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Generalization of the finite difference method in distributions spaces

Stéphane Labbé and Emmanuel Trélat*

Abstract

The aim of this article is to propose a generalization of the finite difference scheme suitable with solutions of Dirac distribution type. This type of solution is for example encountered in earthquake or explosion simulations. In such problems, the difficulty is to catch sharply a moving singular front modeled by a Dirac type distribution. We give a general framework to deal with numerical methods, and use it to build finite difference methods in distribution spaces. Numerical examples are provided for a one-dimensional wave equation.

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1 Introduction

In this article, we propose a generalization of the classical finite difference method suitable with the approximation of solutions in subspaces of the Sobolev spaces H^{-s} , $s > 0$. We implement the proposed method for the one-dimensional wave equation with nonregular initial data of Dirac type. A concrete situation where such a method happens to be relevant is the simulation of an earthquake or an explosion front in which an accurate approximation of the singular front of the solution is required. These objects, from a mathematical point of view, are modeled by Dirac type distributions.

In the classical finite difference method, the Dirac type distributions are usually approximated by smooth functions. This type of approximation, when injected in a temporal processus, becomes swiftly incorrect, due to the fact that the scheme is built on a regular approximation of the solution. Then, the high frequencies are ill-estimated and tend to disperse. The idea is to build a scheme dealing specifically with Dirac type distributions and their derivatives.

To illustrate the problem, consider the one-dimensional boundary value problem

$$\begin{aligned} u''(x) + c(x)u(x) &= f(x), \quad x \in (0, 1), \\ u(0) &= u(1) = 0, \end{aligned} \tag{1}$$

where c and f are continuous functions on $[0, 1]$. The classical finite difference scheme is the following. Let N be a positive integer, $h = 1/N$, and $x_i = ih$, $i = 0, \dots, N$, be discretization points on $[0, 1]$. Note that, if the function u is of class C^2 on $(0, 1)$, then

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2),$$

for every $x \in (0, 1)$. Hence, in order to solve numerically the problem (1), we are naturally led to the numerical finite difference scheme

$$\begin{aligned} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + c(x_i)u_i &= f(x_i), \quad i = 1, \dots, N-1, \\ u_0 &= u_N = 0. \end{aligned}$$

In order to ensure a convergence property of the classical finite difference method, a strong regularity of the solution is usually assumed. For instance, recall that, if $c(x) \geq 0$, for every $x \in [0, 1]$, and if the solution u of (1) is of class C^2 on $(0, 1)$, then there exists a positive real number C , independent on N (and h), such that

$$\max_{0 \leq i \leq N} |u(x_i) - u_i| \leq Ch^2.$$

In this paper, an analysis of this discretization problem suitable for nonregular solutions is achieved, leading to a numerical scheme whose convergence is proved.

The structure of this article is the following.

In Section ??, we recall the classical finite difference method. We then introduce in Section ?? the concept of *discretization process*, and provide a unified mathematical framework for projective discretization methods, ensuring convergence of the methods. We investigate the 1-D wave equation, first recovering the usual finite difference method, and then extending this method to distribution spaces. Finally, in Section ??, the method is implemented and simulations are provided for nonregular initial conditions of Dirac type.

2 Discretization processes: a projective approach

2.1 Generalities

Let W and V be separable Banach spaces, and $\mathcal{A} : W \rightarrow V$ be a bounded linear operator. For $f \in V$, we consider the problem of determining $u \in W$ so that

$$\mathcal{A}u = f. \quad (2)$$

In this section, we introduce an abstract framework in order to define rigorously a discretization process of problem (2), so as to obtain a numerical scheme of the form

$$\mathcal{A}_h u_h = f_h, \quad (3)$$

where $f_h \in V_h$, where $\mathcal{A}_h : W_h \rightarrow V_h$ is a bounded linear operator representing a discretization of the operator \mathcal{A} , in a sense to be made precise next, and where W_h and V_h are suitable vector spaces approximations of W and V . The discretization parameter h is chosen in a given nonempty open subset H of $(0, +\infty)^p$ such that $0 \in \overline{H}$, with p integer (for instance, $p = 2$ for the finite difference method on a time dependent problem on a space interval).

Definition 2.1. A *discretization process* is a triple $\mathcal{D} = (W^*, (W_h, P_h)_{h \in H}, W)$, where:

- W^* and W are separable Banach spaces such that W is a dense subset of W^* ;
- W_h is a vector subspace of W^* , for every $h \in H$;
- $P_h : W^* \rightarrow W_h$ is a projection operator;
- if P_h^* denotes the canonical injection from W_h in W^* , then

$$\lim_{h \rightarrow 0} \|P_h^* \circ P_h u - u\|_{W^*} = 0,$$

for every $u \in W$;

- the norms of the operators P_h and P_h^* (with respect to the norms W^* and W_h) are bounded, uniformly with respect to h .

Remark 2.1. By definition, $P_h \circ P_h^* = \text{id}_{W_h}$ and $P_h \circ P_h = P_h$.

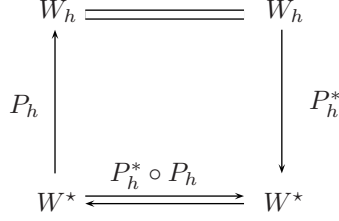


Figure 1: Commutative diagram

The commutative diagram of Figure 2.1, for every $h \in H$, illustrates the definition.

Let $(W^*, (W_h, P_h)_{h \in H}, W)$ and $(V^*, (V_h, Q_h)_{h \in H}, V)$ be two discretization processes. For every $h \in H$, set

$$\mathcal{A}_h = Q_h \circ \mathcal{A}^* \circ P_h^*,$$

where $\mathcal{A}^* : W^* \rightarrow V^*$ is a linear operator extending \mathcal{A} . Set moreover $f_h = Q_h f$. We obtain in this way a discretized version of problem (2), of the form (3).

An important property is the *well posedness* of the discretized problem (3). A scheme is said to be *well posed* if (3) admits a unique solution, for every h . A continuous problem will be said to be well posed if \mathcal{A} is bijective.

The following result is obvious.

Lemma 2.1. *Assume that $\ker Q_h \cap (\mathcal{A}^* \circ P_h^*(W_h)) = \{0\}$, for every $h \in H$. If the extended operator \mathcal{A}^* is surjective, then the discrete problem (3) has a unique solution.*

Note that the well posedness of the continuous problem is not sufficient in general to ensure the well posedness of the discretized problem.

We next recall the definition of a consistent, stable, and convergent scheme.

Definition 2.2. The numerical scheme (3) is said to be *consistent* with (2) if

$$\lim_{h \rightarrow 0} \|Q_h^* \circ \mathcal{A}_h \circ P_h u - \mathcal{A} u\|_{V^*} = 0,$$

where u is the solution of (2).

Definition 2.3. The numerical scheme (3) is said to be *stable* if there exist a connected open subset H_0 of H , with $0 \in \overline{H_0}$, and positive constants C and ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0)$, every $h \in H_0$, and every $e_h \in V_h$ such that $\|e_h\|_{V_h} < \varepsilon$, for all $u_h, \tilde{u}_h \in W_h$ satisfying

$$\begin{aligned} \mathcal{A}_h u_h &= Q_h f, \\ \mathcal{A}_h \tilde{u}_h &= Q_h f + e_h, \end{aligned}$$

there holds

$$\|u_h - \tilde{u}_h\|_{W_h} \leq C\varepsilon.$$

Assume that the problem (2) is well posed.

Definition 2.4. The numerical scheme (3) is said to be *convergent* if there exists a connected open subset H_0 of H , with $0 \in \overline{H_0}$, such that the problem (3) is well posed, for every $h \in H_0$, and

$$\lim_{h \in H_0, h \rightarrow 0} \|u_h - u\|_{W^*} = 0,$$

where u_h is the solution of (3).

If there exist $p > 0$ and $C > 0$, depending only on u , such that there holds moreover

$$\|u_h - u\|_{W^*} \leq C |h|^p,$$

for every $h \in H_0$, then the numerical scheme (3) is said to be *of order p* .

As usually, the consistency and stability properties imply the convergence property, according to the following theorem.

Theorem 2.1. Assume that $\ker(Q_h) \cap (\mathcal{A}^* \circ P_h^*(W_h)) = \{0\}$, for every $h \in H$. If the numerical scheme (3) is consistent with (2) and stable, then it is convergent.

