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# Connection between the 3-Block Generalized Riccati Equation and the Standard Riccati Equation

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**Abstract**—In a recent paper, Nikoukhah et al. [1] developed a maximum likelihood descriptor Kalman filter (DKF) the properties of which are governed by a 3-block generalized Riccati difference equation (GRDE). The convergence of this GRDE has been studied and the generalization of the eigenvector approach to solving the associated generalized algebraic Riccati equation (GARE) has been carried out. In this paper, we connect the GRDE to the standard Riccati equation in order to point out Nikoukhah's results from well known results obtained in the standard Kalman filtering. Moreover, while the previous study made by [1] provides only sufficient conditions on the convergence and stability of the 3-block DKF, the relationship established herein allows us to derive necessary and sufficient conditions. In addition, several descriptor Kalman filtering problems can be treated as in the standard case.

## I. INTRODUCTION

This paper deals with the optimal filtering of generalized state-space systems, also known as descriptor or singular systems [2]-[5]. Such systems may be square and singular, they may even represent under or overconstrained dynamical systems, say not well-posed systems. The general representation of linear discrete-time singular systems has the following form

$$EX(k+1) = AX(k) + Bu(k) + w(k) \quad (1.1)$$

$$Z(k) = HX(k) + v(k) \quad (1.2)$$

where the matrix  $E$  is  $p \times n$ ,  $A$  is  $p \times n$  and  $H$  is  $m \times n$ .  $w(k)$  and  $v(k)$  are zero-mean white gaussian noise sequences with

$$E\left\{ \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w^T(j) & v^T(j) \end{bmatrix} \right\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta(k,j) \quad (1.3)$$

( $\delta(k,j)$  is the Kronecker delta).

For decades, there has been a growing interest on the control theory of such systems. Significant results have been recently obtained on the optimal filters design. The most interesting of them are interpreted as least squares [6], maximum likelihood [1] and minimum variance [7] descriptor Kalman filters. In this paper, we are particularly interested in the maximum likelihood DKF, which has a 3-block form according to the following equations

$$\hat{\mathbf{x}}_{(k+1|k+1)} = \begin{bmatrix} 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}\mathbf{P}_{(k|k)}\mathbf{A}^T + \mathbf{Q} & -\mathbf{S} & \mathbf{E} \\ -\mathbf{S}^T & \mathbf{R} & \mathbf{H} \\ \mathbf{E}^T & \mathbf{H}^T & 0 \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{A}\hat{\mathbf{x}}_{(k|k)} + \mathbf{B}\mathbf{u}_{(k)} \\ \mathbf{z}_{(k+1)} \\ 0 \end{bmatrix} \quad (1.4)$$

$$\mathbf{P}_{(k+1|k+1)} = - \begin{bmatrix} 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}\mathbf{P}_{(k|k)}\mathbf{A}^T + \mathbf{Q} & -\mathbf{S} & \mathbf{E} \\ -\mathbf{S}^T & \mathbf{R} & \mathbf{H} \\ \mathbf{E}^T & \mathbf{H}^T & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ 0 \\ \mathbf{I} \end{bmatrix} \quad (1.5)$$

where the symbol  $^\dagger$  denotes any (1)-inverse of the superscripted matrix (i.e a pseudo-inverse which satisfies the relation  $\mathbf{M}\mathbf{M}^\dagger\mathbf{M} = \mathbf{M}$ ).

The covariance matrices  $\mathbf{Q}$  and  $\mathbf{R}$  may be singular and, therefore, the above DKF fits with many special situations. Equation (1.5) is the so-called 3-block GRDE which, at convergence, yields the following GARE

$$\mathbf{P} = - \begin{bmatrix} 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}\mathbf{P}\mathbf{A}^T + \mathbf{Q} & -\mathbf{S} & \mathbf{E} \\ -\mathbf{S}^T & \mathbf{R} & \mathbf{H} \\ \mathbf{E}^T & \mathbf{H}^T & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ 0 \\ \mathbf{I} \end{bmatrix} \quad (1.6)$$

and the associated steady-state DKF

$$\hat{\mathbf{x}}_{(k+1|k+1)} = \begin{bmatrix} 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}\mathbf{P}\mathbf{A}^T + \mathbf{Q} & -\mathbf{S} & \mathbf{E} \\ -\mathbf{S}^T & \mathbf{R} & \mathbf{H} \\ \mathbf{E}^T & \mathbf{H}^T & 0 \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{A}\hat{\mathbf{x}}_{(k|k)} + \mathbf{B}\mathbf{u}_{(k)} \\ \mathbf{z}_{(k+1)} \\ 0 \end{bmatrix} \quad (1.7)$$

