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► To cite this version:

| Patrick Le Meur. On Galois coverings and tilting modules. 2006. hal-00097962v2

HAL Id: hal-00097962

<https://hal.science/hal-00097962v2>

Preprint submitted on 26 Sep 2006 (v2), last revised 12 Jan 2007 (v4)

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On Galois coverings and tilting modules

Patrick Le Meur ^{*†}

26th September 2006

Abstract

Let A be a basic and connected finite dimensional algebra over an algebraically closed field k . Let T be a basic tilting A -module with arbitrary finite projective dimension. For a finite group G we establish a bijection between the set of isoclasses of connected Galois coverings of A with group G and the set of isoclasses of connected Galois coverings of $\text{End}_A(T)$ with group G . Using the Hasse diagram $\vec{\mathcal{K}}_A$ (see [15] and [21]) of basic tilting A -modules, we give sufficient conditions on T for this bijection to hold in the case G is infinite (these conditions are always verified when A is of finite representation type). Finally, we apply these results to study when the simple connectedness of A implies the one of $\text{End}_A(T)$ (see [4]).

Introduction

Let k be an algebraically closed field and let A be a finite dimensional k -algebra. In order to study the category $\text{mod}(A)$ of finite dimensional (left) A -modules we may assume that A is basic and connected. In the study of $\text{mod}(A)$, tilting theory has proved to be a powerful tool. Indeed, if T is a basic tilting A -module and if we set $B = \text{End}_A(T)$, then A and B have many common properties: Brenner-Butler Theorem establishes an equivalence between certain subcategories of $\text{mod}(A)$ and $\text{mod}(B)$ (see [9], [14] and [19]), A and B have equivalent derived categories (see [13]) and (in particular) they have isomorphic Grothendieck groups and isomorphic Hochschild cohomologies. In this text we will study the following problem relating A and B :

is it possible to compare the Galois coverings of A and those of B ? (P_1)

As an example, if $A = kQ$ with Q a finite quiver without oriented cycle and if T is an APR-tilting module associated to a sink x of Q (see [5]) then $B = kQ'$ where Q' is obtained from Q by reversing all the arrows endind at x . In particular Q and Q' have the same underlying graph and therefore A has a connected Galois covering with group G if and only if the same holds for B .

Recall that in order to consider Galois coverings of A we always consider A as a k -category. When $\mathcal{C} \rightarrow A$ is a Galois covering (with \mathcal{C} a locally bounded k -category), it is possible to describe (part of) $\text{mod}(A)$ in terms of \mathcal{C} -modules (see for example [8] and [12]). This description is useful because $\text{mod}(\mathcal{C})$ is easier to study than $\text{mod}(A)$ especially when \mathcal{C} is simply connected (this last situation may occur when A is of finite representation type, see [12]). Notice that simple connectedness and tilting theory have already been studied together through the following conjecture formulated in [4]:

A is simply connected $\implies B$ is simply connected (P_2)

More precisely, the above implication is true if: A is of finite representation type and T is of projective dimension at most one (see [2]), or if: $A = kQ$ (with Q a quiver) and B is tame (see [4], see also [3] for a generalisation to the case of quasi-tilted algebras). The two problems (P_1) and (P_2) are related because A is simply connected if and only if there is no proper Galois covering $\mathcal{C} \rightarrow A$ with \mathcal{C} connected and locally bounded (see [17]).

In order to study the question (P_1) we will exhibit sufficient conditions for T to be of the first kind w.r.t. a fixed Galois covering $\mathcal{C} \xrightarrow{F} A$. Indeed, if T is of the first kind w.r.t. F , then it is possible to construct a Galois covering of B which is, up to isomorphism of Galois coverings of B , uniquely determined

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by the isomorphism class of F . Here we say that two Galois coverings $F: \mathcal{C} \rightarrow A$ and $F': \mathcal{C}' \rightarrow A$ are isomorphic if and only if there exists a commutative square:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathcal{C}' \\ F \downarrow & & \downarrow F' \\ A & \xrightarrow{\sim} & A \end{array}$$

where horizontal arrows are isomorphisms and where the bottom horizontal arrow restricts to the identity map on the set of objects of A . For simplicity, let us say that A and B have the same connected Galois coverings with group G if there exists a bijection between the sets $\text{Gal}_A(G)$ and $\text{Gal}_B(G)$ where $\text{Gal}_A(G)$ (resp. $\text{Gal}_B(G)$) stands for the set of isomorphism classes of Galois coverings $\mathcal{C} \rightarrow A$ (resp. $\mathcal{C} \rightarrow B$) with group G and with \mathcal{C} connected and locally bounded. With this definition, we prove the following theorem which is the main result of this text and which partially answers (P_1) :

Theorem 1. *Let A and T be as above and let G be group.*

- 1) *If G is finite, then A and B have the same connected Galois coverings with group G .*
- 2) *If $T' \in \vec{\mathcal{K}}_A$ lies in the connected component of $\vec{\mathcal{K}}_A$ containing T , then $\text{End}_A(T)$ and $\text{End}_A(T')$ have the same connected Galois coverings with group G*
- 3) *If T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A , then A and B have the same connected Galois coverings with group G .*

In particular, if $\vec{\mathcal{K}}_A$ is connected (which happens when A is of finite representation type) then A and B have the same Galois coverings with group G , for any group G .

Here $\vec{\mathcal{K}}_A$ is the Hasse diagram associated with the poset \mathcal{T}_A of basic tilting A -modules (see [15] and [21]). Theorem 1 allows us to prove the following corollary related to (P_2) . Notice that in this corollary, the third point has been proved in [2] for tilting modules with projective dimension at most one, and that in the last point, the whole equivalence has been proved in [3] under the additional assumption: B is tame.

Corollary 1. 1) *If $T' \in \vec{\mathcal{K}}_A$ lies in the connected component of $\vec{\mathcal{K}}_A$ containing T , then: $\text{End}_A(T)$ is simply connected if and only if $\text{End}_A(T')$ is simply connected.*

- 2) *If T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A then: A is simply connected if and only if B is simply connected.*
- 3) *(see [2]) If $\vec{\mathcal{K}}_A$ is connected (e.g. A is of finite representation type, see [15]), then: A is simply connected if and only if $\text{End}_A(T)$ is simply connected.*
- 4) *(see [3]) Assume that $A = kQ$ with Q a connected quiver. Then the following implication holds: $\text{End}_A(T)$ is simply connected $\Rightarrow Q$ is a tree (i.e. A is simply connected).*

The text is organised as follows. In Section 1 we give the definition of all the notions mentioned above and which will be used for the proof of Theorem 1. In Section 2 we detail the construction of the Galois covering of B starting from a Galois covering of A such that $T \in \text{mod}_1(A)$ and we study the connectedness of this Galois covering. In Section 3 we prove that for a connected Galois covering $A' \rightarrow A$ with finite group G (the finiteness assumption implies that A' is a k -algebra) the posets \mathcal{T}_A and $\mathcal{T}_{A'}^G$ are isomorphic. The proof of this result shows that any $T \in \mathcal{T}_A$ is of the first kind w.r.t. to any connected Galois covering of A with finite group. In Section 4 we prove that for $T \in \vec{\mathcal{K}}_A$ the full convex subquiver of $\vec{\mathcal{K}}_A$ defined by the vertices A and T is isomorphic to the opposite of the full convex subquiver of $\vec{\mathcal{K}}_B$ defined by the vertices B and T (with $B = \text{End}_A(T)$). This link between $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$ is used in Section 5 in order to give sufficient conditions for T to be of the first kind w.r.t. to all connected Galois coverings of A . It is also used in Section 6 which is devoted to the proof of Theorem 1.

I would like to acknowledge Eduardo N. Marcos for his stimulating remarks concerning the implication (P_2) during the CIMPA school *Homological methods and representations of non-commutative algebras* in Mar del Plata (February 2006).

1 Basic definitions and preparatory lemmata

Reminder on k -categories (see [8] for more details). A k -category is small category \mathcal{C} such that for any $x, y \in \text{Ob}(\mathcal{C})$ the set ${}_y\mathcal{C}_x$ of morphisms from x to y is a k -vector space and such that the composition

of morphisms in \mathcal{C} is k -bilinear. A k -category \mathcal{C} is called connected if and only if there is no non trivial partition $Ob(\mathcal{C}) = E \sqcup F$ such that ${}_y\mathcal{C}_x = {}_x\mathcal{C}_y = 0$ for any $x \in E, y \in F$.

All functors between k -categories are supposed to be k -linear. If $F: \mathcal{E} \rightarrow \mathcal{B}$ and $F': \mathcal{E}' \rightarrow \mathcal{B}$ are functors between k -categories, then F and F' are called isomorphic if there exists a commutative diagram:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\sim} & \mathcal{E}' \\ F \downarrow & & \downarrow F' \\ \mathcal{B} & \xrightarrow{\sim} & \mathcal{B} \end{array}$$

where horizontal arrows are isomorphisms and where the bottom horizontal arrow restricts to the identity map on $Ob(\mathcal{B})$.

A **locally bounded** k -category is a k -category \mathcal{C} verifying the following conditions:

- . distinct objects in \mathcal{C} are not isomorphic,
- . for any $x \in Ob(\mathcal{C})$ the k -vector spaces $\bigoplus_{y \in Ob(\mathcal{C})} {}_y\mathcal{C}_x$ and $\bigoplus_{y \in Ob(\mathcal{C})} {}_x\mathcal{C}_y$ are finite dimensional,
- . for any $x \in Ob(\mathcal{C})$, the k -algebra ${}_x\mathcal{C}_x$ is local.

For example, let A be a basic and finite dimensional k -algebra (basic means that A is the direct sum of pairwise non-isomorphic indecomposable projective A -modules) and let $\{e_1, \dots, e_n\}$ be a complete set of pairwise orthogonal primitive idempotents. Then A can be viewed as a locally bounded k -category as follows: e_1, \dots, e_n are the objects of A , the space of morphisms from e_i to e_j is equal to $e_j A e_i$ for any i, j and the composition of morphisms is induced by the product in A . Notice that different choices for the primitive idempotents e_1, \dots, e_n give rise to isomorphic k -categories. In this text we shall always consider such an algebra A as a locally bounded k -category.

Modules over k -categories. If \mathcal{C} is a k -category, a (left) \mathcal{C} -module is a k -linear functor $M: \mathcal{C} \rightarrow MOD(k)$ where $MOD(k)$ is the category of k -vector spaces. A morphism of \mathcal{C} -modules $M \rightarrow N$ is a k -linear natural transformation of functors. The category of \mathcal{C} -modules is denoted by $MOD(\mathcal{C})$.

Notice that if $\mathcal{C} = \mathcal{C}_1 \amalg \mathcal{C}_2$ where \mathcal{C}_1 and \mathcal{C}_2 are full subcategories of \mathcal{C} , then there is a natural embedding $MOD(\mathcal{C}_i) \hookrightarrow MOD(\mathcal{C})$ for each i . In particular, if $M \in MOD(\mathcal{C})$, then: $M \in MOD(\mathcal{C}_1)$ if and only if $M(x) = 0$ for any $x \in Ob(\mathcal{C}_2)$.

A \mathcal{C} -module M is called locally finite dimensional (resp. finite dimensional) if and only if $M(x)$ is finite dimensional for any $x \in Ob(\mathcal{C})$ (resp. if $\bigoplus_{x \in Ob(\mathcal{C})} M(x)$ is finite dimensional). The category of locally finite dimensional (resp. finite dimensional) \mathcal{C} -modules is denoted by $Mod(\mathcal{C})$ (resp. $mod(\mathcal{C})$). Notice that if $\mathcal{C} = A$ as above, then $Mod(\mathcal{C}) = mod(\mathcal{C})$.

We shall write $IND(\mathcal{C})$ (resp. $Ind(\mathcal{C})$, resp. $ind(\mathcal{C})$) for the full subcategory of $MOD(\mathcal{C})$ (resp. of $Mod(\mathcal{C})$, resp. of $mod(\mathcal{C})$) of indecomposable \mathcal{C} -modules. If $\mathcal{C} = \amalg_{i \in I} \mathcal{C}_i$ where the \mathcal{C}_i 's are the connected components of \mathcal{C} then for any $M \in IND(\mathcal{C})$ there exists a unique $i \in I$ such that $M \in IND(\mathcal{C}_i)$. Finally, if $M = N_1 \oplus \dots \oplus N_t$ with $N_i \in ind(\mathcal{C})$ for any i , then we set $d(M) := t$ and M is called **basic** if and only if M_1, \dots, M_t are pairwise non isomorphic.

Tilting modules. Let A be a finite dimensional and basic k -algebra. A tilting A -module (see [9], [14] and [19]) is a module $T \in mod(A)$ verifying the following conditions:

- (T1) T has finite projective dimension (i.e. $pd_A(T) < \infty$),
- (T2) $Ext_A^i(T, T) = 0$ for any $i > 0$ (i.e. T is selforthogonal),
- (T3) there is an exact sequence in $mod(A)$: $0 \rightarrow A \rightarrow T_1 \rightarrow \dots \rightarrow T_r \rightarrow 0$ with $T_1, \dots, T_r \in add(T)$ (i.e. T_1, \dots, T_r are direct sums of direct summands of T).

A module which satisfies the conditions (T1) and (T2) above is called an exceptionnal module. When T is a tilting A -module, we are usually interested in the k -algebra $End_A(T)$. We refer the reader to [9], [13], [14] and [19] for a deeper study of $End_A(T)$.

Let \mathcal{T}_A be the set of basic tilting A -modules up to isomorphism. Then \mathcal{T}_A is endowed with a partial order introduced in [21] and defined as follows. If $T \in \mathcal{T}_A$, the right perpendicular category T^\perp of T is defined by (see [6]):

$$T^\perp = \{X \in mod(A) \mid (\forall i \geq 1) \ Ext_A^i(T, X) = 0\}$$

If $T' \in \mathcal{T}_A$ is another basic tilting module, we write $T \leq T'$ provided that $T^\perp \subseteq T'^\perp$. In particular, we have $T \leq A$ for any $T \in \mathcal{T}_A$. In [15], D. Happel and L. Unger have proved that the Hasse diagram $\vec{\mathcal{K}}_A$ of

\mathcal{T}_A is as follows. The vertices in $\vec{\mathcal{K}}_A$ are the elements in \mathcal{T}_A and there is an arrow $T \rightarrow T'$ in $\vec{\mathcal{K}}_A$ if and only if: $T = X \oplus \overline{T}$ with $X \in \text{ind}(A)$, $T' = Y \oplus \overline{T}$ with $Y \in \text{ind}(A)$ and there exists an exact sequence $0 \rightarrow X \rightarrow \widehat{M} \rightarrow Y \rightarrow 0$ in $\text{mod}(A)$ with $\widehat{M} \in \text{add}(\overline{T})$. For more details on $\vec{\mathcal{K}}_A$, we refer the reader to [15] and [16].

Galois coverings of k -categories. Let G be a group. A **free G -category** is a k -category \mathcal{E} endowed with a morphism of groups $G \rightarrow \text{Aut}(\mathcal{E})$ such that the induced action of G on $\text{Ob}(\mathcal{E})$ is free. In this case, there exists a (unique) quotient $\mathcal{E} \rightarrow \mathcal{E}/G$ of \mathcal{E} by G in the category of k -categories. With this property, a Galois covering of \mathcal{B} with group G is by definition a functor $F: \mathcal{E} \rightarrow \mathcal{B}$ endowed with a group morphism $G \rightarrow \text{Aut}(F) = \{g \in \text{Aut}(\mathcal{E}) \mid F \circ g = F\}$ and verifying the following facts:

- . the group morphism $G \rightarrow \text{Aut}(F) \hookrightarrow \text{Aut}(\mathcal{E})$ endows \mathcal{E} with a structure of free G -category,
- . the functor $\mathcal{E}/G \xrightarrow{\overline{F}} \mathcal{B}$ induced by F is an isomorphism.