Proof. For every $h \in H$, set

$$e_h = Q_h(Q_h^* \circ \mathcal{A}_h \circ P_h u - \mathcal{A}u),$$

where u is the solution of (2). From the consistency property, there exists $C > 0$ such that, for every $\varepsilon > 0$, there exists $r > 0$ such that, for every $h \in H$ with $|h| < r$, we have

$$\|e_h\|_{V_h} \leq C\varepsilon,$$

where C is a positive real bounding $\|P_h\|$ for every $h \in H$. By definition of e_h , one has

$$\mathcal{A}_h(P_h(u)) = e_h + Q_h f.$$

The stability property implies that, for every h in H_0 ,

$$\|\mathcal{A}_h u - u_h\|_{W_h} \leq C C_1 \varepsilon,$$

where C_1 is the constant introduced in the Definition 2.3, and u_h is the solution of (3). The hypothesis on the kernel of P_h implies, using Lemma (2.1), that u_h is unique. Then, using the properties of the operator $P_h^* \circ P_h$, there holds

$$\begin{aligned} \|P_h^* u_h - u\|_{W^*} &= \|P_h^* u_h - u + P_h^* \circ P_h(u) - P_h^* \circ P_h(u)\|_{W^*} \\ &\leq \|P_h^* \circ P_h(u) - u\|_{W^*} + \|P_h^*(u_h - P_h(u))\|_{W^*} \\ &\leq \|P_h^* \circ P_h(u) - u\|_{W^*} + \|P_h^*\| \|u_h - P_h(u)\|_{W_h^*} \\ &\leq \|P_h^* \circ P_h(u) - u\|_{W^*} + \|P_h^*\| C C_1 \varepsilon. \end{aligned}$$

The term $\|P_h^* \circ P_h(u) - u\|_{W^*}$ converges to zero by hypothesis, and this concludes the proof. \square

Theorem 2.2. *If $\mathcal{A}^* : W^* \rightarrow V^*$ is boundedly invertible, then the numerical scheme (3) built on (2) converges. Let u be the solution of (2), if there exist positive constants C_u and C'_u , and p in \mathbb{N} depending of u , such that*

$$\|Q_h^* \circ \mathcal{A}_h \circ P_h^* u - \mathcal{A}u\|_{V^*} \leq C_u |h|^p,$$

and

$$\|P_h^* \circ P_h u - u\|_{W^*} \leq C'_u |h|^p,$$

for every $h \in H_0$, then the scheme is of order p .

Proof. We first prove that the numerical scheme is consistent. Let u be the solution of (2). There holds

$$\begin{aligned} & \|Q_h^* Q_h \mathcal{A}^* P^* P u - \mathcal{A} u\|_{V^*} \\ &= \|\mathcal{A}^* P_h^* P_h u + (Q_h^* Q_h - id_{V^*}) \mathcal{A}^* P_h^* P_h u - \mathcal{A} u\|_{V^*} \\ &= \|\mathcal{A}^* u + \mathcal{A} u + \mathcal{A}^* (P_h^* P_h - id_{W^*}) u - (Q_h^* Q_h - id_{V^*}) \mathcal{A}^* P_h^* P_h u\|_{V^*} \\ &\leq \|\mathcal{A}^* (P_h^* P_h - id_{W^*}) u\|_{V^*} + \|(Q_h^* Q_h - id_{V^*}) \mathcal{A}^* P_h^* P_h u\|_{V^*} \\ &\leq \|\mathcal{A}^*\|_{L(W^*, V^*)} \|(P_h^* P_h - id_{W^*}) u\|_{W^*} + \|(Q_h^* Q_h - id_{V^*}) \mathcal{A}^*\|_{V^*} \\ &\quad + \|(Q_h^* Q_h - id_{V^*}) \mathcal{A}^* (P_h^* P_h u - u)\|_{V^*}. \end{aligned}$$

Then, from definition 2.1, it is immediate that the discretization process is consistent.

We next prove that the discretization process is stable. Using the fact that V is dense in V^* and that the union of all $Q_h^* V_h$ for $h \in H_0$ contains V we deduce that there exist $r, C > 0$ such that, for every $h \in H$ such that $|h| < r$ and for every e in V_h

$$\|\mathcal{A}^* P_{h_n}^* (u_h - u_{h,e})\|_{W^*} \leq C \|e\|_{V_h},$$

where $u_{h,e}$ is the solution of the Equation (3) when f_h is perturbed by e . Then using the fact \mathcal{A}^* is boundedly invertible, there exists $C^* > 0$ such that

$$C^* \|u_h - u_{h,e}\|_{W_h} \leq \|\mathcal{A}^* P_{h_n}^* (u_h - u_{h,e})\|_{W^*} \leq C \|e\|_{V_h}.$$

Hence the discretization process is stable, and using Theorem 2.1, convergent.

We next prove the second part of the theorem. Let u solution of (2) and u_h solution of (3). One has

$$\mathcal{A}_h u_h = Q_h f,$$

and

$$\mathcal{A}u = f.$$

Hence

$$\mathcal{A}_h P_h u = Q_h f + Q_h (Q_h^* \mathcal{A}_h P_h^* u - \mathcal{A}u)$$

Set $e_h = Q_h^* \mathcal{A}_h P_h^* u - \mathcal{A}u$. By hypothesis, there holds

$$\|e_h\|_{V^*} \leq C_u |h|^p.$$

The stability property implies that there exists $C_1 > 0$ such that, for every $h \in H_0$,

$$\|u_h - P_h u\|_{V^*} \leq C_1 C_u |h|^p.$$

Then,

$$\begin{aligned} \|P^* u_h - u\|_{V^*} &= \|P^* u_h - u + P_h^* \circ P_h u - P_h^* \circ P_h u\|_{V^*} \\ &\leq \|P_h^* \circ P_h u - u\|_{V^*} + \|P_h^* (u_h - P_h u)\|_{V^*} \\ &\leq C'_u |h|^p + \|P_h^*\|_{L(V_h, V^*)} C_1 C_u |h|^p, \end{aligned}$$

and using Definition 2.1 we conclude. \square

Assuming \mathcal{A}^* is boundedly invertible is quite stringent. For example, it holds for linear operators of compact inverse like the Laplacian operator provided the spaces are well chosen. However, this fact cannot be assumed in general. In the case of the finite difference scheme, in order to deal with pointwise values of functions, we need to work in a C^p space. For example, the Laplacian operator sends $W_1 = C^2([0, 1])$ in $W_2 = C^0([0, 1])$. It is then natural to define $W_1^* = W_2^* = L^2([0, 1])$, and thus, the definition of the extension \mathcal{A}^* becomes problematic. A solution is to build $W_{1,h}$ and $W_{2,h}$ so that it is possible to set $W_1^* = W_1$ and $W_2^* = W_2$. Then the extension of the operator \mathcal{A} raises no problem and the convergence theorem is applicable.

In what follows, we describe a finite difference discretization of the wave equation, first on $H^2(I) \times H^1(I)$, and in this case we recover the classical framework of that method, and then on $(H^s(I) \cap H_0^1(I))' \times (H^{s+1}(I) \cap H_0^1(I))'$, $s > 0$.

2.2 The case of evolution problems

It may be convenient to discretize partially a problem. Typically, for evolution problems, one may discretize the spatial variable only. In this section we focus on such evolution problems. We investigate semidiscretization processes, and total discretization processes.

Let T be a positive real number, W^0 and V^0 be separable Banach spaces, and $\mathcal{P} : W^0 \rightarrow V^0$ be a continuous linear operator. For $f \in V^0$, we consider the evolution problem

$$\partial_t u = \mathcal{P} u + f. \quad (4)$$

It is a particular case of the latter section, with $W = C^1(0, T; W^0)$, $V = C^0(0, T; V^0)$, and $\mathcal{A} = \partial_t - \mathcal{P}$. The operator \mathcal{P} extends canonically to $\mathcal{P} : W \rightarrow V$.

Let $(W^{0*}, (W_h^0, P_h)_{h \in H}, W^0)$ and $(V^{0*}, (V_h^0, Q_h)_{h \in H}, V^0)$ be two discretization processes. Set $W^* = C^1(0, T; W^{0*})$, and $V^* = C^0(0, T; V^{0*})$. For every $h \in H$, set $\mathcal{P}_h = Q_h \circ \mathcal{P}^* \circ P_h^*$, where $\mathcal{P}^* : W^{0*} \rightarrow V^{0*}$ is a linear operator extending \mathcal{P} .

As previously, we consider the following canonical extensions:

$$\begin{aligned}\mathcal{P}^* &: W^* \rightarrow V^*, \\ P_h &: W^* \rightarrow C^1(0, T; W_h^0), \\ P_h &: C^0(0, T, W^{0*}) \rightarrow C^0(0, T; W_h^0), \\ Q_h &: V^* \rightarrow C^0(0, T, V_h^0), \\ \mathcal{P}_h &: C^1(0, T; W_h^0) \rightarrow C^0(0, T; V_h^0).\end{aligned}$$

We make the following assumption.

