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# Laguerre polynomials and the inverse Laplace transform using discrete data

Tran Ngoc Lien<sup>1</sup>, Dang Duc Trong<sup>2</sup> and Alain Pham Ngoc Dinh<sup>3</sup>

**Abstract.** We consider the problem of finding a function defined on  $(0, \infty)$  from a countable set of values of its Laplace transform. The problem is severely ill-posed. We shall use the expansion of the function in a series of Laguerre polynomials to convert the problem in an analytic interpolation problem. Then, using the coefficients of Lagrange polynomials we shall construct a stable approximation solution. Error estimate is given.

**Keywords and phrases.** inverse Laplace transform, Laguerre polynomials, Lagrange polynomials, ill-posed problem, regularization.

**Mathematics Subject Classification 2000.** 44A10, 30E05, 33C45.

## 1 Introduction

Let  $L^2_\rho(0, \infty)$  be the space of Lebesgue measurable functions defined on  $(0, \infty)$  such that

$$\|f\|_{L^2_\rho}^2 \equiv \int_0^\infty |f(x)|^2 e^{-x} dx < \infty.$$

We consider the problem of recovering a function  $f \in L^2_\rho(0, \infty)$  satisfying the equations

$$\mathcal{L}f(p_j) \equiv \int_0^\infty e^{-p_j x} f(x) dx = \mu_j \quad (DIL)$$

where  $p_j \in (0, \infty), j = 1, 2, 3, \dots$

As known, we have the classical problem of finding a function  $f(x)$  from its given image  $g(p)$  satisfying

$$\mathcal{L}f(p) \equiv \int_0^\infty e^{-px} f(x) dx = g(p), \quad (1)$$

where  $p$  is in a subset  $\omega$  of the complex plane. We note that  $\mathcal{L}f(p)$  is usually an analytic function on a half plane  $\{Re p > \alpha\}$  for an appropriate real number  $\alpha$ . Frequently, the image of a Laplace transform is known only on a subset  $\omega$  of the right half plane  $\{Re p > \alpha\}$ . Depending on the set  $\omega$ , we shall have appropriate methods to construct the function  $f$  from the values in the set

$$\{\mathcal{L}f(p) : p \in \omega\}.$$

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Hence, there are no universal methods of inversion of the Laplace transform.

If the data  $g(p)$  is given as a function on a line  $(-i\infty + a, +i\infty + a)$  (i.e.,  $\omega = \{p : p = a + iy, y \in \mathbf{R}\}$ ) on the complex plane then we can use the Bromwich inversion formula ([25], p. 67) to find the function  $f(x)$ .

If  $\omega \subset \{p \in \mathbf{R} : p > 0\}$  then we have the problem of real inverse Laplace transform. The right hand side is known only on  $(0, \infty)$  or a subset of  $(0, \infty)$ . In this case, the use of the Bromwich formula is therefore not feasible. The literature on the subject is impressed in both theoretical and computational aspects (see, e.g. [2,3,10,15,17,21]). In fact, if the data  $g(p)$  is given exactly then, by the analyticity of  $g$ , we have many inversion formulas (see, e.g., [3,7,8,19,20]). In [3], the author approximate the function  $f$  by

$$f(t) \cong \sum_{k=0}^N b_k(a) d^k(e^x g(e^x)) / dx^k$$

where  $b_k(a)$  are calculated and tabulated regularization coefficients and  $g$  is the given Laplace transform of  $f$ . Another method is developed by Saitoh and his group ([4,5,19,20]), where the function  $f$  is approximated by integrals having the form

$$u_N(t) = \int_0^\infty g(s) e^{-st} P_N(st) ds, \quad N = 1, 2, \dots$$

and  $P_N$  is known (see [5]). Using the Saitoh formula, we can get directly error estimates.