This definition implies that the group morphism $G \rightarrow \text{Aut}(F)$ is one-to-one (actually one can show that this is an isomorphism when \mathcal{E} is connected). Moreover for any $x \in \text{Ob}(\mathcal{B})$ the set $F^{-1}(x)$ is non empty, it is called the fiber of F at x and verifies: if $\hat{x} \in F^{-1}(x)$, then $F^{-1}(x) = G \cdot \hat{x}$.

If $F: \mathcal{E} \rightarrow \mathcal{B}$ is a Galois covering with group G and with \mathcal{B} connected then \mathcal{E} need not be connected. In such a case, if $\mathcal{E} = \coprod_{i \in I} \mathcal{E}_i$ where the \mathcal{E}_i 's are the connected components of \mathcal{E} , then for each i , the following functor:

$$F_i: \mathcal{E}_i \hookrightarrow \mathcal{E} \rightarrow \mathcal{B}$$

is a Galois covering with group:

$$G_i := \{g \in G \mid g(\text{Ob}(\mathcal{E}_i)) \cap \text{Ob}(\mathcal{E}_i) \neq \emptyset\} = \{g \in G \mid g(\text{Ob}(\mathcal{E}_i)) = \text{Ob}(\mathcal{E}_i)\}$$

Moreover, if $i, j \in I$ then the groups G_i and G_j are conjugated in G and there exists a commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{\sim} & \mathcal{E}_j \\ & \searrow F_i & \swarrow F_j \\ & \mathcal{B} & \end{array}$$

where the horizontal arrow is an isomorphism. This implies in particular that G acts transitively on the set $\{\mathcal{E}_i \mid i \in I\}$ of the connected components of \mathcal{E} . Notice that all these facts may be false if \mathcal{B} is not connected.

For short, a Galois covering with finite group is called a finite Galois covering. Two Galois coverings of \mathcal{B} are called isomorphic if and only if they are isomorphic as functors between k -categories (see above, this implies that the groups of the Galois coverings are isomorphic). Finally, we shall say for short that a Galois covering $\mathcal{E} \rightarrow \mathcal{B}$ is connected if and only if \mathcal{E} is connected and locally bounded (this implies that \mathcal{B} is connected and locally bounded, see [12, 1.2]).

Simply connected locally bounded k -categories. Let \mathcal{B} be a locally bounded k -category. Then \mathcal{B} is called simply connected if and only if there is no proper connected Galois covering of \mathcal{B} (proper means with non trivial group). This definition is equivalent to the original one (see [18] for the triangular case and [17, Prop. 4.1] for the non-triangular case) which was introduced in [1]: \mathcal{B} is simply connected if and only if $\pi_1(Q_{\mathcal{B}}, I) = 1$ for any admissible presentation $kQ_{\mathcal{B}}/I \simeq \mathcal{B}$ of \mathcal{B} (see [18] for the definition of $\pi_1(Q_{\mathcal{B}}, I)$).

Smash-product categories. In [11], C. Cibils and E. N. Marcos have introduced a useful way to describe Galois coverings using smash-products of G -graded categories. A G -graded category is a k -category \mathcal{B} endowed with a vector space decomposition ${}_y\mathcal{B}_x = \bigoplus_{g \in G} {}_y\mathcal{B}_x^g$ (for any $x, y \in \text{Ob}(\mathcal{B})$) such that ${}_x\mathcal{B}_y^g \cdot {}_y\mathcal{B}_x^h \subseteq {}_x\mathcal{B}_x^{gh}$ for any $x, y, z \in \text{Ob}(\mathcal{B}), g, h \in G$. If \mathcal{B} is G -graded, the smash-product category $\mathcal{B} \sharp G$ is defined as follows:

- . $\text{Ob}(\mathcal{B} \sharp G) = \text{Ob}(\mathcal{B}) \times G$,
- . ${}_{(y,t)}(\mathcal{B} \sharp G)_{(x,s)} = {}_y\mathcal{B}_x^{ts^{-1}}$ for any $(x, s), (y, t) \in \text{Ob}(\mathcal{B} \sharp G)$,
- . the composition of morphisms in $\mathcal{B} \sharp G$ is induced by the one in \mathcal{B} .

Notice that the natural projection $\text{Ob}(\mathcal{B} \sharp G) \twoheadrightarrow \text{Ob}(\mathcal{B})$ and the natural injections ${}_{(y,t)}(\mathcal{B} \sharp G)_{(x,s)} \hookrightarrow {}_y\mathcal{B}_x^{ts^{-1}}$ define a functor $\mathcal{B} \sharp G \rightarrow \mathcal{B}$. The following facts have been proved in [11]:

- $\mathcal{B}_\# G \rightarrow \mathcal{B}$ is a Galois covering with group G (the action of G on $\mathcal{B}_\# G$ is given by: $g.(x, s) = (x, sg^{-1})$ and $g.u = u \in {}_y \mathcal{B}_x^{ts^{-1}} = {}_{g.(y,t)}(\mathcal{B}_\# G)_{g.(x,s)}$ for any $(x, s), (y, t) \in \text{Ob}(\mathcal{B}_\# G)$ and any $u \in {}_{(y,t)}(\mathcal{B}_\# G)_{(x,s)}$),
- if $\mathcal{E} \rightarrow \mathcal{B}$ is a Galois covering with group G , there exists a G -grading on \mathcal{B} such that $\mathcal{E} \rightarrow \mathcal{B}$ and $\mathcal{B}_\# G \rightarrow \mathcal{B}$ are isomorphic.

Basic notions on covering techniques. Let $F: \mathcal{E} \rightarrow \mathcal{B}$ be a Galois covering with group G and with \mathcal{E}, \mathcal{B} locally bounded. The covering techniques have been introduced by K. Bongartz, P. Gabriel and C. Riedtmann (see [8] and [20]) and they establish a link between $\text{MOD}(\mathcal{E})$ and $\text{MOD}(\mathcal{B})$. We recall here some features of these techniques. First of all, the action of G on \mathcal{E} gives rise to an action of G on $\text{MOD}(\mathcal{E})$: if $M \in \text{MOD}(\mathcal{E})$ and $g \in G$, then ${}^g M := F \circ g^{-1} \in \text{MOD}(\mathcal{E})$. Moreover, F defines two additive functors $F_\lambda: \text{MOD}(\mathcal{E}) \rightarrow \text{MOD}(\mathcal{B})$ (the push-down functor) and $F.: \text{MOD}(\mathcal{B}) \rightarrow \text{MOD}(\mathcal{E})$ (the pull-up functor) as follows:

- for $M \in \text{MOD}(\mathcal{B})$, we set $F.M := M \circ F \in \text{MOD}(\mathcal{E})$,
- for $M \in \text{MOD}(\mathcal{E})$, we set $F_\lambda M \in \text{MOD}(\mathcal{B})$ to be the unique module verifying:
 - $F_\lambda M(x) = \bigoplus_{\hat{x} \in F^{-1}(x)} M(\hat{x})$ for any $x \in \text{Ob}(\mathcal{B})$,
 - $F_\lambda M(F(u)) = \bigoplus_{g \in G} g.u: \bigoplus_{g \in G} M(g\hat{x}) \rightarrow \bigoplus_{g \in G} M(g\hat{y})$ for any $u \in {}_{\hat{y}} \mathcal{E}_{\hat{x}}$.

We list below some properties of F and F_λ that will be used in the sequel (for more details we refer the reader to [8]):

- F_λ and F are exact and send projective modules to projective modules,
- $F_\lambda(\mathcal{E}) \simeq \bigoplus_{g \in G} \mathcal{B}$ and $F(\mathcal{B}) \simeq \mathcal{E}$,
- $F.F_\lambda \simeq \bigoplus_{g \in G} {}^g \text{Id}_{\text{MOD}(\mathcal{E})}$ and $F_\lambda F \simeq \bigoplus_{g \in G} \text{Id}_{\text{MOD}(\mathcal{B})}$,
- $F_\lambda(\text{mod}(\mathcal{E})) \subseteq \text{mod}(\mathcal{B})$, $F_\lambda(\text{Mod}(\mathcal{E})) \subseteq \text{Mod}(\mathcal{B})$, $F(\text{Mod}(\mathcal{B})) \subseteq \text{Mod}(\mathcal{E})$,
- $D \circ F = F \circ D$ and $D \circ F_\lambda|_{\text{mod}(\mathcal{E})} \simeq F_\lambda \circ D|_{\text{mod}(\mathcal{E})}$ where $D = \text{Hom}_k(?, k)$ is the usual duality,
- F_λ is left adjoint to F ,
- $D \circ F_\lambda \circ D$ is right adjoint to F (in particular, there is a functorial isomorphism $\text{Hom}_{\mathcal{E}}(F.M, N) \simeq \text{Hom}_{\mathcal{B}}(M, F_\lambda(N))$ for any $M \in \text{MOD}(\mathcal{B})$ and any $N \in \text{mod}(\mathcal{E})$).

These properties give the following result which will be used many times in this text:

Lemma 1.1. *If $M \in \text{MOD}(\mathcal{E})$ (resp. $M \in \text{MOD}(\mathcal{B})$) has finite projective dimension, then so does $F_\lambda(M)$ (resp. $F(M)$).*

Let $M \in \text{MOD}(\mathcal{E})$, $N \in \text{MOD}(\mathcal{B})$ and $j \geq 1$. There is an isomorphism of vector spaces:

$$\text{Ext}_{\mathcal{B}}^j(F_\lambda(M), N) \simeq \text{Ext}_{\mathcal{E}}^j(M, F.N)$$

Moreover, if $M \in \text{mod}(\mathcal{E})$ then there is an isomorphism of vector spaces:

$$\text{Ext}_{\mathcal{E}}^j(F.N, M) \simeq \text{Ext}_{\mathcal{B}}^j(N, F_\lambda(M))$$

Proof: The first assertion is due to the fact that F and F_λ are exact and send projective modules to projective modules. For the same reasons, F and F_λ induce $F.: \mathcal{D}(\text{MOD}(\mathcal{B})) \rightarrow \mathcal{D}(\text{MOD}(\mathcal{E}))$ and $F_\lambda: \mathcal{D}(\text{MOD}(\mathcal{E})) \rightarrow \mathcal{D}(\text{MOD}(\mathcal{B}))$ respectively and the adjunctions (F_λ, F) and (F, F_λ) at the level of module categories give rise to adjunctions at the level of derived categories. Since $\text{Ext}_{\mathcal{E}}^j(X, Y) = \text{Hom}_{\mathcal{D}(\text{MOD}(\mathcal{E}))}(Y, X[j])$ we get the announced isomorphisms. \blacksquare

Remark that an isomorphism of k -categories is a particular case of Galois covering. When F is an isomorphism, F and F_λ have additional properties as shows the following lemma.

Lemma 1.2. *Assume that $F: \mathcal{E} \rightarrow \mathcal{B}$ is an isomorphism. Then $F.F_\lambda = \text{Id}_{\text{MOD}(\mathcal{E})}$ and $F_\lambda F = \text{Id}_{\text{MOD}(\mathcal{B})}$. Moreover, if $\mathcal{E} = \mathcal{B}$ (i.e. $F \in \text{Aut}(\mathcal{B})$) and if F restricts to the identity map on $\text{Ob}(\mathcal{B})$, then $F.(X) \simeq X$ for any $X \in \text{MOD}(\mathcal{B})$.*

Proof: The equalities $F.F_\lambda = \text{Id}_{\text{MOD}(\mathcal{E})}$ and $F_\lambda F = \text{Id}_{\text{MOD}(\mathcal{B})}$ are direct consequences of the definition of F and F_λ . Now let us assume that $\mathcal{E} = \mathcal{B}$ and that $F(x) = x$ for any $x \in \text{Ob}(\mathcal{B})$.

For $x \in \text{Ob}(\mathcal{B})$, let $P_x = {}_? \mathcal{B}_x$ be the indecomposable projective module associated to x . Then we have an isomorphism of \mathcal{B} -modules:

$$\begin{array}{ccc} \theta_x: & P_x & \longrightarrow F.P_x \\ u \in P_x(y) = {}_y \mathcal{B}_x & \longmapsto & F(u) \in {}_y \mathcal{B}_x = F.P_x(y) \end{array}$$

If $y \in \text{Ob}(\mathcal{B})$ and if $f: P_y \rightarrow P_x$ is a morphism of \mathcal{B} -modules, then f is defined by $u_0 := f(1_y) \in {}_y\mathcal{B}_x$, i.e. $f(u) = uu_0$ for any u . So $F.f: F.P_y \rightarrow F.P_x$ verifies $F.f(u) = uF(u_0)$ for any u . Thus we have a commutative diagram (in $\text{MOD}(\mathcal{B})$) where vertical arrows are isomorphisms:

$$\begin{array}{ccc} P_y & \xrightarrow{f} & P_x \\ \theta_y \downarrow & & \downarrow \theta_x \\ F.P_y & \xrightarrow{F.f} & F.P_x \end{array}$$

This proves that the functors $\text{Id}_{\text{PROJ}(\mathcal{B})}: \text{PROJ}(\mathcal{B}) \rightarrow \text{PROJ}(\mathcal{B})$ and $F_{|\text{PROJ}(\mathcal{B})}: \text{PROJ}(\mathcal{B}) \rightarrow \text{PROJ}(\mathcal{B})$ are isomorphic (here $\text{PROJ}(\mathcal{B})$ is the full subcategory of $\text{MOD}(\mathcal{B})$ of projective \mathcal{B} -modules). Let $X \in \text{MOD}(\mathcal{B})$. So $X \simeq \text{Ker}(f)$ where $f: P \rightarrow Q$ is a morphism between projective \mathcal{B} -modules. The isomorphism $\text{Id}_{\text{PROJ}(\mathcal{B})} \simeq F_{|\text{PROJ}(\mathcal{B})}$ implies that $\text{Ker}(f) \simeq \text{Ker}(F.f)$. Since F is exact we also have $\text{Ker}(F.f) \simeq F(\text{Ker}(f))$. Therefore, X and $F(X)$ are isomorphic. \blacksquare

Modules of the first kind and graded modules. Let $F: \mathcal{E} \rightarrow \mathcal{B}$ be a Galois covering with group G . A \mathcal{B} -module M is called of the first kind w.r.t. F if and only if for any indecomposable direct summand N of M there exists $\widehat{N} \in \text{MOD}(\mathcal{E})$ such that $N \simeq F_\lambda(\widehat{N})$. We will denote by $\text{ind}_1(\mathcal{B})$ (resp. $\text{mod}_1(\mathcal{B})$) the full subcategory of $\text{ind}(\mathcal{B})$ (resp. of $\text{mod}(\mathcal{B})$) of modules of the first kind w.r.t. F . We define similarly the categories $\text{MOD}_1, \text{Mod}_1, \text{IND}_1, \text{Ind}_1$. Notice the following properties of $\text{IND}_1(\mathcal{B})$:

- . if $M \in \text{IND}_1(\mathcal{B})$ and $N \in \text{MOD}(\mathcal{E})$ verify $M \simeq F_\lambda(N)$, then $N \in \text{IND}(\mathcal{E})$,
- . if $M \in \text{IND}_1(\mathcal{B})$ and $N, N' \in \text{MOD}(\mathcal{E})$ verify $M \simeq F_\lambda(N) \simeq F_\lambda(N')$, then there exists $g \in G$ such that $N' \simeq {}^gN$.