Assumption. We assume that $W^{0*} \subset V^{0*}$, and that $\partial_t W^* \subset V^*$.

For the methods investigated in this paper, this assumption will be verified.

Remark 2.2. The assumption $\partial_t W^* \subset V^*$ is necessary because the operator ∂_t may act as a spatial derivative, for instance in the case of the wave equation.

2.2.1 Semidiscretization processes

The aim of this paragraph is to set a precise framework in order to deal with spatial semidiscretization processes.

Set $f_h = Q_h f$. Then, seeking an approximation solution of (6) amounts to seeking $u_h \in C^1(0, T; W_h^0)$ so that

$$Q_h \partial_t P_h^* u_h = \mathcal{P}_h u_h + f_h. \quad (5)$$

Let $\iota : W^{0*} \rightarrow V^{0*}$ denote the canonical injection. It extends to $\iota : W^* \rightarrow V^*$. Then, for every $h \in H$, the operators ∂_t and P_h^* commute, i.e.,

$$\partial_t P_h^* = \iota P_h^* \partial_t$$

(see the diagram of Figure 2).

$$\begin{array}{ccccc} C^1(0, T; W^{0*}) & \xrightarrow{\partial_t} & C^0(0, T; V^{0*}) & & \\ \downarrow P_h & & \uparrow \iota & & \\ C^1(0, T; W_h^0) & \xrightarrow{\partial_t} & C^0(0, T; W_h^0) & \xrightleftharpoons[P_h]{P_h^*} & C^0(0, T; W^{0*}) \end{array}$$

Figure 2: Diagram

From (5), one gets

$$Q_h \iota P_h^* \partial_t u_h = \mathcal{P}_h u_h + f_h.$$

Define

$$\mathcal{M}_h = Q_h \iota P_h^*$$

as a *filtering operator*. We obtain a semidiscretization scheme of the form

$$\mathcal{M}_h \partial_t u_h = \mathcal{P}_h u_h + f_h, \quad (6)$$

called *semidiscretization process*.

Theorem 2.2, applied to this particular case, yields the following result.

Corollary 2.1. Given two discretization processes $(W^*, (W_h, P_h)_{h>0}, W)$ and $(V^*, (V_h, Q_h)_{h>0}, V)$ and A_h, f_h defined as above, under the assumption that the infinitesimal generator \mathcal{P}^* is a bounded linear operator on W^* , the scheme (4) built on (6) converges.

Moreover, if u denotes the solution of (2), and if there exist $C_u, C'_u \geq 0$, and $p \in \mathbb{N}$ such that

$$\|Q_h^* \circ A_h \circ P_h^* u - Au\|_{V^*} \leq C_u |h|^p,$$

and

$$\|P_h^* \circ P_h u - u\|_{W^*} \leq C'_u |h|^p,$$

for every $h \in H$, then the scheme is of order p .

Proof. This yields the existence of a nonempty open subset of H for which the scheme is stable. Furthermore, the continuity and uniform boundedness properties of operators P_h, P_h^*, Q_h and Q_h^* ensure the consistency. Theorem 2.2 implies to the convergence of the scheme.

The second part of the corollary is deduced from the boundedness of the infinitesimal generator \mathcal{P} . \square

2.2.2 Total discretization processes

A natural question is the following: is it equivalent to discretize first the spatial variable, and then the time variable, or to discretize first the time variable, and then the spatial variable?

The answer is actually positive if we deal with discretization processes, as defined previously.

Let us prove this fact sharply, and without going into details. To make short, the index h (resp. the index k) denotes a discretization with respect to the spatial variable (resp. the time variable). Then, proving the fact amounts to proving that

$$Q_k \circ Q_h \circ \mathcal{A}^* \circ P_h^* \circ P_k^* = Q_h \circ Q_k \circ \mathcal{A}^* \circ P_k^* \circ P_h^*.$$

On the one hand, it is clear that

$$P_h^* \circ P_k^* = P_k^* \circ P_h^*.$$

On the other hand, since Q_h and Q_k are projection operators, and noticing that, up to canonical injections, $Q_h(\text{Im } Q_k) \subset (\text{Im } Q_k)$ and $Q_k(\text{Im } Q_h) \subset (\text{Im } Q_h)$, one gets

$$Q_k \circ Q_h = Q_h \circ Q_k,$$

and the conclusion follows.

This important fact validates our approach by semidiscretization. Indeed, to make numerical simulations on a semidiscretized model, it suffices to choose a time discretization process whose order is greater than the order of the space semidiscretization process. Theorem 2.2 is used to prove the convergence of the method. Let set

$$\begin{aligned} V_{k,h}^0 &= V_{h,k}^0 = Q_k(C^0(0, T; V_h^0)), \\ W_{k,h}^0 &= P_k(C^0(0, T; W_h^0)), \\ W_{h,k}^0 &= P_h(P_k(C^0(0, T; W^*))), \end{aligned}$$

then, one can announce the following corollary

Corollary 2.2. Given the discretization processes $(C^0(0, T; W^*), (W_{h,k}, P_h P_k)_{h>0, k>0}, C^1(0, T; W))$ and $(C^0(0, T; V^*), (V_{h,k}, Q_h Q_k)_{h>0, k>0}, C^1(0, T; V))$ and

$$\mathcal{A}_{h,k} = Q_h \circ Q_k \circ \mathcal{A}^* \circ P_k^* \circ P_h^*,$$

3 Discretization of the 1-D wave equation in $H^2(I) \times H^1(I)$

In this section, we consider the 1-D wave equation with regular data, and show how to recover the classical finite difference scheme within the abstract framework introduced previously.

3.1 Theoretical framework

Consider the Cauchy-Dirichlet problem for the one-dimensional wave equation on $[0, T] \times I$, where $I = [a, b]$,

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(0, \cdot) &= u_0(\cdot), \quad u_t(0, \cdot) = u_1(\cdot), \\ u(t, a) &= g_1, \quad u(t, b) = g_2, \end{aligned} \tag{7}$$

where $u_0 \in H^2(I)$, $u_1 \in H^1(I)$, g_1 and g_2 are real numbers. It is a standard fact that this problem has a unique (weak) solution $u \in C^0(0, T; H^2(I)) \cap C^1(0, T; H^1(I)) \cap C^2(0, T; L^2(I))$.

In order to point out the regularity difference between u and u_t , it is relevant to write (7) in the form

$$\begin{aligned} u_t &= v, \quad v_t = u_{xx}, \\ u(0, \cdot) &= u_0(\cdot), \quad v(0, \cdot) = v_0(\cdot), \\ u(t, a) &= g_1, \quad u(t, b) = g_2, \end{aligned} \tag{8}$$

Let γ_1 (resp. γ_2) denote the left (resp. right) trace operator on $H^1(I)$, and let σ_0 denote the trace operator on $C^0(0, T)$, that is, $\gamma_1 u = u(a)$, $\gamma_2 u = u(b)$, for every $u \in H^1(I)$, and $\sigma_0 u = u(0)$, for every $u \in C^0(0, T)$.

Set

$$W = (C^0(0, T; H^2(I)) \cap C^1(0, T; H^1(I)) \cap C^2(0, T; L^2(I))) \times C^0(0, T; H^1(I)) \cap C^1(0, T; L^2(I)), \quad (9)$$

and

$$V = \mathbb{R} \times (C^0(0, T; H^1(I)) \cap C^1(0, T; L^2(I))) \times C^0(0, T; L^2(I)) \times \mathbb{R} \times H^2(I) \times H^1(I). \quad (10)$$

Define the operator $\mathcal{A} : W \rightarrow V$ by

$$\mathcal{A} = \begin{pmatrix} \gamma_1 & 0 \\ \partial_t & -1 \\ -\partial_{xx} & \partial_t \\ \gamma_2 & 0 \\ \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}.$$

Then, the previous problem is equivalent to determining $U \in W$ so that

$$\mathcal{A}U = \begin{pmatrix} g_1 \\ 0 \\ 0 \\ g_2 \\ u_0 \\ v_0 \end{pmatrix}. \quad (11)$$

In what follows, set

$$W^* = C^0(0, T; L^2(I)) \times C^0(0, T; L^2(I)),$$

and

$$V^* = \mathbb{R} \times C^0(0, T; L^2(I)) \times C^0(0, T; L^2(I)) \times \mathbb{R} \times L^2(I) \times L^2(I).$$

In order to write a finite difference approximation of this operator on $L^2(I)$, we next make precise the spatial and time discretizations.

3.2 Spatial discretization

Let N be a positive integer, $h = (b - a)/N$, and $x_i = a + ih$, $i = 0, \dots, N$, be discretization points on $[a, b]$. For every $i \in \{0, \dots, N\}$, set $\omega_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, where $x_{-\frac{1}{2}} = x_0$, $x_{N+\frac{1}{2}} = x_N$, and $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$, for $i \in \{0, \dots, N-1\}$. Let χ_{ω_i} denote the characteristic function of the interval ω_i .

