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# Reduced-order filter for stochastic bilinear systems with multiplicative noise

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*Abstract*— This paper deals with the design of a reduced-order  $\mathcal{H}_\infty$  filter for a stochastic bilinear systems with a prescribed  $\mathcal{H}_\infty$  norm criterion. The problem is transformed into the search of a unique gain matrix by using a Sylvester-like condition on the drift term. The considered system is bilinear in control and with multiplicative noise in the state and in the measurement equations. The approach is based on the resolution of LMI and is then easily implementable.

*Keywords*— Reduced-order  $\mathcal{H}_\infty$  filter, Itô's formula, Stochastic systems, Bilinear systems, Lyapunov function.

## I. INTRODUCTION

The bilinear systems represent sometimes a good mean to physical systems modeling when the linear representation is not sufficiently significant. The stochastic systems get a great importance during the last decades as shown by numerous references (Kozin, 1969; Has'minskii, 1980; Florchinger, 1995; Mao, 1997; Carravetta *et al.*, 2000; Germani *et al.*, 2002; Xu and Chen, 2003).

Generally, bilinear stochastic system designs a stochastic system with multiplicative noise instead of additive one (Carravetta *et al.*, 2000; Germani *et al.*, 2002). The full and the reduced-order  $\mathcal{H}_\infty$  filtering for stochastic systems with multiplicative noise has been treated in many papers (Hinrichsen and Pritchard, 1998; Gershon *et al.*, 2001; Xu and Chen, 2002; Stoica, 2002). Notice that the measurement equation in (Xu and Chen, 2002; Stoica, 2002) is not corrupted by noise. The problem is solved in terms of two LMIs and a coupling non convex rank constraint.

In this paper the problem of reduced-order  $\mathcal{H}_\infty$  filtering for a larger class of stochastic systems than those studied in the papers cited above is considered since the studied systems are with multiplicative noise and multiplicative control input (the bilinearity is also between the state and the control input). The measurements are subjected to a multiplicative noise too. Notice that, as in the deterministic case, the multiplicative control input affects the observability of the system.

The purpose is to design a reduced-order  $\mathcal{H}_\infty$  filter for such a system. We first use a “unbiasedness” (decoupling) condition on the drift part of the estimation error and a change of variable on the control input. Then applying the Itô formula and LMI method permit to reduce the problem to the search of a unique gain matrix. The reduced-order stochastic filter matrices are then computed using this gain.

Throughout the paper,  $\mathbf{E}$  represents expectation operator with respect to some probability measure  $\mathcal{P}$ .  $\langle X, Y \rangle = X^T Y$  represents the inner product of the vectors  $X, Y \in \mathbb{R}^n$ .

$\text{herm}(A)$  stands for  $A + A^T$ .

$L_2(\Omega, \mathbb{R}^k)$  is the space of square-integrable  $\mathbb{R}^k$ -valued functions on the probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  where  $\Omega$  is the sample space,  $\mathcal{F}$  is a  $\sigma$ -algebra of subsets of the sample space called events and  $\mathcal{P}$  is the probability measure on  $\mathcal{F}$ .  $(\mathcal{F}_t)_{t \geq 0}$  denote an increasing family of  $\sigma$ -algebras  $(\mathcal{F}_t) \in \mathcal{F}$ . We also denote by  $\widehat{L}_2([0, \infty); \mathbb{R}^k)$  the space of non-anticipatory square-integrable stochastic process  $f(\cdot) = (f(t))_{t \in [0, \infty)}$  in  $\mathbb{R}^k$  with respect to  $(\mathcal{F}_t)_{t \in [0, \infty)}$  satisfying

$$\|f\|_{\widehat{L}_2}^2 = \mathbf{E} \left\{ \int_0^\infty \|f(t)\|^2 dt \right\} < \infty$$

where  $\|\cdot\|$  is the well-known Euclidean norm.

## II. PROBLEM STATEMENT

Let us consider the following stochastic bilinear system

$$\begin{cases} dx(t) = (A_{t0}x(t) + u_1(t)A_{t1}x(t)) dt \\ \quad + B_0v(t) dt + A_{w0}x(t) dw_0(t) \\ dy(t) = Cx(t) dt + J_1x(t) dw_1(t) \\ z(t) = Lx(t) \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $y(t) \in \mathbb{R}^p$  is the output,  $u_1(t) \in \mathbb{R}$  is the control input,  $z(t) \in \mathbb{R}^r$  is a linear combination of the state vector with  $r < n$  and  $v(t) \in \mathbb{R}^q$  is the perturbation signal. Without loss of generality  $L$  is assumed to be a full row rank matrix.  $w_i(t)$  is a Wiener process verifying (Has'minskii, 1980)

$$\mathbf{E}(dw_i(t)) = 0, \mathbf{E}(dw_i(t)^2) = dt, \text{ for } i = 0, 1, \quad (2a)$$

$$\mathbf{E}(dw_0(t)dw_1(t)) = \mathbf{E}(dw_1(t)dw_0(t)) = \varphi dt, \quad (2b)$$

with  $|\varphi| < 1$ .