If \mathcal{B} is connected and if $\mathcal{E} = \coprod_{i \in I} \mathcal{E}_i$, where the \mathcal{E}_i 's are the connected components of \mathcal{E} , then an indecomposable \mathcal{B} -module M is of the first kind w.r.t. F if and only if it is of the first kind w.r.t. $F_i: \mathcal{E}_i \hookrightarrow \mathcal{E} \rightarrow \mathcal{B}$ for any $i \in I$. More precisely, we have the following well know lemma where we keep the established notations.

Lemma 1.3. *Let $M \in \text{IND}(\mathcal{E})$. If $\widehat{M} \in \text{IND}(\mathcal{E})$ is such that $F_\lambda(\widehat{M}) \simeq M$, then there is a unique $i \in I$ such that $\widehat{M} \in \text{IND}(\mathcal{E}_i)$. In such a case, we have $M \simeq (F_i)_\lambda(\widehat{M})$. Moreover, if $j \in I$ then there exists $g \in G$ such that $g(\mathcal{E}_i) = \mathcal{E}_j$, and for any such g we have: ${}^g\widehat{M} \in \text{IND}(\mathcal{E}_j)$ and $(F_j)_\lambda({}^g\widehat{M}) \simeq M$.*

When \mathcal{B} is a G -graded category and $F: \mathcal{E} = \mathcal{B} \sharp G \rightarrow \mathcal{B}$ is the natural projection, the modules of the first kind have a useful description in terms of G -graded \mathcal{B} -modules. A G -graded \mathcal{B} -module is by definition a \mathcal{B} -module M endowed with k -vector space decompositions $M(x) = \bigoplus_{g \in G} M(x)^g$ (for any $x \in \text{Ob}(\mathcal{B})$) such that ${}_y\mathcal{B}_x^g.M(x)^h \subseteq M(y)^{gh}$ for any $x, y \in \text{Ob}(\mathcal{B}), g, h \in G$. If $M, N \in \text{MOD}(\mathcal{B})$ are G -graded, a homogeneous morphism $M \rightarrow N$ of degree $g \in G$ is a morphism $f: M \rightarrow N$ in $\text{MOD}(\mathcal{B})$ verifying $f(M(x)^h) \subseteq M(x)^{hg^{-1}}$ for any $x \in \text{Ob}(\mathcal{B})$ and any $h \in G$. Notice that if $f: M \rightarrow N$ is a morphism in $\text{MOD}(\mathcal{B})$ between G -graded \mathcal{B} -modules, then f is the sum of $d \geq 0$ (with d unique) homogeneous morphisms of pairwise different degrees. With these definitions, we have the following well known property:

Proposition 1.4. *Let $M \in \text{MOD}(\mathcal{B})$. Then M is of the first kind w.r.t. F if and only if M admits a G -grading. In such a case, we have $M = F_\lambda(\widetilde{M})$ with $\widetilde{M} \in \text{MOD}(\mathcal{B} \sharp G)$ as follows:*

- . $\widetilde{M}(x, s) = M(x)^s$ for any $(x, s) \in \text{Ob}(\mathcal{B} \sharp G)$,
- . $\widetilde{M}(u): \widetilde{M}(x, s) \rightarrow \widetilde{M}(y, t)$ is the mapping induced by $M(u): M(x) \rightarrow M(y)$, for any $u \in {}_{(y,t)}(\mathcal{B} \sharp G)_{(x,s)} = {}_y\mathcal{B}_x^{ts^{-1}}$.

If $M, N \in \text{MOD}(\mathcal{B})$ are G -graded \mathcal{B} -modules then $f \mapsto F_\lambda(f)$ defines an isomorphism from the space of morphisms ${}^g\widetilde{M} \rightarrow {}^h\widetilde{N}$ in $\text{MOD}(\mathcal{B} \sharp G)$ to the space of homogeneous morphisms $M \rightarrow N$ of degree $h^{-1}g$.

Throughout this text A will denote a finite dimensional basic and connected k -algebra and n will denote the rank of its Grothendieck group $K_0(A)$.

2 Galois coverings associated with modules of the first kind

Throughout this section, $F: \mathcal{C} \rightarrow A$ will be a Galois covering with group G , $T = T_1 \oplus \dots \oplus T_n \in \text{mod}(A)$ (with $T_i \in \text{ind}(A)$) will be a basic A -module of the first kind w.r.t. F and $\lambda_i: F_\lambda(\widehat{T}_i) \rightarrow T_i$ will

be an isomorphism with $\widehat{T}_i \in \text{ind}(\mathcal{C})$, for every $i \in \{1, \dots, n\}$. Recall that $\text{End}_A(T)$ is naturally a locally bounded k -category with objects T_1, \dots, T_n and with space of morphisms from T_i to T_j equal to $\text{Hom}_A(T_i, T_j)$. With this data we wish to construct a Galois covering with group G of $\text{End}_A(T)$ and verifying: the isomorphism class of this Galois covering shall depend only on the isomorphism class of F (and not on $F, \lambda_1, \dots, \lambda_n$).

The above data defines the k -category $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ as follows:

- . the set of objects is $\{{}^g\widehat{T}_i \mid g \in G, i \in \{1, \dots, n\}\}$ (${}^g\widehat{T}_i$ and ${}^{g'}\widehat{T}_j$ are considered as different objects if $(i, g) \neq (j, g')$),
- . the space of morphisms from ${}^g\widehat{T}_i$ to ${}^h\widehat{T}_j$ is equal to $\text{Hom}_{\mathcal{C}}({}^g\widehat{T}_i, {}^h\widehat{T}_j)$,
- . the composition is induced by the composition of morphisms in $\text{MOD}(\mathcal{C})$.

Remark 2.1. 1. The \mathcal{C} -modules $\bigoplus_{g,i} {}^g\widehat{T}_i$ and $F.T$ are isomorphic.

2. If G is a finite group, then \mathcal{C} is a finite dimensional k -algebra. In particular, $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i)$ and $\text{End}_{\mathcal{C}}(F.T)$ are isomorphic as k -algebras.

The isomorphisms $\lambda_1, \dots, \lambda_n$ define the following functor:

$$\begin{array}{ccc} F_{\widehat{T}, \lambda} : \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widehat{T}_i) & \longrightarrow & \text{End}_A(T) \\ & \longmapsto & T_i \\ {}^g\widehat{T}_i \xrightarrow{u} {}^h\widehat{T}_j & \longmapsto & T_i \xrightarrow{\lambda_j \circ F_{\lambda}(u) \circ \lambda_i^{-1}} T_j \end{array}$$

We are going to prove that $F_{\widehat{T}, \lambda}$ is a Galois covering. Therefore it is natural to ask whether $\mathcal{E}nd(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is connected or not. The following lemma partially answers this question.

Lemma 2.2. If \mathcal{C} is not connected, then $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$ is not connected.

Proof: For simplicity let us set $\mathcal{E} := \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\widehat{T}_i)$. Assume that \mathcal{C} is not connected and let $\mathcal{C} = \coprod_{x \in I} \mathcal{C}_x$ where the \mathcal{C}_x 's are the connected components of \mathcal{C} . Let $i \in \{1, \dots, n\}$, since $\widehat{T}_i \in \text{ind}(\mathcal{C})$, there exists a unique $x_i \in I$ such that $\widehat{T}_i \in \text{ind}(\mathcal{C}_{x_i})$. Let us set:

$$G_{x_0} = \{g \in G \mid g(\mathcal{C}_{x_0}) = \mathcal{C}_{x_0}\}$$

Since G acts transitively on $\{\mathcal{C}_x \mid x \in I\}$, for every $i \in \{1, \dots, n\}$ there exists $g_i \in G$ such that $g_i(\mathcal{C}_{x_0}) = \mathcal{C}_{x_i}$ (in particular $g_1 \in G_{x_0}$). Therefore:

$$(\forall i \in \{1, \dots, n\}) \quad {}^{g_i^{-1}}\widehat{T}_i \in \text{mod}(\mathcal{C}_{x_0})$$

Let us set O to be the following set of objects of \mathcal{E} :

$$O := \{{}^g\widehat{T}_i \mid i \in \{1, \dots, n\} \text{ and } gg_i \in G_{x_0}\} \subseteq \text{Ob}(\mathcal{E})$$

Remark that O satisfies the following:

- . $O \neq \emptyset$ for $\widehat{T}_1 \in O$.
- . Since \mathcal{C} is not connected and since G acts transitively on $\{\mathcal{C}_x \mid x \in I\}$ we have $G_{x_0} \subsetneq G$. Let $g \in G \setminus G_{x_0}$, then $gg_1 \notin G_{x_0}$ (recall that $g_1 \in G_{x_0}$) and ${}^g\widehat{T}_1 \notin O$. Hence $O \subsetneq \text{Ob}(\mathcal{C})$.
- . For any ${}^g\widehat{T}_i \in \text{Ob}(\mathcal{E})$, we have ${}^g\widehat{T}_i \in O$ if and only if ${}^g\widehat{T}_i \in \text{ind}(\mathcal{C}_{x_0})$. As a consequence, there is no non zero morphism in \mathcal{E} between an object in O and an object in $\text{Ob}(\mathcal{E}) \setminus O$.

These three facts prove that \mathcal{E} is not connected. ■

The following example shows that if F is a smash-product, then so does $F_{\widehat{T}, \lambda}$.

Example 2.3. Assume that A is G -graded, that $\mathcal{C} = A \sharp G$ and that $F: \mathcal{C} = A \sharp G \rightarrow A$ is the natural projection. Then T_1, \dots, T_n are G -graded A -modules. Fix a G -grading on T_1, \dots, T_n . These gradings define $\widetilde{T}_1, \dots, \widetilde{T}_n \in \text{mod}(A \sharp G)$ verifying $F_{\lambda}(\widetilde{T}_i) = T_i$ for each i (see Proposition 1.4). Let us set $\lambda_i = \text{Id}_{T_i}: F_{\lambda}(\widetilde{T}_i) \rightarrow T_i$. According to the above construction, the data $\widetilde{T}_1, \dots, \widetilde{T}_n, \lambda_1 = \text{Id}_{T_1}, \dots, \lambda_n = \text{Id}_{T_n}$ define the functor:

$$F_{\widetilde{T}, \text{Id}}: \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\widetilde{T}_i) \rightarrow \text{End}_A(T)$$

On the other hand, the G -gradings on T_1, \dots, T_n induce a G -grading on $\text{End}_A(T)$ (where $\text{Hom}_A(T_i, T_j)^g$ is the vector space of homogeneous morphisms $T_i \rightarrow T_j$ of degree g). With this setting, the reader may easily check that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\tilde{T}_i) & \xrightarrow{\sim} & \text{End}_A(T) \sharp G \\ & \searrow F_{\tilde{T}, Id} & \swarrow \\ & \text{End}_A(T) & \end{array}$$

where the horizontal arrow is the following isomorphism (see Proposition 1.4):

$$\begin{array}{ccc} \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g,i} {}^g\tilde{T}_i) & \longrightarrow & \text{End}_A(T) \sharp G \\ {}^g\tilde{T}_i & \longmapsto & (T_i, g^{-1}) \\ {}^g\tilde{T}_i \xrightarrow{u} {}^h\tilde{T}_j & \longmapsto & (T_i, g^{-1}) \xrightarrow{F_{\lambda}(u)} (T_j, h^{-1}) \end{array}$$

Hence, in this particular case, the functor $F_{\tilde{T}, Id}$ is a Galois covering with group G .

Now let us prove that the isomorphism class of $F_{\hat{T}, \lambda}$ depends neither on the choice of $\hat{T}_1, \dots, \hat{T}_n, \lambda_1, \dots, \lambda_n$ nor on the choice of F in its isomorphism class. This will be done through the two following lemmas.

Lemma 2.4. *For each $i \in \{1, \dots, n\}$, let $\mu_i: F_{\lambda}(\overline{T}_i) \rightarrow T_i$ be an isomorphism with $\overline{T}_i \in \text{ind}(\mathcal{C})$. Then $F_{\hat{T}, \lambda}$ and $F_{\overline{T}, \mu}$ are isomorphic.*

Proof: We need to exhibit a commutative square:

$$\begin{array}{ccc} \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\overline{T}_i) & \xrightarrow{\varphi} & \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\hat{T}_i) \\ F_{\overline{T}, \mu} \downarrow & & \downarrow F_{\hat{T}, \lambda} \\ \text{End}_A(T) & \xrightarrow{\psi} & \text{End}_A(T) \end{array} \quad (\star)$$

where φ, ψ are isomorphism and where $\psi(x) = x$ for any $x \in \text{Ob}(\text{End}_A(T)) = \{T_1, \dots, T_n\}$. Let $i \in \{1, \dots, n\}$. We have $F_{\lambda}(\overline{T}_i) \simeq T_i \simeq F_{\lambda}(\hat{T}_i)$, so there exists an isomorphism $\theta_i: \overline{T}_i \xrightarrow{\sim} {}^g\hat{T}_i$ with $g_i \in G$. Let us define φ by:

$$\begin{array}{ccc} \varphi: \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\overline{T}_i) & \longrightarrow & \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{i,g} {}^g\hat{T}_i) \\ {}^g\overline{T}_i & \longmapsto & {}^{gg_i}\hat{T}_i \\ {}^g\overline{T}_i \xrightarrow{u} {}^h\overline{T}_j & \longmapsto & {}^{gg_i}\hat{T}_i \xrightarrow{{}^h\theta_j \circ u \circ {}^g\theta_i^{-1}} {}^{hg_j}\hat{T}_j \end{array}$$

Then φ is an isomorphism of k -categories. Notice that θ_i defines an isomorphism $F_{\lambda}(\theta_i): F_{\lambda}(\overline{T}_i) \rightarrow F_{\lambda}(\hat{T}_i)$. So we can define ψ by:

$$\begin{array}{ccc} \psi: \text{End}_A(T) & \longrightarrow & \text{End}_A(T) \\ T_i & \longmapsto & T_i \\ T_i \xrightarrow{u} T_j & \longmapsto & \psi(u) \end{array}$$

where $\psi(u)$ is the composition:

$$T_i \xrightarrow{\lambda_i^{-1}} F_{\lambda}(\hat{T}_i) \xrightarrow{F_{\lambda}(\theta_i)^{-1}} F_{\lambda}(\overline{T}_i) \xrightarrow{\mu_i} T_i \xrightarrow{u} T_j \xrightarrow{\mu_j^{-1}} F_{\lambda}(\overline{T}_j) \xrightarrow{F_{\lambda}(\theta_j)} F_{\lambda}(\hat{T}_j) \xrightarrow{\lambda_j} T_j$$

So ψ is also an isomorphism of k -categories which restricts to the identity map on $\text{Ob}(\text{End}_A(T))$. Moreover φ and ψ make (\star) commutative. \blacksquare

Lemma 2.5. *Let $F': \mathcal{C}' \rightarrow A$ be a Galois covering with group G and isomorphic to F . Then T is of the first kind w.r.t. F' . For each $i \in \{1, \dots, n\}$ let $\mu_i: F'_{\lambda}(\overline{T}_i) \rightarrow T_i$ be an isomorphism with $\overline{T}_i \in \text{ind}(\mathcal{C}')$. Then $F'_{\overline{T}, \mu}$ and $F_{\hat{T}, \lambda}$ are isomorphic.*

Proof: Let us fix an isomorphism between F and F' :

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\varphi} & \mathcal{C} \\ F' \downarrow & & \downarrow F \\ A & \xrightarrow{\psi} & A \end{array}$$

Let us set $\nu: \text{Aut}(\mathcal{C}') \rightarrow \text{Aut}(\mathcal{C})$ to be the isomorphism of groups (recall that $\text{Aut}(\mathcal{C}') = G$ and $\text{Aut}(\mathcal{C}) = G$):

$$\begin{array}{ccc} \nu: \text{Aut}(\mathcal{C}') & \rightarrow & \text{Aut}(\mathcal{C}) \\ g & \mapsto & \varphi \circ g \circ \varphi^{-1} \end{array}$$

Recall that any $g \in \text{Aut}(\mathcal{C}) = G$ (resp. $g \in \text{Aut}(\mathcal{C}') = G$) defines an automorphism g of $\text{MOD}(\mathcal{C})$ (resp. of $\text{MOD}(\mathcal{C}')$). Therefore we have an equality of functors $\text{MOD}(\mathcal{C}') \rightarrow \text{MOD}(A)$:

$$(\forall g \in \text{Aut}(\mathcal{C}')) \quad \varphi_\lambda \circ g = \nu(g) \circ \varphi_\lambda$$

Let us show that T is of the first kind w.r.t. F' . In this purpose, let us fix, for each i , an isomorphism $\theta_i: \psi.T_i \rightarrow T_i$ (see Lemma 1.2) and let us set $\overline{T}_i = \varphi.(\widehat{T}_i)$. In particular: $\varphi_\lambda(\overline{T}_i) = \widehat{T}_i$ (see Lemma 1.2). Since $\psi.\psi_\lambda = \text{Id}_{\text{MOD}(A)}$ (loc. cit.) and $\psi F' = F\varphi$, we infer that:

$$F'_\lambda(\overline{T}_i) = \psi.\psi_\lambda F'_\lambda(\overline{T}_i) = \psi.F_\lambda \varphi_\lambda(\overline{T}_i) = \psi.F_\lambda(\widehat{T}_i)$$

Therefore, we get for each i an isomorphism $\mu_i: F'_\lambda(\overline{T}_i) \rightarrow T_i$ equal to the composition:

$$\mu_i: F'_\lambda(\widehat{T}_i) = \psi.F_\lambda(\overline{T}_i) \xrightarrow{\psi.(\lambda_i)} \psi.(T_i) \xrightarrow{\theta_i} T_i$$

This proves that T is of the first kind w.r.t. F' and this also defines the functor $F'_{T,\mu}: \text{End}_{\mathcal{C}'}(\bigoplus_{i,g} {}^g \overline{T}_i) \rightarrow \text{End}_A(T)$. Thanks to Lemma 2.4 we only need to prove that $F'_{T,\mu}$ and $F_{\widehat{T},\lambda}$ are isomorphic in order to prove the lemma.