Let $\widetilde{W}_{0,h}$ denote the set of functions on I whose restriction to each subinterval ω_i , $i = 0, \dots, N$, is polynomial with degree less than or equal to two.

For every positive integer m , every strictly ordered vector $X \in \mathbb{R}^m$, and every $Y \in Z^m$, where Z is a separable Banach space, let $\mathcal{J}_X(Y)$ denote the Lagrange interpolation polynom of Y at points X , that is

$$\mathcal{J}_X(Y)(x) = \sum_{i=1}^m y_i \left(\prod_{\substack{j=1 \\ j \neq i}}^m (x - x_j) \right) / \left(\prod_{\substack{j=1 \\ j \neq i}}^m (x_i - x_j) \right)$$

for every $x \in \mathbb{R}$. Note that

$$\mathcal{J}_X(Y)(x_i) = y_i, \quad i = 1, \dots, m.$$

For every $i \in \{1, \dots, N-1\}$, set $V_i = \{i-1, i, i+1\}$, and set $V_0 = \{0\}$, and $V_N = \{N\}$.

For every $X \in \mathbb{R}^{N+1}$, we use the notation $X \cdot \chi_h = \sum_{i=0}^N x_i \chi_{\omega_i}$.

Definition 3.1. • The mean operator $m_h : L^2(I) \rightarrow \mathbb{R}^{N+1}$ is defined by

$$m_h(u) = \left(\frac{1}{|\omega_i|} \int_{\omega_i} u(x) dx \right)_{i \in \{0, \dots, N\}},$$

for every $u \in L^2(I)$.

• Define $\bar{p}_{0,h} : \mathbb{R}^{N+1} \rightarrow \widetilde{W}_{0,h}$ by

$$\bar{p}_{0,h}(U) = \sum_{i=0}^N \mathcal{J}_{(x_j)_{j \in V_i}}((U_j)_{j \in V_i}) \chi_{\omega_i},$$

for every $U \in \mathbb{R}^{N+1}$.

- Set $W_{0,h} = \bar{p}_{0,h}(\mathbb{R}^{N+1})$ and $W_{0,h}^0 = \{u \in W_{0,h} \mid u(0) = u(1) = 0\}$.
- Let $P_{0,h} : L^2(I) \rightarrow W_{0,h}$ denote the linear mapping defined by $u_{hk} \rightarrow u$ et

$$P_{0,h}(u) = \bar{p}_{0,h}(M_h^{-1} m_h(u)),$$

for every $u \in L^2(I)$, where M_h is the $(N+1) \times (N+1)$ tridiagonal matrix

$$M_h = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \frac{1}{24} & \frac{11}{12} & \frac{1}{24} & & \\ 0 & \frac{1}{24} & \frac{11}{12} & \ddots & \\ \vdots & & \ddots & \ddots & \frac{1}{24} & 0 \\ 0 & \dots & \frac{1}{24} & \frac{11}{12} & \frac{1}{24} & 1 \end{pmatrix}.$$

Remark 3.1. It is clear that $\bar{p}_{0,h}$ is an isomorphism from \mathbb{R}^{N+1} onto $W_{0,h}$. Indeed, $\bar{p}_{0,h}$ is surjective by construction, and is easily proved to be one-to-one by iteration.

Remark 3.2. Notice that, for every $U \in \mathbb{R}^{N+1}$, $\bar{p}_{0,h}(U)$ is constant on ω_0 (resp. on ω_N), equal to U_0 (resp. U_N), and is polynomial with degree two on every ω_i , $i = 1, \dots, N-1$. Therefore, $W_{0,h}^0$ is the subset of functions of $W_{0,h}$ that are equal to zero on ω_0 and ω_N (see Figure 3).



Figure 3: Functions of $W_{0,h}$ and $W_{0,h}^0$

Lemma 3.1. The endomorphism $m_h \circ \bar{p}_{0,h}$ on \mathbb{R}^{N+1} is represented by the tridiagonal matrix M_h in the canonical basis of \mathbb{R}^{N+1} .

Proof. For all $U \in \mathbb{R}^{N+1}$ and $i \in \{0, \dots, N\}$, one has

$$\begin{aligned} (m_h \circ \bar{p}_{0,h}(U))_i &= \frac{1}{\omega_i} \int_{\omega_i} \bar{p}_{0,h}(U)(s) ds \\ &= \frac{1}{\omega_i} \int_{\omega_i} \mathbb{I}_{(x_j)_{j \in V_i}}((U_j)_{j \in V_i})(s) ds. \end{aligned}$$

Since $V_0 = \{0\}$ and $V_N = \{N\}$, there holds

$$(m_h \circ \bar{p}_{0,h}(U))_0 = \frac{2}{h} \int_0^{h/2} U_0 ds = U_0,$$

and, similarly, $(m_h \circ \bar{p}_{0,h}(U))_N = U_N$. For every $i \in \{1, \dots, N-1\}$, one has $V_i = \{i-1, i, i+1\}$, and thus,

$$\begin{aligned} (m_h \circ \bar{p}_{0,h}(U))_i &= \frac{1}{h} \int_{x_{i-1}-h/2}^{x_i+h/2} \left(U_{i-1} \frac{(s-x_i)(s-x_{i+1})}{2h^2} \right. \\ &\quad \left. - U_i \frac{(s-x_{i-1})(s-x_{i+1})}{h^2} \right. \\ &\quad \left. + U_{i+1} \frac{(s-x_{i-1})(s-x_i)}{2h^2} \right) ds \\ &= \int_{-1/2}^{1/2} \left(U_{i-1} \frac{s(s-1)}{2} - U_i(s-1)(s+1) + U_{i+1} \frac{s(s+1)}{2} \right) ds \\ &= \frac{1}{12} U_{i-1} + \frac{11}{24} U_i + \frac{1}{12} U_{i+1}. \end{aligned}$$

The conclusion follows. \square

Corollary 3.1. The linear mapping $P_{0,h}$ is a projection operator from $L^2(I)$ onto $W_{0,h}$.

Proof. From Remark 3.1, it suffices to notice that, for every $U \in \mathbb{R}^{N+1}$,

$$P_{0,h}(\bar{p}_{0,h}(U)) = \bar{p}_{0,h}(M_h^{-1}m_h(\bar{p}_{0,h}(U))) = \bar{p}_{0,h}(M_h^{-1}M_h U) = \bar{p}_{0,h}(U).$$

\square

Define $P_{0,h}^* : W_{0,h} \rightarrow L^2(I)$ as the canonical injection.

We extend in a canonical way the operator $P_{0,h}$ on W^* , and $P_{0,h}^*$ on $C^0(0, T; W_{0,h}) \times C^0(0, T; W_{0,h})$.

Let $\tilde{\tilde{V}}_{0,h}$ denote the set of functions on I whose restriction to each subinterval ω_i , $i = 0, \dots, N$, is constant, and moreover is equal to zero on ω_0 and ω_N . Set

$$\tilde{V}_{0,h} = \mathbb{R} \times \tilde{\tilde{V}}_{0,h} \times \mathbb{R},$$

and

$$V_{0,h} = C^0(0, T; \tilde{V}_{0,h}) \times C^0(0, T; \tilde{\tilde{V}}_{0,h}) \times \tilde{V}_{0,h} \times \tilde{\tilde{V}}_{0,h}.$$

Definition 3.2. Let $\tilde{m}_h : \mathbb{R} \times L^2(I) \times \mathbb{R} \rightarrow \tilde{V}_{0,h}$ be defined by

$$\tilde{m}_h(g_1, u, g_2) = \begin{pmatrix} g_1 \\ m_h(u) \\ g_2 \end{pmatrix}.$$

Let $q_{0,h} : \mathbb{R} \times L^2(I) \times \mathbb{R} \rightarrow \tilde{V}_{0,h}$ be defined by

$$q_{0,h}(g_1, u, g_2) = g_1 \chi_{\omega_0} + g_2 \chi_{\omega_N} + \sum_{i=1}^{N-1} \frac{1}{|\omega_i|} \int_{\omega_i} u(x) dx \chi_{\omega_i},$$

and let $Q_{0,h} : V^* \rightarrow V_{0,h}$ be defined by

$$Q_{0,h}(g_1, u, v, g_2, u_0, u_1) = \begin{pmatrix} q_{0,h}(g_1, u, g_2) \\ q_{0,h}(0, v, 0) \\ q_{0,h}(g_1, u_0, g_2) \\ q_{0,h}(0, u_1, 0) \end{pmatrix} = q_{0,h} \begin{pmatrix} g_1 & u & g_2 \\ 0 & v & 0 \\ g_1 & u_0 & g_2 \\ 0 & u_1 & 0 \end{pmatrix}.$$

It is clear that $Q_{0,h}$ is a projection operator from V^* onto $V_{0,h}$.

