



## Limit theorems for coupled interval maps

Jean-Baptiste Bardet, Sébastien Gouëzel, Gerhard Keller

### ► To cite this version:

Jean-Baptiste Bardet, Sébastien Gouëzel, Gerhard Keller. Limit theorems for coupled interval maps. Stochastics and Dynamics, 2007, 7 (1), pp.17-36. hal-00097783

**HAL Id: hal-00097783**

**<https://hal.science/hal-00097783>**

Submitted on 22 Sep 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LIMIT THEOREMS FOR COUPLED INTERVAL MAPS

JEAN-BAPTISTE BARDET, SÉBASTIEN GOUËZEL AND GERHARD KELLER

ABSTRACT. We prove a local limit theorem for Lipschitz continuous observables on a weakly coupled lattice of piecewise expanding interval maps. The core of the paper is a proof that the spectral radii of the Fourier-transfer operators for such a system are strictly less than 1. This extends the approach of [KL06] where the ordinary transfer operator was studied.

## 1. RESULTS

This paper deals with the issue of probabilistic limit theorems in dynamical systems, i.e., limit theorems for the Birkhoff sums  $S_n f = \sum_{k=0}^{n-1} f \circ T^k$ , where  $T$  is a probability preserving transformation of a space  $X$  and  $f : X \rightarrow \mathbb{R}$  is an appropriate measurable function. There are currently many techniques available to prove the central limit theorem  $S_n f / \sqrt{n} \rightarrow \mathcal{N}(0, \sigma^2)$ , let us mention for example elementary techniques, martingales, spectral arguments. On the other hand, if one is interested in the local limit theorem  $\mu\{S_n f \in [a, b]\} \sim \frac{|b-a|}{\sigma\sqrt{2\pi n}}$ , the scope of possible techniques is much more narrow: all known proofs rely on spectral analysis of transfer operators. Therefore, the class of systems for which a local limit theorem is proved is much smaller.

We are interested in limit theorems for coupled map lattices. The only previous result in this context is [Bar02], where central limit theorem, moderate deviations principle and a partial large deviations principle were established under strong analyticity assumptions on the local map and the coupling. In this paper, we establish central and local limit theorems for coupled interval maps under much weaker assumptions. More precisely, we study the same class of systems as in [KL06]. We emphasize on local limit theorem, since it is the most demanding result. But our method, relying on spectral analysis of transfer operators, gives other limit theorems, see Remark 1.4 below.

Let us recall the setup from [KL06]. Given a compact interval  $I \subset \mathbb{R}$  we will consider the phase space  $\Omega := I^{\mathbb{Z}^d}$ . In the following we always assume without loss of generality that  $I = [0, 1]$ .

The single site dynamics is given by a map  $\tau : I \rightarrow I$ . We assume  $\tau$  to be a continuous, piecewise  $\mathcal{C}^2$  map from  $I$  to  $I$  with singularities at  $\zeta_1, \dots, \zeta_{N-1} \in (0, 1)$  in the sense that  $\tau$  is monotone and  $\mathcal{C}^2$  on each component of  $I \setminus \{\zeta_0 =$

---

*Date:* September 22, 2006.

*2000 Mathematics Subject Classification.* 37L60, 60F05.

*Key words and phrases.* Coupled map lattice, piecewise expanding map, spectral gap, local limit theorem.

G.K. thanks the colleagues at the UFR Mathématiques of the University of Rennes 1 for their hospitality during his stay in March and April 2006.

$0, \zeta_1, \dots, \zeta_{N-1}, \zeta_N = 1\}$ . We assume that  $\tau', \tau''$  are bounded and that  $\inf |\tau'| > 2$ . Next, we define the unperturbed dynamics  $T_0 : \Omega \rightarrow \Omega$  by  $[T_0(x)]_{\mathbf{p}} := \tau(x_{\mathbf{p}})$ .

To define the perturbed dynamics we introduce couplings  $\Phi_\varepsilon : \Omega \rightarrow \Omega$  of the form  $\Phi_\varepsilon(\mathbf{x}) := \mathbf{x} + A_\varepsilon(\mathbf{x})$ . We say that such a coupling has range  $r$  and strength  $\varepsilon$  if for all  $\mathbf{k}, \mathbf{p}, \mathbf{q} \in \mathbb{Z}^d$

$$(1.1) \quad |(A_\varepsilon)_{\mathbf{p}}|_\infty \leq 2\varepsilon, \quad |(DA_\varepsilon)_{\mathbf{qp}}|_\infty \leq 2\varepsilon, \quad |\partial_{\mathbf{k}}(DA_\varepsilon)_{\mathbf{qp}}|_\infty \leq 2\varepsilon,$$

and  $\partial_{\mathbf{p}}\Phi_{\varepsilon,\mathbf{q}} = 0$  whenever  $|\mathbf{p} - \mathbf{q}| > r$ . The diffusive nearest neighbor coupling used in [MH93], and in much of the numerical literature, is defined by

$$(1.2) \quad [\Phi_\varepsilon(\mathbf{x})]_{\mathbf{p}} = x_{\mathbf{p}} + \frac{\varepsilon}{2d} \sum_{|\mathbf{p}-\mathbf{q}|=1} (x_{\mathbf{q}} - x_{\mathbf{p}}) \quad (\mathbf{p} \in \mathbb{Z}^d),$$

and it is a trivial example of such a coupling with range  $r = 1$  and strength  $\varepsilon$ . The dynamics  $T_\varepsilon : \Omega \rightarrow \Omega$  that we wish to investigate is then defined as

$$(1.3) \quad T_\varepsilon := \Phi_\varepsilon \circ T_0.$$

Let  $m$  denote Lebesgue measure on the interval  $I$ . The following result is proved in [KL06]:

**Theorem 1.1.** *For each  $r \in \mathbb{N}$ , there exists  $\varepsilon_0(r) > 0$  such that, for any coupling  $\Phi_\varepsilon$  of range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_0(r)$ , there exists a unique measure  $\mu_\varepsilon$  such that, for  $m^{\otimes \mathbb{Z}^d}$ -almost every point  $\mathbf{x}$ ,*

$$(1.4) \quad \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T_\varepsilon^k \mathbf{x}} \rightarrow \mu_\varepsilon.$$

*This measure  $\mu_\varepsilon$  has in fact many additional properties: it is the unique invariant measure in the class  $\mathcal{B}$  of measures of bounded variation (that we will define later), it is exponentially mixing both in time and space, and the convergence (1.4) holds for  $\mu$ -almost every point whenever  $\mu$  is a measure of bounded variation.*

In this paper, we prove the following theorem.

**Theorem 1.2.** *For each  $r \in \mathbb{N}$ , there exists  $0 < \varepsilon_1(r) \leq \varepsilon_0(r)$  satisfying the following property. Let  $\Phi_\varepsilon$  be a coupling of range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_1(r)$ , and let  $\mu_\varepsilon$  denote the corresponding invariant measure given by Theorem 1.1. Let  $f : \Omega \rightarrow \mathbb{R}$  be a Lipschitz function depending on a finite number of coordinates, with  $\int f d\mu_\varepsilon = 0$ .*

**Central limit theorem.** *There exists  $\sigma^2 \geq 0$  such that  $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ T_\varepsilon^k$  converges in distribution to  $\mathcal{N}(0, \sigma^2)$ , with respect to the measure  $\mu_\varepsilon$ . Moreover,  $\sigma^2 = 0$  if and only if there exists a measurable function  $u : \Omega \rightarrow \mathbb{R}$  such that  $f = u - u \circ T_\varepsilon$   $\mu_\varepsilon$ -almost everywhere.*

**Local limit theorem.** *Assume additionally that, whenever  $u : \Omega \rightarrow \mathbb{R}$  is measurable and  $\lambda \in \mathbb{R}^*$ , the function  $f - u + u \circ T_\varepsilon \pmod{\lambda}$  is not  $\mu_\varepsilon$ -almost everywhere constant – we say that  $f$  is aperiodic. In particular, the variance  $\sigma^2$  in the central limit theorem is nonzero. Then, for any compact interval  $I \subset \mathbb{R}$ ,*

$$(1.5) \quad \sigma \sqrt{2\pi n} \cdot \mu_\varepsilon\{x : S_n f(x) \in I\} \rightarrow |I|.$$

*Here,  $|I|$  denotes the length of the interval  $I$ .*

It is probably possible to weaken the assumptions, by replacing the finite range interaction by a short range interaction, and by allowing the function  $f$  to depend on all coordinates but with an exponentially small influence of far away coordinates (by mimicking the techniques of [KL06, Section 5]). On the other hand, it is unclear whether it is possible to remove the continuity assumption on  $\tau$  (notice that, for finite range interactions, this condition is not required in [KL06]).

On the technical level, Theorem 1.2 is a consequence of a spectral description of perturbed transfer operators acting on a suitable Banach space, that we now describe. Denote by  $\mathcal{M}$  the set of complex Borel measures on  $\Omega$  where  $\Omega$  is equipped with the product topology.

Let  $\mathcal{C}$  be a set of objects “acting on functions depending on finitely many coordinates”, defined as follows. An element of  $\mathcal{C}$  is a family  $(\mu_\Lambda)$ , where  $\Lambda$  goes through the finite subsets of  $\mathbb{Z}^d$ , such that  $\mu_\Lambda$  is a complex measure on  $I^\Lambda$ , and such that if  $\Lambda' \subset \Lambda$  then the projection of  $\mu_\Lambda$  on  $I^{\Lambda'}$  is  $\mu_{\Lambda'}$ . Formally,  $\mathcal{C}$  is the projective limit of the spaces of complex measures on  $I^\Lambda$ ,  $\Lambda$  finite subsets of  $\mathbb{Z}^d$ . This is a complex vector space, and we will not use any topology on it. Note that there is a canonical inclusion of  $\mathcal{M}$  in  $\mathcal{C}$ . If  $u$  is a bounded measurable function depending on a finite number of coordinates, and  $\mu \in \mathcal{C}$ , then it is possible to define canonically  $u\mu \in \mathcal{C}$ .