However, in the case of unexact data, we have a severely trouble by the ill-posedness of the problem. In fact, a solution corresponding to the unexact data do not exist if the data is nonsmooth, and in the case of existence, these do not depend continuously on the given data (that are represented by the right hand side of the equalities). Hence, a regularization method is in order. In [7], the authors used the Tikhonov method to regularize the problem. In fact, in this method, we can approximate  $u_0$  by functions  $u_\beta$  satisfying

$$\beta u_\beta + \mathcal{L}^* \mathcal{L} u_\beta = \mathcal{L}^* g, \quad \beta > 0.$$

Since  $\mathcal{L}$  is self-adjoint (cf. [7]), the latter equation can be written as

$$\beta u_\beta + \int_0^\infty \frac{u_\beta(s)}{s+t} ds = \int_0^\infty e^{-st} g(s) ds.$$

The latter problem is well-posed.

Although the inverse Laplace transform has a rich literature, the papers devoted to the problem with discrete data are scarce. In fact, from the analyticity of  $\mathcal{L}f(p)$ , if  $\mathcal{L}f(p)$  is known on a countable subset of  $\omega \subset \{Re p > \alpha\}$  accumulating at a point then  $\mathcal{L}f(p)$  is known on the whole  $\{Re p > \alpha\}$ . Hence, generally, a set of discrete data is enough for constructing an approximation function of  $f$ . It is a moment problem. In [15], the authors presented some theorems on the stabilization of the inverse Laplace transform. The Laplace image is measured at  $N$  points to within some error  $\epsilon$ . This is achieved by proving parallel stabilization results for a related Hausdorff moment problem. For a construction of an approximate solution of (DIL), we note that the sequence of functions  $(e^{-p_j x})$  is (algebraically) linear independent and moreover the vector space generated by the latter sequence is dense in  $L^2(0, \infty)$ . The method of truncated expansion as presented in ([6], Section 2.1) is applicable and we refer the reader to this reference for full details. In [11, 13], the authors convert (DIL) into a moment problem of finding a function in  $L^2(0, 1)$  and, then, they use Muntz polynomials to construct an approximation for  $f$ .

Now, in the present paper, we shall convert (DIL) to an analytic interpolation problem on the Hardy space of the unit disc. After that, we shall use Laguerre polynomials and coefficients of Lagrange polynomials to construct the function  $f$ . An approximation corresponding to the nonexact data and error estimate will be given.

The remainder of the paper divided into two sections. In Section 2, we convert our problem into an interpolation one and give a uniqueness result. In Section 3, we shall give two regularization results in the cases of exact data and nonexact data.

## 2. A uniqueness result

In this paper we shall use Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n).$$

We note that  $\{L_n\}$  is a sequence of orthonormal polynomials on  $L^2_\rho(0, \infty)$ . We note that (see, e.g., [9], page 67)

$$\exp\left(\frac{xz}{z-1}\right) (1-z)^{-1} = \sum_{n=0}^{\infty} L_n(x) z^n.$$

Hence, if we have the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n L_n(x)$$

then

$$\int_0^{\infty} f(x) \exp\left(\frac{xz}{z-1}\right) (1-z)^{-1} e^{-x} dx = \sum_{n=0}^{\infty} a_n z^n.$$

It follows that

$$\sum_{n=0}^{\infty} a_n z^n = \int_0^{\infty} f(x) \exp\left(\frac{x}{z-1}\right) (1-z)^{-1} dx.$$

Put  $\Phi f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $\alpha_j = 1 - 1/p_j$ , one has

$$\Phi f(\alpha_j) = p_j \mu_j,$$

i.e., we have an interpolation problem of finding an analytic function  $\Phi f$  in the Hardy space  $H^2(U)$ . Here, we denote by  $U$  the unit disc of the complex plane and by  $H^2(U)$  the Hardy space. In fact, we recall that  $H^2(U)$  is the space of all functions  $\phi$  analytic in  $U$  and if,  $\phi \in H^2(U)$  has the expansion  $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$  then  $\|\phi\|_{H^2(U)}^2 = \sum_{k=0}^{\infty} |a_k|^2$ . We can verify directly that the linear operator  $\Phi$  is an isometry from  $L^2_\rho$  onto  $H^2(U)$ . In fact, we have

