k$  is an arbitrary matrix which must be chosen such that  $T_k$  is a full row rank matrix and the matrix  $\begin{bmatrix} T_k \\ \Pi_k \end{bmatrix}$  is nonsingular.  $A^+$  represents a pseudo-inverse matrix of  $A$ .

Equation (5) replaced into (4) gives

$$P_{k+1} = (F_k - K_{k+1}\beta_{k+1})P_k(F_k - K_{k+1}\beta_{k+1})^T + Q_{1k} + K_{k+1}\Theta_k K_{k+1}^T - K_{k+1}S_k^T - S_k K_{k+1}^T \quad (6)$$

where

$$Q_{1k} = \Sigma_k Q_k \Sigma_k^T + G_k \Pi_k R_{k+1} \Pi_k^T G_k^T \quad (7)$$

$$\Sigma_k = I - G_k \Pi_k H_{k+1} \quad (8)$$

$$F_k = \Sigma_k \Phi_k \quad (9)$$

$$\beta_{k+1} = T_k H_{k+1} \Phi_k \quad (10)$$

$$S_k = \Sigma_k Q_k H_{k+1}^T T_k^T - G_k \Pi_k R_{k+1} T_k^T \quad (11)$$

and

$$\Theta_k = T_k (H_{k+1} Q_k H_{k+1}^T + R_{k+1}) T_k^T \quad (12)$$

The minimum variance estimation problem is reduced to find the matrix  $K_{k+1}$  which minimizes the trace of the covariance matrix equation (6). The result is given by

$$K_{k+1} = (F_k P_k \beta_{k+1}^T + S_k) (\beta_{k+1} P_k \beta_{k+1}^T + \Theta_k)^{-1} \quad (13)$$

From (10), (12), and (13) the gain matrix  $K_{k+1}$  exists if and only if the matrix  $T_k C_{k+1} T_k^T$

where

$$C_{k+1} = H_{k+1} (Q_k + \Phi_k P_k \Phi_k^T) H_{k+1}^T + R_{k+1}$$

is nonsingular.

The following theorem gives the connection between our results and those obtained by Kitanidis (1987).

*Theorem 1:*

Assume that  $C_{k+1}$  is nonsingular and that  $\text{rank } H_{k+1}G_k = \text{rank } G_k = p$ ,

and let the pseudo-inverse matrix  $(H_{k+1}G_k)^+$  be given by

$$\Pi_k = (H_{k+1}G_k)^+ = (G_k^T H_{k+1}^T C_{k+1}^{-1} H_{k+1} G_k)^{-1} G_k^T H_{k+1}^T C_{k+1}^{-1}$$

and the singular value decomposition of  $C_{k+1}^{-1/2} H_{k+1} G_k$  be

$$C_{k+1}^{-1/2} H_{k+1} G_k = U_k \begin{bmatrix} \Sigma_k^{-1} \\ 0 \end{bmatrix} V_k^T$$

where  $\Sigma_k^{-1} = \text{diag} (\sigma_k^1, \sigma_k^1, \sigma_k^2, \dots, \sigma_k^p)$  with  $\sigma_k^1 \geq \sigma_k^2 \geq \dots \geq \sigma_k^p > 0$ , and  $U_k$  and  $V_k$  are orthogonal matrices.

Then for  $\alpha_k = [0 \quad I] U_k^T C_{k+1}^{-1/2}$  we obtain the Kitanidis filter (Kitanidis, 1987).

*Proof.* See the Appendix. ■

*Remark 1:*

From the above results we can see that the results presented by Kitanidis (1987) can be directly deduced, under restrictive conditions, from ours.

In the sequel we assume that  $\Theta_k$  is nonsingular, this assures the existence of the gain matrix  $K_{k+1}$ . The necessary and sufficient conditions for  $\Theta_k$  to be positive definite are stated in the following lemma.

*Lemma 1:*

Matrix  $\Theta_k$  is positive definite if and only if

$$\text{rank} \begin{bmatrix} I & G_k & Q_k^{1/2} & 0 \\ H_{k+1} & 0 & 0 & R_k^{1/2} \end{bmatrix} = n + m.$$

*Proof:*

We have

$$\begin{aligned} \text{rank} \begin{bmatrix} I & G_k & Q_k^{1/2} & 0 \\ H_{k+1} & 0 & 0 & R_k^{1/2} \end{bmatrix} &= \text{rank} \begin{bmatrix} I & 0 \\ H_{k+1} & -I \end{bmatrix} \begin{bmatrix} I & G_k & Q_k^{1/2} & 0 \\ H_{k+1} & 0 & 0 & R_k^{1/2} \end{bmatrix} \\ &= n + \text{rank} \begin{bmatrix} T_k \\ \Pi_k \end{bmatrix} [H_{k+1}G_k \quad H_{k+1}Q_k^{1/2} \quad -R_k^{1/2}] = n + m \end{aligned}$$

if and only if the matrix  $[T_k H_{k+1} Q_k^{1/2} \quad -T_k R_k^{1/2}]$  is of full row rank or equivalently matrix  $\Theta_k$  is positive definite. ■

