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# Convergence results for a semilinear problem and for a Stokes problem in a periodic geometry

Séverine Baillet\*, Antoine Henrot†, Takéo Takahashi‡

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## Abstract

In this paper, we study the asymptotic behavior of the solution of a semilinear problem and of a Stokes problem, with periodic data, when the size of the domain increases. In particular, we prove exponential convergence to the solution of the corresponding problem with periodic boundary conditions.

## 1 Introduction

At the end of the exploitation of an oil well, surface pumps are no longer efficient enough to extract the oil remaining in the reservoir. Instead of closing the well, oil companies might use well-pumps, introduced deep into the ground in order to maintain the production. Such pumps are composed of a succession of identical stages (typically 15-20 stages, but it could go up to 100 stages) arranged in series. The process optimization needs numerical simulation, but representing the whole pump numerically is impossible (for obvious calculation costs). Thus it is necessary to simplify the model, by representing only one stage. Therefore, a very natural question which arises is the following: does the flow in one standard stage of the pump looks like

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\*Séverine Baillet, IFP, DTIMA, 1 et 4 avenue de Bois-Préau, 92852 Rueil-Malmaison Cedex. [severine.baillet@ifp.fr](mailto:severine.baillet@ifp.fr). Institut Elie Cartan de Nancy, Nancy-Université - CNRS - INRIA, BP 239, 54506 Vandœuvre-lès-Nancy Cedex. [severine.baillet@iecn.u-nancy.fr](mailto:severine.baillet@iecn.u-nancy.fr).

†Antoine Henrot (*corresponding author*), Institut Elie Cartan de Nancy, Nancy-Université - CNRS - INRIA, BP 239, 54506 Vandœuvre-lès-Nancy Cedex. [antoine.henrot@iecn.u-nancy.fr](mailto:antoine.henrot@iecn.u-nancy.fr).

‡Takéo Takahashi, Institut Elie Cartan de Nancy, Nancy-Université - CNRS - INRIA, BP 239, 54506 Vandœuvre-lès-Nancy Cedex. [takahash@iecn.u-nancy.fr](mailto:takahash@iecn.u-nancy.fr).

the one we would obtain by considering periodic boundary conditions at the entrance and exit of the stage. Of course, one cannot hope that this kind of result holds for the first and the last stages which are still influenced by the boundary conditions at the top and the bottom of the pump.

This kind of situation has already been considered in a series of papers by M. Chipot and Y. Xie, see [3], [5], [4], where the authors use variational techniques to prove the desired convergence. See also the book [2] by the first author. Their work is quite general and it definitely inspired us. Our own work differs to the following extent:

- we choose to work with the uniform ( $L^\infty$ ) norm, which might be more natural for the engineers,
- we use a simpler approach, just based on the maximum principle, which allows us to get a stronger convergence result (exponential convergence instead of polynomial convergence),
- in the elliptic case, we consider a semilinear problem instead of a linear one. Conversely, we are restricted to the Laplacian operator and to a simpler geometry (periodicity in only one dimension) than the one used by Chipot and Xie.
- motivated by the initial question, we also consider the Stokes problem in two dimensions (with particular boundary conditions: the so-called Navier slip boundary conditions).

Let us now be more precise in the statement of the problems and the results we get. We consider a domain  $\Omega_k \subset \mathbb{R}^N$ , composed of  $2k$  copies of a bounded open set  $\Omega$  translated in the last ( $N$ -th) space direction. For sake of simplicity, we assume that  $\Omega \subset \mathbb{R}^{N-1} \times (0, L)$  for some positive  $L$  and that

$$\Omega_k = \bigcup_{i=-k}^{k-1} (iLe_N + \Omega).$$

In all the sequel, we assume that  $\Omega_k$  is a domain of class  $C^{1,1}$  for all  $k$ .

Let  $f \in L^2_{loc}(\mathbb{R}^N)$  be a  $L$ -periodic function in the last space direction.