If  $\mu \in \mathcal{C}$  and  $\varphi$  is a bounded measurable function depending on a finite number of coordinates, it is possible to define  $\mu(\varphi)$  as  $\mu_\Lambda(\varphi)$  whenever  $\Lambda$  is large enough. If  $\varphi$  depends on finitely many coordinates, then  $\varphi \circ T_\varepsilon$  also depends on finitely many coordinates. This implies that, for any  $\mu \in \mathcal{C}$ , there exists a unique  $\nu \in \mathcal{C}$  such that, for any  $\varphi$ ,

$$(1.6) \quad \nu(\varphi) = \mu(\varphi \circ T_\varepsilon).$$

We write  $\nu = P_\varepsilon \mu$ . Thus,  $P_\varepsilon$  is a linear operator from  $\mathcal{C}$  to  $\mathcal{C}$ . It is the so-called *transfer operator* of the map  $T_\varepsilon$ . The image under  $P_\varepsilon$  of a measure is still a measure.

**Theorem 1.3.** *There exists a subspace  $\mathcal{D}$  of  $\mathcal{C}$  endowed with a complete norm  $\|\cdot\|$  with the following properties. First,  $\mathcal{D}$  contains the set of measures with bounded variation, and for any  $\mu \in \mathcal{D}$ ,  $|\mu(1)| \leq \|\mu\|$ . Moreover, for any finite subset  $\Lambda$  of  $\mathbb{Z}^d$ , there exists a constant  $C(\Lambda)$  such that, for any  $u : \Omega \rightarrow \mathbb{C}$  depending only on coordinates in  $\Lambda$  and Lipschitz, for any  $\mu \in \mathcal{D}$ ,  $u\mu$  also belongs to  $\mathcal{D}$  and*

$$(1.7) \quad \|u\mu\| \leq C(\Lambda)(\text{Lip}(u) + |u|_\infty) \|\mu\|,$$

where  $\text{Lip}(u)$  denotes the best Lipschitz constant of  $u$ . (The norm  $\|\cdot\|$  is defined in equation (2.20).)

For any  $r \in \mathbb{N}$ , there exists  $\varepsilon_1(r) > 0$  such that, if  $\Phi_\varepsilon$  is a coupling with range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_1(r)$ , then the following holds.

- If  $\mu \in \mathcal{D}$ , then  $P_\varepsilon \mu \in \mathcal{D}$  and  $\|P_\varepsilon \mu\| \leq C \|\mu\|$  for some constant  $C$ . In fact, the operator  $P_\varepsilon$  has a simple eigenvalue at 1 and the rest of its spectrum is contained in a disk of radius  $< 1$ .
- Let  $f$  be a Lipschitz function depending on a finite number of coordinates. Then the map  $t \mapsto P_{t,\varepsilon} = P_\varepsilon(e^{itf})$  is an analytic map from  $\mathbb{R}$  to  $\mathcal{L}(\mathcal{D})$ , the set of continuous linear operators on  $\mathcal{D}$ . If  $f$  is aperiodic, then the spectral radius of  $P_{t,\varepsilon}$  is  $< 1$  for any  $t \neq 0$ .

The derivation of Theorem 1.2 from Theorem 1.3 is classical. Note however that some objects from  $\mathcal{D}$  occurring in the proof are not known to be measures, so that one cannot directly cite [PP90], for example. So we will sketch the details of the

proof in Appendix A, because this seems clearer than applying an abstract result like e.g. [Kat66, Theorem VII.1.8] or [HH01, Corollary III.11].

**Remark 1.4.** *Theorem 1.3 implies even more precise results: the limit theorems of Theorem 1.2 hold not only for  $\mu_\varepsilon$ , but also for any probability measure  $\mu$  which belongs to  $\mathcal{D}$  (and in particular for any probability measure of bounded variation). Additionally, further refinements of the central limit theorem hold. For example, the speed of convergence in the central limit theorem is  $O(1/\sqrt{n})$ , a renewal theorem holds, as well as a large deviation inequality (see again [HH01] for further details). One can also derive in the same way as in [Bar02] the moderate deviations principle.*

The rest of the paper will be devoted to the proof of Theorem 1.3. The main problem will be to get a Lasota-Yorke inequality with compactness, since the space of measures of bounded variation is not compact in the space of finite measures. We will therefore use artificial extensions as in [KL06], but we will lose control in the “central box” due to the factor  $e^{itf}$ . This loss will be compensated by the fact that, in large but finite boxes, the measures of bounded variation form a compact subset of the set finite measures. Technically, we will have to take larger and larger boxes as  $t$  increases, but this causes no harm.

## 2. FUNCTIONAL ANALYTIC CONSTRUCTIONS

**2.1. Abstract tools.** We will need the following lemma.

**Lemma 2.1.** *Let  $T$  be a transformation preserving a probability measure  $\mu$ . Let  $n > 0$ . Then a function  $f$  is aperiodic for  $T$  if and only if  $S_n f$  is aperiodic for  $T^n$ .*

*Proof.* If  $f$  is periodic, then there exist  $c, d > 0$  and  $u$  measurable such that  $f = u - u \circ T + d \pmod{c}$ . Therefore,  $S_n f = u - u \circ T^n + nd \pmod{c}$ , hence  $S_n f$  is periodic.

Conversely, assume that  $S_n f$  is periodic for  $T^n$ , i.e.,  $S_n f = u - u \circ T^n + d \pmod{c}$ . Then  $S_n(f - u + u \circ T) = d \pmod{c}$ . For any function  $v$ ,  $S_n v$  is cohomologous to  $nv$  (since  $v \circ T^k$  is cohomologous to  $v$ ). Therefore, there exists a function  $w$  such that

$$(2.1) \quad n(f - u + u \circ T) = S_n(f - u + u \circ T) + w - w \circ T = d + w - w \circ T \pmod{c}.$$

Therefore,  $f$  is cohomologous to a constant modulo  $c/n$ , and  $f$  is periodic.  $\square$

We will also need the following formula on the essential spectral radius.

**Lemma 2.2.** *Let  $Q$  be a continuous linear operator on a complex Banach space  $(B, \|\cdot\|)$ . Assume that there exists a semi-norm  $\|\cdot\|_w$  on  $B$  such that any sequence  $x_n$  in  $B$  with  $\|x_n\| \leq 1$  contains a Cauchy subsequence for  $\|\cdot\|_w$ . Assume moreover that there exist  $\sigma > 0$  and  $C > 0$  such that, for any  $x \in B$ ,*

$$(2.2) \quad \|Qx\| \leq \sigma \|x\| + C \|x\|_w.$$

*Then the essential spectral radius of  $Q$  is at most  $\sigma$ .*

This is a version of a theorem by Hennion [Hen93], where one does not need to be able to iterate the operator for the weak norm (in the forthcoming application, the operator  $Q$  will indeed not be continuous for the weak norm).

*Proof.* Let  $M > 0$  be such that  $\|Qx\| \leq M \|x\|$ . Notice also that there exists by assumption a constant  $C > 0$  such that  $\|x\|_w \leq C \|x\|$  for all  $x \in B$ . It allows to define a new seminorm on  $B$  by  $\|x\|'_w = \sum_{n \geq 0} (2M)^{-n} \|Q^n x\|_w$ . It satisfies

the same compactness assumptions as  $\|\cdot\|_w$ . Moreover,  $Q$  is continuous for this seminorm. We can therefore iterate the equation  $\|Qx\| \leq \sigma \|x\| + C \|x\|'_w$ , and get an estimate

$$(2.3) \quad \|Q^n x\| \leq \sigma^n \|x\| + C_n \|x\|'_w.$$

The aforementioned theorem of Hennion [Hen93, Corollaire 1] gives the conclusion.  $\square$

**2.2. Measures of bounded variation.** The concept of measures of bounded variation will play a central role. For  $\mu \in \mathcal{M}$  and  $\mathbf{p} \in \mathbb{Z}^d$ , we define

$$(2.4) \quad \text{Var}_{\mathbf{p}} \mu := \sup_{|\varphi|_{C^0(\Omega)} \leq 1} \mu(\partial_{\mathbf{p}} \varphi).$$

Here, the sup is restricted to functions which are  $C^1$  in  $x_{\mathbf{p}}$ , depending only on a finite number of coordinates. Let also

$$(2.5) \quad \text{Var} \mu = \sup_{\mathbf{p} \in \mathbb{Z}^d} \text{Var}_{\mathbf{p}} \mu.$$

The set  $\mathcal{B} := \{\mu \in \mathcal{M} : \text{Var} \mu < \infty\}$  consists of measures whose finite dimensional marginals are absolutely continuous with respect to Lebesgue and the density is a function of bounded variation. In fact, “Var” is a norm and, with this norm,  $\mathcal{B}$  is a Banach space.

We define also in the same way, for any subset  $\Lambda$  of  $\mathbb{Z}^d$  and any measure  $\mu_{\Lambda}$  on  $I^{\Lambda}$ ,

$$(2.6) \quad \text{Var}_{\Lambda} \mu_{\Lambda} = \sup_{\mathbf{p} \in \Lambda} \text{Var}_{\mathbf{p}} \mu_{\Lambda}.$$

We also need the usual total variation norm on complex measures:

$$(2.7) \quad |\mu| := \sup_{|\varphi|_{C^0(\Omega)} \leq 1} \mu(\varphi).$$

Just like in [KL05, Sect. 3.3] one checks easily that

$$(2.8) \quad |\mu| \leq \frac{1}{2} \text{Var}_{\mathbf{p}} \mu \quad (\mathbf{p} \in \mathbb{Z}^d).$$

For  $\mu \in \mathcal{M}$ , let  $A(\mu)$  denote its absolute value, it is a positive measure.

**Lemma 2.3.** *If  $\mu \in \mathcal{B}$ , then  $A(\mu) \in \mathcal{B}$  and  $\text{Var} A(\mu) \leq \text{Var}(\mu)$ .*

*Proof.* When  $\mu$  is a measure with bounded variation on an interval, then the formula  $\text{Var} A(\mu) \leq \text{Var}(\mu)$  is a direct consequence of the formula

$$(2.9) \quad \text{Var}(\mu) = \inf_{f \, d\mu = d\mu} \sup_{x_1 < \dots < x_k} \sum |f(x_{i+1}) - f(x_i)|.$$

Indeed, if  $d\mu = f \, dm$  then  $dA(\mu) = |f| \, dm$ , and the formula  $||f|(x_{i+1}) - |f|(x_i)| \leq |f(x_{i+1}) - f(x_i)|$  implies the conclusion.