As in the most cases for physical processes, we assume that the stochastic bilinear system (1) has known bounded control input, i.e.  $u_1(t) \in \Gamma \subset \mathbb{R}$ , where

$$\Gamma = \{u_1(t) \in \mathbb{R} \mid u_{1\min} \leq u_1(t) \leq u_{1\max}\}. \quad (3)$$

The study made here can be easily generalized for the case where there are  $m$  control inputs.

First, we introduce the following definition and assumption.

**Definition 1.** (Kozin, 1969; Has'minskii, 1980) *The stochastic system (1) with  $v(t) \equiv 0$  is said to be asymptotically mean-square stable if all initial states  $x(0)$  yields*

$$\lim_{t \rightarrow \infty} \mathbf{E} \|x(t)\|^2 = 0, \quad \forall u_1(t) \in \Gamma. \quad (4)$$

**Assumption 1.** *The stochastic bilinear system (1) is assumed to be asymptotically mean-square stable.*

In this paper, the aim is to design a reduced-order filter in the following form

$$d\eta(t) = (M_0 + u_1(t)M_1)\eta(t)dt + (N_0 + u_1(t)N_1)dy(t) \quad (5)$$

where  $\eta(t) \in \mathbb{R}^r$  is the filter state with  $r < n$  and the matrices  $M_i$  and  $N_i$  (for  $i = 0, 1$ ) are to be determined.

Then the following problem is considered.

**Problem 1.** *Given a real  $\gamma > 0$ , the goal is to design a asymptotically mean-square stable reduced-order  $\mathcal{H}_\infty$  filter (5) such that the augmented state  $[x^T(t) \ e^T(t)]^T$  is asymptotically mean-square stable and the following  $\mathcal{H}_\infty$  performance*

$$\|e(t)\|_{\mathcal{L}_2}^2 \leq \gamma \|v(t)\|_{\mathcal{L}_2}^2 \quad (6)$$

is achieved from the disturbance  $v(t)$  to the filtering error  $e(t) = z(t) - \eta(t)$ .

Let us consider the following estimation error

$$e(t) = Lx(t) - \eta(t). \quad (7)$$

So the estimation error dynamics becomes

$$\begin{aligned} d e(t) = & (M_0 + M_1 u_1(t)) e(t) dt + L B_0 v(t) dt \\ & + \{(L A_{t0} - M_0 L - N_0 C) \\ & + (L A_{t1} - M_1 L - N_1 C) u_1(t)\} x(t) dt \\ & + L A_{w0} x(t) d w_0(t) \\ & - ((N_0 + u_1(t) N_1) J_1 x(t) d w_1(t). \end{aligned} \quad (8)$$

In order to suppress the direct effect of the state  $x(t)$  on the drift part of the filtering error, we consider the following Sylvester-like conditions

$$L A_{ti} - M_i L - N_i C = 0, \quad i = 0, 1. \quad (9)$$

Let us consider the following augmented state vector

$$\xi^T(t) = [x^T(t) \ e^T(t)]. \quad (10)$$

Then under (9), the dynamics of the augmented system is given by

$$\begin{aligned} d \xi(t) = & (\mathcal{A}_{t0} + \mathcal{A}_{t1} u_1(t)) \xi(t) dt + \mathcal{B}_0 v(t) dt \\ & + \mathcal{A}_{w0} \xi(t) d w_0(t) \\ & + (\mathcal{A}_{w1} + \mathcal{A}_{w2} u_1(t)) \xi(t) d w_1, \end{aligned} \quad (11)$$

with

$$\begin{aligned} \mathcal{A}_{ti} &= \begin{bmatrix} A_{ti} & 0 \\ 0 & M_i \end{bmatrix}, & \text{for } i = 0, 1, \\ \mathcal{B}_0 &= \begin{bmatrix} B_0 \\ L B_0 \end{bmatrix}, & \mathcal{A}_{w0} = \begin{bmatrix} A_{w0} & 0 \\ L A_{w0} & 0 \end{bmatrix}, \\ \mathcal{A}_{w1} &= \begin{bmatrix} 0 & 0 \\ -N_0 J_1 & 0 \end{bmatrix}, & \mathcal{A}_{w2} = \begin{bmatrix} 0 & 0 \\ -N_1 J_1 & 0 \end{bmatrix}. \end{aligned}$$

In the sequel the relations (9) are used to express the filter matrices through a single gain matrix.

In fact, since  $L$  is a full row rank matrix, relations (9) are equivalent to

$$(L A_{ti} - M_i L - N_i C) [L^\dagger \quad (I_n - L^\dagger L)] = 0, \quad \text{for } i = 0, 1. \quad (13)$$

where  $L^\dagger$  is a generalized inverse of matrix  $L$  satisfying  $L = L L^\dagger L$  (Lancaster and Tismenetsky, 1985) (since  $\text{rank } L = r$ , we have  $L L^\dagger = I_r$ ).