*Remark 2:*

From equations (2) and (5), the proposed filter can be written in the following form

$$\hat{x}_{k+1} = F_k \hat{x}_k + G_k \Pi_k y_{k+1} + K_{k+1} (T_k y_{k+1} - \beta_{k+1} \hat{x}_k) \quad (15)$$

### 3. Convergence and stability of the filter

In this section, we consider the special case where all known matrices are independent of time, and we will use the standard results on convergence of the Riccati equation corresponding to the system that captures the unknown input. First a couple of preliminary lemmas are needed to state the conditions for detectability and reachability of the fictitious system in the original system matrices.

Suppose that  $\Phi_k, G_k, H_k, Q_k$  and  $R_k$  are all independent of time instant  $k$ , i.e.  $\Phi_k = \Phi, G_k = G, H_k = H, Q_k = Q$  and  $R_k = R$ , then we obtain the following results.

The minimum variance estimate becomes

$$\hat{x}_{k+1} = F \hat{x}_k + G \Pi y_{k+1} + K_{k+1} (T y_{k+1} - \beta \hat{x}_k) \quad (16)$$

where

$$K_{k+1} = (F P_k \beta^T + S) (\beta P_k \beta^T + \Theta)^{-1}$$

$$P_{k+1} = (F - K_{k+1} \beta) P_k (F - K_{k+1} \beta)^T + Q_1 + K_{k+1} \Theta K_{k+1}^T - K_{k+1} S^T - S K_{k+1}^T \quad (17)$$

and where

$$Q_1 = \Sigma Q \Sigma^T + G \Pi R \Pi^T G^T \quad (18)$$

$$\Sigma = I - G \Pi H \quad (19)$$

$$F = \Sigma \Phi \quad (20)$$

$$\beta = T H \Phi$$

$$\begin{aligned}
S &= \Sigma Q H^T T^T - G \Pi R T^T \\
\Theta &= T (H Q H^T + R) T^T \\
\Pi &= (H G)^+ \\
T &= \alpha (I - (HG) (HG)^+)
\end{aligned} \tag{21}$$

and  $\begin{bmatrix} T \\ \Pi \end{bmatrix}$  is a nonsingular matrix.

Now define

$$F^s = F - S\Theta^{-1}\beta \tag{22}$$

$$Q^s = Q_1 - S\Theta^{-1}S^T \tag{23}$$

then we have the following lemma.

*Lemma 2:*

The pair  $(F^s, \beta)$  is detectable if and only if

$$\text{rank} \begin{bmatrix} zI - \Phi & -G \\ H & 0 \end{bmatrix} = n + p, \forall z \in \mathbb{C}, |z| \geq 1$$

*Proof:*

The detectability of  $(F^s, \beta)$  is given by

$$\text{rank} \begin{bmatrix} zI - F^s \\ \beta \end{bmatrix} = \text{rank} \begin{bmatrix} I & -S\Theta^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} zI - F \\ \beta \end{bmatrix} = \text{rank} \begin{bmatrix} zI - F \\ \beta \end{bmatrix} = n, \forall z \in \mathbb{C}, |z| \geq 1$$

Now we have

$$\begin{aligned}
&\text{rank} \begin{bmatrix} zI - \Phi & -G \\ H & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} I & 0 \\ -H & zI \end{bmatrix} \begin{bmatrix} zI - \Phi & -G \\ H & 0 \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} zI - \Phi & -G \\ H\Phi & HG \end{bmatrix} = \text{rank} \begin{bmatrix} I & 0 \\ 0 & T \\ 0 & \Pi \end{bmatrix} \begin{bmatrix} zI - \Phi & -G \\ H\Phi & HG \end{bmatrix} \\
&= \text{rank} \begin{bmatrix} zI - \Phi & -G \\ \beta & 0 \\ \Pi H\Phi & I \end{bmatrix} = \text{rank} \begin{bmatrix} zI - F \\ \beta \end{bmatrix} + p = n + p \quad \forall z \in \mathbb{C}, |z| \geq 1
\end{aligned}$$

if and only if the pair  $(F, \beta)$  is detectable or as shown  $(F^s, \beta)$  is detectable. ■

*Lemma 3:*

The existence of unreachable mode  $\lambda$  of  $(F^s, Q^{s1/2})$  is equivalent to the existence of a row vector  $\varpi \neq 0$  such that

$$\varpi \begin{bmatrix} -\lambda I + \Phi & G & Q^{1/2} & 0 \\ \lambda H & 0 & 0 & R^{1/2} \end{bmatrix} = 0$$