Implementing the above DKF requires the computation of a pseudo-inverse or the inverse of an augmented matrix, probably of large dimension with regard to the original system's dimension, and not necessarily nonnegative definite. Beyond the computational considerations, and notwithstanding the study carried out by Nikoukhah et al. [1] about the convergence and stability properties of the above DKF, it is still tempting to relate the 3-block GRDE to the standard Riccati difference equation

(SRDE). So, the main objective of this paper is to realize such a connection which allows the 3-block DKF to be studied from the existing results on the standard Riccati equation. In particular, this relation enables us to give necessary and sufficient conditions on the convergence and stability of the DKF, whereas in [1], there are only sufficient conditions. The study presented here is restricted to the generalized conditions of convergence and stability of the filter. However, other significant considerations, such as the numerical issues of the 3-block DKF, can be undertaken from the relationship established. In the next section, we transform the 3-block GRDE into an SRDE and, in section III, we show how the main results established in [1] can be explicated from those obtained by [8] in the standard case.

## II. TRANSFORMATION OF THE 3-BLOCK GRDE

Let us consider the general case where the 3-block matrix in (1.4) and (1.5) may be singular. Under the full-column rank condition on matrix  $\begin{bmatrix} E \\ H \end{bmatrix}$ , which is necessary and sufficient for the DKF to exist, the development carried out below lies on the singular value decomposition (SVD) of this matrix. We construct a (1)-inverse which puts the 3-block DKF into the Kalman filter equations. Indeed, let the SVD of  $\begin{bmatrix} E \\ H \end{bmatrix}$  be written as

$$\begin{bmatrix} E \\ H \end{bmatrix} = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T \quad (2.1)$$

where  $U$  and  $V$  are orthogonal matrices and  $\Sigma$  is the diagonal matrix of the singular values of  $\begin{bmatrix} E \\ H \end{bmatrix}$ , all of which are strictly positive in this case. Let's accordingly define

$$U^T \begin{bmatrix} A \\ 0 \end{bmatrix} = \begin{bmatrix} \Sigma V^T \bar{A} \\ \bar{H} \end{bmatrix} \quad (2.2)$$

$$U^T \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Sigma V^T \bar{B} \\ G \end{bmatrix} \quad (2.3)$$

$$U^T \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix} U = \begin{bmatrix} \Sigma V^T \bar{Q} V \Sigma & \Sigma V^T \bar{S} \\ \bar{S}^T V \Sigma & \bar{R} \end{bmatrix} \quad (2.4)$$

where matrices  $\bar{A}$ ,  $\bar{B}$ ,  $G$ ,  $\bar{H}$ ,  $\bar{Q}$ ,  $\bar{R}$  and  $\bar{S}$  have appropriate dimensions. Then, we have the following result.

*Theorem 2.1:* The 3-block DKF equations (1.4)-(1.5) are equivalent to the following recursions

$$\hat{\mathbf{x}}_{(k+1|k+1)} = \bar{\mathbf{A}}\hat{\mathbf{x}}_{(k|k)} + \bar{\mathbf{B}}\bar{\mathbf{u}}_{(k+1)} + \mathbf{K}_{(k)}(\bar{\mathbf{z}}_{(k+1)} - \bar{\mathbf{H}}\hat{\mathbf{x}}_{(k|k)}) \quad (2.5)$$

$$\mathbf{P}_{(k+1|k+1)} = \bar{\mathbf{A}}\mathbf{P}_{(k|k)}\bar{\mathbf{A}}^T - (\bar{\mathbf{A}}\mathbf{P}_{(k|k)}\bar{\mathbf{H}}^T + \bar{\mathbf{S}})(\bar{\mathbf{H}}\mathbf{P}_{(k|k)}\bar{\mathbf{H}}^T + \bar{\mathbf{R}})^\dagger(\bar{\mathbf{A}}\mathbf{P}_{(k|k)}\bar{\mathbf{H}}^T + \bar{\mathbf{S}})^T + \bar{\mathbf{Q}} \quad (2.6)$$