In section 2, we are interested in the following semilinear problem posed on  $\Omega_k$ :

$$\begin{aligned} -\Delta u_k + g(x, u_k) &= f && \text{in } \Omega_k, \\ u_k &= 0 && \text{on } \partial\Omega_k, \end{aligned}$$

where the function  $g$  satisfies usual assumptions (see section 2 for more details) and let us denote by  $u_\infty$  the solution of the periodic problem posed

on  $\Omega$ :

$$\begin{aligned} -\Delta u_\infty + g(x, u_\infty) &= f && \text{in } \Omega, \\ u_\infty &= 0 && \text{on } \Gamma, \\ u_\infty &&& Le_N\text{-periodic.} \end{aligned}$$

where  $\Gamma$  is the lateral boundary of  $\Omega$ . Then, we prove the convergence result:

**Theorem 1.1.** *For all  $k_0 \in \mathbb{N}^*$ , there exist two constants  $K > 0$  and  $\alpha > 0$ , depending only on  $\Omega_{k_0}$ , such that for all  $k \geq k_0$ ,*

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})} \leq K e^{-\alpha k}.$$

In section 3, we work in two dimensions ( $N = 2$ ) and we consider the following Stokes problem:

$$\begin{aligned} -\Delta u_k + \nabla p_k &= f && \text{in } \Omega_k, \\ \nabla \cdot u_k &= 0 && \text{in } \Omega_k, \\ \text{curl } u_k &= 0 && \text{on } \partial\Omega_k, \\ u_k \cdot n &= 0 && \text{on } \partial\Omega_k. \end{aligned}$$

We are interested in proving the convergence of the solution to the previous problem, towards the solution of the following periodic Stokes problem:

$$\begin{aligned} -\Delta u_\infty + \nabla p_\infty &= f && \text{in } \Omega, \\ \nabla \cdot u_\infty &= 0 && \text{in } \Omega, \\ \text{curl } u_\infty &= 0 && \text{on } \Gamma, \\ u_\infty \cdot n &= 0 && \text{on } \Gamma, \\ u_\infty &&& Le_2\text{-periodic.} \end{aligned}$$

We will prove that:

**Theorem 1.2.** *For all  $k_0 \in \mathbb{N}^*$ , there exist some positive constants  $K, K', \alpha$  such that, for all  $k \geq k_0$ , we have*

$$\begin{aligned} \|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})^2} &\leq K e^{-\alpha k}, \\ \|p_k - p_\infty\|_{L^2(\Omega_{k_0})} &\leq K' e^{-\alpha k}. \end{aligned}$$

At last, section 4 is devoted to some concluding remarks and possible extensions.

**Notation:**

$x$	$x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$
$x'$	$x' = (x_1, \dots, x_{N-1})$ where $x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N$
$\Omega$	bounded open set of $\mathbb{R}^N$ such that $\Omega \subset \mathbb{R}^{N-1} \times [0, L]$ , $L \in \mathbb{R}^{+*}$
$\Omega_k$	union of $2k$ open set $\Omega$ , such that $\Omega_k \subset \mathbb{R}^{N-1} \times [-kL, kL]$
$\partial\Omega$	boundary of $\Omega$
$\partial\Omega_k$	boundary of $\Omega_k$
$\Sigma_+$	$\Sigma_+ = \partial\Omega \cap \{x_N = L\}$
$\Sigma_-$	$\Sigma_- = \partial\Omega \cap \{x_N = 0\}$
$\Sigma$	$\Sigma = \Sigma_+ \cup \Sigma_-$
$\Sigma_{k+}$	$\Sigma_{k+} = \partial\Omega_k \cap \{x_N = kL\}$
$\Sigma_{k-}$	$\Sigma_{k-} = \partial\Omega_k \cap \{x_N = -kL\}$
$\Sigma_k$	$\Sigma_k = \Sigma_{k+} \cup \Sigma_{k-}$
$\Gamma$	$\Gamma = \partial\Omega \setminus \Sigma$
$\Gamma_k$	$\Gamma_k = \partial\Omega_k \setminus \Sigma_k$

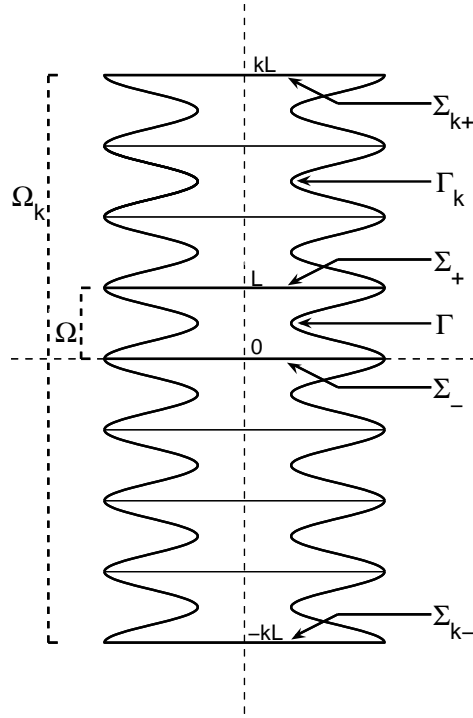


Figure 1: Periodic geometry as studied in this paper.