First, we have the following functor induced by φ_λ :

$$\begin{array}{ccc} \overline{\varphi}: \text{End}_{\mathcal{C}'}(\bigoplus_{i,g} {}^g \overline{T}_i) & \longrightarrow & \text{End}_{\mathcal{C}}(\bigoplus_{i,g} {}^g \widehat{T}_i) \\ {}^g \overline{T}_i & \longmapsto & \nu(g)\widehat{T}_i = \varphi_\lambda({}^g \overline{T}_i) \\ {}^g \overline{T}_i \xrightarrow{u} {}^h \overline{T}_j & \longmapsto & \nu(g)\widehat{T}_i \xrightarrow{\varphi_\lambda(u)} \nu(h)\widehat{T}_j \end{array}$$

Since $\nu: G \rightarrow G$ is an isomorphism and because of the equalities $\varphi_\lambda \varphi. = \text{Id}_{\text{MOD}(\mathcal{C})}$ and $\varphi.\varphi_\lambda = \text{Id}_{\text{MOD}(\mathcal{C})}$ (see Lemma 1.2), the functor $\overline{\varphi}$ is an isomorphism.

Secondly, we have the following functor induced by ψ_λ :

$$\begin{array}{ccc} \overline{\psi}: \text{End}_A(T) & \longrightarrow & \text{End}_A(T) \\ T_i & \longmapsto & T_i \\ T_i \xrightarrow{u} T_j & \longmapsto & T_i \xrightarrow{\psi_\lambda(\theta_j u \theta_i^{-1})} T_j \end{array}$$

Since $\psi_\lambda \psi. = \psi.\psi_\lambda = \text{Id}_{\text{MOD}(A)}$, the functor $\overline{\psi}$ is a well defined isomorphism and restricts to the identity map on $\text{Ob}(\text{End}_A(T))$. Therefore, we have a diagram whose horizontal arrows are isomorphisms and whose bottom horizontal arrow restricts to the identity map on the set of objects:

$$\begin{array}{ccc} \text{End}_{\mathcal{C}'}(\bigoplus_{i,g} {}^g \overline{T}_i) & \xrightarrow{\overline{\varphi}} & \text{End}_{\mathcal{C}}(\bigoplus_{i,g} {}^g \widehat{T}_i) \\ F'_{T,\mu} \downarrow & & \downarrow F_{\widehat{T},\lambda} \\ \text{End}_A(T) & \xrightarrow{\overline{\psi}} & \text{End}_A(T) \end{array}$$

This diagram is commutative, indeed, for any ${}^g \overline{T}_i \xrightarrow{u} {}^h \overline{T}_j$ we have:

$$\begin{aligned} \overline{\psi} F'_{T,\mu}(u) &= \overline{\psi}(\mu_j F'_\lambda(u) \mu_i^{-1}) = \psi_\lambda(\theta_j^{-1} \mu_j F'_\lambda(u) \mu_i^{-1} \theta_i) \\ &= \psi_\lambda(\theta_j^{-1} \theta_j \psi.(\lambda_j) F'_\lambda(u) \psi.(\lambda_i)^{-1} \theta_i^{-1} \theta_i) \\ &= \lambda_j (\psi_\lambda F'_\lambda(u) \lambda_i^{-1}) && \text{because } \psi_\lambda \psi. = \text{Id}_{\text{MOD}(\mathcal{C})} \\ &= \lambda_j (F_\lambda \varphi_\lambda(u) \lambda_i^{-1}) && \text{because } F\varphi = \psi F' \\ &= F_{\widehat{T},\lambda}(\varphi_\lambda(u)) = F_{\widehat{T},\lambda} \overline{\varphi}(u) \end{aligned}$$

This proves that $F'_{T,\mu}$ and $F_{\widehat{T},\lambda}$ are isomorphic. ■

Remark 2.6. The preceding lemma shows that if T is of the first kind w.r.t. F then T is of the first kind with respect to any Galois covering F' isomorphic to F . Hence, one may talk about a module which is of the first kind w.r.t. an isomorphism class of Galois coverings of A .

Example 2.3, Lemma 2.4, Lemma 2.5 and the fact that $\bigoplus_{i,g} {}^g\widehat{T}_i \simeq F.T$ justify the following definition:

Definition 2.7. The isomorphism class of F and the basic A -module $T = T_1 \oplus \dots \oplus T_n$ of the first kind w.r.t. F (with $T_i \in \text{ind}(A)$) uniquely define an isomorphism class of Galois covering of $\text{End}_A(T)$ which admits $F_{\widehat{T},\lambda}$ as a representative. This isomorphism class will be denoted by $F_T: \mathcal{E}nd_{\mathcal{C}}(F.T) \rightarrow \text{End}_A(T)$.

Starting from T and F we have constructed F_T . Therefore one may try to perform the same construction starting from T and F_T in order to get $(F_T)_T$. The end of this section is devoted to the comparison of $(F_T)_T$ and A . Until the end of the section we shall write B for $\text{End}_A(T)$. Remark that for any $x \in \text{Ob}(A)$ the vector space $T(x)$ is a B -submodule of T and that $T = \bigoplus_{x \in \text{Ob}(A)} T(x)$ is a decomposition into a direct sum of B -modules. In particular, if $T(x)$ is indecomposable for every $x \in \text{Ob}(A)$ (which happens if T is a basic tilting A -module, see [19, Thm.1.5]) then $\text{End}_B(T)$ is naturally a k -category with set of objects equal to $\{T(x) \mid x \in \text{Ob}(A)\}$ and with space of morphisms from $T(x)$ to $T(y)$ equal to $\text{Hom}_B(T(x), T(y))$. Thus, we have a well defined functor between k -categories:

$$\begin{aligned} \rho: \quad A &\longrightarrow \text{End}_B(T) \\ x \in \text{Ob}(A) &\longmapsto T(x) \in \text{Ob}(\text{End}_B(T)) \\ a \in {}_y A_x &\longmapsto (t \mapsto at) \in {}_{T(y)} \text{End}_B(T)_{T(x)} \end{aligned}$$

Recall from [19, Thm.1.5] that ρ is an isomorphism of k -categories if T is a basic tilting A -module. These facts motivate the following lemma where we keep the setting of the beginning of this section.

Lemma 2.8. Assume that for every $x \in \text{Ob}(A)$, the B -module $T(x)$ is indecomposable and that $T = \bigoplus_{x \in \text{Ob}(A)} T(x)$ is a basic B -module (i.e. the $T(x)$'s are pairwise non isomorphic). Then T is of the first kind w.r.t. F_T (see Remark 2.6). Moreover, if $\rho: A \rightarrow \text{End}_B(T)$ (as defined above) is an isomorphism of k -categories, then the isomorphism class of F coincides with $\rho^{-1} \circ (F_T)_T$. As a consequence, if \mathcal{C} is connected, then $\mathcal{E}nd_{\mathcal{C}}(F.T)$ is connected.

Proof: For simplicity, we shall consider F_T as a Galois covering rather than as an isomorphism class of Galois coverings. Thanks to Example 2.3, to Lemma 2.4 and to Lemma 2.5 we may assume that:

- . A is G -graded, $\mathcal{C} = A\sharp G$ and $F: A\sharp G \rightarrow A$ is the natural projection,
- . T_1, \dots, T_n are G -graded A -modules,
- . F_T is the natural projection $B\sharp G \rightarrow B$, where the G -grading on B is given by the homogeneous morphisms $T_i \rightarrow T_j$ between G -graded A -modules.

In order to prove that T is a B -module of the first kind w.r.t. to F_T it thus suffices to exhibit a G -grading on the B -module $T(x)$, for every $x \in \text{Ob}(A)$. Let $x \in \text{Ob}(A)$. Let us set, for $g \in G$:

$$T(x)_g := T_1(x)^{g^{-1}} \bigoplus \dots \bigoplus T_n(x)^{g^{-1}}$$

Using the fact that $T(x) = T_1(x) \bigoplus \dots \bigoplus T_n(x)$, it is easily verified that:

$$T(x) = \bigoplus_{g \in G} T(x)_g \tag{i}$$

On the other hand, if $f \in {}_{T_j} B_{T_i}^g$, then $f \in \text{Hom}_A(T_i, T_j)$ is homogeneous of degree g , so:

$$(\forall h \in G) \quad f(T_i(x)^h) \subseteq T_j(x)^{hg^{-1}}$$

Therefore:

$$(\forall h \in G) \quad f.T(x)_h \subseteq T(x)_{gh} \tag{ii}$$

From (i) and (ii) we deduce that $T(x)$ is a G -graded B -module for any $x \in \text{Ob}(A)$. In other words, T is of the first kind w.r.t. F_T (here we implicitly use the assumption $T(x) \in \text{ind}(B)$).

Now let us assume that $\rho: A \rightarrow \text{End}_B(T)$ is an isomorphism of k -categories and let us prove that F and $\rho^{-1} \circ (F_T)_T$ are isomorphic. Recall that since the $T(x)$'s are G -graded B -modules, $\text{End}_B(T)$ is also endowed with a structure of G -graded k -category as follows: ${}_{T(y)} \text{End}_B(T)_{T(x)}^g$ is the vector space of homogeneous morphisms $f: T(x) \rightarrow T(y)$ of B -modules and of degree g , i.e. verifying:

$$(\forall h \in G) \quad f(T(x)_h) \subseteq T(y)_{hg^{-1}}, \text{ i.e. } f(T(x)^h) \subseteq T(y)^{gh} \tag{iii}$$

Therefore, using Example 2.3, Lemma 2.4 and Lemma 2.5 we may assume that $(F_T)_T$ is the natural projection:

$$(F_T)_T: \text{End}_B(T) \sharp G \rightarrow \text{End}_B(T)$$

On the other hand, if $a \in {}_y A_x^g$, then $a.T(x)^h \subseteq T(y)^{gh}$ for any $h \in G$. Thanks to (iii), this means that $\rho(a) \in {}_{T(y)} \text{End}_B(T)_{T(x)}^g$. Thus:

$$\rho: A \rightarrow \text{End}_B(T) \text{ is an isomorphism of } G\text{-graded } k\text{-categories}$$

Therefore, the following commutative diagram is an isomorphism between F and $\rho^{-1} \circ (F_T)_T$:

$$\begin{array}{ccc} A \sharp G & \xrightarrow{\rho \sharp G} & \text{End}_B(T) \sharp G \\ F \downarrow & & \downarrow \rho^{-1} \circ (F_T)_T \\ A & \xrightarrow{\text{Id}_A} & A \end{array}$$

where $\rho \sharp G$ is the isomorphism defined by:

$$\begin{aligned} A \sharp G &\longrightarrow \text{End}_B(T) \sharp G \\ (x, g) &\longrightarrow (T(x), g) \\ u \in (y, t) A \sharp G_{(x, s)} = {}_y A_x^{ts^{-1}} &\longrightarrow \rho(u) \in {}_{T(y)} \text{End}_B(T)_{T(x)}^{ts^{-1}} = {}_{(T(y), t)} \text{End}_B(T) \sharp G_{(T(x), s)} \end{aligned}$$

This proves that F and $\rho^{-1} \circ (F_T)_T$ are isomorphic. The last assertion of the lemma is a consequence of Lemma 2.2. \blacksquare

3 Comparison of \mathcal{T}_A and $\mathcal{T}_{A'}$ when $A' \rightarrow A$ is a finite Galois covering

Throughout this section, A' will be a basic finite dimensional k -algebra and $F: A' \rightarrow A$ will be a Galois covering with (finite) group G . The aim of this section is to compare the posets \mathcal{T}_A and $\mathcal{T}_{A'}$. In this purpose, we begin with the following proposition:

Proposition 3.1. *Let $T \in \mathcal{T}_A$, then $F.T \in \mathcal{T}_{A'}$. Moreover T is of the first kind w.r.t. F . Finally, if X is an indecomposable summand of T and if $\hat{T} \in \text{mod}(A')$ verifies $F_\lambda(\hat{X}) \simeq X$ then $G_{\hat{X}} = 1$.*

Proof: From Lemma 1.1 applied to T , we have $\text{pd}_{A'}(F.T) < \infty$ together with:

$$(\forall i \geq 1) \quad \text{Ext}_{A'}^i(F.T, F.T) \simeq \text{Ext}_A^i(F_\lambda F.T, T) \simeq \text{Ext}_A^i\left(\bigoplus_{g \in G} T, T\right) \simeq \bigoplus_{g \in G} \text{Ext}_A^i(T, T) = 0$$

Thus $F.T$ is an exceptionnal A' -module. Moreover, we have an exact sequence in $\text{mod}(A)$:

$$0 \rightarrow A \rightarrow T_1 \rightarrow \dots \rightarrow T_r \rightarrow 0$$

where $T_i \in \text{add}(T)$ for every i . Applying F . to this exact sequence gives an exact sequence in $\text{mod}(A')$:

$$0 \rightarrow F.A \simeq A' \rightarrow F.T_1 \rightarrow \dots \rightarrow F.T_r \rightarrow 0$$

where $F.T_i \in \text{add}(F.T)$ for every i . Hence $F.T$ is a tilting A' -module.