Relations (13) give

$$0 = L A_{ti} L^\dagger - M_i - N_i C L^\dagger \quad \text{for } i = 0, 1, \quad (14a)$$

$$0 = L \bar{A}_i - N_i \bar{C} \quad \text{for } i = 0, 1, \quad (14b)$$

where

$$\bar{A}_i = A_{ti} (I_n - L^\dagger L) \quad \text{for } i = 0, 1, \quad (15a)$$

$$\bar{C} = C (I_n - L^\dagger L). \quad (15b)$$

The relation (14a) gives

$$M_i = \bar{A}_i - N_i \bar{C}, \quad \text{for } i = 0, 1, \quad (16)$$

where

$$\bar{\bar{A}}_i = L A_{ti} L^\dagger, \quad \text{for } i = 0, 1, \quad (17a)$$

$$\bar{\bar{C}} = C L^\dagger. \quad (17b)$$

The relation (14b) becomes

$$\mathcal{K} \Sigma = L \bar{A}, \quad (18)$$

where

$$\mathcal{K} = \begin{bmatrix} N_0 & N_1 \end{bmatrix}, \quad (19)$$

$$\bar{A} = \begin{bmatrix} \bar{A}_0 & \bar{A}_1 \end{bmatrix}, \quad (20)$$

$$\Sigma = \begin{bmatrix} \bar{C} & 0 \\ 0 & \bar{C} \end{bmatrix}, \quad (21)$$

and a general solution to equation (18), if it exists, is given by

$$\mathcal{K} = L \bar{A} \Sigma^\dagger + Z (I_{2p} - \Sigma \Sigma^\dagger), \quad (22)$$

where

$$Z = \begin{bmatrix} Z_0 & Z_1 \end{bmatrix}, \quad (23)$$

is an arbitrary matrix of appropriate dimensions.

### III. TRANSFORMATION OF THE BILINEAR SYSTEM FILTERING PROBLEM INTO AN UNCERTAIN ONE

As in (Zasadzinski *et al.*, 2003), let us introduce a change of variable on the control  $u_1(t)$  as follows

$$u_1(t) = \alpha_1 + \sigma_1 \varepsilon_1(t) \quad (24)$$

where  $\alpha_1 \in \mathbb{R}$  and  $\sigma_1 \in \mathbb{R}$  are given by

$$\alpha_1 = \frac{1}{2}(u_{1\min} + u_{1\max}), \quad \sigma_1 = \frac{1}{2}(u_{1\max} - u_{1\min}). \quad (25)$$

The new ‘‘uncertain’’ variable is  $\varepsilon_1(t) \in \bar{\Gamma} \subset \mathbb{R}$  where the polytope  $\bar{\Gamma}$  is defined by

$$\bar{\Gamma} = \{\varepsilon_1(t) \in \mathbb{R} \mid \varepsilon_{1\min} = -1 \leq \varepsilon_1(t) \leq \varepsilon_{1\max} = 1\}. \quad (26)$$

Then the error dynamics (8) can be rewritten as

$$\begin{aligned} d e(t) = & \left( \mathbb{A}_t - Z \mathbb{C}_t + (\tilde{\mathbb{A}}_t - Z \tilde{\mathbb{C}}_t) \Delta_\varepsilon(\varepsilon_1(t)) H_e \right) e(t) dt \\ & + \mathbb{B}_0 v(t) dt + \mathbb{A}_{w0} x(t) dt + \mathbb{A}_{w(11)} x(t) dt + Z \mathbb{A}_{w(12)} x(t) dt \\ & + (\tilde{\mathbb{A}}_{w(11)} - Z \tilde{\mathbb{A}}_{w(12)}) \Delta_x(\varepsilon_1(t)) H_x x(t) dt \end{aligned} \quad (27)$$

where

$$\begin{aligned} \mathbb{A}_t &= \bar{\mathbb{A}}_0 + \alpha_1 \bar{\mathbb{A}}_1 - L \bar{\mathbb{A}} \Sigma^\dagger \Lambda, & \mathbb{C}_t &= (I_{2p} - \Sigma \Sigma^\dagger) \Lambda, \\ \tilde{\mathbb{A}}_t &= \sigma_1 \bar{\mathbb{A}}_1 - L \bar{\mathbb{A}} \Sigma^\dagger \Lambda, & \tilde{\mathbb{C}}_t &= (I_{2p} - \Sigma \Sigma^\dagger) \Lambda, \\ \mathbb{B}_{01} &= L B_0, & \mathbb{A}_{w0} &= L A_{w0}, \\ \mathbb{A}_{w(11)} &= L \bar{\mathbb{A}} \Sigma^\dagger \Psi_\alpha, & \mathbb{A}_{w(12)} &= (I_{2p} - \Sigma \Sigma^\dagger) \Psi_\alpha, \\ \tilde{\mathbb{A}}_{w(11)} &= L \bar{\mathbb{A}} \Sigma^\dagger \Psi_\sigma, & \tilde{\mathbb{A}}_{w(12)} &= (I_{2p} - \Sigma \Sigma^\dagger) \Psi_\sigma, \end{aligned}$$

and

$$\begin{aligned} \Lambda &= \begin{bmatrix} CL^\dagger \\ \alpha_1 CL^\dagger \end{bmatrix}, & \Psi_\alpha &= \begin{bmatrix} -J_1 \\ -\alpha J_1 \end{bmatrix}, & \Psi_\sigma &= \begin{bmatrix} 0 \\ -\sigma J_1 \end{bmatrix}, \\ H_e &= I_r, & H_x &= I_n, \end{aligned}$$

$$\Delta_\varepsilon(\varepsilon_1(t)) = \varepsilon_1(t) I_r, \quad \Delta_x(\varepsilon_1(t)) = \varepsilon_1(t) I_n.$$