**Lemma 1** *Let  $f \in L^2_\rho(0, \infty)$ . Then  $\mathcal{L}f(z)$  is analytic on  $\{z \in \mathbf{C} \mid \operatorname{Re} z > 1/2\}$ . If we have an expansion*

$$f = \sum_{n=0}^{\infty} a_n L_n$$

*then one has  $\Phi f \in H^2(U)$  and*

$$\|\Phi f\|_{H^2(U)}^2 = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|_{L^2_\rho(0, \infty)}^2.$$

Moreover, If we have in addition that  $xe^{-x}|f'|^2 \in L^1(0, \infty)$  then

$$\sum_{n=0}^{\infty} n|a_n|^2 \leq \|xe^{-x}|f'|^2\|_{L^1(0, \infty)}.$$

**Proof**

Putting  $F_z(t) = e^{-zt}f(t)$ , we have  $F_z \in L^2(0, \infty)$  for every  $\text{Re}z > 1/2$ . Hence  $\mathcal{L}f(z) = \int_0^{\infty} F_z(t)dt$  is analytic for  $\text{Re}z > 1/2$ . From the definitions of  $L^2_\rho(0, \infty)$  and  $H^2(U)$ , we have the isometry equality. Now we prove the second inequalities. We first consider the case  $f', f''$  in the space

$$B = \{g \text{ Lebesgue measurable on } (0, \infty) \mid \sqrt{x}g \in L^2_\rho(0, \infty)\}.$$

We have the expansion

$$f = \sum_{n=0}^{\infty} a_n L_n.$$

The function  $y = L_n$  satisfies the following equation

$$xy'' + (1-x)y' + ny = 0$$

which gives

$$(xe^{-x}y')' + ny e^{-x} = 0.$$

It follows that

$$\begin{aligned} \int_0^{\infty} f(xe^{-x}L'_m)' dx &= \sum_{n=0}^{\infty} a_n \int_0^{\infty} (xe^{-x}L'_m)' L_n dx \\ &= \sum_{n=0}^{\infty} a_n \int_0^{\infty} (xe^{-x}L'_n)' L_m dx \\ &= - \sum_{n=0}^{\infty} na_n \int_0^{\infty} L_n L_m e^{-x} dx. \end{aligned}$$

Hence, the orthonormality of  $(L_m)$  gives

$$\int_0^{\infty} (xe^{-x}f'(x))' a_m L_m(x) dx = -ma_m^2 \quad \text{for } m = 0, 1, \dots$$

It follows that

$$\int_0^{\infty} (xe^{-x}f'(x))' f(x) dx = - \sum_{n=0}^{\infty} na_n^2.$$

Integrating by parts, we get

$$\int_0^{\infty} xe^{-x}|f'(x)|^2 dx = \sum_{n=0}^{\infty} na_n^2.$$

Now, for  $f' \in B$  we choose  $(f_k)$  such that  $f'_k, f''_k \in B$  for every  $k = 1, 2, \dots$  and  $f_k \rightarrow f$  in  $L^2_\rho$  as  $k \rightarrow \infty$ . Assume that

$$f_k = \sum_{n=0}^{\infty} a_{kn} L_n.$$

Then we have

$$\int_0^\infty x e^{-x} |f'_k(x)|^2 dx = \sum_{n=0}^\infty n a_{kn}^2.$$

Let  $k \rightarrow \infty$  in the latter equality, we complete the proof of Lemma 1.