Now let us prove that $F.T$ is basic and that $T \in \text{mod}_1(A)$. Since $T \in \mathcal{T}_A$, we have $T = T_1 \oplus \dots \oplus T_n$ with $T_i \in \text{ind}(A)$ for each i and $T_i \not\simeq T_j$ if $i \neq j$. For each i , let us write $F.T_i = \bigoplus_{l=1}^{m_i} X_{i,l}$ with $X_{i,l} \in \text{ind}(A')$. Hence $F.T = \bigoplus_{i=1}^n X_{i,l}$ is a tilting A' -module and is the direct sum of $\sum_{i=1}^n m_i$ indecomposable A' -modules, so (see [19, Thm 1.19]):

$$\sum_{i=1}^n m_i \geq \text{rk } K_0(A') = n \cdot |G| \tag{i}$$

Moreover, we have for each i :

$$\bigoplus_{g \in G} T_i = F_\lambda F.(T_i) = \bigoplus_{l=1}^{m_i} F_\lambda(X_{i,l}) \tag{ii}$$

where the last term (resp. the first term) of the equality is the direct sum of at least m_i (resp. exactly $|G|$) indecomposable A -modules, so:

$$(\forall i \in \{1, \dots, n\}) \quad |G| \geq m_i \tag{iii}$$

From (i), (ii) and (iii) we deduce that:

(iv) $m_i = |G|$ for each i ,

(v) $F_\lambda(X_{i,l}) \simeq T_i$ for each i, l (in particular, for any i , T_i is of the first kind w.r.t. F),

Hence $F.T = \bigoplus_{i,l} X_{i,l}$ is a tilting A' -module and is the direct sum of $\sum_i m_i = n \cdot |G| = \text{rk } K_0(A')$ indecomposable A' -modules. This shows that (see [19, Thm 1.19]) $F.T$ is basic (and therefore $F.T \in \mathcal{T}_{A'}$), and (v) shows that $T \in \text{mod}_1(A)$.

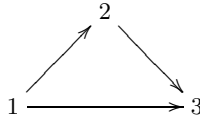
Finally, let $X \in \text{ind}(A)$ be a direct summand of T and let $\hat{X} \in \text{ind}(A')$ verify $F_\lambda(\hat{X}) \simeq X$. Thus, we have $F(X) \simeq \bigoplus_{g \in G} {}^g \hat{X}$. Moreover, $F.X$ is a direct summand of $F.T$ which is basic. So $\bigoplus_{g \in G} {}^g \hat{X}$ is basic, i.e. $G_{\hat{X}} = 1$. ■

Remark 3.2. . Proposition 3.1 proves that F sends tilting A -modules to tilting A' -modules.

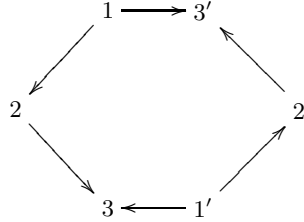
. In general, given $X \in \text{ind}(A')$ (with G non necessarily finite), the property $G_X = 1$ (like in Proposition 3.1) implies that $F_\lambda(X)$ is indecomposable (see [12, 3.5]).

The following example shows that F_λ does not always preserve tilting modules.

Example 3.3. Let A be the path algebra of the quiver Q equal to:



Let A' be the path algebra of the quiver Q' equal to:



The mapping $i, i' \mapsto i$ ($i \in \{1, 2, 3\}$) uniquely defines a Galois covering $A' \rightarrow A$ with group $\mathbb{Z}/2\mathbb{Z}$. For x a vertex of Q or Q' , let P_x be the associated indecomposable projective module and let τ_A (resp. $\tau_{A'}$) be the Auslander-Reiten translation (see [7]) of A (resp. of A'). Let $T = P_1 \oplus P_2 \oplus P_3 \oplus P_{1'} \oplus P_{2'} \oplus \tau_{A'}^{-1}(P_{3'})$ be the APR-tilting A' -module associated with the sink $3'$ of Q' (see [5]). Then:

- . $P_3 \oplus \tau_{A'}^{-1}(P_{3'})$ is a selforthogonal A' -module.
- . $F_\lambda(P_3 \oplus \tau_{A'}^{-1}(P_{3'})) = P_3 \oplus \tau_A^{-1}(P_3)$ is not selforthogonal because of the almost split sequence in $\text{mod}(A)$:

$$0 \rightarrow P_3 \rightarrow P_1 \bigoplus P_2 \rightarrow \tau_A^{-1}(P_3) \rightarrow 0$$

This shows that $F_\lambda(T)$ is not selforthogonal and therefore not tilting.

Although F_λ does not always transform a tilting A' -module into a tilting A -module, it is possible to characterise those tilting A' -modules T such that $F_\lambda(T)$ is tilting. We establish this characterisation in the following proposition.

Proposition 3.4. 1) If $T \in \mathcal{T}_A$ then $G_{F.T} = G$.

2) Let $X \in \mathcal{T}_{A'}$ verify $G_X = G$. Let $T \in \text{mod}(A)$ be a basic A -module such that $\text{add}(T) = \text{add}(F.X)$. Then $T \in \mathcal{T}_A$ and $X \simeq F.T$.

3) Let $X \in \mathcal{T}_{A'}$, then $F_\lambda(X)$ is tilting if and only if $G_X = G$.

Proof: 1) is a consequence of the definition of F . Let us prove 2). From Lemma 1.1 we get $\text{pd}_A(F_\lambda(X)) \leq \text{pd}_A(T) < \infty$. Since X is selforthogonal and G is finite, Lemma 1.1 implies that:

$$(\forall i \geq 1) \quad \text{Ext}_A^i(F_\lambda(X), F_\lambda(X)) \simeq \text{Ext}_{A'}^i(X, F.F_\lambda(X)) \simeq \text{Ext}_{A'}^i(X, \bigoplus_{g \in G} {}^g X) \simeq \bigoplus_{g \in G} \text{Ext}_{A'}^i(X, {}^g X) = 0$$

So $F_\lambda(X)$ and T are selforthogonal. This shows that T is an exceptionnal and basic A -module. Hence, in order to prove that $T \in \mathcal{T}_A$, it suffices to prove that $d(T) \geq n$ (see [19, Thm 1.19], recall that $d(T)$ is the

number of indecomposables whose direct sum is T). The equalities $\text{add}(T) = \text{add}(F_\lambda(X))$ and $G_X = G$ imply that:

$$\text{add}(F.T) = \text{add}(F.F_\lambda(X)) = \text{add}\left(\bigoplus_{g \in G} {}^g X\right) = \text{add}(X) \quad (i)$$

Since X is basic, we infer that X is a direct summand of $F.T$. In particular, we have:

$$d(F.T) \geq d(X) = n \cdot |G| \quad (ii)$$

the last equality of (ii) is due to the fact that $X \in \mathcal{T}_{A'}$. Using (ii) we get:

$$d(T) = \frac{1}{|G|} d\left(\bigoplus_{g \in G} T\right) = \frac{1}{|G|} d(F_\lambda F.(T)) \geq \frac{1}{|G|} d(F.T) \geq \frac{1}{|G|} n|G| = n$$

Hence T is an exceptional basic A -module satisfying $d(T) \geq n$. So $T \in \mathcal{T}_A$ and $d(T) = n$. Now let us prove that $F.T \simeq X$. From Proposition 3.1 we know that $F.T$ is basic. Since X is basic as well and since $\text{add}(X) = \text{add}(F.T)$ (see (i) above) we infer that $X \simeq F.T$. This proves 2).

Let us prove 3). If $G_X = G$, then $\text{add}(F_\lambda(X)) = \text{add}(T)$ with $T \in \mathcal{T}_A$ (see 2)), so $F_\lambda(X)$ is tilting. Conversely, suppose that $F_\lambda(X)$ is tilting. Then we get from Lemma 1.1 (recall that G is finite):

$$(\forall i \geq 1) \quad 0 = \text{Ext}_A^i(F_\lambda(X), F_\lambda(X)) \simeq \text{Ext}_{A'}^i(X, F.F_\lambda(X)) \simeq \bigoplus_{g \in G} \text{Ext}_{A'}^i(X, {}^g X)$$

So:

$$(\forall i \geq 1) \quad (\forall g \in G) \quad \begin{cases} \text{Ext}_{A'}^i(X, {}^g X) = 0 \\ \text{Ext}_{A'}^i({}^g X, X) \simeq \text{Ext}_{A'}^i(X, {}^{g^{-1}} X) = 0 \end{cases}$$

Since ${}^g X \in \mathcal{T}_{A'}$ for any $g \in G$ we infer that ${}^g X \simeq X$ for any $g \in G$. ■

Before stating the theorem comparing \mathcal{T}_A and $\mathcal{T}_{A'}$, we remark that the action of G on $\text{mod}(A')$ naturally defines an action on the poset $\mathcal{T}_{A'}$ (the action is compatible with the order: $X \geq X' \Rightarrow {}^g X \geq {}^g X'$).

Theorem 3.5. *The mapping:*

$$\begin{array}{ccc} \iota: & \mathcal{T}_A & \longrightarrow \mathcal{T}_{A'} \\ & T & \longmapsto F.T \end{array}$$

is a well defined injective morphism of posets. Its image is equal to:

$$\mathcal{T}_{A'}^G = \{T \in \mathcal{T}_{A'} \mid (\forall g \in G) \quad {}^g X \simeq X\}$$

Proof: Thanks to Proposition 3.1 and Proposition 3.4 we only need to prove that ι is injective and increasing. If $T, T' \in \mathcal{T}_{A'}$ are such that $F.T \simeq F.T'$ then $\bigoplus_{g \in G} T \simeq F_\lambda F.(T) \simeq F_\lambda F.(T') \simeq \bigoplus_{g \in G} T'$ and therefore $T \simeq T'$ (recall that T and T' are basic). So ι is injective. Now assume that $T \leq T'$ in \mathcal{T}_A . Let $X \in (F.T)^\perp$, then Lemma 1.1 implies that:

$$(\forall i \geq 1) \quad 0 = \text{Ext}_{A'}^i(F.T, X) \simeq \text{Ext}_A^i(T, F_\lambda(X))$$

So $F_\lambda(X) \in T^\perp \subseteq T'^\perp$. Using once more Lemma 1.1 we infer that:

$$(\forall i \geq 1) \quad 0 = \text{Ext}_A^i(T', F_\lambda(X)) \simeq \text{Ext}_{A'}^i(F.T', X)$$

So $X \in (F.T')^\perp$. This shows that ι is a morphism of posets. ■

In Proposition 3.1 we have proved that any basic tilting A -module is of the first kind w.r.t. F . Therefore, Definition 2.7 and Remark 2.1 give a Galois covering $F_T: \text{End}_{A'}(F.T) \rightarrow \text{End}_A(T)$ with group G . In view of the proof of Theorem 1 we now verify that this Galois covering is connected.

Proposition 3.6. *Let T be a basic tilting A -module. Then the Galois covering $F_T: \text{End}_{A'}(F.T) \rightarrow \text{End}_A(T)$ is connected.*

Proof: Let us write $T = T_1 \oplus \dots \oplus T_n$ (with $T_i \in \text{ind}(A)$). For each i let us fix $\widehat{T}_i \in \text{ind}(A')$ verifying $F_\lambda(\widehat{T}_i) \simeq T_i$. In order to prove that the Galois covering F_T is connected we thus need to prove that the k -category $\text{End}_{A'}(\bigoplus_{g \in G} {}^g \widehat{T}_i)$ is locally bounded and connected. From Proposition 3.1 we know that $F.T \simeq \bigoplus_{g \in G} {}^g \widehat{T}_i$ is a basic tilting A' -module. So $\text{End}_{A'}(\bigoplus_{g \in G} {}^g \widehat{T}_i)$ is a finite dimensional basic and connected k -algebra, i.e. a locally bounded and connected. ■

4 Comparison of $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_{\text{End}_A(T)}$ for a tilting A -module T

Throughout this section, T will be a basic tilting A -module. Let $B = \text{End}_A(T)$, in particular, T is a basic tilting B -module (see [19, Thm 1.5]). The aim of the present section is to establish a link between $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$. This link will be used in the comparison of the Galois coverings of A and B . For simplicity, if $X \in \text{mod}(A)$ (resp. $u \in \text{Hom}_A(X, Y)$) we shall write X_T (resp. u_T) for the B -module (resp. the morphism of B -modules) $\text{Hom}_A(X, T)$ (resp. $\text{Hom}_A(u, T): \text{Hom}_A(Y, T) \rightarrow \text{Hom}_A(X, T)$). Also, whenever f is a morphism of modules, we shall write f_* (resp. f^*) for the mapping $g \mapsto fg$ (resp. $g \mapsto gf$). We begin with a useful lemma.

Lemma 4.1. *Let $X \in \text{mod}(A)$ and let $T' \in \vec{\mathcal{K}}_A$ be a predecessor of T (i.e. there is a path in $\vec{\mathcal{K}}_A$ starting at T' and ending at T). Then, for any direct summand Y of T' , the natural morphism:*

$$\begin{array}{ccc} \theta_{X,Y}: \text{Hom}_A(X, Y) & \longrightarrow & \text{Hom}_B(Y_T, X_T) \\ u & \longmapsto & u_T \end{array}$$

is an isomorphism. In particular: $Y \in \text{ind}(A) \Leftrightarrow Y_T \in \text{ind}(B)$.

Proof: Remark that for $Y_1, Y_2 \in \text{mod}(A)$ we have $\theta_{X, Y_1 \oplus Y_2} = \begin{bmatrix} \theta_{X, Y_1} & 0 \\ 0 & \theta_{X, Y_2} \end{bmatrix}$. Thus, $\theta_{X, Y}$ is an isomorphism for any direct summand Y of T' if and only if $\theta_{X, T'}$ is an isomorphism.

By assumption on T' , there exists a path in $\vec{\mathcal{K}}_A$ starting at T' and ending at T . Let us prove by induction on the length l of this path that $\theta_{X, T'}$ is an isomorphism.

If $l = 0$ then $T = T'$. So $\theta_{X, T'} = \theta_{X, T}$ is equal to:

$$\begin{array}{ccc} \text{Hom}_A(X, T) = X_T & \longrightarrow & \text{Hom}_B(T_T, X_T) = \text{Hom}_B(B, X_T) \\ u & \longmapsto & (f \mapsto fu) \end{array}$$

So $\theta_{X, T'}$ is an isomorphism (with inverse $\varphi \mapsto \varphi(1_B)$). This proves the lemma when $l = 0$.

Now assume that $n > 0$ and assume that $\theta_{X, T''}$ is an isomorphism whenever T'' is the source of a path in $\vec{\mathcal{K}}_A$ ending at T and with length equal to $l - 1$. We have a path $T' \rightarrow T'' \rightarrow \dots \rightarrow T$ of length l in $\vec{\mathcal{K}}_A$. Therefore:

$$\theta_{X, Y} \text{ is an isomorphism for any direct summand } Y \text{ of } T'' \quad (i)$$

Moreover, thanks to the arrow $T' \rightarrow T''$ in $\vec{\mathcal{K}}_A$, we have:

- (ii) $T' = \overline{T} \oplus Y'$ with $Y' \in \text{ind}(A)$,
- (iii) $T'' = \overline{T} \oplus Y''$ with $Y'' \in \text{ind}(A)$,
- (iv) a non split exact sequence $0 \rightarrow Y' \rightarrow \tilde{T} \rightarrow Y'' \rightarrow 0$ with $\tilde{T} \in \text{add}(\overline{T})$.