In dimension  $n$ , the variation of a measure can be written as the integral of one-dimensional variations (see e.g. (43) in [KL05]). Hence, the result is implied by the one-dimensional result.

Consider now a measure  $\mu \in \mathcal{B}$ . If  $A(\mu) = 0$ , there is nothing to do. Otherwise, we can assume without loss of generality that  $A(\mu)$  is a probability measure. There exists a measurable function  $\varphi$ , of absolute value almost everywhere equal to one, such that  $\mu = \varphi A(\mu)$ . Let  $\psi$  be a  $C^1$  test function depending on a finite number

of coordinates and bounded by 1, and let  $\mathbf{q} \in \mathbb{Z}^d$ . For any finite box  $\Lambda$  (containing all the coordinates on which  $\psi$  depends), the finite dimensional result implies

$$(2.10) \quad A(\pi_\Lambda \mu)(\partial_{\mathbf{q}} \psi) \leq \text{Var}(A(\pi_\Lambda \mu)) \leq \text{Var}(\pi_\Lambda \mu) \leq \text{Var}(\mu).$$

Let  $\varphi_\Lambda$  denote the conditional expectation (for the measure  $A(\mu)$ ) of the function  $\varphi$  with respect to the  $\sigma$ -algebra of sets depending only on coordinates in  $\Lambda$ . Then  $\pi_\Lambda(\mu) = \pi_\Lambda(\varphi A\mu) = \varphi_\Lambda \pi_\Lambda(A\mu)$ . Therefore,  $A(\pi_\Lambda \mu) = |\varphi_\Lambda| \pi_\Lambda(A\mu)$ . Hence, (2.10) reads

$$(2.11) \quad \int |\varphi_\Lambda| \partial_{\mathbf{q}} \psi d(A\mu) \leq \text{Var}(\mu).$$

When the box  $\Lambda$  increases, the sequence of functions  $\varphi_\Lambda$  converges in  $L^1(A\mu)$  to  $\varphi$ , by the martingale convergence theorem. Therefore,  $|\varphi_\Lambda|$  converges to  $|\varphi| = 1$ . Taking the limit in (2.11), we get

$$(2.12) \quad \int \partial_{\mathbf{q}} \psi d(A\mu) \leq \text{Var}(\mu). \quad \square$$

An element  $\mu$  of  $\mathcal{B}$  gives canonically rise to an element  $(\mu_\Lambda)_\Lambda$  of  $\mathcal{C}$  by taking the induced measure on every finite subset of  $\mathbb{Z}^d$ . It satisfies

$$(2.13) \quad \sup_{\Lambda} \text{Var}_\Lambda(\mu_\Lambda) < \infty.$$

**Lemma 2.4.** *Conversely, consider an element  $(\mu_\Lambda)_\Lambda$  of  $\mathcal{C}$  satisfying (2.13). Then it comes from an element of  $\mathcal{B}$ .*

*Proof.* Let  $\Lambda_n$  be an increasing sequence of boxes. Define a measure  $\mu_n$  on  $I^{\mathbb{Z}^d}$  by  $\mu_n = \mu_{\Lambda_n} \otimes m^{\otimes \mathbb{Z}^d \setminus \Lambda_n}$ . The sequence  $\mu_n$  has uniformly bounded variation. Let  $\mu$  be one of its weak limits. Its marginal on each box  $\Lambda$  coincides with  $\mu_\Lambda$  by construction.  $\square$

For  $u : \Omega \rightarrow \mathbb{R}$  and  $\mathbf{p} \in \mathbb{Z}^d$ , let

$$(2.14) \quad \text{Lip}_{\mathbf{p}}(u) = \sup_{\mathbf{x} \in I^{\mathbb{Z}^d \setminus \{\mathbf{p}\}}} \sup_{x_{\mathbf{p}} \neq x'_{\mathbf{p}} \in I} \frac{u(x_{\mathbf{p}}, \mathbf{x}) - u(x'_{\mathbf{p}}, \mathbf{x})}{|x_{\mathbf{p}} - x'_{\mathbf{p}}|}.$$

**Lemma 2.5.** *For any  $u : \Omega \rightarrow \mathbb{R}$  depending on a finite number of coordinates, any  $\mu \in \mathcal{B}$  and any  $\mathbf{p} \in \mathbb{Z}^d$ ,*

$$(2.15) \quad \text{Var}_{\mathbf{p}}(u\mu) \leq \sup |u| \text{Var}_{\mathbf{p}}(\mu) + \text{Lip}_{\mathbf{p}}(u)|\mu|.$$

*Proof.* In one dimension, this is a consequence of [KL05, Lemma 2.2(b)] and the fact that a Lipschitz function is differentiable almost everywhere and is equal to the integral of its derivative. This extends to finite boxes by (43) in [KL05]. Taking the supremum over finite boxes yields the conclusion of the lemma.  $\square$

**2.3. A family of extensions.** For  $\mathbf{p} \in \mathbb{Z}^d$ , denote by  $\mathcal{B}_{\mathbf{p}}$  the set of measures  $\mu$  in  $\mathcal{B}$  such that, whenever a test function  $\varphi$  does not depend on the coordinate  $\mathbf{p}$ , then  $\mu(\varphi) = 0$ .

We can now define a family of extensions. We adapt the construction of [KL06, Section 3], the main difference being that we keep a central part of the measure on a finite subset of  $\mathbb{Z}^d$ .

Let  $\Lambda$  be a finite subset of  $\mathbb{Z}^d$ , we define a space  $\mathcal{E}(\Lambda)$  as follows. An element of  $\mathcal{E}(\Lambda)$  is a family  $\mu = (\mu_c, (\mu_{\mathbf{p}})_{\mathbf{p} \in \mathbb{Z}^d \setminus \Lambda})$  such that  $\mu_c$  is a measure of the form

$\nu \otimes m^{\mathbb{Z}^d \setminus \Lambda}$  where  $\nu$  is a measure on  $I^\Lambda$ , and  $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$ . Here,  $m$  denotes Lebesgue measure on  $I$ . We assume moreover

$$(2.16) \quad \|\mu\| := \max(\text{Var}(\mu_c), \sup_{\mathbf{p} \in \mathbb{Z}^d \setminus \Lambda} \text{Var}(\mu_{\mathbf{p}})) < \infty.$$

On  $\mathcal{E}(\Lambda)$ , we also define a “weak norm” by

$$(2.17) \quad \|\mu\|_w = |\mu_c|.$$

The unit ball of  $(\mathcal{E}(\Lambda), \|\cdot\|)$  is relatively compact for the seminorm  $\|\cdot\|_w$ .

There is a canonical projection from  $\mathcal{E}(\Lambda)$  to  $\mathcal{C}$ , given by the sum of the measures  $\mu_c$  and  $(\mu_{\mathbf{p}})_{\mathbf{p} \in \mathbb{Z}^d \setminus \Lambda}$ . We will denote it by  $\pi_{\mathcal{E}(\Lambda)}$  or simply by  $\pi$ .

We describe now a (non-canonical) redistribution process introduced in [KL06]. Let  $B$  be a subset of  $\mathbb{Z}^d$ , of cardinality  $J \in [0, \infty]$ . Let  $\sigma : [0, J] \rightarrow B$  be an enumeration of the points in  $B$ . For  $j \leq J$ , let  $B_j = \sigma[0, j]$ . In particular,  $B_0 = \emptyset$  and  $B_J = B$ . If  $\mu \in \mathcal{B}$ , define measures  $\mu_{\mathbf{p}}$ , for  $\mathbf{p} \in B$ , by

$$(2.18) \quad \mu_{\mathbf{p}} = \pi_{\mathbb{Z}^d \setminus B_j} \mu \otimes m^{\otimes B_j} - \pi_{\mathbb{Z}^d \setminus B_{j+1}} \mu \otimes m^{\otimes B_{j+1}}, \quad \text{where } j = \sigma^{-1}(\mathbf{p}).$$

By construction,  $\mu = \pi_{\mathbb{Z}^d \setminus B} \mu \otimes m^{\otimes B} + \sum_{\mathbf{p} \in B} \mu_{\mathbf{p}}$ , and  $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$  satisfies  $\text{Var}(\mu_{\mathbf{p}}) \leq 2 \text{Var} \mu$ . We say that  $\pi_{\mathbb{Z}^d \setminus B} \mu \otimes m^{\otimes B}$  is the part of  $\mu$  remaining at the end of the redistribution process.

Using this process for  $B = \mathbb{Z}^d \setminus \Lambda$ , we obtain a map  $H_\Lambda$  which associates to any  $\mu \in \mathcal{B}$  an element  $H_\Lambda(\mu) \in \mathcal{E}(\Lambda)$ . It satisfies  $\|H_\Lambda(\mu)\| \leq 2 \text{Var}(\mu)$ , and  $\pi \circ H_\Lambda = \text{Id}$ .

Finally, let  $f$  be a Lipschitz function depending on a finite number of coordinates, and let  $t \in \mathbb{R}$ . Assume that the function  $tf$  depends only on coordinates in  $\Lambda$ . For each  $n \in \mathbb{N}$ , we define on  $\mathcal{E}(\Lambda)$  an operator  $Q_{t,\varepsilon,n,\Lambda}$ , which is a (non-canonical) lift of  $P_{t,\varepsilon}^n$  on  $\mathcal{C}$ . Starting from  $\mu = (\mu_c, (\mu_{\mathbf{p}})) \in \mathcal{E}(\Lambda)$ , apply first  $P_{t,\varepsilon}^n$  to each measure  $\mu_c$  and  $\mu_{\mathbf{p}}$ . Then, redistribute the mass as follows:

- For  $\text{dist}(\mathbf{p}, \Lambda) > nr$ , distribute  $P_{t,\varepsilon}^n \mu_{\mathbf{p}}$  to  $B = \{\mathbf{q} : |\mathbf{q} - \mathbf{p}| \leq nr\}$ . The points of  $B$  are all outside of  $\Lambda$ . Moreover, since  $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$  and  $tf$  depends only on coordinates in  $\Lambda$ , we have  $\pi_{\mathbb{Z}^d \setminus B}(P_{t,\varepsilon}^n \mu_{\mathbf{p}}) = 0$ , i.e., there is no mass remaining at the end of this redistribution process.