*Proof:*

Let  $\lambda$  be an unreachable mode of  $(F^s, Q^{s1/2})$ , then there exists a row vector  $q \neq 0$  such that

$$\begin{cases} q F^s = \lambda q \\ q Q^{s1/2} = 0 \end{cases} \tag{24}$$

From (18) and (23), one can easily verify that

$$Q^{s1/2} = [ I \quad -S\Theta^{-1} ] \begin{bmatrix} \Sigma Q^{1/2} & G\Pi R^{1/2} \\ THQ^{1/2} & -TR^{1/2} \end{bmatrix}$$

Using (19), (20), and (21) equation (24) can be written as

$$q_1 \begin{bmatrix} \Sigma\Phi - \lambda I & \Sigma Q^{1/2} & G\Pi R^{1/2} \\ \beta & THQ^{1/2} & -TR^{1/2} \end{bmatrix} = 0$$

where

$$q_1 = q [ I \quad -S\Theta^{-1} ]$$

which leads to

$$\varpi \begin{bmatrix} -\lambda I + \Phi & G & Q^{1/2} & 0 \\ \lambda H & 0 & 0 & R^{1/2} \end{bmatrix} = 0$$

where

$$\varpi = q_1 \begin{bmatrix} \Sigma & -G\Pi \\ TH & T \\ \Pi H & \Pi \end{bmatrix} \quad \blacksquare$$

Now the associated algebraic Riccati equation (ARE) can be written as

$$(F^s - K^s\beta) P (F^s - K^s\beta)^T + K^s \Theta K^{sT} + Q^s = P \quad (25)$$

with  $K^s = K - S \Theta^{-1}$ .

Then we can give the properties of the obtained filter from the well known results on the stability and convergence of the standard Kalman filter De Souza et al. (1986), Bitmead et al. (1990), Anderson and Moore (1979).

Before doing this, we recall that the strong solutions of the (ARE) are those that give rise to a filter having poles inside or on the stability boundary, we call the corresponding solution of the ARE a stabilizing solution.

Using the above results, the convergence and stability conditions of the filter for the time invariant system can be given by the following theorem.

*Theorem 2:*

Under the assumptions

A-1)

$$\text{rank } HG = \text{rank } G = p$$

A-2)

$$\text{rank} \begin{bmatrix} I & G & Q^{1/2} & 0 \\ H & 0 & 0 & R^{1/2} \end{bmatrix} = n + m$$

then, the ARE (25) has a unique stabilizing solution  $P$  and, subject to  $P_0 > 0$ , the sequence  $P_k$ , generated by the RDE (17), converges exponentially to  $P$  if and only if

$$i) \quad \text{rank} \begin{bmatrix} zI - \Phi & -G \\ H & 0 \end{bmatrix} = n + p, \quad \forall z \in \mathbb{C}, |z| \geq 1$$

$$\text{ii) } \text{rank} \begin{bmatrix} -e^{j\omega}I + \Phi & G & Q^{1/2} & 0 \\ e^{j\omega}H & 0 & 0 & R^{1/2} \end{bmatrix} = n + m, \forall \omega \in [0, 2\pi].$$

#### 4. Conclusion

In this paper we have presented a new minimum variance method for the state estimation of linear discrete-time stochastic system in presence of unknown inputs. Conditions for stability and convergence for the time invariant system case have been given.

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*Appendix-* Connections with Kitanidis method (Kitanidis, 1987).

From Kitanidis (1987), under the assumption that  $C_{k+1}$  is positive definite and

$$\Pi_k = (H_{k+1}G_k)^+ = (G_k^T H_{k+1}^T C_{k+1}^{-1} H_{k+1} G_k)^{-1} G_k^T H_{k+1}^T C_{k+1}^{-1} \quad (\text{A-1})$$

the optimal solution for  $L_{k+1}$  can be written as

$$L_{k+1} = G_k \Pi_k + (\Phi_k P_k \Phi_k^T + Q_k) H_{k+1}^T C_{k+1}^{-1} P_r \quad (\text{A-2})$$

with

$$P_r = I - H_{k+1} G_k (H_{k+1} G_k)^+$$

By comparing equation (5) and (A-2) it is clear that, the equality holds if and only if



$$K_{k+1} \alpha_k P_r = (\Phi_k P_k \Phi_k^T + Q_k) H_{k+1}^T C_{k+1}^{-1} P_r \quad (A-3)$$

From (8)-(11), (A-1) and since

$$G_k \Pi_k C_{k+1} T_k^T = 0$$

equation (A-3) holds if

$$P_r = C_{k+1} T_k^T (T_k C_{k+1} T_k^T)^{-1} \alpha_k P_r$$

Using the singular value decomposition and choosing  $\alpha_k$  as given in theorem 1 it can be seen that

$$P_r = C_{k+1} T_k^T (T_k C_{k+1} T_k^T)^{-1} \alpha_k P_r = C_{k+1}^{1/2} U_k \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U_k^T C_{k+1}^{-1/2}$$

This proves theorem 1.