Thanks to (i), (ii) and (iii) we only need to prove that $\theta_{X, Y'}$ is an isomorphism. Remark that by assumption on T' and T'' we have $T \in T^\perp \subseteq T''^\perp$. This implies in particular that $\text{Ext}_A^1(Y'', T) = 0$. Therefore, (iv) yields an exact sequence in $\text{mod}(A)$:

$$0 \rightarrow Y'_T \rightarrow \tilde{T}_T \rightarrow Y''_T \rightarrow 0$$

This gives rise to the exact sequence:

$$0 \rightarrow \text{Hom}_B(Y'_T, X_T) \rightarrow \text{Hom}_B(\tilde{T}_T, X_T) \rightarrow \text{Hom}_B(Y''_T, X_T)$$

On the other hand, (iv) yields the following exact sequence:

$$0 \rightarrow \text{Hom}_A(X, Y') \rightarrow \text{Hom}_A(X, \tilde{T}) \rightarrow \text{Hom}_A(X, Y'')$$

Therefore, we have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(X, Y') & \longrightarrow & \text{Hom}_A(X, \tilde{T}) & \longrightarrow & \text{Hom}_A(X, Y'') \\ & & \theta_{X, Y'} \downarrow & & \theta_{X, \tilde{T}} \downarrow & & \theta_{X, Y''} \downarrow \\ 0 & \longrightarrow & \text{Hom}_A(Y'_T, X_T) & \longrightarrow & \text{Hom}_A(\tilde{T}_T, X_T) & \longrightarrow & \text{Hom}_A(Y''_T, X_T) \end{array}$$

where the rows are exact and where $\theta_{X, \tilde{T}}$ and $\theta_{X, Y''}$ are isomorphisms (because of (i), (iii) and (iv)). This shows that $\theta_{X, Y'}$ is an isomorphism. So $\theta_{X, T'}$ is an isomorphism and the induction is finished. This proves the first assertion of the lemma. The second assertion is due to the functoriality of $\theta_{X, Y}$. \blacksquare

Remark 4.2. 1. Lemma 4.1 holds for any $Y \in \text{add}(T')$.

2. Assume that A is hereditary. Then Lemma 4.1 still holds if one replaces the hypothesis " T' is a predecessor of T " by " $T' \geq T$ " (i.e. $T^\perp \subseteq T'^\perp$). The proof is then a classical application of left- $\text{add}(T)$ approximations (in the above proof, one replaces the exact sequence ε by a coresolution, which necessarily a short exact sequence, of T' in $\text{add}(T)$).

The following proposition is the base of the link between $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$: it explains how to associate suitable tilting B -modules with tilting A -modules.

Proposition 4.3. Let $X \rightarrow Y$ be an arrow in $\vec{\mathcal{K}}_A$ where X and Y are predecessors of T . Then:

$$X_T \in \vec{\mathcal{K}}_B \Leftrightarrow Y_T \in \vec{\mathcal{K}}_A$$

When the two conditions of the above equivalence are satisfied, there is an arrow $Y_T \rightarrow X_T$ in $\vec{\mathcal{K}}_B$.

Proof: Let us assume that $Y_T \in \vec{\mathcal{K}}_B$ and let us show that $X_T \in \vec{\mathcal{K}}_B$ and that there is an arrow $Y_T \rightarrow X_T$ in $\vec{\mathcal{K}}_B$ (the proof of the remaining implication is then obtained by exchanging the roles of X and Y). The arrow $X \rightarrow Y$ in $\vec{\mathcal{K}}_A$ gives the following data:

- . $X = M \oplus \overline{X}$ with $M \in \text{ind}(A)$,
- . $Y = N \oplus \overline{X}$ with $N \in \text{ind}(A)$,
- . $\varepsilon: 0 \rightarrow M \xrightarrow{i} X' \xrightarrow{p} N \rightarrow 0$ is a non split exact sequence in $\text{mod}(A)$ with $X' \in \text{add}(\overline{X})$.

The tilting A -module Y is a predecessor of T . Hence $T \in T^\perp \subseteq Y^\perp$ and therefore $\text{Ext}_A^1(N, T) = 0$. We infer that $\text{Hom}_A(\varepsilon, T)$ gives an exact sequence in $\text{mod}(B)$:

$$0 \rightarrow N_T \xrightarrow{p_T} X'_T \xrightarrow{i_T} M_T \rightarrow 0 \quad (\varepsilon_T)$$

Notice that we also have:

- . $X_T = M_T \oplus \overline{X}_T$,
- . $Y_T = N_T \oplus \overline{X}_T$,
- . $X'_T \in \text{add}(\overline{X}_T)$.

Hence, in order to prove that $X_T \in \vec{\mathcal{K}}_B$ and that there is an arrow $Y_T \rightarrow X_T$ in $\vec{\mathcal{K}}_B$, we only need to prove the following facts:

- 1) ε_T does not split,
- 2) $M_T \in \text{ind}(B)$ and $N_T \in \text{ind}(B)$,
- 3) $\text{pd}_B(X_T) < \infty$,
- 4) X_T is selforthogonal,
- 5) $d(X_T) = n$ and X_T is basic.

1) Let us prove that ε_T does not split. If ε_T splits, then i_T is a retraction:

$$(\exists \lambda \in \text{Hom}_B(M_T, X'_T)) \quad \text{Id}_{M_T} = i_T \circ \lambda$$

Since M is a direct summand of $Y \in \vec{\mathcal{K}}_A$ and since Y is a predecessor of T , Lemma 4.1 implies that $\lambda = \pi_T$ with $\pi \in \text{Hom}_A(X', M)$. Thus we have $(\pi \circ i)_T = (\text{Id}_M)_T$. Using again Lemma 4.1 we deduce that $\pi \circ i = \text{Id}_M$ which is impossible because ε does not split. So ε_T does not split.

2) Lemma 4.1 implies that $M_T, N_T \in \text{ind}(B)$.

3) Since we assumed that $Y_T \in \vec{\mathcal{K}}_B$, we have $\text{pd}_B(\overline{X}_T) < \infty$, $\text{pd}_B(X'_T) < \infty$ and $\text{pd}_B(N_T) < \infty$. Hence ε_T gives $\text{pd}_B(M_T) < \infty$. So $\text{pd}_B(X_T) < \infty$.

4) Let us prove that X_T is selforthogonal. In this purpose, we use the following lemma.

Lemma 4.4. The following morphisms induced by $p_T: N_T \rightarrow X'_T$ are surjective:

$$(p_T)^* \quad \begin{array}{ccc} \text{Hom}_B(X'_T, \overline{X}_T) & \longrightarrow & \text{Hom}_B(N_T, \overline{X}_T) \\ f & \longmapsto & f \circ p_T \end{array}$$

$$(p_T)^* \quad \begin{array}{ccc} \text{Hom}_B(X'_T, M_T) & \longrightarrow & \text{Hom}_B(N_T, M_T) \\ f & \longmapsto & f \circ p_T \end{array}$$

Proof: We only prove that the first morphism is surjective, the second morphism is dealt with the same way after replacing \overline{X} by M . Since $X = M \oplus \overline{X} \in \overrightarrow{\mathcal{K}}_A$, we have $Ext_A^1(\overline{X}, M) = 0$. Hence $Hom_A(\overline{X}, \varepsilon)$ gives rise to a surjective morphism induced by p :

$$\begin{array}{ccc} p_*: & Hom_A(\overline{X}, X') & \twoheadrightarrow Hom_A(\overline{X}, N) \\ & f \longmapsto & p \circ f \end{array}$$

Let us apply Lemma 4.1 and Remark 4.2 to $X' \in add(Y)$ and $N \in add(Y)$. We get the following commutative diagram where vertical arrows are isomorphisms:

$$\begin{array}{ccc} Hom_A(\overline{X}, X') & \xrightarrow{p_*} & Hom_A(\overline{X}, N) \\ \theta_{\overline{X}, X'} \downarrow & & \theta_{\overline{X}, N} \downarrow \\ Hom_B(X'_T, \overline{X}_T) & \xrightarrow{(p_T)^*} & Hom_B(N_T, \overline{X}_T) \end{array}$$

Since p_* is surjective, we infer that so is $(p_T)^*$. So the lemma is proved. \blacksquare

Now we can prove that $X_T = \overline{X}_T \oplus M_T$ is selforthogonal. Since $\overline{X}_T \in add(Y_T)$ and $Y_T \in \overrightarrow{\mathcal{K}}_B$, we get:

$$(\forall i \geq 1) \quad Ext_B^i(\overline{X}_T, \overline{X}_T) = 0 \quad (i)$$

For each $i \geq 1$, $Hom_B(\overline{X}_T, \varepsilon_T)$ gives the following exact sequence:

$$Ext_B^i(\overline{X}_T, X'_T) \rightarrow Ext_B^i(\overline{X}_T, M_T) \rightarrow Ext_B^{i+1}(\overline{X}_T, N_T)$$

Since $\overline{X}_T, X'_T, N_T \in add(Y_T)$ and $Y_T \in \overrightarrow{\mathcal{K}}_B$, we get:

$$(\forall i \geq 1) \quad Ext_B^i(\overline{X}_T, M_T) = 0 \quad (ii)$$

On the other hand, $Hom_B(\varepsilon_T, \overline{X}_T)$ gives the following exact sequences:

$$\begin{aligned} & Hom_B(X'_T, \overline{X}_T) \xrightarrow{(p_T)^*} Hom_B(N_T, \overline{X}_T) \rightarrow Ext_B^1(M_T, \overline{X}_T) \rightarrow Ext_B^1(X'_T, \overline{X}_T) \\ & Ext_B^i(N_T, \overline{X}_T) \rightarrow Ext_B^{i+1}(M_T, \overline{X}_T) \rightarrow Ext_B^{i+1}(X'_T, \overline{X}_T) \text{ for } i \geq 1 \end{aligned}$$

These exact sequences together with Lemma 4.4 and the selforthogonality of Y_T imply that:

$$(\forall i \geq 1) \quad Ext_B^i(M_T, \overline{X}_T) = 0 \quad (iii)$$

In order to get the selforthogonality of $X_T = M_T \oplus \overline{X}_T$ it only remains to prove that M_T is selforthogonal (because of (i), (ii) and (iii)). Notice that $Hom_B(N_T, \varepsilon_T)$ gives the following exact sequence for each $i \geq 1$:

$$Ext_B^i(N_T, X'_T) \rightarrow Ext_B^i(N_T, M_T) \rightarrow Ext_B^{i+1}(N_T, N_T)$$

Using $Y_T \in \overrightarrow{\mathcal{K}}_B$ and $X'_T, N_T \in add(Y_T)$ we deduce that:

$$(\forall i \geq 1) \quad Ext_B^i(N_T, M_T) = 0 \quad (iv)$$

Finally $Hom_B(\varepsilon_T, M_T)$ gives the following exact sequences:

$$\begin{aligned} & Hom_B(X'_T, M_T) \xrightarrow{(p_T)^*} Hom_B(N_T, M_T) \rightarrow Ext_B^1(M_T, M_T) \rightarrow Ext_B^1(X'_T, M_T) \\ & Ext_B^i(N_T, M_T) \rightarrow Ext_B^{i+1}(M_T, M_T) \rightarrow Ext_B^{i+1}(X'_T, M_T) \text{ for } i \geq 1 \end{aligned}$$

These exact sequences together with Lemma 4.4, (ii) and (iv) imply that (recall that $X'_T \in add(\overline{X}_T)$):

$$(\forall i \geq 1) \quad Ext_B^i(M_T, M_T) = 0 \quad (v)$$

From (i), (ii), (iii) and (v) we deduce that $X_T = M_T \oplus \overline{X}_T$ is selforthogonal.

5) To finish, let us prove that X_T is basic and that $d(X_T) = n$. Since $Y_T = N_T \oplus \overline{X}_T \in \overrightarrow{\mathcal{K}}_B$ we infer that \overline{X}_T is basic. Therefore, if $X_T = M_T \oplus \overline{X}_T$ is not basic, then $M_T \in add(\overline{X}_T) \subseteq Y_T^\perp$ which implies that $Ext_B^1(M_T, N_T) = 0$. This last property contradicts the fact that ε_T does not split. This proves that X_T is basic. Moreover, $Y_T \in \overrightarrow{\mathcal{K}}_B$ and $N_T \in ind(B)$ imply that:

$$d(\overline{X}_T) = d(Y_T) - d(N_T) = n - 1$$

Since $M_T \in \text{ind}(B)$ we deduce that:

$$d(X_T) = d(M_T) + d(\overline{X}_T) = n$$

This finishes the proof of the proposition. \blacksquare

Remark 4.5. As in Remark 4.2 concerning Lemma 4.1, when A is hereditary, Proposition 4.3 has the following generalisation: Let $X \in \overrightarrow{\mathcal{K}}_A$ be such that $X \geq T$, then $X_T \in \overrightarrow{\mathcal{K}}_B$. The proof of this generalisation is obtained by replaing the use of the exact sequence ε by a coresolution of X in $\text{add}(T)$.

Proposition 4.3 gives the following proposition which will be used in the comparison of the Galois coverings of A and B . We omit the proof which is immediate using Proposition 4.3.

Proposition 4.6. Let $X \in \overrightarrow{\mathcal{K}}_A$ be such that there exists a path in $\overrightarrow{\mathcal{K}}_A$ starting at X and ending at T . Then $X_T \in \overrightarrow{\mathcal{K}}_B$ and there exists in $\overrightarrow{\mathcal{K}}_B$ a path starting at B and ending at X_T . In particular, X_T and B lie in the same connected component of $\overrightarrow{\mathcal{K}}_B$.

Proposition 4.3 also allows us to prove the main result of this section. Recall that for a quiver Q , we write Q^{op} for the opposite quiver (obtained from Q by reversing the arrows).

Theorem 4.7. Let $\overrightarrow{\mathcal{K}}_A(T)$ (resp. $\overrightarrow{\mathcal{K}}_B(T)$) be the full convex subquiver of $\overrightarrow{\mathcal{K}}_A$ (resp. $\overrightarrow{\mathcal{K}}_B$) generated by $\{A, T\}$ (resp. $\{B, T\}$). Then we have an isomorphism of quivers:

$$\begin{array}{ccc} \alpha: & \overrightarrow{\mathcal{K}}_A(T) & \xrightarrow{\sim} \overrightarrow{\mathcal{K}}_B(T)^{op} \\ & X & \mapsto X_T = \text{Hom}_A(X, T) \end{array}$$

Under this correspondance, $A \in \overrightarrow{\mathcal{K}}_A(T)$ (resp. $T \in \overrightarrow{\mathcal{K}}_A(T)$) is associated with $T \in \overrightarrow{\mathcal{K}}_B(T)$ (resp. $B \in \overrightarrow{\mathcal{K}}_B(T)$).

Proof: Thanks to Proposition 4.3, the mapping α is a well defined morphism of quivers. Thus, it only remains to exhibit an inverse morphism. From [19, Thm 1.5] we know that T is a basic tilting $\text{End}_B(T)$ -module and that we have an isomorphism of k -algebras:

$$\begin{array}{ccc} A & \longrightarrow & \text{End}_B(T) \\ a & \longmapsto & (t \mapsto at) \end{array}$$

Henceforth, we shall consider A -modules as $\text{End}_B(T)$ -modules and vice-versa using the above isomorphism. In particular, we have an identification of quivers:

$$\begin{array}{ccc} \overrightarrow{\mathcal{K}}_A(T) & \xrightarrow{\sim} & \overrightarrow{\mathcal{K}}_{\text{End}_B(T)}(T) \\ X & \mapsto & X \end{array}$$

Therefore, we also have a well defined morphism of quivers:

$$\begin{array}{ccc} \alpha': & \overrightarrow{\mathcal{K}}_B(T)^{op} & \rightarrow \overrightarrow{\mathcal{K}}_A(T) \\ & X & \mapsto X_T = \text{Hom}_B(X, T) \end{array}$$

Let us prove that $\alpha'\alpha$ is an isomorphism. Let $X \in \overrightarrow{\mathcal{K}}_A(T)$. Then X is a predecessor of T . Therefore, Lemma 4.1 implies that:

$$\text{Hom}_B(\text{Hom}_A(X, T), T) \simeq \text{Hom}_B(\text{Hom}_A(X, T), \text{Hom}_A(A, T)) \simeq \text{Hom}_A(A, X) \simeq X$$

This proves that $\alpha'\alpha$ is an isomorphism of quivers. With the same arguments one also shows that $\alpha\alpha'$ is an isomorphism. So does $\alpha: \overrightarrow{\mathcal{K}}_A(T) \rightarrow \overrightarrow{\mathcal{K}}_B(T)^{op}$. \blacksquare

Remark 4.8. Assume that A is hereditary, then Theorem 4.7 has the following generalisation, thanks to Remark 4.5: Let Q_A (resp. Q_B) be the full subquiver of $\overrightarrow{\mathcal{K}}_A$ (resp. $\overrightarrow{\mathcal{K}}_B$) made of the tilting modules $X \geq T$. Then $X \mapsto X_T$ induces an isomorphism of quivers $Q_A \xrightarrow{\sim} Q_B^{op}$.