Using the definition (26), the matrix  $\Delta_\varepsilon(\varepsilon_1(t))$  and  $\Delta_x(\varepsilon_1(t))$  satisfy

$$\|\Delta_\varepsilon(\varepsilon_1(t))\| \leq 1, \quad \text{and} \quad \|\Delta_x(\varepsilon_1(t))\| \leq 1. \quad (28)$$

Using (24), the system state equation (see (1)) becomes

$$\begin{aligned} dx(t) = & (A_{t0} + \alpha_1 A_{t1} + \sigma_1 \varepsilon_1(t) A_{t1}) x(t) dt \\ & + B_0 v(t) dt + A_{w0} x(t) dt + \mathbb{A}_{w(11)} x(t) dt + Z \mathbb{A}_{w(12)} x(t) dt. \end{aligned} \quad (29)$$

So the augmented system (11) is rewritten as

$$\begin{aligned} d\xi(t) = & \left( \hat{A}_{t0} + \Delta \hat{A}_{t0}(t) \right) \xi(t) dt + \hat{B}_0 v(t) dt \\ & + \hat{A}_{w0} \xi(t) dt + \mathbb{A}_{w(11)} x(t) dt + Z \mathbb{A}_{w(12)} x(t) dt \\ & + \left( \hat{A}_{w1} + \Delta \hat{A}_{w1}(t) \right) \xi(t) dt + \mathbb{A}_{w(11)} x(t) dt + Z \mathbb{A}_{w(12)} x(t) dt \end{aligned} \quad (30)$$

where

$$\begin{aligned} \hat{A}_{t0} &= \begin{bmatrix} A_{t0} + \alpha_1 A_{t1} & 0 \\ 0 & \mathbb{A}_t - Z \mathbb{C}_t \end{bmatrix}, \\ \Delta \hat{A}_{t0}(t) &= H_1 \Delta_\varepsilon(\varepsilon_1(t)) H_t, \\ \hat{B}_0 &= \begin{bmatrix} B_0 \\ \mathbb{B}_{01} \end{bmatrix}, & \hat{A}_{w0} &= \begin{bmatrix} A_{w0} & 0 \\ \mathbb{A}_{w0} & 0 \end{bmatrix}, \\ \hat{A}_{w1} &= \begin{bmatrix} 0 & 0 \\ \mathbb{A}_{w(11)} - Z \mathbb{A}_{w(12)} & 0 \end{bmatrix}, \\ \Delta \hat{A}_{w1}(t) &= H_2 \Delta_\varepsilon(\varepsilon_1(t)) H_w, \\ H_1 &= \begin{bmatrix} \sigma_1 A_{t1} & 0 \\ 0 & \tilde{\mathbb{A}}_t - Z \tilde{\mathbb{C}}_t \end{bmatrix}, & H_2 &= \begin{bmatrix} 0 \\ \tilde{\mathbb{A}}_{w(11)} - Z \tilde{\mathbb{A}}_{w(12)} \end{bmatrix}, \\ H_t &= \begin{bmatrix} I_n & 0 \\ 0 & I_r \end{bmatrix}, & H_w &= [I_n \ 0], & \Delta_\varepsilon(\varepsilon_1(t)) &= \varepsilon_1(t). \end{aligned}$$

Notice that from (26),  $\Delta_\varepsilon(\varepsilon_1(t))$  satisfy

$$\|\Delta_\varepsilon(\varepsilon_1(t))\| \leq 1. \quad (31)$$

#### IV. SYNTHESIS OF THE REDUCED-ORDER FILTER

Consider the following system obtained from (30)

$$\begin{cases} d\xi(t) = \left( \hat{A}_{t0} + \Delta \hat{A}_{t0}(t) \right) \xi(t) dt + \hat{B}_0 v(t) dt \\ + \hat{A}_{w0} \xi(t) dt + \mathbb{A}_{w(11)} x(t) dt + Z \mathbb{A}_{w(12)} x(t) dt \\ e(t) = \hat{C} \xi(t) \end{cases} \quad (32)$$

where  $\hat{C} = [0 \ I_r]$ .

Then the following theorem is given for the filter synthesis.