3.3 Recovering the usual finite difference method

Consider the discretization processes $\mathcal{D}_1 = (W^*, (W_{0,h}, P_{0,h})_{h>0}, W)$ and $\mathcal{D}_2 = (V^*, (V_{0,h}, Q_{0,h})_{h>0}, V)$. As we will see, they actually permit to recover the usual finite difference discretization of the Laplacian operator. However, due to the fact that Δu belongs to a subspace of $H^{-2}(I)$ whenever $u \in W_{0,h}$, the discretization process \mathcal{D}_1 should be built on a subspace of $H^{-2}(I)$ instead of $L^2(I)$. We are thus faced with the following alternative: either we should extend the previous theory to subspaces of H^{-s} , $s > 0$, and this is the object of the next section; or we use these discretization processes and in this case the Laplacian operator has to be "truncated", in the sense that it should be replaced by the operator $\Delta^* : L^2(I) \rightarrow L^2(I)$ defined by

$$\Delta^* u(x) = \begin{cases} \Delta u(x) & \text{if } u \text{ is } C^2 \text{ at } x, \\ 0 & \text{otherwise.} \end{cases}$$

Let us investigate the second choice, and show that it leads to the classical finite difference method. The first step is to project the truncated Laplacian operator Δ^* on the arrival discretization space $V_{0,h}$, using the projection $Q_{0,h}$. Notice that $W_{0,h} \subset L^2(I)$, and that $\Delta^*(W_{0,h}) \subset V_{0,h}$. This justifies the following definition.

Definition 3.3. The *finite difference discretization operator* $\Delta_h : W_{0,h} \rightarrow \tilde{V}_{0,h}$ of the operator

$$\begin{pmatrix} \gamma_1 \\ \Delta \\ \gamma_2 \end{pmatrix} : H^2(I) \rightarrow \mathbb{R} \times L^2(I) \times \mathbb{R}$$

is defined by

$$\Delta_h = \tilde{m}_h \circ \begin{pmatrix} \gamma_1 \\ \Delta^* \\ \gamma_2 \end{pmatrix} \circ P_{0,h}^*.$$

A straightforward computation leads to the following result.

Proposition 3.1. For every $u_h \in W_{0,h}$, one has

$$\Delta_h(u_h) = \begin{pmatrix} u_h(a) \\ \overline{\Delta}_h \overline{p}_{0,h}^{-1}(u_h) \cdot \tilde{\chi}_h \\ u_h(b) \end{pmatrix},$$

where $\overline{\Delta}_h$ denotes the $(N-1) \times (N+1)$ matrix

$$\overline{\Delta}_h = \frac{1}{h^2} \begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & 1 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \end{pmatrix}.$$

Here we use the notation $X \cdot \tilde{\chi}_h = \sum_{i=1}^{N-1} x_i \chi_{\omega_i}$, for every $X \in \mathbb{R}^{N-1}$.

In this way, we recover the classical discretization of the Laplacian operator with Dirichlet boundary conditions using the finite difference method.

Remark 3.3. This method requires $m_h \circ \Delta^* \cdot \tilde{\chi}(W_{0,h})$ to be equal to $\tilde{V}_{0,h}$. This requirement is natural because $\Delta(H^2(I)) = L^2(I)$. Therefore, if $\tilde{V}_{0,h}$ is defined as the set of functions on I whose restriction on each subinterval ω_i , $i = 0 \dots, N$, is polynomial with degree p , and moreover is equal to zero on ω_0 and ω_N , then $\tilde{W}_{0,h}$ has to be defined as the set of functions on I whose restriction on each subinterval ω_i , $i = 0 \dots, N$, is polynomial with degree $p + 2$.

The method described previously actually consists in choosing $p = 0$. It is built on local second-order interpolation processes.

Remark 3.4. Here, we dealt with Dirichlet boundary conditions. The trace operator has degree zero. In these conditions, the interpolation polynomial function corresponding to the extremities can be chosen with degree zero. If we deal with Neumann boundary conditions, corresponding to the trace of an operator of degree one, the interpolation polynomial function corresponding to the extremities has to be chosen with degree greater than or equal to one.

3.4 Discretization of the time variable

We introduce the framework for the discretization of the time variable $t \in [0, T]$. As it will be applied to discretize the wave equation, and since solutions of this equation are continuous with respect to t , we restrict the following definitions to the simpler case of continuous functions.

Let N be a positive integer, and let $k = (k_j)_{j \in \{0, \dots, N\}}$ be a finite sequence of positive real numbers so that $\sum_{j=1}^N k_j = T$. Set $t_0 = 0$, and $t_i = \sum_{j=1}^i k_j$, for $i = 1, \dots, N$. The subintervals $\tau_i = (t_i, t_{i+1})$, $i = 0, \dots, N - 1$, form a subdivision of the interval $[0, T]$. Let χ_{τ_i} denote the characteristic function of the interval τ_i .

For every $i \in \{0, \dots, N - 1\}$, let B_i a set of distinct integers of $\{0, \dots, N\}$, such that:

- i and $i + 1$ belong to B_i ;
- $\max(B_i) = i + 1$;
- there exists $C > 0$ such that $\max_{i,l \in B_j} |t_i - t_l| < Ck_i$, for every $j \in \{0, \dots, N - 1\}$,

Set $M = \max_j \#B_j$.

Let $\tilde{\Xi}_k$ denote the set of functions on I whose restriction to each subinterval τ_i , $i = 0, \dots, N - 1$, is polynomial.

Definition 3.4. Let W be a Banach space.

- The *value operator* $v_k : C^0([0, T], W) \rightarrow W^N$ is defined by

$$v_k(u) = (u(t_i))_{i \in \{1, \dots, N\}},$$

for every $u \in C^0(0, T; W)$.

- Define $\bar{q}_k : W^N \rightarrow \tilde{\Xi}_k$ by

$$\bar{q}_k(U) = \sum_{i=0}^{N-1} \mathcal{J}_{j \in B_i}((U_j)_{j \in B_i}) \chi_{\tau_i},$$

for every $U \in W^N$.

- Let $\Xi_k(W) = \bar{q}_k(W^N)$, and $S_k : C^0(0, T; W) \rightarrow \Xi_k(W)$ be defined by

$$S_k(u) = \bar{q}_k(v_k(u)),$$

for every $u \in C^0(0, T; W)$.

- The *mean operator* $m_k : L^2(0, T; W) \rightarrow L^2(0, T; W)$ is defined by

$$\forall u \in L^2(0, T; W), \quad m_k(u) = \sum_{i=0}^{N-1} \left(\frac{1}{|\tau_i|} \int_{\tau_i} u(t) dt \right) \chi_{\tau_i}.$$

Denote by \tilde{m}_k a quadrature approximation formula of order β of m_k .

- Set $\Lambda_k(W) = \tilde{m}_k(L^2(0, T; W))$.

Obviously, the linear mapping S_k is a projection of $C^0(0, T; W)$ onto $\Xi_k(W)$. Define S_k^* (resp. \tilde{m}_k^*) as the canonical injection of $\Xi_k(W)$ in $C^0(0, T; W)$ (resp. of $\Lambda_k(W)$ in $L^2(0, T; W)$). Discretization processes built on these continuous and discrete spaces verify the assumptions of Definition 2.1.

Proposition 3.2. *Consider the extension $\partial_t^* : C^0(0, T; W) \rightarrow L^2(0, T; W)$ of the operator $\partial_t : C^1(0, T; W) \rightarrow C^0(0, T; W)$ defined as follows: for every u in $C^0(0, T; W)$, set $\partial_t^* u(t) = \partial_t u(t)$ whenever u is C^1 at t , and $\partial_t^* u(t) = 0$ else.*

The numerical scheme built on (6) with the extended operator ∂_t^ , associated with the discretization processes*

$$(C^0(0, T; W), (\Xi_k(W), S_k)_{k \geq 0}, C^1(0, T; W))$$

and

$$(L^2(0, T; W), (\Lambda_k(W), \tilde{m}_k)_{k \geq 0}, C^0(0, T; W)),$$

is consistent and stable, and thus convergent.