5 Tilting modules of the first kind w.r.t. infinite Galois coverings

Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G and with \mathcal{C} locally bounded. The aim of this section is to give sufficient conditions for $T \in \vec{\mathcal{K}}_A$ to satisfy the following conditions (compare with Proposition 3.1 in the case G is finite):

- . T is of the first kind w.r.t. F ,
- . for each indecomposable summand N of T , any $\hat{N} \in \text{mod}(\mathcal{C})$ such that $F_\lambda(\hat{N}) \simeq N$ has trivial stabiliser: $G_{\hat{N}} = G$.

We begin with the following proposition.

Proposition 5.1. *Let $T \rightarrow T'$ be an arrow in $\vec{\mathcal{K}}_A$. Then:*

$$T \in \text{mod}_1(A) \Leftrightarrow T' \in \text{mod}_1(A)$$

Proof: Let us assume that $T \in \text{mod}_1(A)$. Since $T \rightarrow T'$ is an arrow in $\vec{\mathcal{K}}_A$, we have the following data:

- . $T = X \oplus \bar{T}$ with $X \in \text{ind}(A)$,
- . $T' = Y \oplus \bar{T}$ with $Y \in \text{ind}(A)$,
- . $\varepsilon: 0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0$ a non split exact sequence in $\text{mod}(A)$ with $M \in \text{add}(\bar{T})$.

Let us write:

- . $M = M_1 \oplus \dots \oplus M_t$ with $M_i \in \text{ind}(A)$ for each i ,
- . $u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}: X \rightarrow M = M_1 \oplus \dots \oplus M_t$,
- . $v = [v_1 \ \dots \ v_t]: M = M_1 \oplus \dots \oplus M_t \rightarrow Y$.

Since $T' = Y \oplus \bar{T}$, since $T = X \oplus \bar{T}$ and since $T \in \text{mod}_1(A)$, we only need to prove that $Y \in \text{mod}_1(A)$ in order to get $T' \in \text{mod}_1(A)$. Without loss of generality, we may assume the following:

- . A is G -graded, $\mathcal{C} = A\sharp G$ and $F: \mathcal{C} \rightarrow A$ is the natural projection,
- . X, M_1, \dots, M_t are G -graded A -modules.

In order to prove that $Y \in \text{mod}_1(A)$, we will need the following lemma.

Lemma 5.2. *Assume that A is G -graded and let:*

$$0 \rightarrow X \xrightarrow{u} M \xrightarrow{v} Y \rightarrow 0 \tag{\varepsilon}$$

be an exact sequence in $\text{mod}(A)$ verifying the following hypotheses:

- . $X \in \text{ind}(A)$ is G -graded and $Y \in \text{ind}(A)$,
- . $M = M_1 \oplus \dots \oplus M_t$ where $M_i \in \text{ind}(A)$ is G -graded for every i ,
- . $\text{Ext}_A^1(Y, M) = 0$.

Then (ε) is isomorphic to an exact sequence in $\text{mod}(A)$:

$$0 \rightarrow X \xrightarrow{\begin{bmatrix} u'_1 \\ \vdots \\ u'_t \end{bmatrix}} M_1 \oplus \dots \oplus M_t \rightarrow Y \rightarrow 0$$

where each $u'_i: X \rightarrow M_i$ is a homogeneous morphism (of some degree).

Proof of Lemma 5.2: Let us write $u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}$ with $u_i: X \rightarrow M_i$ for each i . We may assume that

$u_1: X \rightarrow M_1$ is not homogeneous. Thus:

$$u_1 = h_1 + \dots + h_d$$

where $d \geq 2$ and $h_1, \dots, h_d: X \rightarrow M_1$ are non zero homogeneous morphisms of pairwise different degree (recall that d is uniquely determined by u_1). In order to prove the lemma, it suffices to prove the following property which we denote by (\mathcal{P}) :

“ (ε) is isomorphic to an exact sequence of the form:

$$0 \rightarrow X \xrightarrow{\begin{bmatrix} u'_1 \\ u_2 \\ \vdots \\ u_t \end{bmatrix}} M_1 \bigoplus \dots \bigoplus M_t \rightarrow Y \rightarrow 0 \quad (\varepsilon')$$

where u'_1 is the sum of at most $d-1$ non zero homogeneous morphisms $X \rightarrow M_1$ of pairwise different degree. “

For simplicity we adopt the following notations:

$$\begin{aligned} \cdot \quad \overline{M} &= M_2 \bigoplus \dots \bigoplus M_t \text{ (so } M = M_1 \bigoplus \overline{M}\text{),} \\ \cdot \quad \overline{u} &= \begin{bmatrix} u_2 \\ \vdots \\ u_t \end{bmatrix} : X \rightarrow \overline{M} \text{ (so } u = \begin{bmatrix} u_1 \\ \overline{u} \end{bmatrix} : X \rightarrow M_1 \bigoplus \overline{M}\text{),} \\ \cdot \quad \overline{h} &= h_2 + \dots + h_d : X \rightarrow M_1 \text{ (so } u_1 = h_1 + \overline{h}\text{).} \end{aligned}$$

From $\text{Hom}_A(\varepsilon, M_1)$ we get the exact sequence:

$$\text{Hom}_A(M_1 \bigoplus \overline{M}, M_1) \xrightarrow{u^*} \text{Hom}_A(X, M_1) \rightarrow \text{Ext}_A^1(Y, M_1) = 0$$

So there exists $[\lambda, \mu] : M_1 \bigoplus \overline{M} \rightarrow M_1$ such that $h_1 = [\lambda, \mu]u$. Hence:

$$h_1 = \lambda u_1 + \mu \overline{u} = \lambda h_1 + \lambda \overline{h} + \mu \overline{u} \quad (i)$$

Let us distinguish two cases whether $\lambda \in \text{End}_A(M_1)$ is invertible or nilpotent (recall that $M_1 \in \text{ind}(A)$):

• If λ is invertible then:

$$\theta := \begin{bmatrix} \lambda & \mu \\ 0 & Id_{\overline{M}} \end{bmatrix} : M_1 \bigoplus \overline{M} \rightarrow M_1 \bigoplus \overline{M}$$

is invertible. Using (i) we deduce an isomorphism of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{\begin{bmatrix} u_1 \\ \overline{u} \end{bmatrix}} & M_1 \bigoplus \overline{M} & \longrightarrow & Y \longrightarrow 0 \\ & & \parallel & & \downarrow \theta & & \downarrow \sim \\ 0 & \longrightarrow & X & \xrightarrow{\begin{bmatrix} h_1 \\ \overline{u} \end{bmatrix}} & M_1 \bigoplus \overline{M} & \longrightarrow & Y \longrightarrow 0 \end{array} \quad \begin{array}{l} (\varepsilon) \\ (\varepsilon') \end{array}$$

Since $h_1 : X \rightarrow M_1$ is homogeneous, (ε') fits property (\mathcal{P}) . So (\mathcal{P}) is satisfied in this case.

• If $\lambda \in \text{End}_A(M_1)$ is nilpotent, let $p \geq 0$ be such that $\lambda^p = 0$. Using (i) we get the following equalities:

$$\begin{aligned} h_1 &= \lambda^2 h_1 + (\lambda^2 + \lambda) \overline{h} + (\lambda + Id_{M_1}) \mu \overline{u} \\ \vdots & \quad \vdots \\ h_1 &= \lambda^t h_1 + (\lambda^t + \lambda^{t-1} + \dots + \lambda) \overline{h} + (\lambda^{t-1} + \dots + \lambda + Id_{M_1}) \mu \overline{u} \\ \vdots & \quad \vdots \\ h_1 &= \lambda^p h_1 + (\lambda^p + \lambda^{p-1} + \dots + \lambda) \overline{h} + (\lambda^{p-1} + \dots + \lambda + Id_{M_1}) \mu \overline{u} \end{aligned}$$

Since $\lambda^p = 0$ and $u_1 = h_1 + \overline{h}$ we infer that:

$$u_1 = \lambda' \overline{h} + \lambda' \mu \overline{u}$$

where $\lambda' := Id_{M_1} + \lambda + \dots + \lambda^{p-1} \in \text{End}_A(M_1)$ is invertible. So we have an isomorphism:

$$\theta := \begin{bmatrix} \lambda' & \lambda' \mu \\ 0 & Id_{\overline{M}} \end{bmatrix} : M_1 \bigoplus \overline{M} \rightarrow M_1 \bigoplus \overline{M}$$

and consequently we have an isomorphism of exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{\begin{bmatrix} \bar{h} \\ \bar{u} \end{bmatrix}} & M_1 \oplus \bar{M} & \longrightarrow & Y \longrightarrow 0 & (\varepsilon') \\
& & \downarrow \sim & & \downarrow \theta & & \downarrow \sim & \\
0 & \longrightarrow & X & \xrightarrow{\begin{bmatrix} u_1 \\ \bar{u} \end{bmatrix}} & M_1 \oplus \bar{M} & \longrightarrow & Y \longrightarrow 0 & (\varepsilon)
\end{array}$$

where $\bar{h} = h_2 + \dots + h_p$ is the sum of $p-1$ homogeneous morphisms of pairwise different degrees. So (\mathcal{P}) is satisfied in this case. This finishes the proof of the lemma. \blacksquare

Now we can prove that $Y \in \text{mod}_1(A)$. Thanks to the preceding lemma we know that (ε) is isomorphic to an exact sequence in $\text{mod}(A)$:

$$0 \rightarrow X \xrightarrow{\begin{bmatrix} u'_1 \\ \vdots \\ u'_t \end{bmatrix}} M_1 \bigoplus \dots \bigoplus M_t \rightarrow Y \rightarrow 0 \quad (\varepsilon')$$

where each $u'_i: X \rightarrow M_i$ is homogeneous of degree $g_i \in G$. From Proposition 1.4 we get:

$$(\forall i \in \{1, \dots, n\}) \quad (\exists \tilde{u}'_i \in \text{Hom}_{\mathcal{C}}(\tilde{X}, {}^{g_i^{-1}}\tilde{M}_i)) \quad u'_i = F_\lambda(\tilde{u}'_i)$$

This gives (recall that F_λ is exact):

$$Y \simeq \text{Coker} \left(\begin{bmatrix} u'_1 \\ \vdots \\ u'_t \end{bmatrix} \right) = \text{Coker} \left(\begin{bmatrix} F_\lambda(\tilde{u}'_1) \\ \vdots \\ F_\lambda(\tilde{u}'_t) \end{bmatrix} \right) \simeq F_\lambda \left(\text{Coker} \begin{bmatrix} \tilde{u}'_1 \\ \vdots \\ \tilde{u}'_t \end{bmatrix} \right)$$

This proves that $Y \in \text{mod}_1(A)$. Therefore $T' = Y \oplus \bar{T} \in \text{mod}_1(A)$.

The proof of the implication $T' \in \text{mod}_1(A) \Rightarrow T \in \text{mod}_1(A)$ is identical to the one we have just written except that instead of using Lemma 5.2 we use a dual version:

Lemma 5.3. *Assume that A is G -graded and let:*

$$0 \rightarrow X \rightarrow M \xrightarrow{v} Y \rightarrow 0 \quad (\varepsilon)$$

be an exact sequence in $\text{mod}(A)$ verifying the following hypotheses:

- . $Y \in \text{ind}(A)$ is G -graded and $X \in \text{ind}(A)$,
- . $M = M_1 \oplus \dots \oplus M_t$ where $M_i \in \text{ind}(A)$ is G -graded for every i ,
- . $\text{Ext}_A^1(M, X) = 0$.

Then (ε) is isomorphic to an exact sequence in $\text{mod}(A)$:

$$0 \rightarrow X \rightarrow M_1 \bigoplus \dots \bigoplus M_T \rightarrow Y \xrightarrow{\begin{bmatrix} v'_1 \\ \vdots \\ v'_t \end{bmatrix}} 0$$

where each $v'_i: X \rightarrow M_i$ is a homogeneous morphism (of some degree).

This finishes the proof of Proposition 5.1. \blacksquare

Remark 5.4. *The proof of Proposition 5.1 shows that for an arrow $T \rightarrow T'$ in $\vec{\mathcal{K}}_A$ such that $T, T' \in \text{mod}_1(A)$ there exists an exact sequence in $\text{mod}(\mathcal{C})$:*

$$0 \rightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} Y \rightarrow 0$$

with the following properties:

- . $T = F_\lambda(X) \oplus \overline{T}$ and $F_\lambda(X) \in \text{ind}(A)$,
- . $T' = F_\lambda(Y) \oplus \overline{T}$ and $F_\lambda(Y) \in \text{ind}(A)$,
- . $F_\lambda(M) \in \text{add}(\overline{T})$.

Proposition 5.1 gives immediately the following result.

Proposition 5.5. *Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with \mathcal{C} locally bounded. If $T, T' \in \overrightarrow{\mathcal{K}}_A$ lie in a same connected component of $\overrightarrow{\mathcal{K}}_A$ then:*

$$T \in \text{mod}_1(A) \Leftrightarrow T' \in \text{mod}_1(A)$$

In particular, any $T \in \overrightarrow{\mathcal{K}}_A$ lying in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing A is of the first kind w.r.t. F .

Remark 5.6. *The preceding proposition is similar to part of [12, Thm 3.6] where P. Gabriel proves the following: if $F: \mathcal{C} \rightarrow A$ is a Galois covering with group G , with \mathcal{C} locally bounded and such that G acts freely on $\text{ind}(\mathcal{C})$, then for any connected component C of the Auslander-Reiten quiver of A , all indecomposable modules of C lie in $\text{ind}_1(A)$ as soon as any one of them does.*

Recall that $\overrightarrow{\mathcal{K}}_A$ has a Brauer-Thrall type property (see [15, Cor. 2.2]): $\overrightarrow{\mathcal{K}}_A$ is finite and connected if it has a finite connected component. In particular, $\overrightarrow{\mathcal{K}}_A$ is finite and connected if A is of finite representation type. Using Proposition 5.5, we get the following corollary.

Corollary 5.7. *If $\overrightarrow{\mathcal{K}}_A$ is finite (e.g. A is of finite representation type), then any $T \in \overrightarrow{\mathcal{K}}_A$ is of the first kind w.r.t. F .*

Now we turn to the second goal of this section: for $T \in \text{mod}_1(A)$ a basic tilting A -module, give sufficient conditions for T to verify: if $X \in \text{ind}(A) \cap \text{add}(T)$ and if $\widehat{X} \in \text{ind}(\mathcal{C})$ verifies $F_\lambda(\widehat{X}) \simeq X$ then $G_{\widehat{X}} = 1$. Notice that T verifies this property if and only if $F.T$ is a basic \mathcal{C} -module.