with  $\mathbf{K}_{(k)} = (\bar{\mathbf{A}}\mathbf{P}_{(k|k)}\bar{\mathbf{H}}^T + \bar{\mathbf{S}})(\bar{\mathbf{H}}\mathbf{P}_{(k|k)}\bar{\mathbf{H}}^T + \bar{\mathbf{R}})^\dagger$

(2.7)

where  $\bar{\mathbf{u}}_{(k+1)} = \begin{bmatrix} \mathbf{u}_{(k)} \\ \mathbf{z}_{(k+1)} \end{bmatrix}$  and  $\bar{\mathbf{z}}_{(k+1)} = -\mathbf{G}\bar{\mathbf{u}}_{(k+1)}$ .

*Proof:* From the SVD of  $\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}$ , we have

$$\begin{bmatrix} \mathbf{A}\mathbf{P}_{(k|k)}\mathbf{A}^T + \mathbf{Q} & -\mathbf{S} & \mathbf{E} \\ -\mathbf{S}^T & \mathbf{R} & \mathbf{H} \\ \mathbf{E}^T & \mathbf{H}^T & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathbf{U}^T \left( \begin{bmatrix} \mathbf{A} \\ \mathbf{0} \end{bmatrix} \mathbf{P}_{(k|k)} \begin{bmatrix} \mathbf{A}^T & | & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{Q} & -\mathbf{S} \\ -\mathbf{S}^T & \mathbf{R} \end{bmatrix} \right) \mathbf{U} & | & \begin{bmatrix} \Sigma \\ \mathbf{0} \end{bmatrix} \\ & & \begin{bmatrix} \Sigma & | & \mathbf{0} \\ & & \mathbf{0} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathbf{U}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^T \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{bmatrix} \alpha & \beta & \Sigma \\ \beta^T & \gamma & \mathbf{0} \\ \Sigma & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^T \end{bmatrix}$$

with  $\alpha = \Sigma\mathbf{V}^T(\bar{\mathbf{Q}} + \bar{\mathbf{A}}\mathbf{P}_{(k|k)}\bar{\mathbf{A}}^T)\mathbf{V}\Sigma$ ,  $\beta = \Sigma\mathbf{V}^T(\bar{\mathbf{S}} + \bar{\mathbf{A}}\mathbf{P}_{(k|k)}\bar{\mathbf{H}}^T)$  and  $\gamma = \bar{\mathbf{R}} + \bar{\mathbf{H}}\mathbf{P}_{(k|k)}\bar{\mathbf{H}}^T$ .

Now, it can be checked that  $\begin{bmatrix} \mathbf{0} & \mathbf{0} & \Sigma^{-1} \\ \mathbf{0} & \gamma^\dagger & -\gamma^\dagger\beta^T\Sigma^{-1} \\ \Sigma^{-1} & -\Sigma^{-1}\beta\gamma^\dagger & -\Sigma^{-1}(\alpha - \beta\gamma^\dagger\beta^T)\Sigma^{-1} \end{bmatrix}$  is a (1)-inverse of  $\begin{bmatrix} \alpha & \beta & \Sigma \\ \beta^T & \gamma & \mathbf{0} \\ \Sigma & \mathbf{0} & \mathbf{0} \end{bmatrix}$

. It follows that

$$\begin{bmatrix} \mathbf{A}\mathbf{P}_{(k|k)}\mathbf{A}^T + \mathbf{Q} & -\mathbf{S} & \mathbf{E} \\ -\mathbf{S}^T & \mathbf{R} & \mathbf{H} \\ \mathbf{E}^T & \mathbf{H}^T & \mathbf{0} \end{bmatrix}^\dagger = \begin{bmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \Sigma^{-1} \\ \mathbf{0} & \gamma^\dagger & -\gamma^\dagger\beta^T\Sigma^{-1} \\ \Sigma^{-1} & -\Sigma^{-1}\beta\gamma^\dagger & -\Sigma^{-1}(\alpha - \beta\gamma^\dagger\beta^T)\Sigma^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^T \end{bmatrix}$$

and hence

$$\mathbf{P}_{(k+1|k+1)} = \mathbf{V}\Sigma^{-1}(\alpha - \beta\gamma^\dagger\beta^T)\Sigma^{-1}\mathbf{V}^T \quad (2.8)$$