**Theorem 1.** *The reduced-order  $\mathcal{H}_\infty$  filtering problem 1 is solved for the system (1) with the filter (5) such that the augmented system (32) is asymptotically mean-square stable and verifies the  $\mathcal{H}_\infty$  performance (6) if, for some reals  $\mu_1 > 0, \mu_2 > 0$  and  $\mu_3 > 0$  there exist matrices  $P_1 = P_1^T > 0 \in \mathbb{R}^{n \times n}, P_2 = P_2^T > 0 \in \mathbb{R}^{r \times r}, P_3 \in \mathbb{R}^{n \times r}, G_2 \in \mathbb{R}^{r \times 2p}$  and  $G_3 \in \mathbb{R}^{n \times 2p}$  such that*

$$\begin{bmatrix} \begin{matrix} (1,1) & (1,2) & P_1 B_0 + P_3 \mathbb{B}_{01} & \sigma_1 P_1 A_{t1} \\ (1,2)^T & (2,2) & P_3^T B_0 + P_2 \mathbb{B}_{01} & \sigma_1 P_3^T A_{t1} \\ B_0^T P_1 + \mathbb{B}_{01}^T P_3^T & B_0^T P_3 + \mathbb{B}_{01}^T P_2 & -\gamma^2 I_q & 0 \\ \sigma_1 A_{t1}^T P_1 & \sigma_1 A_{t1}^T P_3^T & 0 & -\mu_1 I_n \\ \tilde{\mathbb{A}}_t^T P_3^T - \tilde{\mathbb{C}}_t^T G_3^T & \tilde{\mathbb{A}}_t^T P_2 - \tilde{\mathbb{C}}_t^T G_2^T & 0 & 0 \\ (1,6) & 0 & 0 & 0 \\ (1,7) & 0 & 0 & 0 \\ (1,8)^T & 0 & 0 & 0 \\ (1,9)^T & 0 & 0 & 0 \\ (1,10)^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} (1,6) & (1,7) & (1,8) & (1,9) & (1,10) & 0 \\ P_3 \tilde{\mathbb{A}}_t - G_3 \tilde{\mathbb{C}}_t & 0 & 0 & 0 & 0 & 0 \\ P_2 \tilde{\mathbb{A}}_t - G_2 \tilde{\mathbb{C}}_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\mu_1 I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & -P_1 & -P_3 & 0 & 0 & 0 \\ 0 & -P_3^T & -P_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_3 I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_1 & -P_3 \\ 0 & 0 & 0 & 0 & -P_3^T & -P_2 \\ 0 & 0 & 0 & 0 & (11,9)^T & (11,10)^T \\ 0 & 0 & 0 & 0 & (11,9)^T & (11,10)^T - \mu_2 I_n \end{matrix} \end{bmatrix} < 0 \quad (33)$$

where

$$\begin{aligned} (1,1) &= (\mu_1 + \mu_2 + \varphi \mu_3) I_n + \text{herm} \{ P_1 A_{\alpha 1} \\ &+ \varphi \left( A_{w0}^T (P_3 \mathbb{A}_{w(11)} - G_3 \mathbb{A}_{w(12)}) \right. \\ &\left. + \mathbb{A}_{w0}^T (P_2 \mathbb{A}_{w(11)} - G_2 \mathbb{A}_{w(12)}) \right) \}, \\ (1,2) &= A_{\alpha 1}^T P_3 + P_3 \mathbb{A}_t - G_3 \mathbb{C}_t, \\ (2,2) &= \text{herm} (P_2 \mathbb{A}_t - G_2 \mathbb{C}_t) + (1 + \mu_1) I_r, \\ (1,6) &= A_{w0}^T P_1 + \mathbb{A}_{w0}^T P_3^T, \\ (1,7) &= A_{w0}^T P_3 + \mathbb{A}_{w0}^T P_2, \\ (1,8) &= \varphi^{\frac{1}{2}} \left( A_{w0}^T \left( P_3 \tilde{\mathbb{A}}_{w(11)} - G_3 \tilde{\mathbb{A}}_{w(12)} \right) \right. \\ &\left. + \mathbb{A}_{w0}^T \left( P_2 \tilde{\mathbb{A}}_{w(11)} - G_2 \tilde{\mathbb{A}}_{w(12)} \right) \right), \\ (1,9) &= \mathbb{A}_{w(11)}^T P_3^T - \mathbb{A}_{w(12)}^T G_3^T, \\ (1,10) &= \mathbb{A}_{w(11)}^T P_2 - \mathbb{A}_{w(12)}^T G_2^T, \\ (11,9) &= P_3 \tilde{\mathbb{A}}_{w(11)} - G_3 \tilde{\mathbb{A}}_{w(12)}, \\ (11,10) &= P_2 \tilde{\mathbb{A}}_{w(11)} - G_2 \tilde{\mathbb{A}}_{w(12)}, \\ A_{\alpha 1} &= A_{t0} + \alpha_1 A_{t1}, \end{aligned}$$

and such that the gain matrices  $G_2$  and  $G_3$  are the solution of the following equation

$$\begin{bmatrix} G_2 \\ G_3 \end{bmatrix} = \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} Z. \quad (34)$$

*Proof.* Consider the following Lyapunov function

$$V(\xi) = \xi^T \mathcal{P} \xi, \quad (35)$$

where

$$\mathcal{P} = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix}. \quad (36)$$

Applying Itô formula (Mao, 1997) to the system (30) (or (32)), we get

$$dV(\xi(t)) = \mathcal{L}V(\xi(t)) dt + 2\xi^T(t) \mathcal{P} \Psi(t) \xi(t) \quad (37)$$

where

$$\Psi(t) = \widehat{A}_{w0} dw_0(t) + \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right) dw_1(t), \quad (38)$$

and

$$\begin{aligned} \mathcal{L}V(\xi(t)) dt = & 2\xi^T(t) \left( \left( \widehat{A}_{t0} + \Delta \widehat{A}_{t0}(t) \right) \xi(t) + \widehat{B}_0 v(t) \right) dt \\ & + \xi^T(t) \langle \mathcal{P} \Psi(t), \Psi(t) \rangle \xi(t). \end{aligned} \quad (39)$$