- For the other measures, use  $H_\Lambda$ .

We get as in [KL06, Lemma 3.1]

$$(2.19) \quad \|Q_{t,\varepsilon,n,\Lambda} \mu\| \leq 2B(\Lambda, n, r) \sup \left( \text{Var}(P_{t,\varepsilon}^n \mu_c), \sup_{\mathbf{p} \in \mathbb{Z}^d \setminus \Lambda} \text{Var}(P_{t,\varepsilon}^n \mu_{\mathbf{p}}) \right)$$

with  $B(\Lambda, n, r) = \#\{\mathbf{q} \in \mathbb{Z}^d : \text{dist}(\mathbf{q}, \Lambda) \leq nr\} + \#\{\mathbf{q} \in \mathbb{Z}^d : |\mathbf{q}| \leq nr\}$ , since every new measure receives a contribution from a number of sites bounded by  $B(\Lambda, n, r)$ . Note that we have written  $n$  as an index and not an exponent, in  $Q_{t,\varepsilon,n,\Lambda}$ , since these operators are not the powers of a single operator due to the (non-canonical) redistribution process.

Note that the extension  $\mathcal{E}(\emptyset)$  is at the heart of the proof of [KL06].

**2.4. Construction of a canonical extension.** Let  $\Lambda$  be a finite subset of  $\mathbb{Z}^d$ . Let  $\mathcal{E}(\Lambda)_0 \subset \mathcal{E}(\Lambda)$  be the kernel of  $\pi_{\mathcal{E}(\Lambda)}$ , i.e., the elements of  $\mathcal{E}(\Lambda)$  which induce the zero measure on the basis. This is a closed subspace of  $\mathcal{E}(\Lambda)$ , we can therefore consider the quotient space  $\mathcal{D}(\Lambda) := \mathcal{E}(\Lambda)/\mathcal{E}(\Lambda)_0$  with its canonical norm. The map  $\pi_{\mathcal{E}(\Lambda)}$  induces a map  $\pi_{\mathcal{D}(\Lambda)} : \mathcal{D}(\Lambda) \rightarrow \mathcal{C}$ , which is injective. In this way, we can therefore consider  $\mathcal{D}(\Lambda)$  as a subspace of  $\mathcal{C}$ .



Since  $\pi_{\mathcal{E}(\Lambda)} \circ Q_{t,\varepsilon,n,\Lambda} = P_{t,\varepsilon}^n \circ \pi_{\mathcal{E}(\Lambda)}$ , the operator  $Q_{t,\varepsilon,n,\Lambda}$  leaves  $\mathcal{E}(\Lambda)_0$  invariant, and induces therefore a map  $\bar{Q}_{t,\varepsilon,n,\Lambda}$  on  $\mathcal{D}(\Lambda)$ . An interesting consequence of this construction is that  $\bar{Q}_{t,\varepsilon,n,\Lambda} = \bar{Q}_{t,\varepsilon,1,\Lambda}^n$ , i.e., we are really dealing with the powers of a single operator. This is due to the fact that the non-canonicity in the redistribution process is killed by the quotient, any redistribution would induce the same map on  $\mathcal{D}(\Lambda)$ .

**Proposition 2.6.** *If  $\Lambda, \Lambda'$  are two finite subsets of  $\mathbb{Z}^d$ , then the subsets  $\pi_{\mathcal{D}(\Lambda)}(\mathcal{D}(\Lambda))$  and  $\pi_{\mathcal{D}(\Lambda')}(\mathcal{D}(\Lambda'))$  of  $\mathcal{C}$  are equal, and the induced norms are equivalent.*

*Proof.* It is sufficient to prove this for  $\Lambda' \subset \Lambda$ . Consider  $\Lambda' \subset \Lambda$ , and construct a continuous linear map from  $\mathcal{E}(\Lambda)$  to  $\mathcal{E}(\Lambda')$  by redistributing the mass of  $\mu_c$  in any convenient way. This induces a map from  $\mathcal{D}(\Lambda)$  to  $\mathcal{D}(\Lambda')$ . Conversely, starting from an element of  $\mathcal{E}(\Lambda')$ , we can consider  $\mu_c + \sum_{\mathbf{p} \in \Lambda \setminus \Lambda'} \mu_{\mathbf{p}}$  and redistribute it in any way, to get an element of  $\mathcal{E}(\Lambda)$ . Going to the quotient gives a canonical map from  $\mathcal{D}(\Lambda')$  to  $\mathcal{D}(\Lambda)$ , which is inverse to the previous one. Hence, we have constructed a canonical isomorphism between  $\mathcal{D}(\Lambda)$  and  $\mathcal{D}(\Lambda')$ , which commutes with the projections  $\pi_{\mathcal{E}(\Lambda)}$  and  $\pi_{\mathcal{E}(\Lambda')}$ . We get the proposition by projecting everything in  $\mathcal{C}$ .  $\square$

Let  $\mathcal{D} \subset \mathcal{C}$  be obtained by projecting any  $\mathcal{D}(\Lambda)$ . It is independent of the choice of  $\Lambda$ . We consider on it the norm given by the projection of the norm on  $\mathcal{D}(\emptyset)$  – any other choice would give an equivalent norm. This is the space described in Theorem 1.3.

In a pedestrian way, the norm of an element of  $\mu \in \mathcal{C}$  is the infimum of the quantity

$$(2.20) \quad \max(|\mu(1)|, \sup_{\mathbf{p} \in \mathbb{Z}^d} \text{Var}(\mu_{\mathbf{p}}))$$

over all decompositions  $\mu = \mu(1)m^{\otimes \mathbb{Z}^d} + \sum_{\mathbf{p} \in \mathbb{Z}^d} \mu_{\mathbf{p}}$  where  $\mu_{\mathbf{p}} \in \mathcal{B}_{\mathbf{p}}$ . The elements of  $\mathcal{D}$  are exactly those elements of  $\mathcal{C}$  for which such a decomposition exists with finite (2.20).

### 3. PROOF OF THE MAIN THEOREM

From this point on, we fix a range  $r$ . The next lemma gives a contraction estimate for the action of  $P_{\varepsilon}^n$  on the measures  $\mu \in \mathcal{B}_{\mathbf{p}}$ . This is essentially contained in the paper [KL06], one has simply to check that the only variations involved in the computation are those of points close to  $\mathbf{p}$ .

**Lemma 3.1.** *There exist  $\alpha, \rho \in (0, 1)$ ,  $\varepsilon_1 = \varepsilon_1(r) > 0$  and  $C = C(r) > 0$  such that, for any coupling  $\Phi_{\varepsilon}$  of range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_1$ , for all  $\mathbf{p} \in \mathbb{Z}^d$ , for all  $n \in \mathbb{N}$ , and for all  $\mu \in \mathcal{B}_{\mathbf{p}}$ ,*

$$(3.1) \quad |P_{\varepsilon}^n \mu| \leq C \alpha^{2n} \sup_{\mathbf{j} \in \mathbb{Z}^d} \rho^{|\mathbf{j} - \mathbf{p}|} \text{Var}_{\mathbf{j}}(\mu).$$

The precise choice of the constants and the details of the proof are provided in Section 4.

**Lemma 3.2.** *There exist  $\alpha \in (0, 1)$  and  $\varepsilon_1 = \varepsilon_1(r) > 0$  such that, for any coupling  $\Phi_{\varepsilon}$  of range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_1$ , and for any Lipschitz function  $f$  depending*

only on a finite number of coordinates, there exists  $C > 0$  such that, for all  $\mu \in \mathcal{B}$ , for all  $n \in \mathbb{N}$ , and for all  $t \in \mathbb{R}$ ,

$$(3.2) \quad \text{Var}(P_{t,\varepsilon}^n \mu) \leq \alpha^{2n} \text{Var}(\mu) + C(1 + |t|)|\mu|.$$

*Proof.* By Lemma 2.5, we have

$$(3.3) \quad \text{Var}(e^{itf} \mu) \leq \text{Var}(\mu) + C|t||\mu|.$$

Moreover, [KL05, Proposition 4.1 for  $\theta = 1$ ] implies that

$$(3.4) \quad \text{Var}(P_\varepsilon \mu) \leq \alpha^2 \text{Var}(\mu) + C|\mu|.$$

Using these two equations and a geometric series, we get the conclusion.<sup>1</sup>  $\square$

We fix from now on the value of  $\varepsilon_1(r)$  as the minimum of those given in the two previous lemmas, it will satisfy the conclusion of Theorem 1.3. We denote also by  $\alpha$  the maximum of the values given in the previous lemmas. Fix now a coupling  $\Phi_\varepsilon$  of range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_1(r)$ , as well as a Lipschitz function  $f$  depending only on a finite number of coordinates, say coordinates in a box  $[-A, A]^d$ . All the constants that we will construct from this point on are allowed to depend on  $f$  as well as  $r, \tau$ .