Finally, replacing  $\alpha$ ,  $\beta$  and  $\gamma$  by their expressions in (2.8) gives (2.6), and similar algebraic manipulations give equation (2.5) for the state estimate recursion. ■

*Remark 2.1:* At this point, some comment might be made concerning equations (2.5)-(2.7). In fact, should the 3-block matrix in (1.4) and (1.5) be singular, so will also be matrix  $\gamma$ , i.e the 3-block DKF

would not be unique. Obviously, this is an undesirable situation from the practical point of view. Therefore, for practical reasons, the 3-block DKF must be unique and, in turn, this implies the positive definiteness of matrix  $\gamma$ , so that equations (2.5)-(2.7) appear as a standard Kalman filter. In the sequel, we focus only on the case where matrix  $\gamma$  is nonsingular.

### III. DERIVATION OF THE GENERALIZED CONVERGENCE AND STABILITY CONDITIONS

Considering the singular system (1.1)-(1.2), let us recall the following structural properties, which are useful in the study of the steady-state DKF.

*Definition 3.1:* The triplet  $(A, E, H)$  is said to be detectable if the pencil  $\begin{bmatrix} \lambda E - \mu A \\ H \end{bmatrix}$  has full-column rank for all complex pairs  $(\lambda, \mu) \neq (0, 0)$  satisfying  $|\lambda| \geq |\mu|$ .

*Definition 3.2:* The quartet  $(A, E, H, \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix})$  is said to be stabilizable if the pencil  $\begin{bmatrix} \lambda E - \mu A & Q & -S \\ \lambda H & -S^T & R \end{bmatrix}$  has full-row rank for all complex pairs  $(\lambda, \mu) \neq (0, 0)$  satisfying  $|\lambda| \geq |\mu|$ .

The above definitions can be found in [1]. In the light of this definition, two interesting points may be mentioned. At infinity, the detectability reduces to the full-column rank condition on matrix  $\begin{bmatrix} E \\ H \end{bmatrix}$ , which also stands for the definition of the observability at infinity of singular systems (for  $n = p$ ) [4], [9], while the stabilizability reduces to the full-row rank condition on matrix  $\begin{bmatrix} E & Q & -S \\ H & -S^T & R \end{bmatrix}$ . Hence, the finite detectability and the finite stabilizability are subject respectively to the full-column rank condition on the pencil  $\begin{bmatrix} \lambda E - A \\ H \end{bmatrix}$  and the full-row rank condition on the pencil  $\begin{bmatrix} \lambda E - A & Q & -S \\ \lambda H & -S^T & R \end{bmatrix}$ , for all finite complex numbers  $\lambda$  such that  $|\lambda| \geq 1$ . This complex domain separation for the detectability and stabilizability is helpful to understand which conditions govern the existence and uniqueness of the DKF, and which ones govern its convergence and stability. Indeed, while the observability at infinity is necessary and sufficient for the existence of the DKF [1], [6], [7], we'll show below that the stabilizability at infinity is sufficient (but not necessary) for its uniqueness. Furthermore, the convergence and stability of the 3-block DKF are suspended to the finite detectability and the finite stabilizability.

Now, let's turn to the subject matter of this study. We have the following results.

*Lemma 3.1:* If system (1.1)-(1.2) is stabilizable at infinity, then matrix  $\bar{R}$  is positive definite, and equations (2.5)-(2.7) are equivalent to the following:

$$\hat{X}_{(k+1|k+1)} = \bar{A}^s \hat{X}_{(k|k)} + \bar{B}^s \bar{u}_{(k+1)} + K^s_{(k)}(\bar{Z}_{(k+1)} - \bar{H} \hat{X}_{(k|k)}) \quad (3.1)$$