Proposition 5.8. *Let $T, T' \in \overrightarrow{\mathcal{K}}_A \cap \text{mod}_1(A)$ lie in a same connected component of $\overrightarrow{\mathcal{K}}_A$, then:*

$$F.T \text{ is a basic } \mathcal{C}\text{-module} \Leftrightarrow F.T' \text{ is a basic } \mathcal{C}\text{-module}.$$

Proof: Without loss of generality, we may assume that there is an arrow $T \rightarrow T'$ in $\overrightarrow{\mathcal{K}}_A$. Let us assume that $F.T$ is basic and let us prove that so is $F.T'$. We will use Remark 5.4 from which we adopt the notations, in particular, the exact sequence $0 \rightarrow X \xrightarrow{\iota} M \xrightarrow{\pi} Y \rightarrow 0$ in $\text{mod}(\mathcal{C})$ will be denoted by (ε) . Because $F.T$ is basic and because of the properties verified by (ε) , we only need to prove that $G_Y = 1$. Let $\varphi: Y \rightarrow {}^g Y$ be an isomorphism in $\text{mod}(\mathcal{C})$ (with $g \in G$), and let us prove that $g = 1$. To do this we will exhibit an isomorphism $\theta: X \rightarrow {}^g X$. Notice that:

$$(\forall h \in G) \quad \begin{cases} {}^h X, {}^h M \in \text{add}(F.T) \\ {}^h Y, {}^h M \in \text{add}(F.T') \end{cases} \quad (1)$$

Moreover, thanks to $T \in \overrightarrow{\mathcal{K}}_A$ and to $F_\lambda F.T = \bigoplus_{h \in G} T$, we have:

$$(\forall i \geq 1) \quad \text{Ext}_{\mathcal{C}}^i(F.T, F.T) \simeq \text{Ext}_A^i(F_\lambda F.(T), T) \simeq \prod_{h \in G} \text{Ext}_A^i(T, T) = 0$$

In particular:

$$\text{Ext}_{\mathcal{C}}^1({}^g M, X) = \text{Ext}_{\mathcal{C}}^1(M, {}^g X) = 0 \quad (i)$$

With $\text{Hom}_{\mathcal{C}}(M, {}^g \varepsilon)$, this last equality gives the exact sequence:

$$\text{Hom}_{\mathcal{C}}(M, {}^g M) \xrightarrow{({}^g \pi)_*} \text{Hom}_{\mathcal{C}}(M, {}^g Y) \rightarrow \text{Ext}_{\mathcal{C}}^1(M, {}^g X) = 0$$

From this exact sequence, we deduce the existence of $\psi \in \text{Hom}_{\mathcal{C}}(M, {}^g M)$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\pi} & Y \\ \psi \downarrow & & \downarrow \varphi \\ {}^g M & \xrightarrow{{}^g \pi} & {}^g Y \end{array}$$

This implies the existence of $\theta \in \text{Hom}_{\mathcal{C}}(X, {}^g X)$ making commutative the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\ & & \downarrow \theta & & \downarrow \psi & & \downarrow \varphi & & \\ 0 & \longrightarrow & {}^g X & \xrightarrow{{}^g \iota} & {}^g M & \xrightarrow{{}^g \pi} & {}^g Y & \longrightarrow & 0 \end{array} \quad (ii)$$

We claim that $\theta: X \rightarrow {}^g X$ is an isomorphism. The arguments that have been used to get (ii) may be adapted (just replace the use of $\text{Hom}_{\mathcal{C}}(M, {}^g \varepsilon)$ and of $\varphi: Y \rightarrow {}^g Y$ by $\text{Hom}_{\mathcal{C}}({}^g M, \varepsilon)$ and $\varphi^{-1}: {}^g Y \rightarrow Y$) to get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^g X & \xrightarrow{{}^g \iota} & {}^g M & \xrightarrow{{}^g \pi} & {}^g Y & \longrightarrow & 0 \\ & & \downarrow \theta' & & \downarrow \psi' & & \downarrow \varphi^{-1} & & \\ 0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0 \end{array} \quad (iii)$$

In order to show that $\theta: X \rightarrow {}^g X$ is an isomorphism, let us show that $\theta'\theta \in \text{End}_{\mathcal{C}}(X)$ is an isomorphism. Notice (ii) and (iii) give the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0 \\ & & \downarrow \theta'\theta - \text{id}_X & & \downarrow \psi'\psi - \text{id}_M & & \downarrow 0 & & \\ 0 & \longrightarrow & X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \longrightarrow & 0 \end{array} \quad (iv)$$

In particular we have $\pi(\psi'\psi - \text{id}_M) = 0$. Thus, $\psi'\psi - \text{id}_M$ factorises through ι . So there exists $\lambda \in \text{Hom}_{\mathcal{C}}(M, X)$ such that:

$$\psi'\psi - \text{id}_M = \iota\lambda$$

This last property implies that:

$$\iota(\theta'\theta - \text{id}_X) = \iota\lambda\iota$$

and since ι is one-to-one, we get $\theta'\theta - \text{id}_X = \lambda\iota$, i.e.:

$$\theta'\theta = \text{id}_X + \lambda\iota$$

If $\lambda\iota \in \text{End}_{\mathcal{C}}(X)$ was an isomorphism, then $\iota: X \rightarrow M$ would be a section. This would imply that $F_{\lambda}(X)$ is a direct summand of $F_{\lambda}(M)$. This last property is impossible because: $T = F_{\lambda}(X) \oplus \overline{T}$, $F_{\lambda}(M) \in \text{add}(\overline{T})$ and T is basic. This contradiction proves that $\lambda\iota \in \text{End}_{\mathcal{C}}(X)$ is nilpotent. Therefore $\theta'\theta = \text{id}_X + \lambda\iota \in \text{End}_{\mathcal{C}}(X)$ is invertible. As a consequence, $\theta: X \rightarrow {}^g X$ is a section. Since $X, {}^g X \in \text{ind}(\mathcal{C})$, we deduce that $\theta: X \rightarrow {}^g X$ is an isomorphism. Since we assumed that $F.T$ is basic, we get $g = 1$. This finishes the proof of the implication:

$$F.T \text{ is basic} \Rightarrow F.T' \text{ is basic.}$$

under the assumption that $T \rightarrow T'$ is an arrow in $\vec{\mathcal{K}}_A$. After exchanging the roles of T and T' in the above arguments, we also prove that:

$$F.T \text{ is basic} \Leftarrow F.T' \text{ is basic.}$$

under the assumption that $T \rightarrow T'$ is an arrow in $\vec{\mathcal{K}}_A$. This achieves the proof of the proposition. \blacksquare

We end this section with a result which be useful to prove Theorem 1. Recall from Definition 2.7 that to $T \in \text{mod}(A)$ of the first kind w.r.t. F it is possible to associate a Galois covering F_T of $\text{End}_A(T)$. The following proposition gives sufficient conditions on $T \in \text{mod}_1(A) \cap \vec{\mathcal{K}}_A$ for F_T to be a connected Galois covering.

Proposition 5.9. *Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G and with \mathcal{C} locally bounded. Let $T, T' \in \vec{\mathcal{K}}_A$ lie in a same connected component of $\vec{\mathcal{K}}_A$. Assume that $T, T' \in \text{mod}_1(A)$. Then:*

$$F_T \text{ is a connected Galois covering} \Leftrightarrow F_{T'} \text{ is a connected Galois covering.}$$

Proof: Without loss of generality, we may assume that there is an arrow $T \rightarrow T'$ in $\vec{\mathcal{K}}_A$. Let us assume that $F_T: \text{End}_{\mathcal{C}}(F.T) \rightarrow \text{End}_A(T)$ is a connected Galois covering, i.e. $F.T$ is basic \mathcal{C} -module and $\text{End}_{\mathcal{C}}(F.T)$ is connected. Thanks to Proposition 5.8 we already know that $F.T'$ is a basic \mathcal{C} -module. On the other hand, since \mathcal{C} is connected, Lemma 2.8 proves that $\text{End}_{\mathcal{C}}(F.T')$ is connected. \blacksquare

6 Comparison of the Galois coverings of A and $End_A(T)$ for T basic tilting A -module

This section is devoted to the proof of Theorem 1. Let $T \in \vec{\mathcal{K}}_A$ and let $B = End_A(T)$. In Section 3 and Section 5 we have given sufficient conditions for T to be of the first kind w.r.t. a given Galois covering of A . Moreover, when $\mathcal{C} \rightarrow A$ is a Galois covering with group G (and with \mathcal{C} locally bounded) and such that $T \in mod_1(A)$, we have constructed (see Definition 2.7) an isomorphism class of Galois covering $F_T: \mathcal{E}nd_{\mathcal{C}}(F.T) \rightarrow End_A(T)$ with group G . Recall that we are mainly interested in Galois coverings $\mathcal{C}' \rightarrow B$ where \mathcal{C}' is connected and locally bounded. For this reason, we introduce the following property depending on A, T and a fixed group G :

$\mathcal{P}(A, T, G)$ = "For any connected Galois covering $F: \mathcal{C} \rightarrow A$ with group G , we have: T is of the first kind w.r.t. F and F_T is a connected Galois covering"

Notice that $\mathcal{P}(A, A, G)$ is true for any G . The above definition of $\mathcal{P}(A, T, G)$ is relevant because of the following proposition.

Proposition 6.1. *Let G be a group. Assume that $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ are true. Then A and B have the same connected Galois coverings with group G .*

Proof: Let us denote by $Gal_G(A)$ (resp. $Gal_G(B)$) the set of isomorphism classes of connected Galois coverings of A (resp. of B) with group G . So we need to exhibit a bijective map $Gal_G(A) \rightarrow Gal_G(B)$. For simplicity we shall consider F_T both as a Galois covering and as an isomorphism class.

From Definition 2.7 and because $\mathcal{P}(A, T, G)$ is true, we have a well defined mapping:

$$\begin{array}{ccc} \varphi: & Gal_G(A) & \longrightarrow Gal_G(B) \\ & F & \longmapsto F_T \end{array} \quad (i)$$

Similarly, $\mathcal{P}(B, T, G)$ is true so we have a well defined mapping:

$$\begin{array}{ccc} & Gal_G(B) & \longrightarrow Gal_G(End_B(T)) \\ & F & \longmapsto F_T \end{array} \quad (ii)$$

Recall from [19, Thm. 1.5] that we have an isomorphism of k -categories:

$$\begin{array}{ccc} \rho: & A & \longrightarrow End_B(T) \\ x \in Ob(A) & \longmapsto & T(x) \in Ob(End_B(T)) \\ a \in {}_y A_x & \longmapsto & (t \mapsto at) \in {}_{T(y)} End_B(T)_{T(x)} \end{array}$$

So (ii) gives the well defined mapping:

$$\begin{array}{ccc} \psi: & Gal_G(B) & \longrightarrow Gal_G(A) \\ & F & \longmapsto \rho^{-1} \circ F_T \end{array} \quad (iii)$$

From Lemma 2.8 we know that $\psi\varphi = Id_{Gal_G(A)}$. Similarly, $\varphi\psi = Id_{Gal_G(B)}$ (the situation is symmetrical between A and B because $B = End_A(T)$ and because of the isomorphism $\rho: A \rightarrow End_B(T)$). This proves that $\varphi: Gal_G(A) \rightarrow Gal_G(B)$ is a bijection. ■

Thanks to Proposition 6.1 we are reduced to find sufficient conditions for $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ to be simultaneously true. Proposition 3.1 and Proposition 3.6 give immediately the following sufficient condition.

Proposition 6.2. *Let G be a finite group. Then $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ are true.*

When dealing with an infinite group G , we need more steps in order to know if $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ are true. The following proposition is a direct consequence of: Proposition 5.5, Proposition 5.9 and the fact that $\mathcal{P}(A, A, G)$ is true.

Proposition 6.3. *Let G be a group. Let $T' \in \vec{\mathcal{K}}_A$ lying in the connected component of $\vec{\mathcal{K}}_A$ containing T . Then:*

$$\mathcal{P}(A, T, G) \text{ is true} \Leftrightarrow \mathcal{P}(A, T', G) \text{ is true}$$

In particular, if T lies in the connected component of $\vec{\mathcal{K}}_A$ containing A then $\mathcal{P}(A, T, G)$ is true.

Thanks to Proposition 6.3, we may look for conditions for T to lie in both connected components of $\vec{\mathcal{K}}_A$ and $\vec{\mathcal{K}}_B$ containing A and B respectively. Such a condition is given by the following proposition.

Proposition 6.4. *Let G be a group and assume that there exists a path in $\vec{\mathcal{K}}_A$ starting at A and ending at T . Then T lies in the connected component of $\vec{\mathcal{K}}_A$ (resp. $\vec{\mathcal{K}}_B$) containing A (resp. B). Consequently, $\mathcal{P}(A, T, G)$ and $\mathcal{P}(B, T, G)$ are true.*

Proof: Theorem 4.7 implies that there exists a path in $\vec{\mathcal{K}}_B$ starting at $\text{Hom}_A(T, T) = B$ and ending at $\text{Hom}_A(A, T) = T$. Using Proposition 6.3 we get the desired conclusion. ■

Now we can prove Theorem 1:

Proof of Theorem 1: 1) is a consequence of Proposition 6.1 and Proposition 6.2.

2) We may assume that there exists a path in $\vec{\mathcal{K}}_A$ starting at T' and ending at T . Using Lemma 4.1 and Proposition 4.6 we infer that:

- (i) $\text{End}_A(T')$ and $\text{End}_B(\text{Hom}_A(T', T))$ are isomorphic as k -algebras,
- (ii) there exists a path in $\vec{\mathcal{K}}_B$ starting at $\text{Hom}_A(T, T) = B$ and ending at $\text{Hom}_A(T', T)$.

This implies (thanks to Proposition 6.4 and to Proposition 6.1) that $\text{End}_A(T')$ and $\text{End}_A(T) = B$ have the same connected Galois coverings with group G .

3) is a consequence of 2), of the fact that $\text{End}_A(A) \simeq A^{op}$ and of the fact that A and A^{op} have the same Galois coverings ($F: \mathcal{C} \rightarrow A$ is a Galois covering if and only if $F^{op}: \mathcal{C}^{op} \rightarrow A$ is a Galois covering and \mathcal{C}^{op} is connected and locally bounded if and only if \mathcal{C}^{op} is). ■

Now we can use Theorem 1 to prove Corollary 1.

Proof of Corollary 1: 1) and 2) are consequences of Theorem 1 and of the fact that A is simply connected if and only if it has no proper connected Galois covering (see [17, Cor. 4]).

3) is a consequence of 2).

4) If $\text{End}_A(T)$ is simply connected, then $A = kQ$ has no proper connected Galois covering with finite group (see Theorem 1). On the other hand, A admits a connected Galois covering with group $\pi_1(Q)$ which is a free group (see [18]). So A admits connected Galois coverings with group arbitrary factor groups of $\pi_1(Q)$. Note that if $\pi_1(Q) \neq 1$, then $\pi_1(Q)$ admits non trivial finite factor groups. Therefore, we necessarily have $\pi_1(Q) = 1$ (i.e. A is simply connected). ■

Final remark

The Hasse diagram $\vec{\mathcal{K}}_A$ of basic tilting A -modules describes the combinatoric relations between tilting modules. When A is hereditary (i.e. $A = kQ$ with Q a finite quiver with no oriented cycle) these combinatorics are also described by the cluster category \mathcal{C}_Q of the quiver Q (see [10]). In particular, the underlying graph of $\vec{\mathcal{K}}_A$ is embedded into the graph of tilting objects in \mathcal{C}_Q . Since the latter is always connected (see [10, 3.5]) it is natural to ask if it is possible to remove all conditions concerning connected components in Theorem 1 and Corollary 1 (in the hereditary case). As an example, it is indeed possible to adapt the proof of Proposition 5.1 to the framework of the cluster category. Therefore, using the connectedness of the graph of tilting objects in \mathcal{C}_Q one can prove that any tilting kQ -module is of the first kind w.r.t. any connected Galois covering of kQ . These developpements will be detailed in a forecoming version of the present text.

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