**Lemma 3.3.** *There exist  $C_0 > 1$ ,  $\ell > 0$  such that, for all  $\mathbf{p} \in \mathbb{Z}^d$ , for all  $n \in \mathbb{N}$  with  $|\mathbf{p}| > \ell n$ , for all  $\mu \in \mathcal{B}_{\mathbf{p}}$ ,*

$$(3.5) \quad \text{Var}(P_{t,\varepsilon}^n \mu) \leq C_0(1 + |t|)^2 \alpha^n \text{Var}(\mu).$$

*Proof.* Write  $n = a + b$  where  $a = n/2$  or  $(n-1)/2$  depending on whether  $n$  is even or odd. By (3.2),

$$(3.6) \quad \text{Var}(P_{t,\varepsilon}^n \mu) \leq \alpha^{2a} \text{Var}(P_{t,\varepsilon}^b \mu) + C(1 + |t|)|P_{t,\varepsilon}^b \mu|.$$

For the first term, (3.2) again gives  $\text{Var}(P_{t,\varepsilon}^b \mu) \leq C(1 + |t|) \text{Var}(\mu)$ . For the second one, by (3.1) and Lemma 2.5,

$$\begin{aligned} |P_{t,\varepsilon}^b \mu| &= |P_\varepsilon^b(e^{itS_b f} \mu)| \leq C\alpha^{2b} \sup_{\mathbf{j} \in \mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}|} \text{Var}_{\mathbf{j}}(e^{itS_b f} \mu) \\ &\leq C\alpha^{2b} \sup_{\mathbf{j} \in \mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}|} (\text{Var} \mu + \text{Lip}_{\mathbf{j}}(e^{itS_b f})|\mu|). \end{aligned}$$

Define a distance on  $\Omega$  by  $d(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{q} \in \mathbb{Z}^d} |x_{\mathbf{q}} - y_{\mathbf{q}}|$ . It does not define the product topology but, nevertheless, there exists a constant  $C$  such that  $|f(\mathbf{x}) - f(\mathbf{y})| \leq Cd(\mathbf{x}, \mathbf{y})$  (since  $f$  is Lipschitz and depends on a finite number of coordinates). Moreover, we have  $d(T_0 \mathbf{x}, T_0 \mathbf{y}) \leq Cd(\mathbf{x}, \mathbf{y})$ , since  $\tau$  is Lipschitz, as well as  $d(\Phi_\varepsilon \mathbf{x}, \Phi_\varepsilon \mathbf{y}) \leq Cd(\mathbf{x}, \mathbf{y})$ . Hence,

$$(3.7) \quad \text{Lip}_{\mathbf{j}}(e^{itS_b f}) \leq |t| \sum_{k=0}^{b-1} \text{Lip}_{\mathbf{j}}(f \circ T_\varepsilon^k) \leq |t| \sum_{k=0}^{b-1} C^k \leq |t|C^b.$$

Moreover, if  $|\mathbf{j}| > rb + A$ , the function  $e^{itS_b f}$  does not depend on the coordinate  $\mathbf{j}$ , hence  $\text{Lip}_{\mathbf{j}}(e^{itS_b f}) = 0$ . Finally,

$$(3.8) \quad |P_{t,\varepsilon}^b \mu| \leq C\alpha^{2b} \text{Var} \mu + C\alpha^{2b} \sup_{|\mathbf{j}| \leq rb+A} |t| \rho^{|\mathbf{j}-\mathbf{p}|} C^b |\mu|.$$

<sup>1</sup>Referring to [KL05, Proposition 4.1] in this proof we make use of the assumption that  $\inf |\tau'| > 2$ . However,  $\text{var}(P_{t,\varepsilon}^\ell \mu)$  could be estimated, for each fixed  $\ell$ , just as for  $\ell = 1$  in (3.3) and (3.4), and with  $\alpha \in (0, 1)$  chosen such that  $\alpha^{-2\ell} < \inf |(\tau^\ell)'|$  one would recover (3.2).

If  $|\mathbf{p}| > \ell n$  for some large enough  $\ell$ , we have  $\rho^{|\mathbf{p}| - rb - A} C^b \leq 1$ , and we get  $|P_{t,\varepsilon}^b \mu| \leq C(1 + |t|) \alpha^{2b} \text{Var } \mu$ . Together with (3.6), this proves the lemma.  $\square$

Fix  $t \in \mathbb{R}$ . Let  $\Lambda_N$  be the box  $[-\ell N, \ell N]^d$ , with  $\ell$  from the previous Lemma. Let  $N$  be large enough so that  $C_0(1 + |t|)^2 \alpha^{N/2} \leq \frac{1}{4B(\Lambda_N, N, r)}$ . We will now work in the extension  $\mathcal{E} = \mathcal{E}(\Lambda_N)$ , and study the operator  $Q = Q_{t,\varepsilon,N,\Lambda_N}$ .

**Lemma 3.4.** *There exists a constant  $C > 0$  such that, for all  $\mu \in \mathcal{E}$ ,*

$$(3.9) \quad \|Q\mu\| \leq \frac{1}{2} \|\mu\| + C \|\mu\|_w.$$

*Proof.* By (2.19), we have

$$(3.10) \quad \|Q\mu\| \leq 2B(\Lambda_N, N, r) \sup \left( \text{Var}(P_{t,\varepsilon}^N \mu_c), \sup_{\mathbf{p} \in \mathbb{Z}^d \setminus \Lambda_N} \text{Var}(P_{t,\varepsilon}^N \mu_{\mathbf{p}}) \right).$$

Moreover, (3.5) shows that  $\text{Var}(P_{t,\varepsilon}^N \mu_{\mathbf{p}}) \leq C_0(1 + |t|)^2 \alpha^{N/2} \|\mu\|$ , while (3.2) gives  $\text{Var}(P_{t,\varepsilon}^N \mu_c) \leq \alpha^N \|\mu\| + C(1 + |t|) \|\mu\|_w$ . We get

$$(3.11) \quad \|Q\mu\| \leq 2B(\Lambda_N, N, r) \max(C_0(1 + |t|)^2 \alpha^{N/2} \|\mu\|, \alpha^N \|\mu\| + C(1 + |t|) \|\mu\|_w),$$

which yields the desired conclusion by the choice of  $N$ .  $\square$

This is a Lasota-Yorke inequality for the operator  $Q$ . The main advantage of this construction is that, since the unit ball of  $(\mathcal{E}(\Lambda), \|\cdot\|)$  is relatively compact for the seminorm  $\|\cdot\|_w$ , we get from Lemma 2.2 that the essential spectral radius of  $Q$  is at most  $1/2$ . To show that the spectral radius of  $Q$  is less than 1, it is therefore sufficient to check that there is no eigenvalue of modulus  $\geq 1$ .

**Lemma 3.5.** *Let  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq 1$ . Let  $\mu \in \mathcal{E}$  satisfy  $Q\mu = \lambda\mu$ . Then  $\mu = 0$ , or  $|\lambda| = 1$  and  $t\mathbf{f}$  is periodic.*

*Proof.* Let  $\nu = \pi(\mu) \in \mathcal{C}$ . We will first check that it belongs to  $\mathcal{B}$ . By Lemma 2.4, it is sufficient to check that the variations of the measures  $\nu_\Lambda$  are uniformly bounded.

Let  $\varphi$  be a smooth function depending on a finite number of coordinates, bounded by 1, and fix  $\mathbf{q}$ . Fix  $K \geq A$  such that  $|\mathbf{q}| \leq K$  and that  $\varphi$  depends only on the coordinates  $\mathbf{p}$  with  $|\mathbf{p}| \leq K$ . For  $n \in \mathbb{N}$  which is a multiple of  $N$  we have, since  $\mu$  is an eigenfunction of  $Q$ ,

$$|\nu(\partial_{\mathbf{q}} \varphi)| \leq |P_{t,\varepsilon}^n \mu_c(\partial_{\mathbf{q}} \varphi)| + \sum_{|\mathbf{p}| \leq K+rn} |P_{t,\varepsilon}^n \mu_{\mathbf{p}}(\partial_{\mathbf{q}} \varphi)|.$$

Indeed, if  $|\mathbf{p}| > K + rn$ , then

$$(3.12) \quad P_{t,\varepsilon}^n \mu_{\mathbf{p}}(\partial_{\mathbf{q}} \varphi) = \mu_{\mathbf{p}}(e^{itS_n \mathbf{f}}(\partial_{\mathbf{q}} \varphi) \circ T_\varepsilon^n) = 0$$

since  $e^{itS_n \mathbf{f}}(\partial_{\mathbf{q}} \varphi) \circ T_\varepsilon^n$  does not depend on  $x_{\mathbf{p}}$ . We get

$$(3.13) \quad |\nu(\partial_{\mathbf{q}} \varphi)| \leq \text{Var}(P_{t,\varepsilon}^n \mu_c) + \sum_{|\mathbf{p}| \leq K+rn} \text{Var}(P_{t,\varepsilon}^n \mu_{\mathbf{p}}).$$

Let  $\ell' = \max(\ell, r) + 1$  with  $\ell$  as in Lemma 3.3, and let  $k(\mathbf{p})$  be the integer part of  $|\mathbf{p}|/\ell'$ . If  $n \geq K$ , then  $k(\mathbf{p}) \leq n$  whenever  $|\mathbf{p}| \leq K + rn$ . Then, by (3.2),

$$(3.14) \quad \text{Var}(P_{t,\varepsilon}^n \mu_{\mathbf{p}}) \leq C(1 + |t|) \text{Var}(P_{t,\varepsilon}^{k(\mathbf{p})} \mu_{\mathbf{p}}).$$

We can then use (3.5) since  $|\mathbf{p}| > \ell k(\mathbf{p})$ . We get

$$(3.15) \quad \text{Var}(P_{t,\varepsilon}^n \mu_{\mathbf{p}}) \leq C(1 + |t|)^3 \alpha^{k(\mathbf{p})} \text{Var}(\mu_{\mathbf{p}}).$$

Finally,

$$(3.16) \quad |\nu(\partial_{\mathbf{q}} \varphi)| \leq C(1 + |t|) \text{Var}(\mu_c) + C(1 + |t|)^3 \sum_{|p| \leq K+rn} \alpha^{k(\mathbf{p})} \text{Var}(\mu_{\mathbf{p}}).$$

This last sum is bounded uniformly in  $K$  and  $n$ . This proves that the variation of the measures  $\nu_\Lambda$  are uniformly bounded, i.e.  $\nu \in \mathcal{B}$ .

If  $\nu = 0$ , the marginal of  $\nu$  on  $\Lambda_N$  vanishes, i.e.,  $\mu_c = 0$ . Therefore,  $\|\mu\|_w = 0$ . The Lasota-Yorke inequality (3.9) then gives  $\mu = 0$ .