## 2 Semilinear problems in $N$ -dimension

In this section, we consider a semilinear problem in a periodic geometry immersed in  $\mathbb{R}^N$  where  $N \in \mathbb{N}$ ,  $N \geq 2$ . Let  $f \in L^2_{loc}(\mathbb{R}^N)$  be a  $L$ -periodic function in the  $N$ -th space direction and let  $g$  be a function such that

$$x \mapsto g(x, \cdot)$$

is measurable and is a  $L$ -periodic function in the  $N$ -th space direction, and

$$s \mapsto g(\cdot, s) \text{ is continuous and increasing, } g(\cdot, 0) = 0. \quad (1)$$

Let us remark that the assumption  $g(\cdot, 0) = 0$  is not a restriction since, otherwise, we could consider  $g_1 = g - g(\cdot, 0)$  and transfer  $g(\cdot, 0)$  in the right-hand side with  $f$ .

Moreover, let us assume that  $g$  satisfies the following property:

- if  $N = 2$ , there exist  $r \in (1, \infty)$ ,  $a_0 \in L^{\frac{r}{r-1}}(\Omega)$  and  $b_0 \geq 0$ , such that

$$|g(\cdot, s)| \leq a_0(\cdot) + b_0|s|^{r-1} \quad \text{in } \Omega; \quad (2)$$

- if  $N \geq 3$ , there exist  $a_0 \in L^{\frac{2N}{N+2}}(\Omega)$  and  $b_0 \geq 0$ , such that

$$|g(\cdot, s)| \leq a_0(\cdot) + b_0|s|^{\frac{N+2}{N-2}} \quad \text{in } \Omega. \quad (3)$$

For all  $k \in \mathbb{N}^*$ , we denote by  $u_k$  the solution of the problem

$$\begin{aligned} -\Delta u_k + g(x, u_k) &= f && \text{in } \Omega_k, \\ u_k &= 0 && \text{on } \partial\Omega_k, \end{aligned} \quad (4)$$

and by  $u_\infty$  the solution of the problem

$$\begin{aligned} -\Delta u_\infty + g(x, u_\infty) &= f && \text{in } \Omega, \\ u_\infty &= 0 && \text{on } \Gamma, \\ u_\infty &&& L e_N\text{-periodic.} \end{aligned} \quad (5)$$

We recall that with the above hypotheses on  $g$ , the systems (4) and (5) are well-posed. More precisely, we have that

**Proposition 2.1.** *Assume that  $\Omega_k$  is of class  $C^{1,1}$  and assume that  $g$  satisfies the above hypotheses. Then for any  $f \in L^2_{loc}(\mathbb{R}^N)$  and for any  $k \in \mathbb{N}$ , there exists a unique solution  $u_k \in H^1_0(\Omega_k)$  to the problem (4). Moreover  $u_k$  belongs to the Sobolev space  $W^{2,q}(\Omega_k)$  for some  $q > 1$  (depending on the dimension  $N$ ).*

The same result holds for problem (5). The proof of this proposition is very classical. The standard way to prove existence and uniqueness consists in looking for minimizers of the functional

$$I(v) = \int_{\Omega_k} \frac{1}{2} |\nabla v(x)|^2 + G(x, v(x)) - f(x)v(x) \, dx,$$

where

$$G(x, u) = \int_0^u g(x, s) \, ds.$$

We easily check that  $I$  is strictly convex, coercive and lower semicontinuous on  $H_0^1(\Omega_k)$  so that  $I$  attains its minimum only at  $u_k$ . Moreover, from assumption (2), (3) together with Lemma 17.1 and Corollary 17.2 of [11, pp. 64,65] we see that  $I$  is differentiable with

$$I'(v) = -\Delta v + g(x, v) - f.$$

Consequently, we have obtained the existence and uniqueness of a function  $u_k \in V$  such that