Using Lemma 1, one has a uniqueness result

**Theorem 1.** *Let  $p_j > 1/2$  for every  $j = 1, 2, \dots$ . If*

$$\sum_{p_j > 1} \frac{1}{p_j} + \sum_{1/2 < p_j < 1} \frac{2p_j - 1}{p_j} = \infty$$

then Problem (DIL) has at most one solution in  $L^2_\rho(0, \infty)$ .

**Proof**

Let  $f_1, f_2 \in L^2_\rho(0, \infty)$  be two solutions of (DIL). Putting  $g = f_1 - f_2$  then  $g \in L^2_\rho(0, \infty)$  and  $\mathcal{L}g(p_j) = 0$ . It follows that  $\Phi g(1 - 1/p_j) = 0$ ,  $j = 1, 2, \dots$ . It follows that  $\alpha_j = 1 - 1/p_j$  are zeroes of  $\Phi g$ . We have  $\Phi g \in H^2(U)$  and

$$\sum_{j=1}^\infty (1 - |\alpha_j|) = \sum_{p_j > 1} \frac{1}{p_j} + \sum_{1/2 < p_j < 1} \frac{2p_j - 1}{p_j} = \infty.$$

Hence we get  $\Phi g \equiv 0$  (see, e.g., [18]). It follows that  $g \equiv 0$ . This completes the proof of Theorem.

### 3. Regularization and error estimates

In the section, we assume that  $(p_j)$  is a convergent sequence. Without loss of generality, we shall assume that  $\lim_{j \rightarrow \infty} p_j = 1$ . In fact, if  $\lim_{j \rightarrow \infty} p_j = \rho_0 > 1$  then, by putting  $\tilde{f}(x) = e^{-(\rho_0 - 1)x} f(x)$  and  $p'_j = p_j - \rho_0 + 1$ , we can transform the problem to the one of finding  $\tilde{f} \in L^2_\rho(0, \infty)$  such that

$$\int_0^\infty e^{-p'_j x} \tilde{f}(x) dx = \mu_j, \quad j = 1, 2, \dots$$

in which  $\lim_{j \rightarrow \infty} p'_j = 1$ .

We denote by  $\ell_k^{(m)}(\nu)$  the coefficient of  $z^k$  in the expansion of the Lagrange polynomial  $L_m(\nu)$  ( $\nu = (\nu_1, \dots, \nu_m)$ ) of degree (at most)  $m - 1$  satisfying

$$L_m(\nu)(z_k) = \nu_k, \quad 1 \leq k \leq m,$$

where  $z_k = \alpha_k$ . We denote by

$$L_m^\theta(\nu)(z) = \sum_{0 \leq k \leq \theta(m-1)} \ell_k^{(m)}(\nu) z^k.$$

The polynomial  $L_m^\theta(\nu)$  is called a truncated Lagrange polynomial (see also [24]). For every  $g \in L^2_\rho(0, \infty)$ , we put

$$T_n g = (p_1 \mathcal{L}g(p_1), \dots, p_n \mathcal{L}g(p_n)).$$

Here, we recall that  $\alpha_n = 1 - 1/p_n$ . We shall approximate the function  $f$  by

$$F_m = \Phi^{-1} L_m^\theta(T_m f) = \sum_{0 \leq k \leq \theta(m-1)} \ell_k^{(m)}(T_m f) L_k.$$

We shall prove that  $F_m$  is an approximation of  $f$ . More precisely, one has

**Theorem 2** *Let  $\sigma \in (0, 1/3)$ , let  $f$  be as in Lemma 1 and let  $p_j > 1/2$  for  $j = 1, 2, \dots$  satisfy  $\lim_{j \rightarrow \infty} p_j = 1$  and*

$$\left| 1 - \frac{1}{p_j} \right| \leq \sigma.$$

*Put  $\theta_0$  be the unique solution of the equation (unknown  $x$ )*

$$\frac{2\sigma^{1-x}}{1-\sigma} = 1.$$

*Then for  $\theta \in (0, \theta_0)$ , one has*

$$\|f - F_m\|_{L_\rho^2(0, \infty)}^2 \longrightarrow 0 \quad \text{as } m \rightarrow \infty.$$