Assume now that  $\nu \neq 0$ . We will prove that  $|\lambda| = 1$  and that  $tf$  is periodic. The measure  $\nu$  satisfies  $P_{t,\varepsilon}^N \nu = \lambda \nu$ . The absolute value  $A(\nu)$  of the measure  $\nu$  belongs to  $\mathcal{B}$ , by Lemma 2.3. It satisfies

$$(3.17) \quad A(\nu) = |\lambda|^{-1} A(P_{t,\varepsilon}^N \nu) \leq P_\varepsilon^N A(\nu),$$

where the last inequality is obtained by the following direct computation:

$$(3.18) \quad \begin{aligned} A(P_{t,\varepsilon}^N \nu)(\varphi) &= \sup_{|g| \leq 1} |(P_{t,\varepsilon}^N \nu)(g \cdot \varphi)| = \sup_{|g| \leq 1} |\nu(e^{itS_N f} \cdot g \circ T_\varepsilon^N \cdot \varphi \circ T_\varepsilon^N)| \\ &\leq \sup_{|g| \leq 1} |\nu(g \cdot \varphi \circ T_\varepsilon^N)| = P_\varepsilon^N A(\nu)(\varphi). \end{aligned}$$

Since  $A(\nu)$  and  $P_\varepsilon^N A(\nu)$  have the same mass, this yields  $A(\nu) = P_\varepsilon^N A(\nu)$  and  $|\lambda| = 1$ . Since  $A(\nu)$  belongs to  $\mathcal{B}$ , it has to be a scalar multiple of the SRB measure  $\mu_\varepsilon$ , see Theorem 1.1. In particular,  $\nu$  is absolutely continuous with respect to  $\mu_\varepsilon$ , and the Radon-Nikodym derivative  $g = \frac{d\nu}{d\mu_\varepsilon}$  is a function of almost everywhere constant modulus. Since we assume  $\nu$  to be nonzero, we have  $|g| \neq 0$  almost everywhere. Then

$$(3.19) \quad P_\varepsilon^N \left( e^{itS_N f} \frac{g}{g \circ T_\varepsilon^N} \mu_\varepsilon \right) = \frac{1}{g} P_\varepsilon^N (e^{itS_N f} g \mu_\varepsilon) = \frac{1}{g} P_{t,\varepsilon}^N (\nu) = \lambda \frac{1}{g} \nu = \lambda \mu_\varepsilon.$$

In particular,

$$(3.20) \quad 1 = \left| \int e^{itS_N f} \frac{g}{g \circ T_\varepsilon^N} d\mu_\varepsilon \right| \leq \int \left| e^{itS_N f} \frac{g}{g \circ T_\varepsilon^N} \right| d\mu_\varepsilon = 1.$$

Therefore, we have equality in the inequality, and  $e^{itS_N f} \frac{g}{g \circ T_\varepsilon^N}$  is almost everywhere equal to a constant of modulus 1. This shows that  $tS_N f$  is periodic for  $T_\varepsilon^N$ . By Lemma 2.1,  $tf$  is periodic for  $T_\varepsilon$ .  $\square$

*Proof of Theorem 1.3.* The extension  $\mathcal{D}$  is as described in Paragraph 2.4. The formula (2.20) for the norm clearly gives  $|\mu(1)| \leq \|\mu\|$ .

Let  $\Lambda$  be a fixed box, we want to check equation (1.7), i.e.

$$(3.21) \quad \|u\mu\| \leq C(\Lambda)(\text{Lip}(u) + |u|_\infty) \|\mu\|$$

whenever  $u$  is Lipschitz continuous and depends only on coordinates in  $\Lambda$ . Let us first work in the extension  $\mathcal{E}(\Lambda)$ . If  $\mu = (\mu_c, (\mu_{\mathbf{p}})_{\mathbf{p} \in \mathbb{Z}^d \setminus \Lambda}) \in \mathcal{E}(\Lambda)$ , then all the measures  $u\mu_{\mathbf{p}}$  still belong to  $\mathcal{B}_{\mathbf{p}}$ , and they satisfy  $\text{Var}(u\mu_{\mathbf{p}}) \leq (\text{Lip } u + |u|_\infty) \text{Var } \mu_{\mathbf{p}}$  by Lemma 2.5. Moreover,  $u\mu_c$  is still of the form  $\nu \otimes m^{\mathbb{Z}^d \setminus \Lambda}$ , and its variation is at most  $(\text{Lip } u + |u|_\infty) \text{Var } \mu_c$ , again by Lemma 2.5. Hence, the multiplication by  $u$  is well defined on  $\mathcal{E}(\Lambda)$  and its norm is at most  $(\text{Lip } u + |u|_\infty)$ . This multiplication leaves  $\mathcal{E}(\Lambda)_0$  invariant, hence induces an operator on the quotient space  $\mathcal{D}(\Lambda)$  with

the same bound on its norm (see section 2.4). Since  $\mathcal{D}(\Lambda)$  is isomorphic to  $\mathcal{D}$  by Proposition 2.6, this proves (3.21).

In order to get analyticity of  $t \mapsto P_{t,\varepsilon}$ , it is enough to prove that the map  $M_t(\mu) = e^{itf}\mu$  depends analytically on  $t$ . For this, we only have to check that the series expansion

$$(3.22) \quad \sum_{n \geq 0} \frac{(itf)^n}{n!} \mu$$

is well defined for any  $\mu \in \mathcal{D}$ . But this is a direct consequence of (3.21), since

$$(3.23) \quad \begin{aligned} \left\| \frac{(itf)^n}{n!} \mu \right\| &\leq C(\text{Lip}(f^n) + |f^n|_\infty) \|\mu\| \frac{|t|^n}{n!} \\ &\leq C(n \text{Lip}(f) + |f|_\infty) |f|_\infty^{n-1} \|\mu\| \frac{|t|^n}{n!}. \end{aligned}$$

This gives analyticity of  $M_t$ , and its series expansion.

In [KL06], it is proved that, in the extension  $\mathcal{E}(\emptyset)$ , the operator  $Q_{0,\varepsilon,N,\emptyset}$  (which is a lift of  $P_{0,\varepsilon}^N$  on  $\mathcal{C}$ ) has a simple eigenvalue at 1 for sufficiently large  $N$ , the rest of its spectrum being contained in a disk of radius  $< 1$ : this is indeed an easy consequence of Lemma 3.1. After a quotient by  $\mathcal{E}(\emptyset)_0$  (which is left invariant by  $Q_{0,\varepsilon,N,\emptyset}$ ), this implies that  $P_\varepsilon^N$  acts continuously on  $\mathcal{D}$ , has a simple eigenvalue at 1 and the rest of its spectrum is contained in a disk of radius  $< 1$ . The same is then true for the operator  $P_\varepsilon$  itself.

Consider now  $t \neq 0$  and assume that  $f$  is aperiodic. For a suitable  $N$  and a suitable box  $\Lambda_N$ , Lemma 3.5 shows that the spectral radius of  $Q_{t,\varepsilon,N,\Lambda_N}$  is  $< 1$  on  $\mathcal{E}(\Lambda_N)$ . On the quotient  $\mathcal{D} \cong \mathcal{E}(\Lambda_N)/\mathcal{E}(\Lambda_N)_0$ , this implies that the spectral radius of  $P_{t,\varepsilon}^N$  is  $< 1$ . Therefore,  $P_{t,\varepsilon}$  also has a spectral radius  $< 1$ .  $\square$

#### 4. PROOF OF LEMMA 3.1

We introduce a family of additional “local” norms: for  $\rho \in (0, 1)$  (to be chosen later), for any  $\mathbf{p} \in \mathbb{Z}^d$ ,  $\Lambda \subset \mathbb{Z}^d$  and  $\mu \in \mathcal{B}$  let

$$(4.1) \quad \text{Var}^{\mathbf{p}}(\mu) = \sup_{\mathbf{j} \in \mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}|} \text{Var}_{\mathbf{j}}(\mu),$$

$$(4.2) \quad \text{Var}^\Lambda(\mu) = \sup_{\mathbf{p} \in \Lambda} \text{Var}^{\mathbf{p}}(\mu).$$

Observe that  $\text{Var}(\mu) = \sup_{\mathbf{p}} \text{Var}^{\mathbf{p}}(\mu) = \text{Var}^{\mathbb{Z}^d}(\mu)$ . In this section we denote  $\Lambda(\mathbf{p}, n) = \{\mathbf{q} : |\mathbf{q} - \mathbf{p}| \leq rn\}$ , so the range  $r$  will often be suppressed in the notation.

For the proof of Lemma 3.1 we need two further lemmas that will be proved later. Let  $\lambda_1 = \frac{1}{2} \inf |\tau'| > 1$  and denote by  $\alpha_0 \in (0, 1)$  the mixing rate of  $\tau$ .

**Lemma 4.1** (Localized Lasota-Yorke type estimate). *For any  $\lambda \in (1, \lambda_1)$ , for any range  $r$  and any  $\rho \in (0, 1)$ , there are  $\varepsilon_2 > 0$  and  $C > 0$  such that, for any coupling  $\Phi_\varepsilon$  of range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_2$ , for all  $m \in \mathbb{N}$ , for all  $\mathbf{p} \in \mathbb{Z}^d$ , and for all  $\nu \in \mathcal{B}$ ,*

$$(4.3) \quad \text{Var}^{\mathbf{p}}(P_\varepsilon^m \nu) \leq C (\lambda^{-m} \text{Var}^{\mathbf{p}}(\nu) + |\nu|) \leq C \text{Var}^{\mathbf{p}}(\nu).$$

**Lemma 4.2.** *For any range  $r$  and any  $\rho \in (0, 1)$ , there are  $\varepsilon_3 > 0$  and  $C > 0$  such that, for any coupling  $\Phi_\varepsilon$  of range  $r$  and strength  $0 \leq \varepsilon \leq \varepsilon_3$ , for all  $m \in \mathbb{N}$ , for all  $\mathbf{p} \in \mathbb{Z}^d$ , and for all  $\nu \in \mathcal{B}_{\mathbf{p}}$ ,*

$$(4.4) \quad |P_\varepsilon^m \nu| \leq C (\alpha_0^m \text{Var}_{\mathbf{p}}(\nu) + m\varepsilon \text{Var}^{\mathbf{p}}(\nu)) .$$

*Proof of Lemma 3.1.* We can precise the choice of the constants appearing in the lemma: let  $\lambda \in (1, \lambda_1)$  be fixed, then choose  $\alpha, \alpha_1, \rho \in (0, 1)$  such that

$$(4.5) \quad \sqrt{\max\{\lambda^{-1}, \alpha_0\}} < \alpha_1 < \rho^{2r} \alpha .$$

The maximal coupling strength  $\varepsilon_1$  will have to be taken smaller than  $\varepsilon_2$  and  $\varepsilon_3$  from the previous lemmas, and even smaller in the calculation below.