$$-\Delta u_k + g(x, u_k) = f.$$

For the regularity of the solution, in dimension 2, since  $u_k$  belongs to  $L^r$  space, we have  $g(x, u_k) \in L^{\frac{r}{r-1}}(\Omega_k)$  and then  $u_k \in W^{2,q}(\Omega_k)$  with  $q = \min\left(\frac{r}{r-1}, 2\right)$  by using the classical  $L^q$  regularity results (see e.g. [9, Theorem 9.15]). In the case  $N \geq 3$ , thanks to assumption (3), we have that  $g(x, u_k) \in L^{2N/(N+2)}(\Omega_k)$ . Therefore, using again the  $L^q$  regularity results (see e.g. [9, Theorem 9.15]),  $u_k$  belongs to the Sobolev space  $W^{2,q}(\Omega_k)$  with  $q = 2N/(N+2)$  and then  $-\Delta u_k + g(x, u_k(x)) = f(x)$  holds a.e.

The main result of this section is the following theorem.

**Theorem 2.1.** *Assume that  $\Omega_1$  is of class  $C^{1,1}$ . Then for all  $k_0 \in \mathbb{N}^*$ , there exist two constants  $K > 0$  and  $\alpha > 0$ , depending only on  $\Omega_{k_0}$ , such that for all  $k \geq k_0$ ,*

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})} \leq K e^{-\alpha k}.$$

*Proof.* By periodicity assumption, the function  $u_\infty$  satisfies

$$-\Delta u_\infty + g(x, u_\infty) = f$$

in the whole domain  $\Omega_k$ . Then, the function  $v_k = u_k - u_\infty$  is solution of the following boundary value problem

$$\begin{aligned} -\Delta v_k + \left[ \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty} \right] v_k &= 0 && \text{in } \Omega_k, \\ v_k &= 0 && \text{on } \Gamma_k, \\ v_k &= -u_\infty && \text{on } \Sigma_k. \end{aligned} \tag{6}$$

Let  $B_{N-1}$  be a ball of  $\mathbb{R}^{N-1}$  such that  $\Omega_k \subset\subset B_{N-1} \times (-kL, kL)$ . We denote by  $\lambda_1$  the first eigenvalue of the Laplacian with Dirichlet boundary conditions in  $B_{N-1}$  and we consider an associated eigenfunction  $\varphi_1$ . We can assume that  $\varphi_1 > 0$  in  $B_{N-1}$ . We define the function  $v$  in  $B_{N-1} \times [-kL, kL]$  by

$$v(x) = M\varphi_1(x_1, \dots, x_{N-1}) \frac{\cosh(\sqrt{\lambda_1}x_N)}{\cosh(\sqrt{\lambda_1}kL)},$$

where  $M$  is a positive constant such that

$$M\varphi_1 \geq |u_\infty| \text{ on } \Sigma_k.$$

The function  $v$  satisfies the following system

$$\begin{aligned} -\Delta v &= 0 && \text{in } \Omega_k, \\ v &\geq 0 && \text{on } \Gamma_k, \\ v &\geq |u_\infty| && \text{on } \Sigma_k. \end{aligned} \tag{7}$$

By definition, we can notice that

$$v \geq 0 \text{ in } \Omega_k. \tag{8}$$

Using the fact that  $s \mapsto g(\cdot, s)$  is an increasing function, we also have

$$\frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty} \geq 0 \text{ in } \Omega_k. \tag{9}$$

Therefore, from (6), (7), (8) and (9), we obtain

$$\begin{aligned} -\Delta(v - v_k) + \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}(v - v_k) \\ = \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}v \geq 0 \text{ in } \Omega_k. \end{aligned}$$

Consequently, the function  $v - v_k$  satisfies

$$\begin{aligned} -\Delta(v - v_k) + \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}(v - v_k) &\geq 0 \text{ in } \Omega_k, \\ v - v_k &\geq 0 \text{ on } \partial\Omega_k. \end{aligned} \tag{10}$$

Likewise, the function  $-v - v_k$  satisfies

$$\begin{aligned} -\Delta(-v - v_k) + \frac{g(x, u_k) - g(x, u_\infty)}{u_k - u_\infty}(-v - v_k) &\leq 0 \text{ in } \Omega_k, \\ -v - v_k &\leq 0 \text{ on } \partial\Omega_k. \end{aligned} \tag{11}$$