Proof. For every $u \in (C^0(0, T; W))$, on has

$$\|S_k \partial_t^* u\|_{L^2(0, T; W)}^2 = \sum_{i=0}^{N-1} \|S_k \partial_t^* u\|_{L^2(0, T; W)}^2 = \sum_{i=0}^{N-1} \|S_k \partial_t u\|_{L^2(0, T; W)}^2 = \|S_k \partial_t u\|_{L^2(0, T; W)}^2.$$

Given u solution of (6), we have

$$\begin{aligned}
& \|\mathcal{M}_h m_k^* m_k \partial_t^* S_k^* S_k u - m_k^* m_k \mathcal{P}_h S_k^* S_k u - \mathcal{M}_h \partial_t^* u + \mathcal{P}_h u\|_{L^2(0,T;W)} \\
= & \|\mathcal{M}_h (m_k \partial_t^* S_k u - \partial_t u) - m_k \mathcal{P}_h S_k u + \mathcal{P}_h u\|_{L^2(0,T;W)} \\
\leq & \|\mathcal{M}_h\| \|m_k \partial_t^* S_k u - \partial_t u\|_{L^2(0,T;W)} + \|m_k \mathcal{P}_h S_k u - \mathcal{P}_h u\|_{L^2(0,T;W)} \\
\leq & C_h \|m_k \partial_t^* S_k u - \partial_t S_k u + \partial_t S_k u - \partial_t u\|_{L^2(0,T;W)} \\
& + \|m_k \mathcal{P}_h S_k u - \mathcal{P}_h S_k u + \mathcal{P}_h S_k u - \mathcal{P}_h u\|_{L^2(0,T;W)} \\
\leq & C_h \|(m_k - Id)(\partial_t S_k u)\|_{L^2(0,T;W)} + \|\partial_t (S_k u - u)\|_{L^2(0,T;W)} \\
& + \|(m_k - Id)(\mathcal{P}_h S_k u)\|_{L^2(0,T;W)} + \|\mathcal{P}_h (S_k - Id)u\|_{L^2(0,T;W)}
\end{aligned}$$

Then, using the properties of the discretization processes given in Definition 2.1, the consistency property follows. In order to prove the stability, for every $e \in \Xi_k(W)$, consider u and u_e solution of

$$\begin{aligned}
(\mathcal{M}_h m_k \partial_t^* S_k - m_k S_k^* \mathcal{P}_h)u &= m_k f_h, \\
(\mathcal{M}_h m_k \partial_t^* S_k - m_k S_k^* \mathcal{P}_h)u_e &= m_k f_h + e.
\end{aligned}$$

Set $v = u - u_e$. On every subinterval τ_i of $(0, T)$, one has

$$\mathcal{M}_h S_k^* v(t_{i+1}) - \mathcal{M}_h S_k^* v(t_i) - \int_{\tau_i} S_k^* \mathcal{P}_h S_k^* v(s) ds = \int_{\tau_i} S_k^* e(s) ds,$$

and thus,

$$\mathcal{M}_h (S_k^* v(t_{i+1}) - S_k^* v(t_i)) - S_k^* \mathcal{P}_h \int_{\tau_i} S_k^* v(s) ds = \int_{\tau_i} S_k^* e(s) ds.$$

Hence,

$$S_k^* v_{i+1} = S_k^* v_i + \mathcal{M}_h^{-1} S_k^* \mathcal{P}_h \int_{\tau_i} S_k^* v(s) ds + \int_{\tau_i} S_k^* e(s) ds.$$

By construction, there exists $(\alpha_{i,j})_{i \in \{0, \dots, N\}, j \in B_i}$ such that

$$S_k^* v_{i+1} = S_k^* v_i + \mathcal{M}_h^{-1} S_k^* \mathcal{P}_h \sum_{j \in B_i} \alpha_{i,j} v_j + \int_{\tau_i} S_k^* e(s) ds.$$

Then, choosing h such that the operator

$$S_k^* - \alpha_{i,i+1} \mathcal{M}_h^{-1} S_k^* \mathcal{P}_h$$

is invertible, we have

$$v_{i+1} = (S_k^* - \alpha_{i,i+1} \mathcal{M}_h^{-1} S_k^* \mathcal{P}_h)^{-1} ((S_k^* v_i - \mathcal{M}_h^{-1} S_k^* \mathcal{P}_h \sum_{j \in B_i, j \neq i+1} \alpha_{i,j} v_j) + \int_{\tau_i} S_k^* e(s) ds).$$

The stability property follows from the boundedness of $(\alpha_{i,j})_{i \in \{0, \dots, N\}, j \in B_i}$ and from the discrete version of Gronwall's Lemma. \square

3.5 Application to the wave equation on $H^2(I) \times H^1(I)$

For $k, h > 0$, set

$$W_{k,h} = \Xi_k(W_{0,h} \times W_{0,h}), \quad V_{k,h} = \Lambda_k(V_{0,h}).$$

The operator $\mathcal{A}_{k,h} : W_{k,h} \rightarrow V_{k,h}$ is defined by

$$\mathcal{A}_{k,h} = m_k \circ Q_{0,h} \circ \mathcal{A}^* \circ P_{0,h}^* \circ S_k.$$

From Proposition 3.2, we can set

$$\mathcal{A}_h = \lim_{k \rightarrow +\infty} \mathcal{A}_{k,h}.$$

Proposition 3.3. *For all $u_h, v_h \in C^0(0, T; W_{0,h})$, one has*

$$\mathcal{A}_h(u_h, v_h) = \bar{\mathcal{A}}_h(\bar{p}_{0,h}^{-1}(u_h), \bar{p}_{0,h}^{-1}(v_h)) \cdot \chi_h,$$

where

$$\bar{\mathcal{A}}_h = \begin{pmatrix} \gamma_1 & 0 \\ \partial_t & -1 \\ -\bar{\Delta}_h & \partial_t M_h \\ \gamma_2 & 0 \\ \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}.$$

In other words, the semidiscretization of the operator \mathcal{A} is $\bar{\mathcal{A}}_h$. If the mass matrix M_h is approximated by the identity matrix (this can be realized by approximating the operator m_h by the one-point Gauss quadrature formula), then we recover the classical framework of the finite difference method applied to the wave equation.

4 Discretization of the 1-D wave equation in distribution spaces

In this section, we provide a space semidiscretization process of the 1-D wave equation

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(0, \cdot) &= u_0(\cdot), \quad v(0, \cdot) = v_0(\cdot), \\ u(t, a) &= u(t, b) = 0, \end{aligned} \tag{12}$$

for nonregular initial data u_0 and u_1 .

4.1 Theoretical framework

Let a, b two real numbers, and $I = (a, b)$. We first recall well known existence and uniqueness results for initial data u_0 and u_1 belonging to subspaces of $H^{-s}(I)$, $s > 0$, where the Sobolev space $H^{-s}(I)$ is defined as the topological dual space of $H_0^s(I)$, with the agreement that $H^0(I) = L^2(I)$ (see [5]).

First of all, recall the following fact. Let A be a densely defined operator on a reflexive Banach space X , of compact inverse, of domain $D(A)$. Let β be a real number belonging to the resolvent set of A . Define the Banach space X_{-1} as the completion of X for the norm $\|(\beta I - A)^{-1}x\|_X$. Then, X_{-1} is isomorphic to the dual space $D(A^*)'$. Then, the operator A extends to an operator on X_{-1} , of domain X . By induction, one defines Banach spaces X_{-n} , for n integer, and the operator A extends to an operator on X_{-n-1} , of domain X_{-n} . On the other hand, define X_1 as the Banach space X_1 , equipped with the graph norm. In the same way, one defined the Banach spaces X_n , for n integer. It is clear that $X_n = D(A^n)$, for every integer n . The family of Banach spaces $(X_n)_{n \in \mathbb{Z}}$, is called *scale of rigged spaces* in [7] (see also [6, pp. 18–20] or [4] for this standard construction).

In the case of the wave equation with Dirichlet boundary conditions, set $X_0 = L^2(I)$ and $X_1 = H_0^1(I)$. The scale of rigged spaces is built with respect to the operator $A = (-\Delta)^{1/2}$ of X_0 , defined on the domain X_1 . It is clear that

$$\begin{aligned} & \vdots \\ X_2 &= H^2(I) \cap H_0^1(I), \\ X_1 &= H_0^1(I), \\ X_0 &= L^2(I), \\ X_{-1} &= H^{-1}(I), \\ X_{-2} &= (H^2(I) \cap H_0^1(I))' \quad (\text{w.r.t. the pivot space } L^2(I)), \\ & \vdots \end{aligned}$$

With these notations, for $T > 0$, if $u_0 \in X_n$ and $u_1 \in X_{n-1}$ for some $n \in \mathbb{Z}$, then (12) has a unique solution

$$u \in C^0(0, T; X_n) \cap C^1(0, T; X_{n-1}) \cap C^2(0, T; X_{n-2}).$$

4.2 Discretization of the 1-D Laplacian operator

Let $\mathcal{D}(I)$ denote the set of smooth functions on I having a compact support on (a, b) , and let $\mathcal{D}'(I)$ denote the topological dual space of $\mathcal{D}(I)$ (set of distributions on I). For all p integer, $u \in \mathcal{D}'(I)$ and $\phi \in \mathcal{D}(I)$, set

$$\langle (\partial_x^p)^t u, \phi \rangle = (-1)^p \langle u, \partial_x^p \phi \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the duality bracket between $\mathcal{D}'(I)$ and $\mathcal{D}(I)$.