Before getting into the proof of Lemma 3.1, let us establish a preliminary inequality in the extension  $\mathcal{E}(\emptyset)$ , using Lemmas 4.1 and 4.2. Let  $Q := Q_{t,\varepsilon,2m,\emptyset}$  be the lift of  $P_{t,\varepsilon}^{2m}$  described in section 2.2. It redistributes mass from a site  $\mathbf{q}$  to sites in  $\Lambda(\mathbf{q}, 2m)$  only. We claim that there exist  $m \in \mathbb{N}$  and  $\varepsilon_1 > 0$  such that, whenever the coupling strength is at most  $\varepsilon_1$ , for each  $\Gamma \subseteq \mathbb{Z}^d$  and each  $\tilde{\nu} \in \mathcal{E}(\emptyset)$  with  $\tilde{\nu}_c = 0$ ,

$$(4.6) \quad \sup_{\mathbf{j} \in \mathbf{q} + \Gamma} \text{Var}^{\mathbf{q} + \Gamma}((Q\tilde{\nu})_{\mathbf{j}}) \leq \alpha_1^{2m} \sup_{\mathbf{j} \in \Lambda(\mathbf{q}, 2m) + \Gamma} \text{Var}^{\Lambda(\mathbf{q}, 2m) + \Gamma}(\tilde{\nu}_{\mathbf{j}})$$

In view of the redistribution mechanism described in (2.19), we have

$$(4.7) \quad \begin{aligned} & \sup_{\mathbf{j} \in \mathbf{q} + \Gamma} \text{Var}^{\mathbf{q} + \Gamma}((Q\tilde{\nu})_{\mathbf{j}}) \\ & \leq C m^d \sup_{\mathbf{j} \in \Lambda(\mathbf{q}, 2m) + \Gamma} \text{Var}^{\mathbf{q} + \Gamma}(P_\varepsilon^{2m} \tilde{\nu}_{\mathbf{j}}) \\ & \leq C m^d \sup_{\mathbf{i}, \mathbf{j} \in \Lambda(\mathbf{q}, 2m) + \Gamma} \left( \lambda^{-m} \text{Var}^{\mathbf{i}}(P_\varepsilon^m \tilde{\nu}_{\mathbf{j}}) + |P_\varepsilon^m \tilde{\nu}_{\mathbf{j}}| \right) \\ & \leq C m^d \sup_{\mathbf{i}, \mathbf{j} \in \Lambda(\mathbf{q}, 2m) + \Gamma} \left( \lambda^{-m} \text{Var}^{\mathbf{i}}(\tilde{\nu}_{\mathbf{j}}) \right. \\ & \quad \left. + \alpha_0^m \text{Var}_{\mathbf{j}}(\tilde{\nu}_{\mathbf{j}}) + m\varepsilon \text{Var}^{\mathbf{j}}(\tilde{\nu}_{\mathbf{j}}) \right) \end{aligned}$$

where we used the Lasota-Yorke type inequality (4.3) and the estimate (4.4). Hence,

$$(4.8) \quad \begin{aligned} & \sup_{\mathbf{j} \in \mathbf{q} + \Gamma} \text{Var}^{\mathbf{q} + \Gamma}((Q\tilde{\nu})_{\mathbf{j}}) \\ & \leq C m^d \sup_{\mathbf{i}, \mathbf{j} \in \Lambda(\mathbf{q}, 2m) + \Gamma} \left( \lambda^{-m} \text{Var}^{\mathbf{i}}(\tilde{\nu}_{\mathbf{j}}) + (\alpha_0^m + m\varepsilon) \text{Var}^{\mathbf{j}}(\tilde{\nu}_{\mathbf{j}}) \right) \\ & \leq C m^d (\lambda^{-m} + \alpha_0^m + m\varepsilon) \sup_{\mathbf{j} \in \Lambda(\mathbf{q}, 2m) + \Gamma} \text{Var}^{\Lambda(\mathbf{q}, 2m) + \Gamma}(\tilde{\nu}_{\mathbf{j}}). \end{aligned}$$

Choosing  $m$  sufficiently large and then  $\varepsilon_1$  sufficiently small, (4.6) follows.

Let us now prove Lemma 3.1. As  $P_\varepsilon$  contracts the total variation norm, it suffices to prove the lemma for multiples  $n = k2m$  of the fixed integer  $2m$  which satisfies (4.6). So let  $\mu \in \mathcal{B}_{\mathbf{p}}$ , define an element  $\tilde{\mu}$  which has only zero components except for  $\tilde{\mu}_{\mathbf{p}} = \mu$  and observe first that

$$(4.9) \quad \begin{aligned} |P_\varepsilon^{k2m} \mu| & \leq \sum_{\mathbf{q} \in \mathbb{Z}^d} \left| (Q^k \tilde{\mu})_{\mathbf{q}} \right| \leq \frac{1}{2} \sum_{\mathbf{q} \in \Lambda(\mathbf{p}, k2m)} \text{Var}_{\mathbf{q}} \left( (Q^k \tilde{\mu})_{\mathbf{q}} \right) \\ & \leq C \cdot (k2m)^d \cdot \sup_{\mathbf{q} \in \Lambda(\mathbf{p}, k2m)} \text{Var}_{\mathbf{q}} \left( (Q^k \tilde{\mu})_{\mathbf{q}} \right) , \end{aligned}$$

where we used that each application of  $Q := Q_{t,\varepsilon,2m,\emptyset}$  redistributes mass from a site  $\mathbf{q}$  to sites in  $\Lambda(\mathbf{q}, 2m)$  only.

Applying (4.6) repeatedly and observing that  $\tilde{\mu}_{\mathbf{j}} = 0$  if  $\mathbf{j} \neq \mathbf{p}$  and  $\tilde{\mu}_{\mathbf{p}} = \mu$ , we obtain

$$\begin{aligned}
 \sup_{\mathbf{q} \in \Lambda(\mathbf{p}, k2m)} \text{Var}_{\mathbf{q}}((Q^k \tilde{\mu})_{\mathbf{q}}) &\leq \sup_{\mathbf{q} \in \Lambda(\mathbf{p}, k2m)} \alpha_1^{k2m} \sup_{\mathbf{j} \in \Lambda(\mathbf{q}, k2m)} \text{Var}^{\Lambda(\mathbf{q}, k2m)}(\tilde{\mu}_{\mathbf{j}}) \\
 (4.10) \qquad \qquad \qquad &= \alpha_1^{k2m} \sup_{\mathbf{q} \in \Lambda(\mathbf{p}, k2m)} \text{Var}^{\Lambda(\mathbf{q}, k2m)}(\tilde{\mu}_{\mathbf{p}}) \\
 &\leq \alpha_1^{k2m} \rho^{-2rk2m} \text{Var}^{\mathbf{P}}(\mu) .
 \end{aligned}$$

Together with (4.9) this yields  $|P_\varepsilon^n \mu| \leq C n^d (\alpha_1 \rho^{-2r})^n \text{Var}^{\mathbf{P}}(\mu)$ , which finishes the proof of Lemma 3.1 in view of the choice of the constants in (4.5).  $\square$

*Proof of Lemma 4.1.* We will prove

$$(4.11) \qquad \qquad \qquad \text{Var}^{\mathbf{P}}(P_\varepsilon \nu) \leq \lambda^{-1} \text{Var}^{\mathbf{P}}(\nu) + C|\nu| .$$

From this (4.3) follows by induction.<sup>2</sup>

Observe first that

$$(4.12) \qquad \qquad \qquad \text{Var}^{\mathbf{P}}(P_0 \nu) \leq \lambda_1^{-1} \text{Var}^{\mathbf{P}}(\nu) + C|\nu| ,$$

where  $\lambda_1 = \frac{1}{2} \inf |\tau'|$ . This is a simple consequence of the Lasota-Yorke inequality for the single site map, compare e.g. the proof of Lemma 3.2 in [KL05]. We will show that<sup>3</sup>

$$(4.13) \qquad \qquad \qquad \text{Var}^{\mathbf{P}}((\Phi_\varepsilon)_* \nu) \leq (1 + C\varepsilon) \text{Var}^{\mathbf{P}}(\nu) .$$

We first notice that under a mild bound on the coupling strength  $\varepsilon$ , the coupling assumption (1.1) ensures that the infinite matrix  $D\Phi_\varepsilon(\mathbf{x})$  is invertible. Moreover, taking  $C$  large enough to get  $1_{|\mathbf{i}-\mathbf{j}| \leq r} \leq \frac{C}{2} \rho^{4|\mathbf{i}-\mathbf{j}|}$  for all  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ , the second and third part of this assumption can be rewritten as

$$(4.14) \qquad \qquad \qquad |(DA_\varepsilon)_{\mathbf{ij}}|_\infty \leq C\varepsilon \rho^{4|\mathbf{i}-\mathbf{j}|}, \quad |\partial_{\mathbf{k}}(DA_\varepsilon)_{\mathbf{ij}}|_\infty \leq C\varepsilon \rho^{4|\mathbf{i}-\mathbf{j}|} .$$

A direct computation using these estimates (see for example page 300 in [JP98]) gives that  $B(\mathbf{x}) := (D\Phi_\varepsilon(\mathbf{x}))^{-1}$  satisfies

$$(4.15) \qquad \qquad \qquad |b_{\mathbf{ii}}|_\infty \leq 1 + C\varepsilon, \quad |b_{\mathbf{ij}}|_\infty \leq C\varepsilon \rho^{2|\mathbf{j}-\mathbf{i}|}, \quad |\partial_{\mathbf{i}} b_{\mathbf{ij}}|_\infty \leq C\varepsilon \rho^{2|\mathbf{j}-\mathbf{i}|} .$$

<sup>2</sup>It is only here where we use the assumption  $\inf |\tau'| > 2$ . For  $1 < \inf |\tau'| \leq 2$  this reduction to the case  $m = 1$  is not possible, see also the remarks in [KL06, Footnote 12].

<sup>3</sup>We write  $F_* \nu$  for the push-forward of the measure  $\nu$  under the map  $F$ , i.e., the measure given by  $(F_* \nu)(A) = \nu(F^{-1}A)$ . Note that this object is sometimes denoted by  $F^* \nu$  in [KL05, KL06].