Then we apply the maximum principle [9, Theorem 8.1, p. 179] to the systems (10) and (11) and we get:

$$-v \leq v_k \leq v \text{ in } \Omega_k.$$

As a consequence, there exist two constants  $M'$  and  $K$  such that for all  $k \geq k_0$ ,

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})} \leq M' \frac{\cosh(\sqrt{\lambda_1} k_0 L)}{\cosh(\sqrt{\lambda_1} k L)} \leq K \exp(-\sqrt{\lambda_1} k L). \quad (12)$$

□

**Remark 2.1.** According to (12), the rate of decay (measuring the speed of convergence of  $u_k$  to  $u_\infty$ ) is given by  $\alpha = \sqrt{\lambda_1} L$ . If we want to get the best rate of decay, we have to choose a ball  $B_{N-1}$  as small as possible. Of course, the greater  $\lambda_1$ , the greater the constant  $K$  appearing in (12). Besides, the optimal ball  $B_{N-1}$ , which would make the cylinder  $B_{N-1} \times (-kL, kL)$  tangent to  $\Omega_k$ , would let  $K$  going to  $+\infty$ .

### 3 The Stokes problem with Navier slip boundary conditions

In this section, we consider the Stokes problem in dimension 2 in space (i.e.  $N = 2$ ). Let  $f \in L^2_{loc}(\mathbb{R}^2)^2$  be a  $L$ -periodic function in the second space direction. For any  $k \in \mathbb{N}$ , we consider the solution  $(u_k, p_k)$  of the following Stokes problem in  $\Omega_k$  ( $u_k$  is of course a vector-valued function):

$$\begin{aligned} -\Delta u_k + \nabla p_k &= f & \text{in } \Omega_k, \\ \nabla \cdot u_k &= 0 & \text{in } \Omega_k, \\ \text{curl } u_k &= 0 & \text{on } \partial\Omega_k, \\ u_k \cdot n &= 0 & \text{on } \partial\Omega_k. \end{aligned} \quad (13)$$

We also consider the solution  $(u_\infty, p_\infty)$  of the following periodic Stokes problem in  $\Omega$ :

$$\begin{aligned} -\Delta u_\infty + \nabla p_\infty &= f & \text{in } \Omega, \\ \nabla \cdot u_\infty &= 0 & \text{in } \Omega, \\ \text{curl } u_\infty &= 0 & \text{on } \Gamma, \\ u_\infty \cdot n &= 0 & \text{on } \Gamma, \\ u_\infty & \text{Le}_2\text{-periodic.} \end{aligned} \quad (14)$$

We recall that for a vector function  $u = (u_1, u_2)$ , its curl is a scalar function defined by  $\text{curl } u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ . Note that in the above systems, we have not

considered the classical no-slip condition but instead we have chosen the Navier slip boundary condition introduced by Navier in [12]

$$u \cdot n = 0 \quad \text{and} \quad \text{curl } u = 0 \quad \text{on } \Gamma.$$

The above condition can also be written under the form

$$u \cdot n = 0 \quad \text{and} \quad D(u)n \cdot \tau = 0 \quad \text{on } \Gamma,$$

where  $\tau$  is the unit tangent vector field on  $\Gamma$ . This boundary condition corresponds to the physical hypothesis that the fluid slips at the border without friction. This slip boundary condition was derived from the Boltzmann equation in [6] and was already used by [8] and [7].

In the systems (13) and (14), the pressures  $p_k$  and  $p_\infty$  are unique up to a constant. In the sequel, we impose that

$$\int_{\Omega_k} p_k \, dx = 0, \quad \int_{\Omega} p_\infty \, dx = 0.$$

Let us also remark that, since  $f \in L^2(\Omega)$ , by classical ellipticity results the solution  $u_\infty$  belongs to  $H^2(\Omega)$  and then to  $L^\infty(\Omega)$ . The same holds for  $u_k$ . We are now in position to state the main result of the paper.