By replacing (38) and (39), the relation (37) becomes

$$\begin{aligned} dV(\xi(t)) = & 2\xi^T(t) \mathcal{P} \left( \left( \widehat{A}_{t0} + \Delta \widehat{A}_{t0}(t) \right) \xi(t) + \widehat{B}_0 v(t) \right) dt \\ & + \xi^T(t) \widehat{A}_{w0}^T \mathcal{P} \widehat{A}_{w0} \xi(t) dw_0(t)^2 \\ & + \xi^T(t) \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right)^T \mathcal{P} \\ & \quad \times \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right) \xi(t) dw_1(t)^2 \\ & + \xi^T(t) \widehat{A}_{w0}^T \mathcal{P} \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right) \xi(t) dw_0(t) dw_1(t) \\ & + \xi^T(t) \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right)^T \mathcal{P} \widehat{A}_{w0} \xi(t) dw_1(t) dw_0(t) \\ & + 2\xi^T(t) \mathcal{P} \widehat{A}_{w0} \xi(t) dw_0(t) \\ & + 2\xi^T(t) \mathcal{P} \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right) \xi(t) dw_1(t). \end{aligned} \quad (40)$$

Using the majoration lemma (Wang *et al.*, 1992), it can be shown that

$$\begin{aligned} 2\xi^T(t) \mathcal{P} \Delta \widehat{A}_{t0}(t) \xi(t) \\ \leq \xi^T(t) \left( \mu_1^{-1} \mathcal{P} H_1 H_1^T \mathcal{P} + \mu_1 H_t^T H_t \right) \xi(t), \end{aligned} \quad (41)$$

$$\begin{aligned} \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right)^T \mathcal{P} \left( \widehat{A}_{w1} + \Delta \widehat{A}_{w1}(t) \right) \\ \leq \widehat{A}_{w1}^T \left( \mathcal{P}^{-1} - \mu_2^{-1} H_2 H_2^T \right)^{-1} \widehat{A}_{w1} + \mu_2 H_w^T H_w, \end{aligned} \quad (42)$$

$$\begin{aligned} 2\xi^T(t) \widehat{A}_{w0}^T \mathcal{P} \Delta \widehat{A}_{w1}(t) \xi(t) \\ \leq \xi^T(t) \left( \mu_3^{-1} \widehat{A}_{w0}^T \mathcal{P} H_2 H_2^T \mathcal{P} \widehat{A}_{w0} + \mu_3 H_w^T H_w \right) \xi(t). \end{aligned} \quad (43)$$

Now, taking the expectation of (40) (see (Mao, 1997)) and using the relations (2) and the last three inequalities, then  $\mathbf{E}\{dV(\xi(t))\}$  can be bounded as

$$\mathbf{E}\{dV(\xi(t))\} \leq \mathbf{E}\left\{ \begin{bmatrix} \xi(t)^T & v(t)^T \end{bmatrix} \Theta_1 \begin{bmatrix} \xi(t) \\ v(t) \end{bmatrix} dt \right\} \quad (44)$$

where

$$\begin{aligned} \Theta_1 = & \begin{bmatrix} \mathcal{P} \widehat{A}_{t0} + \widehat{A}_{t0}^T \mathcal{P} + \mathcal{C}^T \mathcal{C} & \mathcal{P} \widehat{B}_0 \\ \widehat{B}_0^T \mathcal{P} & -\gamma^2 I_q \end{bmatrix} \\ & + \mu_1 \begin{bmatrix} H_t^T H_t & 0 \\ 0 & 0 \end{bmatrix} + \mu_1^{-1} \begin{bmatrix} \mathcal{P} H_1^T H_1 \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} \widehat{A}_{w1}^T (\mathcal{P}^{-1} - \mu_2^{-1} H_2 H_2^T)^{-1} \widehat{A}_{w1} & 0 \\ 0 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} H_w^T H_w & 0 \\ 0 & 0 \end{bmatrix} \\ & + \varphi \mu_3 \begin{bmatrix} H_w^T H_w & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \widehat{A}_{w0}^T \mathcal{P} \widehat{A}_{w0} & 0 \\ 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} \varphi \mu_3^{-1} \left( \widehat{A}_{w0}^T \mathcal{P} H_2 H_2^T \mathcal{P} \widehat{A}_{w0} \right) & 0 \\ 0 & 0 \end{bmatrix} \\ & + \varphi \begin{bmatrix} \widehat{A}_{w0}^T \mathcal{P} \widehat{A}_{w1} + \widehat{A}_{w1}^T \mathcal{P} \widehat{A}_{w0} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (45)$$