$$P_{(k+1|k+1)} = \bar{A}^s P_{(k|k)} \bar{A}^{sT} - \bar{A}^s P_{(k|k)} \bar{H}^T (\bar{H} P_{(k|k)} \bar{H}^T + \bar{R})^{-1} \bar{H} P_{(k|k)} \bar{A}^{sT} + \bar{Q}^s \quad (3.2)$$

$$\text{with } K^s_{(k)} = \bar{A}^s P_{(k|k)} \bar{H}^T (\bar{H} P_{(k|k)} \bar{H}^T + \bar{R})^{-1} \quad (3.3)$$

where  $\bar{A}^s = \bar{A} - \bar{S} \bar{R}^{-1} \bar{H}$ ,  $\bar{B}^s = \bar{B} + \bar{S} \bar{R}^{-1} G$  and  $\bar{Q}^s = \bar{Q} - \bar{S} \bar{R}^{-1} \bar{S}^T$ .

*Proof:* Let's write  $U^T = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$  according to the partitioning of  $\begin{bmatrix} \Sigma \\ 0 \end{bmatrix}$ . We have

$$\text{rank} \left( \begin{bmatrix} E & Q & -S \\ H & -S^T & R \end{bmatrix} \right) = \text{rank} \left( U^T \begin{bmatrix} E & Q & -S \\ H & -S^T & R \end{bmatrix} \right) = \text{rank} \left( \begin{array}{c|c} \Sigma V^T & U_1 \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix} \\ \hline 0 & U_2 \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix} \end{array} \right)$$

If system (1.1)-(1.2) is stabilizable at infinity, then matrix  $U_2 \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix}$  must have full-row rank.

By noting that  $\begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix} \geq 0$ , the stabilizability at infinity implies also that  $U_2 \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix}^{\frac{1}{2}}$  has full-row rank. It follows that

$$\bar{R} = U_2 \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix}^{\frac{1}{2}T} U_2^T > 0.$$

Finally, equations (3.1)-(3.3) are obtained simply by applying the usual decorrelation method to equations (2.5)-(2.7) [10]. ■

From the above transformations, the study of the convergence and stability of the 3-block DKF reduces to that of the SRDE (3.2) which can be rewritten as

$$P_{(k+1|k+1)} = (\bar{A}^s - K^s_{(k)} \bar{H}) P_{(k|k)} (\bar{A}^s - K^s_{(k)} \bar{H})^T + K^s_{(k)} \bar{R} K^{sT}_{(k)} + \bar{Q}^s \quad (3.4)$$

where  $(\bar{A}^s - K^s_{(k)} \bar{H})$  is the DKF transition matrix, as it is shown by the following filter equation

$$\hat{X}_{(k+1|k+1)} = (\bar{A}^s - K^s_{(k)} \bar{H}) \hat{X}_{(k|k)} + (\bar{B}^s - K^s_{(k)} G) \bar{u}_{(k+1)} \quad (3.5)$$

When the DKF has converged, it obviously satisfies the following steady-state equations

$$\hat{X}_{(k+1|k+1)} = (\bar{A}^s - K^s \bar{H}) \hat{X}_{(k|k)} + (\bar{B}^s - K^s G) \bar{u}_{(k+1)} \quad (3.6)$$

$$P = (\bar{A}^s - K^s \bar{H}) P (\bar{A}^s - K^s \bar{H})^T + K^s \bar{R} K^{sT} + \bar{Q}^s \quad (3.7)$$

where  $P$  and  $K^s$  denote the steady-state error covariance and gain matrices respectively. The conditions under which the time-invariant filter (3.6)-(3.7) is stable are those for existence of a *stabilizing solution* of the algebraic Riccati equation (3.7). A stabilizing solution of this equation is a nonnegative definite solution  $P_s$  for which the steady-state transition matrix  $(\bar{A}^s - K^s \bar{H})$  has all its eigenvalues inside the unit circle (if all the eigenvalues lie on or inside the unit circle, then the solution is called a *strong solution*). The stability conditions, together with those conditions for the recursions (3.4)-(3.5) to converge to (3.6)-(3.7), have been established by De Souza et al. [8, Theorem 4.1] in the case of nonstabilizable systems, with possible singular state transition matrices, and are given by the following theorem.