**Theorem 3.1.** *For all  $k_0 \in \mathbb{N}^*$ , there exist three constants  $K > 0$ ,  $K' > 0$  and  $\alpha > 0$ , depending only on  $\Omega_{k_0}$ , such that for all  $k \geq k_0$ ,*

$$\|u_k - u_\infty\|_{L^\infty(\Omega_{k_0})^2} \leq K e^{-\alpha k} \quad \text{and} \quad \|u_k - u_\infty\|_{H_0^1(\Omega_{k_0})^2} \leq K e^{-\alpha k},$$

$$\|p_k - p_\infty\|_{L^2(\Omega_{k_0})} \leq K' e^{-\alpha k}.$$

*Proof.* According to [10], there exist two functions  $\psi_k$  and  $\psi_\infty$  such that

$$\text{curl } \psi_k = \begin{pmatrix} \frac{\partial \psi_k}{\partial x_2} \\ -\frac{\partial \psi_k}{\partial x_1} \end{pmatrix} = u_k \quad \text{in } \Omega_k, \\ \psi_k = 0 \quad \text{on } \partial\Omega_k,$$

and

$$\text{curl } \psi_\infty = u_\infty \quad \text{in } \Omega, \\ \psi_\infty = 0 \quad \text{on } \Gamma.$$

Moreover, if we define

$$\omega_k = \text{curl } u_k \quad \text{and} \quad \omega_\infty = \text{curl } u_\infty,$$

then we have

$$\begin{aligned} -\Delta\psi_k &= \omega_k & \text{in } \Omega_k, \\ \psi_k &= 0 & \text{on } \partial\Omega_k, \end{aligned}$$

and

$$\begin{aligned} -\Delta\psi_\infty &= \omega_\infty & \text{in } \Omega_k, \\ \psi_\infty &= 0 & \text{on } \Gamma_k, \\ \psi_\infty & & Le_2\text{-periodic.} \end{aligned}$$

Besides, the function  $\omega_k$ , respectively  $\omega_\infty$ , satisfies the following boundary value problem:

$$\begin{aligned} -\Delta\omega_k &= \text{curl } f & \text{in } \Omega_k, \\ \omega_k &= 0 & \text{on } \partial\Omega_k, \end{aligned}$$

respectively,

$$\begin{aligned} -\Delta\omega_\infty &= \text{curl } f & \text{in } \Omega, \\ \omega_\infty &= 0 & \text{on } \Gamma, \\ \omega_\infty & & Le_2\text{-periodic.} \end{aligned}$$

The results of the previous section (with  $g = 0$ ) apply here for these two systems. Nevertheless, we need to be more precise since we want to apply it to a right-hand side which is not periodic (but almost).

Let  $R_a = (-a, a) \times [-kL, kL]$  be a rectangle which strictly contains  $\overline{\Omega_k}$ . In this rectangle, we consider the function

$$\phi : (x_1, x_2) \mapsto K_1 \cos(\pi x_1/2a) \frac{\cosh(\pi x_2/2a)}{\cosh(\pi kL/2a)}.$$

As we did in the previous section, it is possible to choose  $K_1$  large enough so that, using the maximum principle, we get

$$|\omega_k - \omega_\infty| \leq \phi \text{ in } \Omega_k, \tag{15}$$

and then

$$\forall k_0 \in \mathbb{N}^*, \forall k \geq k_0, \|\omega_k - \omega_\infty\|_{L^\infty(\Omega_{k_0})} \leq K_1 \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)}.$$

We now want the same kind of inequality for  $\psi_k$ . We define  $\phi_k = \psi_k - \psi_\infty$ . Then

$$\begin{aligned} -\Delta\phi_k &= (\omega_k - \omega_\infty) & \text{in } \Omega_k, \\ \phi_k &= 0 & \text{on } \Gamma_k, \\ \phi_k &= -\psi_\infty & \text{on } \Sigma_k. \end{aligned} \tag{16}$$

We consider  $\epsilon > 0$  such that

$$\overline{\Omega_k} \subset \subset (-a + \epsilon, a - \epsilon) \times [-kL, kL], \tag{17}$$

and we define the function

$$\phi_\epsilon : (x_1, x_2) \mapsto K_2 \cos\left(\frac{\pi x_1}{2(a-\epsilon)}\right) \frac{\cosh(\pi x_2/2a)}{\cosh(\pi kL/2a)}, \quad (18)$$

with a positive  $K_2 \in \mathbb{R}$  to be specified further. We have

$$-\Delta\phi_\epsilon = \left(\frac{\pi}{2(a-\epsilon)}\right)^2 \phi_\epsilon - \left(\frac{\pi}{2a}\right)^2 \phi_\epsilon = \frac{\pi^2\epsilon(2a-\epsilon)}{4a^2(a-\epsilon)^2} \phi_\epsilon.$$