Now, applying the Schur lemma (Boyd *et al.*, 1994),  $\Theta_1$  can be rewritten as

$$\begin{bmatrix} (1,1) & \mathcal{P} \widehat{B}_0 & \mathcal{P} H_1 & \widehat{A}_{w0}^T \mathcal{P} \\ \widehat{B}_0^T \mathcal{P} & -\gamma^2 I_q & 0 & 0 \\ H_1^T \mathcal{P} & 0 & -\mu_1 I_{n+r} & 0 \\ \mathcal{P} \widehat{A}_{w0} & 0 & 0 & -\mathcal{P} \\ \varphi^{\frac{1}{2}} H_2^T \mathcal{P} \widehat{A}_{w0} & 0 & 0 & 0 \\ \mathcal{P} \widehat{A}_{w1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varphi^{\frac{1}{2}} \widehat{A}_{w0}^T \mathcal{P} H_2 & \widehat{A}_{w1}^T \mathcal{P} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mu_3 I_{n+r} & 0 & 0 & 0 \\ 0 & -\mathcal{P} & \mathcal{P} H_2 & 0 \\ 0 & H_2^T \mathcal{P} & -\mu_2 I_n & 0 \end{bmatrix} \quad (46)$$

where

$$\begin{aligned} (1,1) = & \mathcal{P} \widehat{A}_{t0} + \widehat{A}_{t0}^T \mathcal{P} + \widehat{C}^T \widehat{C} + (\mu_2 + \varphi \mu_3) H_w^T H_w \\ & + \mu_1 H_t^T H_t + \varphi \left( \widehat{A}_{w0}^T \mathcal{P} \widehat{A}_{w1} + \widehat{A}_{w1}^T \mathcal{P} \widehat{A}_{w0} \right). \end{aligned} \quad (47)$$

Once the LMI (33) is verified, the asymptotic mean-square stability of the system (32), for  $v(t) \equiv 0$ , can be proved using Schur lemma and the same method of (Souley Ali *et al.*, 2005).

Now consider the following performance index

$$J_{\xi v} = \int_0^\infty \mathbf{E} \left( \xi^T(t) \widehat{C}^T \widehat{C} \xi(t) - \gamma^2 v^T(t) v(t) \right) dt. \quad (48)$$

Writting  $J_{\xi v}$  as

$$\begin{aligned} J_{\xi v} = & \int_0^\infty \left\{ \mathbf{E} \left( \left( \xi^T(t) \widehat{C}^T \widehat{C} \xi(t) - \gamma^2 v^T(t) v(t) \right) dt \right. \right. \\ & \left. \left. + dV(\xi(t)) \right) \right\} - \mathbf{E}(V(\xi(t)))_{t=\infty} + \mathbf{E}(V(\xi(t)))_{t=0}. \end{aligned} \quad (49)$$

Or, since  $\mathbf{E}(V(\xi(t)))_{t=0} = 0$  because  $\xi(0) = 0$  and  $\mathbf{E}(V(\xi(t)))_{t=\infty} \geq 0$ , this implies

$$J_{\xi v} \leq \int_0^\infty \left\{ \mathbf{E} \left( \left( \xi^T(t) \widehat{C}^T \widehat{C} \xi(t) - \gamma^2 v^T(t) v(t) \right) dt + dV(\xi(t)) \right) \right\}. \quad (50)$$

Now if the LMI (33) holds, then applying Schur lemma yields

$$\underbrace{\begin{bmatrix} \Theta & \mathcal{P}\widehat{B}_0 \\ \widehat{B}_0^T \mathcal{P} & 0 \end{bmatrix}}_{\Pi} + \begin{bmatrix} \widehat{C}^T \widehat{C} & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} < 0 \quad (51)$$

with

$$\begin{aligned} \Theta = & \mathcal{P}\widehat{A}_{t0} + \widehat{A}_{t0}^T \mathcal{P} + \mu_1 H_t^T H_t + (\mu_2 + \varphi\mu_3) H_w^T H_w \\ & + \varphi(\widehat{A}_{w0}^T \mathcal{P}\widehat{A}_{w1} + \widehat{A}_{w1}^T \mathcal{P}\widehat{A}_{w0}) + \mu_1^{-1} \mathcal{P} H_1^T H_1 \mathcal{P} + \\ & \widehat{A}_{w0}^T \mathcal{P}\widehat{A}_{w0} + \varphi\mu_3^{-1} (\widehat{A}_{w0}^T \mathcal{P} H_2 H_2^T \mathcal{P}\widehat{A}_{w0}). \end{aligned}$$

Therefore

$$\begin{aligned} J_{\xi v} \leq & \int_0^\infty \mathbf{E} \left( [\xi(t)^T \ v(t)^T] \Pi \begin{bmatrix} \xi(t) \\ v(t) \end{bmatrix} dt \right. \\ & \left. + [\xi(t)^T \ v(t)^T] \begin{bmatrix} \widehat{C}^T \widehat{C} & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \begin{bmatrix} \xi(t) \\ v(t) \end{bmatrix} dt \right) < 0, \end{aligned}$$

so, if the LMI (33) holds the asymptotic mean-square stability and the  $\mathcal{H}_\infty$  performance are proved.  $\square$