*If, we assume in addition that  $x e^{-x} |f'|^2 \in L^1(0, \infty)$  then*

$$\|f - F_m\|_{L_\rho^2(0, \infty)}^2 \leq (1 + m\theta)^3 \|f\|_{L_\rho^2}^2 \left( \frac{2\sigma^{1-\theta}}{1-\sigma} \right)^{2m} + \frac{1}{m\theta} \|x e^{-x} |f'|^2\|_{L^1(0, \infty)}.$$

### Proof

We have in view of Lemma 1

$$\|f - F_m\|_{L_\rho^2(0, \infty)}^2 = \sum_{0 \leq k \leq \theta(m-1)} |\delta_k^{(m)}|^2 + \sum_{k > \theta(m-1)} |a_k|^2 \quad (2)$$

where  $\delta_k^{(m)} = a_k - \ell_k^{(m)}(T_m f)$ . We shall give an estimate for  $\delta_k^{(m)}$ . In fact, we have

$$\|\Phi f - L_m(T_m f)\|_{H^2(U)}^2 = \sum_{k=0}^m -1 |\delta_k^{(m)}|^2 + \sum_{k=m}^{\infty} |a_k|^2.$$

On the other hand, the Hermite representation (see, e.g. [12]) gives

$$\Phi f(z) - L_m(T_m f)(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\omega_m(z)(\Phi f)(\zeta) d\zeta}{\omega_m(\zeta)(\zeta - z)}$$

where  $\omega_m(z) = (z - \alpha_1) \dots (z - \alpha_m)$ . Now, if we denote by  $\sigma_{-1}^{(m)} = \sigma_{-2}^{(m)} = \dots = 0$  and

$$\begin{aligned} \sigma_0^{(m)} &= 1 \\ \sigma_r^{(m)} &= \sum_{1 \leq j_1 < \dots < j_r \leq m} \alpha_{j_1} \dots \alpha_{j_r} \quad (1 \leq r \leq m), \\ \beta_s^{(m)} &= \frac{1}{2\pi i} \int_{\partial U} \frac{\Phi f(\zeta) d\zeta}{\zeta^{s+1} \omega_m(\zeta)} \end{aligned}$$

then we can write in view of the Hermite representation

$$\Phi f(z) - L_m(T_m f)(z) = \sum_{k=0}^{\infty} \left( \sum_{r=0}^k (-1)^r \sigma_{m-r}^{(m)} \beta_{k-r}^{(m)} \right) z^k.$$

From the latter representation, one gets

$$\delta_k^{(m)} = \sum_{r=0}^k (-1)^r \sigma_{m-r}^{(m)} \beta_{k-r}^{(m)}, \quad 0 \leq k \leq m-1.$$

Now, by direct computation, one has

$$|\beta_s^{(m)}| \leq \|\phi_f\|_{H^2(U)} (1-\sigma)^{-m}$$

and

$$|\sigma_{m-r}^{(m)}| \leq \sigma^{m-r} C_m^r \leq \sigma^{m-k} 2^m,$$

where  $C_m^k = \frac{m!}{k!(m-k)!}$ . Hence, we have

$$|\delta_k^{(m)}| \leq (1+m\theta) \|f\|_{L_\rho^2} \left( \frac{2\sigma^{1-\theta}}{1-\sigma} \right)^m.$$

From the latter inequality, one has in view of (2)

$$\|f - F_m\|_{L_\rho^2(0,\infty)}^2 \leq (1+m\theta)^3 \|f\|_{L_\rho^2}^2 \left( \frac{2\sigma^{1-\theta}}{1-\sigma} \right)^{2m} + \sum_{k>m\theta}^{\infty} |a_k|^2.$$

For  $\theta \in (0, \theta_0)$ , one has

$$0 < \frac{2\sigma^{1-\theta}}{1-\sigma} < \frac{2\sigma^{1-\theta_0}}{1-\sigma} = 1.$$

Hence, we have

$$\lim_{m \rightarrow \infty} \|f - F_m\|_{L_\rho^2(0,\infty)}^2 = 0$$

as desired.