From (17) and (18), we can choose  $K_2$  so that

$$\begin{aligned} -\Delta\phi_\epsilon &\geq \phi && \text{in } \Omega_k, \\ \phi_\epsilon &\geq |\psi_\infty| && \text{on } \partial\Omega_k. \end{aligned}$$

Combining (15) and (16) in the above system yields that

$$\begin{aligned} -\Delta(\phi_\epsilon - \phi_k) &\geq 0 && \text{in } \Omega_k, \\ \phi_\epsilon - \phi_k &\geq 0 && \text{on } \partial\Omega_k. \end{aligned}$$

In the same way, we also have that

$$\begin{aligned} -\Delta(-\phi_\epsilon - \phi_k) &\leq 0 && \text{in } \Omega_k, \\ -\phi_\epsilon - \phi_k &\leq 0 && \text{on } \partial\Omega_k. \end{aligned}$$

Using the maximum principle, it finally comes that  $|\phi_k| \leq \phi_\epsilon$  i.e.

$$\forall k_0 \in \mathbb{N}^*, \forall k \geq k_0, \|\psi_k - \psi_\infty\|_{L^\infty(\Omega_{k_0})} \leq K'_2 \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)}, \quad (19)$$

for some positive constant  $K'_2$ .

For  $1 < \beta < \infty$ , by applying a classical regularity result on elliptic equations (see [9, Theorem 9.14] for instance), we have that

$$\begin{aligned} \|\psi_k - \psi_\infty\|_{W^{2,\beta}(\Omega_{k_0})} &\leq c(\|\psi_k - \psi_\infty\|_{L^\beta(\Omega_{k_0})} + \|\omega_k - \omega_\infty\|_{L^\beta(\Omega_{k_0})}) \\ &\leq C \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)}. \end{aligned}$$

Since  $u_k = \text{curl } \psi_k$  and  $u_\infty = \text{curl } \psi_\infty$ , the above inequality implies that

$$\|u_k - u_\infty\|_{W^{1,\beta}(\Omega_{k_0})^2} \leq C \frac{\cosh(\pi k_0 L/2a)}{\cosh(\pi k L/2a)} \leq K e^{-(\frac{\pi}{2a}L)k}. \quad (20)$$

For  $\beta > 2$ , we can use the Sobolev embedding

$$W^{1,\beta}(\Omega_{k_0}) \subset L^\infty(\Omega_{k_0}),$$

to get the  $L^\infty$  estimate for the velocity  $u_k$ .

We have now to estimate the difference  $p_k - p_\infty$ . To achieve this, we write the variational formulation corresponding to (13):

$$\forall v \in H_0^1(\Omega_{k_0})^2, \quad \mu \int_{\Omega_{k_0}} \nabla u_k : \nabla v \, dx - \int_{\Omega_{k_0}} p_k (\nabla \cdot v) \, dx = \int_{\Omega_{k_0}} f \cdot v \, dx \quad (21)$$

In the above equation, for any smooth  $v : \Omega_k \rightarrow \mathbb{R}^2$ , we have denoted by  $\nabla v$  the matrix

$$(\nabla v)_{i,j} = \frac{\partial v_i}{\partial x_j}, \quad (i, j \in \{1, 2\})$$

and, for two matrices  $A = (a_{i,j})_{i,j \in \{1,2\}}$  and  $B = (b_{i,j})_{i,j \in \{1,2\}}$ , we have denoted by  $A : B$  the scalar product

$$A : B = \sum_{i,j \in \{1,2\}} a_{i,j} b_{i,j}.$$

Now, the periodic solution  $(u_\infty, p_\infty)$  defined on  $\Omega$  can be naturally extended to  $\Omega_k$  (see also [5, Lemma 2.4]) in such a way that it satisfies on  $\Omega_k$ :

$$\forall v \in H_0^1(\Omega_{k_0})^2, \quad \mu \int_{\Omega_{k_0}} \nabla u_\infty : \nabla v \, dx - \int_{\Omega_{k_0}} p_\infty (\nabla \cdot v) \, dx = \int_{\Omega_{k_0}} f \cdot v \, dx. \quad (22)$$