*Theorem 3.1:* Subject to  $P(0|0) > 0$ , then the detectability of the pair  $(\bar{H}, \bar{A}^s)$  and the nonexistence of unreachable modes of  $(\bar{A}^s, \bar{Q}^{\frac{1}{s^2}})$  on the unit circle are necessary and sufficient conditions for

$$\lim_{k \rightarrow \infty} P_{(k|k)} = P_s \text{ (exponentially fast)}$$

where  $P_{(k|k)}$  is the solution of (3.4) with initial condition  $P(0|0)$  and  $P_s$  is the unique stabilizing solution of (3.7).

Our purpose now is to relate the above results to the original singular system, and this is done by the results established below.

*Lemma 3.2:* The pair  $(\bar{H}, \bar{A}^s)$  is detectable if and only if system (1.1)-(1.2) is finite detectable, i.e the pencil  $\begin{bmatrix} \lambda E - A \\ H \end{bmatrix}$  has full-column rank for all finite complex  $\lambda$  satisfying  $|\lambda| \geq 1$ .

*Proof:* From Wonham [11], the detectability of  $(\bar{H}, \bar{A}^s)$  is equivalent to that of  $(\bar{H}, \bar{A}^s + L\bar{H})$ , where  $L$  is any matrix. By choosing  $L = \bar{S}\bar{R}^{-1}$ , it follows that  $(\bar{H}, \bar{A})$  is detectable and, by definition,

this is equivalent to  $\begin{bmatrix} \lambda I - \bar{A} \\ \bar{H} \end{bmatrix}$  having full-column rank for all finite  $\lambda$  satisfying  $|\lambda| \geq 1$ . From the

expressions of  $\bar{A}$  and  $\bar{H}$ , we have

$$\begin{aligned}
\text{rank}\left(\begin{bmatrix} \lambda I - \bar{A} \\ \bar{H} \end{bmatrix}\right) &= \text{rank}\left(\begin{bmatrix} \lambda I - V\Sigma^{-1}U_1 \begin{bmatrix} A \\ 0 \end{bmatrix} \\ U_2 \begin{bmatrix} A \\ 0 \end{bmatrix} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \lambda\Sigma V^T \\ 0 \end{bmatrix} - \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix}\right) \\
&= \text{rank}\left(\lambda U^T \begin{bmatrix} E \\ H \end{bmatrix} - U^T \begin{bmatrix} A \\ 0 \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \lambda E - A \\ \lambda H \end{bmatrix}\right) \\
&= \text{rank}\left(\begin{bmatrix} I & 0 \\ 0 & \lambda I \end{bmatrix} \begin{bmatrix} \lambda E - A \\ H \end{bmatrix}\right) \text{ with } |\lambda| > 0
\end{aligned}$$

It comes

$$\text{rank}\left(\begin{bmatrix} \lambda I - \bar{A} \\ \bar{H} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \lambda E - A \\ H \end{bmatrix}\right), \forall \lambda \in \mathbb{C}, |\lambda| \geq 1. \quad \blacksquare$$

*Lemma 3.3:*  $\lambda$  is an unreachable mode of the pair  $(\bar{A}^s, \bar{Q}^{s^{\frac{1}{2}}})$  if and only if there exists a row-vector  $q \neq 0$  such that

$$q \begin{bmatrix} \lambda E - A & Q & -S \\ \lambda H & -S^T & R \end{bmatrix} = 0.$$

*Proof:* Let  $\lambda$  be an unreachable mode of  $(\bar{A}^s, \bar{Q}^{s^{\frac{1}{2}}})$ . Then, there exists a row-vector  $e \neq 0$  such that

$$e\bar{A}^s = \lambda e \text{ and } e\bar{Q}^{s^{\frac{1}{2}}} = 0 \text{ or } e\bar{Q}^s = 0$$

From the expressions of  $\bar{A}^s$  and  $\bar{Q}^s$ , we have

$$eV\Sigma^{-1}(\Sigma V^T \bar{A} - \Sigma V^T \bar{S} \bar{R}^{-1} \bar{H}) = \lambda e$$

$$eV\Sigma^{-1}(\Sigma V^T \bar{Q} V \Sigma - \Sigma V^T \bar{S} \bar{R}^{-1} \bar{S}^T V \Sigma) \Sigma^{-1} V^T = 0$$