Now if  $x e^{-x} |f'|^2 \in L^1(0, \infty)$  then one has

$$\begin{aligned} \sum_{k>m\theta}^{\infty} |a_k|^2 &\leq \frac{1}{m\theta} \sum_{k=0}^{\infty} k |a_k|^2 \\ &= \frac{1}{m\theta} \|x e^{-x} |f'|^2\|_{L^1(0,\infty)}. \end{aligned}$$

This completes the proof of Theorem 1.

Now, we consider the case of non-exact data. Put

$$D_m = \max_{1 \leq n \leq m} \left( \max_{|z| \leq R} \left| \frac{\omega_m(z)}{(z - \alpha_n) \omega'_m(z)} \right| \right).$$



Let  $\psi : [0, \infty) \rightarrow \mathbf{R}$  be an increasing function satisfying

$$\psi(m) \geq (m+1)D_m, \quad m = 1, 2, \dots$$

and

$$m(\epsilon) = [\psi(\epsilon^{-1/2})] + 1$$

where  $[x]$  is the greatest integer  $\leq x$ .

**Theorem 2.** *Let the assumptions of Theorem 1 hold. Let  $\epsilon > 0$  and let  $(\mu_j^\epsilon)$  be a measured data of  $(\mathcal{L}f(p_j))$  satisfying*

$$\sup_j |p_j(\mathcal{L}f(p_j) - \mu_j^\epsilon)| < \epsilon.$$

Then we have

$$\|f - \Phi^{-1}L_{m(\epsilon)}^\theta(\nu^\epsilon)\|_{L_\rho^2}^2 \leq 2(1 + m(\epsilon)\theta)^3 \|f\|_{L_\rho^2}^2 \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m(\epsilon)} + \frac{1}{m(\epsilon)\theta} \|xe^{-x}|f'|^2\|_{L^1(0,\infty)} + \epsilon^{1/2}.$$

where  $\nu_j^\epsilon = p_j\mu_j^\epsilon$  for  $j = 1, 2, \dots$

**Proof**

We note that

$$L_m(T_m f)(z) - L_m(\nu^\epsilon)(z) = \sum_{j=0}^m (p_j\mu_j - \nu_j^\epsilon) \frac{\omega_m(z)}{(z - \alpha_j)\omega'_m(z)}.$$

It follows that

$$\|L_m(T_m f) - L_m(\nu^\epsilon)\|_\infty \leq \epsilon(m+1)D_m.$$

Hence

$$\|L_m^\theta(T_m f) - L_m^\theta(\nu^\epsilon)\|_{H^2(U)} \leq \|L_m(T_m f) - L_m(\nu^\epsilon)\|_\infty \leq \epsilon(m+1)D_m.$$

It follows by the isometry property of  $\Phi$

$$\begin{aligned} \|f - \Phi^{-1}L_m^\theta(\nu^\epsilon)\|_{L_\rho^2}^2 &\leq 2\|f - F_m\|_{L_\rho^2}^2 + 2\|\Phi^{-1}L_m^\theta(T_m f) - \Phi^{-1}L_m^\theta(\nu^\epsilon)\|_{L_\rho^2}^2 \\ &\leq (1 + m\theta)^3 \|f\|_{L_\rho^2}^2 \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m} + \frac{1}{m\theta} \|xe^{-x}|f'|^2\|_{L^1(0,\infty)} \\ &\quad + \epsilon(m+1)D_m. \end{aligned}$$

By choosing  $m = m(\epsilon)$  we get the desired result. This completes the proof of Theorem 2.

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