For $h > 0$, set

$$W_{s,h}^0 = \sum_{p=0}^s (\partial_x^p)^t W_{0,h}^0,$$

where $W_{0,h}^0$ is defined in Definition 3.1.

Lemma 4.1. *For every $h > 0$, there holds*

$$W_{s,h}^0 = \bigoplus_{p=0}^s (\partial_x^p)^t W_{0,h}^0.$$

Proof. This result is obtained directly using the independence of $\delta^{(i)}$ and $\delta^{(j)}$ (the derivatives of respective orders i and j of the Dirac distribution δ), for $i \neq j$, in $\mathcal{D}'(\mathbb{R})$. □

Remark 4.1. It is clear that $W_{s,h}^0 \subset X_{-s}$, for every integer s , and every $h > 0$.

In the following definition, the notation g_1 (resp. g_2) stands for the Dirichlet condition $u(t, a) = g_1$ (resp. $u(t, b) = g_2$) when dealing with the wave equation.

Definition 4.1. • For every positive integer s , set

$$W_{s,h} = \{u + g_1\chi_{\omega_0} + g_2\chi_{\omega_N} \mid u \in W_{s,h}^0\}.$$

- Let $P_{s,h}^*$ denote the canonical injection from $W_{s,h}$ into X_{-s} .
- Set

$$V_{s,h} = \sum_{p=0}^s (\partial_x^p)^t V_{0,h}.$$

- Let $Q_{s,h} : W_{s,h} + V_{s,h} \rightarrow V_{s,h}$ be defined as follows. For $u \in W_{s,h}$ and $v \in V_{s,h}$, there exists, from Lemma 4.1, a unique $(u_0, \dots, u_s) \in W_{0,h}^0$ so that

$$u = \sum_{i=0}^s (\partial_x^i)^t u_i + g_1\chi_{\omega_0} + g_2\chi_{\omega_N},$$

and we set

$$Q_{s,h}(u + v) = \sum_{i=0}^s (\partial_x^i)^t (Q_{0,h}u_i) + g_1\chi_{\omega_0} + g_2\chi_{\omega_N} + v.$$

- Let $Q_{s,h}^*$ denote the canonical injection from $V_{s,h}$ into $H^{-s}(I)$.

Remark 4.2. The operator $Q_{s,h}$ restricted to $V_{s,h}$ is the identical mapping.

Remark 4.3. The previous definition is related to a choice of representation formula in $H^{-s}(I)$. Due to the non-uniqueness of the representation on $L^2(I)$ of $H^{-s}(I)$, $Q_{s,h}$ and $P_{s,h}$ cannot be explicitly defined, although Hahn-Banach's Theorem asserts their existence. More precisely, from this theorem, for every $u \in H^{-s}(I)$, there exist $u_0, u_1, \dots, u_s \in L^2(I)$ so that

$$\langle u, \varphi \rangle = \int u_0 \varphi + \int u_1 \varphi' + \dots + \int f_s \varphi^{[s]},$$

for every $\varphi \in H_0^s(I)$, where $\langle \cdot, \cdot \rangle$ stands for the duality bracket between $H^{-s}(I)$ and $H_0^s(I)$ (see [5, Th. 12.1 p. 78]). In the previous notations, this means that

$$u = u_0 + \partial_x^t u_1 + \dots + (\partial_x^s)^t u_s.$$

This decomposition is however not unique. This explains why operators $P_{s,h}$ and $Q_{s,h}$ are not canonical.

In view of constructing a discretization of the Laplacian operator $\Delta : X_{-s} \rightarrow X_{-s-2}$, introduce the discretization processes

$$\mathcal{D}_1 = (X_{-s}, (W_{s,h}, P_{s,h})_{h>0}, X_{-s}),$$

and

$$\mathcal{D}_2 = (X_{-s-2}, (V_{s+2,h}, Q_{s+2,h})_{h>0}, X_{-s-2}).$$

Definition 4.2. The *discretized Laplacian operator* $\Delta_{s,h} : W_{s,h} \rightarrow V_{s+2,h}$ of the Laplacian operator $\Delta : X_{-s} \rightarrow X_{-s-2}$ is defined by

$$\Delta_{s,h} = Q_{s+2,h} \circ \Delta \circ P_{s,h}^*.$$

Remark 4.4. Contrarily to the previous section, we use here the complete Laplacian operator, and not a truncated one.

The following lemma is straightforward.

Lemma 4.2. *For every $u \in W_{s,h}$, there exists a unique $(U_0, U_1, \dots, U_s) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N-1} \times \dots \times \mathbb{R}^{N-1}$ such that*

$$u = \bar{p}_{0,h}(U_0) + \sum_{p=1}^s (\partial_x^p)^t \bar{p}_{0,h}(\theta U_p) + \frac{1}{b-a}((x-a)g_2 - (x-b)g_1) + g_2 - g_1,$$

where

$$\theta U = \begin{pmatrix} 0 \\ U \\ 0 \end{pmatrix},$$

for every $U \in \mathbb{R}^{N-1}$.

It is clear by construction that $W_{s,h} \subset W_{s+2,h}$. Define $\pi_{s,h}$ as the canonical projection of $W_{s+2,h}$ onto $W_{s,h}$. More precisely, for $u \in W_{s+2,h}$ there exists a unique $(u_0, \dots, u_{s+2}) \in W_{0,h}^0$ so that

$$u = \sum_{i=0}^{s+2} (\partial_x^i)^t u_i + \frac{1}{b-a} ((x-a)g_2 - (x-b)g_1) + g_2 - g_1,$$

and we set

$$\pi_{s,h} u = \sum_{i=0}^s (\partial_x^i)^t u_i + \frac{1}{b-a} ((x-a)g_2 - (x-b)g_1) + g_2 - g_1.$$

The next proposition describes the matrix $\overline{\Delta}_{s+2,h}$ associated to the operator $\Delta_{s,h} \circ \pi_{s,h} : W_{s+2,h} \rightarrow V_{s+2,h}$.

Proposition 4.1. *For every $u_h \in W_{s+2,h}$, there exists a unique $U = (U_0, \dots, U_{s+2}) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N-1} \times \dots \times \mathbb{R}^{N-1}$ so that*

$$\Delta_{s,h} \circ \pi_{s,h}(u_h) = (\overline{\Delta}_{s+2,h} U) \cdot \chi_h,$$

where $\overline{\Delta}_{s+2,h}$ denotes the matrix

$$\overline{\Delta}_{s+2,h} = \left(\begin{array}{c|c|c|c|c|c} \Delta'_{0,h} & 0 & & \dots & & 0 \\ \hline 0 & \Delta_{0,h}^0 & 0 & & & \\ \hline E^{-1}\theta_0 K_1 & 0 & \Delta_{0,h}^0 & 0 & & \vdots \\ \hline 0 & E^{-1}\theta_0 K_1 \theta & 0 & \Delta_{0,h}^0 & & \\ \hline \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \hline 0 & \dots & 0 & E^{-1}\theta_0 K_1 \theta & 0 & \Delta_{0,h}^0 \end{array} \right)$$

with

$$\Delta_{0,h} = \frac{1}{h^2} \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix} \in \mathfrak{M}_{N-1, N+1}(\mathbb{R}),$$

$$\Delta'_{0,h} = \left(\begin{array}{c|c} \frac{1}{h} & 0 \quad \dots \quad \dots \quad 0 \\ \hline \Delta_{0,h} \\ \hline 0 & \dots & \dots & 0 & \frac{1}{h} \end{array} \right) \in \mathfrak{M}_{N+1, N+1}(\mathbb{R}),$$

$$\begin{aligned}
\Delta_{0,h}^0 &= \Delta_{0,h} \theta_0 = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \in \mathfrak{M}_{N-1,N-1}(\mathbb{R}), \\
\theta_0 &= \begin{pmatrix} 0 & \cdots & 0 \\ 1 & 0 & \vdots \\ 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathfrak{M}_{N-1,N-2}(\mathbb{R}), \quad \theta = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & & 0 \end{pmatrix} \in \mathfrak{M}_{N+1,N-1}(\mathbb{R}), \\
K_1 &= \begin{pmatrix} \frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & -\frac{1}{8} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \frac{1}{8} & -\frac{3}{8} & \frac{3}{8} & -\frac{1}{8} \end{pmatrix} \in \mathfrak{M}_{N-2,N+1}(\mathbb{R}), \\
E &= \begin{pmatrix} 1 & 0 & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & -1 & 1 \end{pmatrix} \in \mathfrak{M}_{N-1,N-1}(\mathbb{R}),
\end{aligned}$$