## V. NUMERICAL EXAMPLE

Consider the stochastic bilinear system (1) and suppose that the matrices have the following numerical value

$$\begin{aligned} A_{t0} &= \begin{bmatrix} -1.5 & 1 & -1 \\ 0.5 & -2.5 & 1 \\ 0 & -0.6 & -3.5 \end{bmatrix}, & B_0 &= \begin{bmatrix} -0.1 & 0.3 \\ -1 & 0.2 \\ 0.6 & 0.5 \end{bmatrix}, \\ A_{t1} &= \begin{bmatrix} -0.01 & 0.1 & 0 \\ 0 & -0.05 & 0 \\ 0.15 & 0 & -0.02 \end{bmatrix}, \\ A_{w0} &= \begin{bmatrix} 1 & 0 & 0.2 \\ 0.5 & 0.3 & -0.1 \\ -0.2 & 0 & 0.2 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & L &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \\ J_1 &= \begin{bmatrix} -0.03 & 0 & -0.03 \\ 0 & -0.01 & 0 \end{bmatrix}. \end{aligned}$$

The control  $u_1(t)$  is defined as in (3), with

$$u_{1 \min} = -5 \leq u_1(t) \leq u_{1 \max} = 6,$$

and the initial state  $\xi(0) = [x^T(0) \ e^T(0)]^T$  is

$$\xi(0) = [-1 \ 0.5 \ 1 \ 0.5 \ -2]^T.$$

The gain  $Z$  is obtained for  $\gamma = 22$  and  $\mu_1 = 7.4628, \mu_2 = 0.0057$  and  $\mu_3 = 0.2476$  and is then given by

$$Z = \begin{bmatrix} -6179.565 & -6191.2468 & 47.380 & 47.2190 \\ -1810.724 & -1819.847 & 20.760 & 20.796 \end{bmatrix}.$$

Finally, the matrices of the reduced-order filter (5) are

$$\begin{aligned} M_0 &= \begin{bmatrix} -9.291 & -4.791 \\ -4.862 & -7.362 \end{bmatrix}, & M_1 &= \begin{bmatrix} -0.041 & -0.020 \\ -0.114 & -0.146 \end{bmatrix}, \\ N_0 &= \begin{bmatrix} 4.291 & -7.391 \\ 5.862 & -3.262 \end{bmatrix}, & N_1 &= \begin{bmatrix} 0.171 & 0.009 \\ -0.002 & -0.002 \end{bmatrix}. \end{aligned}$$

The following figures show the simulation results of the augmented system (32). The state  $x(t)$  and the estimation error  $e(t)$  are plotted. The disturbance signal  $v(t)$  is presented with the error plots. The simulation is made for the control  $u_1(t) = 0.5 \sin(3t) + 2$ , and the covariance factor between the Wiener processes defined in (2b)  $\varphi = 0.0215$ .

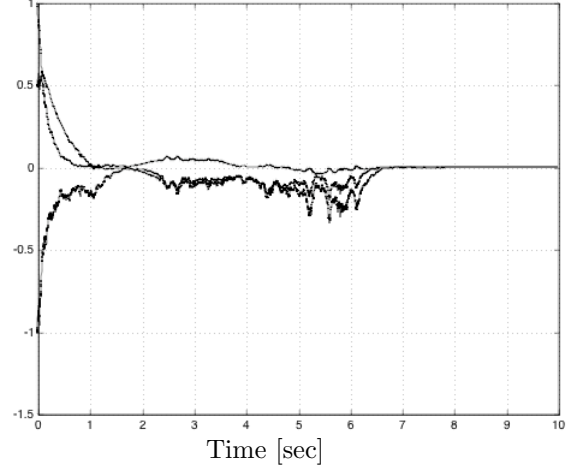


Fig. 1. The actual state  $x(t)$ .

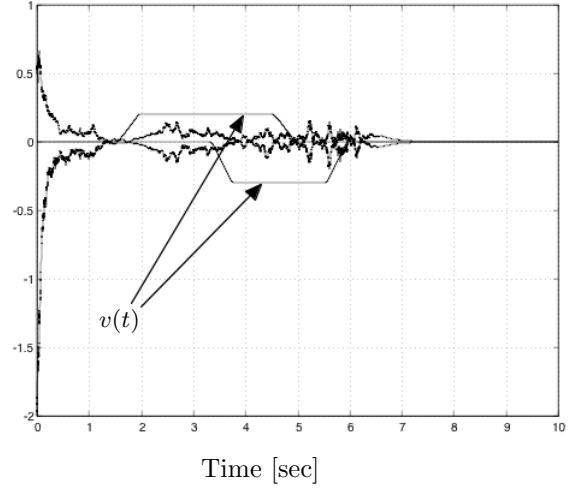


Fig. 2. The error  $e(t)$  and the disturbance  $v(t)$ .

## VI. CONCLUSION

This paper provided a solution to the reduced-order  $\mathcal{H}_\infty$  filtering problem for bilinear stochastic systems with multiplicative noise. The approach is based on a change of variable on the control input and on the using of a Sylvester-like condition on the drift term to transform the problem into a robust reduced-order stochastic filtering one. Using the LMI method and the Itô formula we reduced the problem to the search of a unique gain matrix. Then the filter matrices are computed through this gain.

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