The last two equations are equivalent to the following

$$eV\Sigma^{-1} [I \mid -\Sigma V^T \bar{S} \bar{R}^{-1}] \begin{bmatrix} \Sigma V^T \bar{A} & \Sigma V^T \bar{Q} V \Sigma & \Sigma V^T \bar{S} \\ \bar{H} & \bar{S}^T V \Sigma & \bar{R} \end{bmatrix} = \lambda e V \Sigma^{-1} [I \mid -\Sigma V^T \bar{S} \bar{R}^{-1}] \begin{bmatrix} \Sigma V^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which can be rewritten as

$$q \begin{bmatrix} \begin{bmatrix} A \\ 0 \end{bmatrix} & \begin{bmatrix} Q & -S \\ -S^T & R \end{bmatrix} U \end{bmatrix} = \lambda q \begin{bmatrix} E & 0 & 0 \\ H & 0 & 0 \end{bmatrix}$$

or, since  $U$  is non singular,

$$q \begin{bmatrix} A & Q & -S \\ 0 & -S^T & R \end{bmatrix} = \lambda q \begin{bmatrix} E & 0 & 0 \\ H & 0 & 0 \end{bmatrix}$$

with  $q = eV\Sigma^{-1}[I \mid -\Sigma V^T \bar{S} \bar{R}^{-1}] U^T$ . This yields

$$q \begin{bmatrix} \lambda E - A & Q & -S \\ \lambda H & -S^T & R \end{bmatrix} = 0. \quad \blacksquare$$

Using the above results, the convergence and stability conditions of the 3-block DKF are summarized in the following theorem.

*Theorem 3.2:* (i) If matrices  $\begin{bmatrix} E \\ H \end{bmatrix}$  and  $\begin{bmatrix} E & Q & -S \\ H & -S^T & R \end{bmatrix}$  have full-column and full-row rank

respectively, then the 3-block GRDE (1.4) is equivalent to the SRDE (3.2).

(ii) The 3-block GARE (1.6) has a unique stabilizing solution  $P$  and, subject to  $P_{(0|0)} > 0$ , the sequence

$\{P_{(k|k)}\}$ , generated by the 3-block GRDE (1.4), converges exponentially to this stabilizing solution if and only if the pencil  $\begin{bmatrix} \lambda E - A \\ H \end{bmatrix}$  has full-column rank for all finite complex  $\lambda$  satisfying  $|\lambda| \geq 1$ , and

the pencil  $\begin{bmatrix} \lambda E - A & Q & -S \\ \lambda H & -S^T & R \end{bmatrix}$  has full-row rank for all finite complex  $\lambda$  satisfying  $|\lambda| = 1$ .

#### IV. CONCLUDING REMARKS

In this paper, the relationship between the 3-block DKF, developed by Nikoukhah et al. [1], and the standard Kalman filter has been carried out. Although this study has dealt with the asymptotic behavior of the 3-block DKF, it allows also further interesting developments on this DKF as though it were the standard Kalman filter.

Writing the 3-block GRDE into the SRDE form does not need any change of basis and, as mentioned before, this transformation is attractive from several points of view. In particular, it allows the determination of the solution of the GARE from known methods of computing the solution of the ARE, which are well documented in the literature. Furthermore, numerical issues of the 3-block DKF, such as efficient sequential, information and square-root implementations can be carried out easily.

It is worth saying that the results established here allow a good understanding of the descriptor Kalman filtering, since it is closely related to the standard Kalman filter. Theorem 3.2 is less restrictive than theorem 4.3 in [1]. Indeed Nikoukhah et al. [1] give sufficient conditions for the convergence of the 3-block GRDE (1.4) to the solution of the 3-block GARE (1.6) and for the stability of the associated steady-state DKF (1.7). In addition, if we replace  $P_{(0|0)} > 0$  by  $P_{(0|0)} \geq 0$  in theorem 3.2, the GARE (1.6) has a stabilizing solution  $P$ . This follows immediately from theorem 3.2 and [8, Theorem

3.2]. In this case, we implement the steady-state DKF (1.7) or, equivalently, the steady-state Kalman filter associated to (3.1).

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