and thus

$$E^{-1} = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}.$$

Proof. One has $W_{0,h} = \bar{p}_{0,h}(\mathbb{R}^{N+1})$, and for every $U \in \mathbb{R}^{N+1}$,

$$\bar{p}_{0,h}(U) = \chi_{\omega_0} U_0 + \chi_{\omega_{N+1}} U_N + \sum_{i=1}^{N-1} (U_{i-1} p_1(x - x_i) + (U_i p_2(x - x_i) + (U_{i+1} p_3(x - x_i))),$$

where

$$\begin{aligned}
p_1(x) &= \chi_{[-h/2, h/2]} \frac{x(x-h)}{2h^2}, \\
p'_1(x) &= \frac{3}{8}\delta_{-h/2} + \chi_{[-h/2, h/2]} \frac{x-h/2}{h^2} + \frac{1}{8}\delta_{h/2}, \\
p''_1(x) &= \frac{3}{8}\delta'_{-h/2} + \frac{1}{h}\delta_{-h/2} + \chi_{[-h/2, h/2]} \frac{1}{h^2} + \frac{1}{8}\delta'_{h/2}, \\
p_2(x) &= \chi_{[-h/2, h/2]} \frac{(h-x)(x-h)}{h^2}, \\
p'_2(x) &= \frac{3}{4}\delta_{-h/2} - 2\chi_{[-h/2, h/2]} \frac{x}{h^2} - \frac{2}{4}\delta_{h/2}, \\
p''_2(x) &= \frac{3}{4}\delta'_{-h/2} + \frac{1}{h}\delta_{-h/2} + \frac{1}{h}\delta_{h/2} - 2\chi_{[-h/2, h/2]} \frac{1}{h^2} - \frac{3}{4}\delta'_{h/2}, \\
p_3(x) &= \chi_{[-h/2, h/2]} \frac{x(x+h)}{2h^2}, \\
p'_3(x) &= -\frac{1}{8}\delta_{-h/2} + \chi_{[-h/2, h/2]} \frac{x+h/2}{h^2} - \frac{3}{8}\delta_{h/2}, \\
p''_3(x) &= -\frac{1}{8}\delta'_{-h/2} - \frac{1}{h}\delta_{-h/2} + \chi_{[-h/2, h/2]} \frac{1}{h^2} - \frac{3}{8}\delta'_{h/2}.
\end{aligned}$$

Here, the derivatives are considered in the sense of distributions on $(-h, h)$. In the following computations, we consider the derivatives in the sense of distributions on $(h/2, 1-h/2)$, i.e., the first and last cells of the discretizations are omitted in terms of derivatives as in the continuous case on $(0, 1)$. Then, no Dirac corrections will appear in $h/2$ and $1-h/2$.

We compute the derivatives of $\bar{p}_{0,h}(U)$. Here, the notation δ stands for the element of $(\mathcal{D}'(0, 1))^{N-2}$ such that, for every $i \in \{1, \dots, N-2\}$, one has $\delta_i = \delta_{x=x_i+h/2}$. The notation χ stands for the element of $L^2(0, 1)^{N+1}$ such that, for every $i \in \{0, \dots, N+1\}$, one has $\chi_i = \chi_{\omega_i}$. There holds

$$\frac{\partial}{\partial x} \bar{p}_{0,h}(U) = (K_1 U) \cdot \delta + \sum_{i=1}^{N-1} \left(U_{i-1} \frac{x-x_{i+1/2}}{h^2} - U_i \frac{x-x_i}{h^2} + U_{i+1} \frac{x-x_{i-1/2}}{h^2} \right),$$

and

$$\frac{\partial^2}{\partial x^2} \bar{p}_{0,h}(U) = (K_1 U) \cdot \delta' + (\Delta'_{0,h} U) \cdot \chi.$$

We next compute the second derivative of elements of $W_{0,h}^0$. For every $v \in W_{0,h}^0$, there exists $V \in \mathbb{R}^{N-1}$ such that $v = \bar{p}_{0,h}(\theta V)$. Then,

$$\frac{\partial^2 v}{\partial x^2} \bar{p}_{0,h}(\theta V) = (K_1 \theta V) \cdot \delta' + (\Delta'_{0,h} \theta V) \cdot \chi.$$

The second derivative of elements of $W_{0,h}^0$ and of $W_{0,h}$ belongs to $V_{-2,h}$.

Hence, there exist U_0, U_1, U_2 and V_0, V_1, V_2 in \mathbb{R}^{N-1} such that

$$\frac{\partial^2 u}{\partial x^2} \bar{p}_{0,h}(U) = (\theta U_0) \cdot \chi + ((\theta U_1) \cdot \chi)' + ((\theta U_2) \cdot \chi)'',$$

and

$$\frac{\partial^2 u}{\partial x^2} \bar{p}_{0,h}(\theta V) = (\theta V_0) \cdot \chi + ((\theta V_1) \cdot \chi)' + ((\theta V_2) \cdot \chi)''.$$

Then,

$$\begin{aligned} U_0 &= \Delta'_{0,h} U, \\ U_1 &= 0, \\ U_2 &= E^{-1} \theta_0 K_1 U, \\ V_0 &= \Delta^0_{0,h} V, \\ V_1 &= 0, \\ V_2 &= E^{-1} \theta_0 K_1 \theta V, \end{aligned}$$

and the conclusion follows. \square

4.3 Numerical simulations

In this section, we provide numerical simulations of the Dirac propagation phenomenon for the 1-D wave equation investigated above. To discretize the time variable, we use the framework of Section 3.4. To compare the multi-level scheme and the classical scheme, we compare solution for the two discretisations and the optimised time discretisation provided by `matlab`, `ode45`.

In the following results, the solution is soften in order to have a good visualisation of the solution. This is achieved using a filtering operator. The mean is taken over seven successive discretization cells.

The results provided here are obtained for initial data in $H^{-1}(I)$.

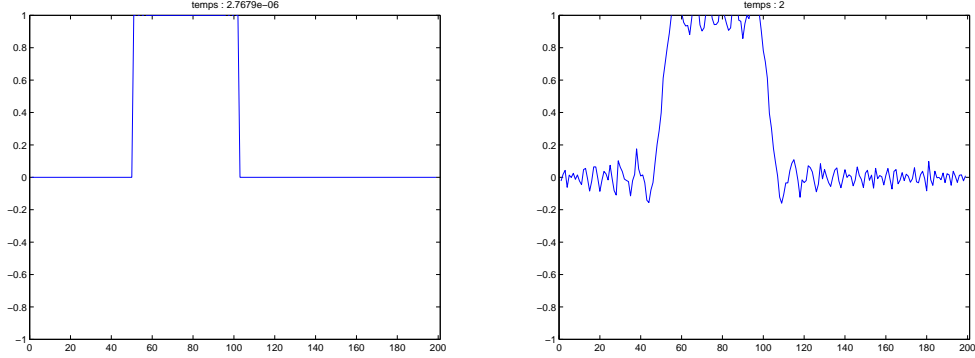


Figure 4: initial and final data on level 1 for the double Dirac derivative (non filtered solution).

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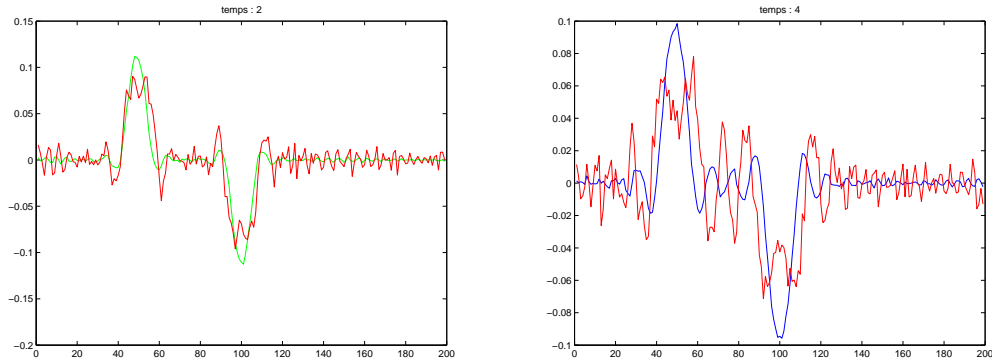


Figure 5: Solution comparison for the double Dirac between the classical approach and the multi-level approach (filtered solutions), $t = 2$ and $t = 4$.

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