Combining (21) and (22) yields that

$$\mu \int_{\Omega_{k_0}} (\nabla u_k - \nabla u_\infty) : \nabla v \, dx = \int_{\Omega_{k_0}} (p_k - p_\infty) \nabla \cdot v \, dx.$$

From (20) and the above equation, we get that

$$\int_{\Omega_{k_0}} (p_k - p_\infty) \nabla \cdot v \, dx \leq C e^{-(\frac{\pi}{2a}L)k} \|v\|_{H_0^1(\Omega_k)}$$

for any  $v \in H_0^1(\Omega_k)^2$ . This inequality and the inf-sup property (or LBB condition, see [10] or [1] for instance) imply that

$$\forall k_0 \in \mathbb{N}^*, \quad \forall k \geq k_0, \quad \|p_k - p_\infty\|_{L^2(\Omega_{k_0})} \leq C' e^{-(\frac{\pi}{2a}L)k}$$

with  $C'$  only depending on  $\Omega_{k_0}$ . □

## 4 Conclusion

This paper does not completely answer to the question raised in the introduction. Indeed, the physical situation considered in the oil extraction would necessitate to extend the present work in three directions:

- consider the three-dimensional case,
- consider more general boundary conditions,
- consider the Navier-Stokes model instead of the Stokes problem.

It seems to us that each of these extensions is really challenging.

However, numerical simulations tend to confirm, in the Navier-Stokes case, the results proved for the Stokes equations. We use Fluent<sup>®</sup> to solve the Navier-Stokes equations in three dimensions for a gasoil flow in a pump. We study two cases: the first one is a six stages pump with velocity inlet and pressure outlet; the second one is a one stage pump with periodic conditions at the entrance and exit. We plot the velocity magnitude of the fluid in the pump, at a given radius. On Figure 2 the velocity profile appears to be the same in the fourth and fifth stages of the six stages pump. It indicates that the velocity profile in a stage seems also to converge in the 3D Navier-Stokes model.

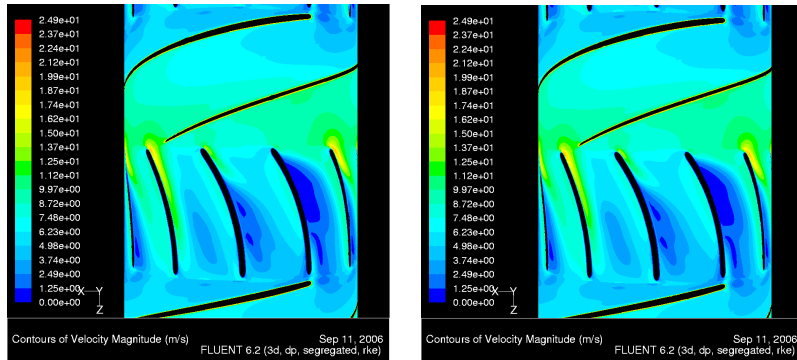


Figure 2: Velocity magnitude profile in the fourth (left) and fifth (right) stage of a six stages pump.

Figure 3 shows that the velocity profile in the fifth stage of a six stages pump and in a single stage with periodic conditions are very alike. It is a strong indication that our convergence result should also hold for the

Navier-Stokes problem (maybe with another rate of convergence). Though our simulation is limited to six stages, because of calculation costs, we could expect even better results with a bigger geometry.

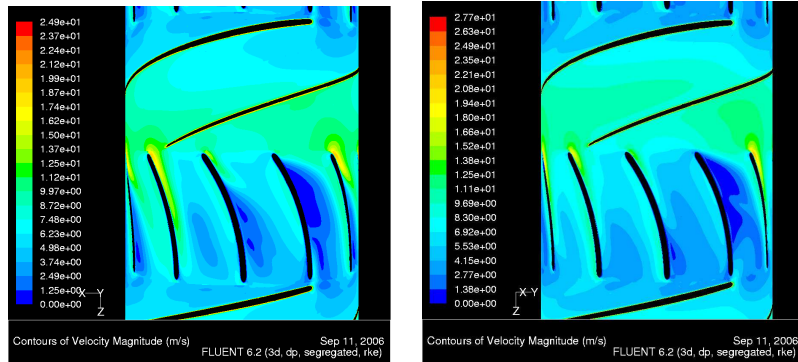


Figure 3: Velocity magnitude profile in the fifth stage of a six stages pump (left) and in a single stage with periodic conditions (right, the stage is displayed periodically).

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