We can then follow the proof of Lemma 3.3 in [KL05] with some modifications. For all  $\mathbf{j}, \mathbf{p} \in \mathbb{Z}^d$ ,

$$\begin{aligned}
\rho^{|\mathbf{j}-\mathbf{p}|}((\Phi_\varepsilon)_*\nu)(\partial_{\mathbf{j}}\varphi) &= \rho^{|\mathbf{j}-\mathbf{p}|} \sum_{\mathbf{i} \in \mathbb{Z}^d} \nu(\partial_{\mathbf{i}}(\varphi \circ \Phi_\varepsilon) b_{\mathbf{ij}}) \\
&= \rho^{|\mathbf{j}-\mathbf{p}|} \sum_{\mathbf{i} \in \mathbb{Z}^d} \nu(\partial_{\mathbf{i}}(\varphi \circ \Phi_\varepsilon \cdot b_{\mathbf{ij}})) - \rho^{|\mathbf{j}-\mathbf{p}|} \sum_{\mathbf{i} \in \mathbb{Z}^d} \nu(\varphi \circ \Phi_\varepsilon \cdot \partial_{\mathbf{i}} b_{\mathbf{ij}}) \\
&\leq \rho^{|\mathbf{j}-\mathbf{p}|} \sum_{\mathbf{i} \in \mathbb{Z}^d} \text{Var}_{\mathbf{i}}(\nu) \cdot |b_{\mathbf{ij}}|_\infty + |\nu| \rho^{|\mathbf{j}-\mathbf{p}|} \cdot \sum_{\mathbf{i} \in \mathbb{Z}^d} |\partial_{\mathbf{i}} b_{\mathbf{ij}}|_\infty \\
&\leq \rho^{|\mathbf{j}-\mathbf{p}|} \text{Var}_{\mathbf{j}}(\nu) + C\varepsilon \left( \text{Var}^{\mathbf{p}}(\nu) \sum_{\mathbf{i} \in \mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}| - |\mathbf{i}-\mathbf{p}| + 2|\mathbf{j}-\mathbf{i}|} + |\nu| \sum_{\mathbf{i} \in \mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{p}| + 2|\mathbf{j}-\mathbf{i}|} \right) \\
&\leq (1 + C\varepsilon) \text{Var}^{\mathbf{p}}(\nu).
\end{aligned}$$

This yields (4.13) and finishes the proof of Lemma 4.1.  $\square$

*Proof of Lemma 4.2.* The proof follows closely the corresponding one in [KL06]. For each  $\mathbf{p} \in \mathbb{Z}^d$  define a coupling map  $\Phi_{\varepsilon, \mathbf{p}} : \Omega \rightarrow \Omega$  where site  $\mathbf{p}$  is decoupled from all other sites,

$$(4.16) \quad (\Phi_{\varepsilon, \mathbf{p}}(\mathbf{x}))_{\mathbf{q}} = \begin{cases} x_{\mathbf{p}} & \text{if } \mathbf{q} = \mathbf{p} \\ (\Phi_\varepsilon(\mathbf{x}_{\mathbb{Z}^d \setminus \{\mathbf{p}\}}, a))_{\mathbf{q}} & \text{if } \mathbf{q} \neq \mathbf{p} \end{cases}$$

where  $a$  is an arbitrary point in  $I$ . Denote by  $P_{\varepsilon, \mathbf{p}}$  the Perron-Frobenius operator of  $\Phi_{\varepsilon, \mathbf{p}} \circ T_0$ . We will show that, for each  $\nu \in \mathcal{B}$ ,

$$(4.17) \quad |(\Phi_\varepsilon)_*\nu - (\Phi_{\varepsilon, \mathbf{p}})_*\nu| \leq C\varepsilon \text{Var}^{\mathbf{p}}(\nu).$$

Then, making use of the fact that  $|P_\varepsilon| = |P_{\varepsilon, \mathbf{p}}| = 1$  and of estimate (4.3), a simple telescoping argument yields

$$(4.18) \quad |P_\varepsilon^m \nu - P_{\varepsilon, \mathbf{p}}^m \nu| \leq C m \varepsilon \text{Var}^{\mathbf{p}}(\nu),$$

and (4.4) follows once we have shown that  $|P_{\varepsilon, \mathbf{p}}^m \nu| \leq C \alpha_0^m \text{Var}_{\mathbf{p}}(\nu)$  for any  $\nu \in \mathcal{B}_{\mathbf{p}}$ . But this is proved precisely as in [KL06, p.40/41], where  $\alpha_0$  is the mixing rate for the single site map.

It remains to prove (4.17). Here we can follow closely the proof of Lemma 3.2a) in [KL06]. Indeed, let  $F_t := t\Phi_{\varepsilon, \mathbf{p}} + (1-t)\Phi_\varepsilon$  and  $\Delta_{\mathbf{q}} := (\Phi_{\varepsilon, \mathbf{p}} - \Phi_\varepsilon)_{\mathbf{q}}$ . Just as in [KL06] one shows that, for each test function  $\varphi$ ,

$$(4.19) \quad ((\Phi_{\varepsilon, \mathbf{p}})_*\nu - (\Phi_\varepsilon)_*\nu)(\varphi) = \int_0^1 \sum_{\mathbf{q} \in \mathbb{Z}^d} (F_t)_*(\Delta_{\mathbf{q}} \cdot \nu)(\partial_{\mathbf{q}}\varphi) dt.$$

As, in our case,  $\Delta_{\mathbf{q}} = 0$  if  $|\mathbf{q} - \mathbf{p}| > r$ , we conclude

$$\begin{aligned}
|(\Phi_{\varepsilon, \mathbf{p}})_*\nu - (\Phi_\varepsilon)_*\nu| &\leq \sum_{|\mathbf{q}-\mathbf{p}| \leq r} \sup_{0 \leq t \leq 1} \text{Var}_{\mathbf{q}}((F_t)_*(\Delta_{\mathbf{q}} \cdot \nu)) \\
(4.20) \quad &\leq C \sum_{|\mathbf{q}-\mathbf{p}| \leq r} \text{Var}^{\mathbf{q}}(\Delta_{\mathbf{q}} \cdot \nu),
\end{aligned}$$



where we used (4.13) (which applies as well to  $(F_t)_*$ ) for the second inequality. Hence, by Lemma 2.5,

$$\begin{aligned} |(\Phi_{\varepsilon, \mathbf{p}})_* \nu - (\Phi_\varepsilon)_* \nu| &\leq C \sum_{|\mathbf{q}-\mathbf{p}| \leq r} \left( |\Delta_{\mathbf{q}}|_\infty \text{Var}^{\mathbf{q}}(\nu) + \sup_{\mathbf{j} \in \mathbb{Z}^d} \rho^{|\mathbf{j}-\mathbf{q}|} \text{Lip}_{\mathbf{j}}(\Delta_{\mathbf{q}}) |\nu| \right) \\ &\leq C\varepsilon \text{Var}^{\mathbf{q}}(\nu) \end{aligned}$$

in view of assumption (1.1). This is (4.17) and finishes the proof of the lemma.  $\square$

#### APPENDIX A. PROOF OF THEOREM 1.2 ASSUMING THEOREM 1.3

The operator  $P_\varepsilon$  has a simple eigenvalue at 1, and the corresponding eigenfunction is the invariant measure  $\mu_\varepsilon$  obtained in [KL06]. By classical analytic perturbation theory, the operator  $P_{t,\varepsilon}$  has for small  $t$  a unique eigenvalue  $\lambda(t)$  close to 1, which is still simple. Let  $\Pi_t$  denote the corresponding spectral projection, and  $\mu_{t,\varepsilon} = \Pi_t(\mu_\varepsilon)$ . There exist  $\delta < 1$  and  $C > 0$  such that, for all small enough  $t$ , for all  $n \in \mathbb{N}$ ,

$$(A.1) \quad \left| \int e^{itS_n f} d\mu_\varepsilon - \lambda(t)^n \mu_{t,\varepsilon}(1) \right| \leq C\delta^n.$$

Hence, a precise description of the eigenvalue  $\lambda(t)$  will imply a central limit theorem for the Birkhoff sums  $S_n f$ .

Let  $\nu_t = \mu_{t,\varepsilon}/\mu_{t,\varepsilon}(1)$ . Differentiating the equality  $P_{t,\varepsilon}\nu_t = \lambda(t)\nu_t$  and using  $\frac{dP_{t,\varepsilon}}{dt}\Big|_{t=0} = P_\varepsilon(if\cdot)$ , we get

$$(A.2) \quad P_\varepsilon(if\mu_\varepsilon) + P_\varepsilon(\nu'_0) = \lambda'(0)\mu_\varepsilon + \nu'_0.$$

Integrating the function 1 with respect to this equality, we obtain

$$(A.3) \quad i \int f d\mu_\varepsilon + \int d\nu'_0 = \lambda'(0) + \int d\nu'_0.$$

Since  $\int f d\mu_\varepsilon = 0$ , we therefore have  $\lambda'(0) = 0$ .

Differentiating twice  $P_{t,\varepsilon}\nu_t = \lambda(t)\nu_t$  yields

$$(A.4) \quad P_\varepsilon(-f^2\mu_\varepsilon) + 2P_\varepsilon(if\nu'_0) + P_\varepsilon(\nu''_0) = \nu''_0 + \lambda''(0)\mu_\varepsilon.$$

Integrating the function 1 yields

$$(A.5) \quad \lambda''(0) = - \int f^2 d\mu_\varepsilon + 2i \int f d\nu'_0.$$

From (A.2), we have  $\nu'_0 = P_\varepsilon\nu'_0 + P_\varepsilon(if\mu_\varepsilon)$ . Iterating this equation gives

$$(A.6) \quad \nu'_0 = P_\varepsilon^n \nu'_0 + i \sum_{k=1}^n P_\varepsilon^k(f\mu_\varepsilon).$$

Since  $\nu_t(1) = 1$ , we have  $\nu'_0(1) = 0$ . The space  $\{\mu \in \mathcal{D} : \mu(1) = 0\}$  is closed and  $P_\varepsilon$  leaves this space invariant, therefore its spectral radius on this space is  $< 1$ . This implies that  $P_\varepsilon^n \nu'_0$  converges exponentially fast to 0. In the same way,  $(f\mu_\varepsilon)(1) = 0$ , hence  $P_\varepsilon^k(f\mu_\varepsilon)$  converges exponentially fast to 0 in  $\mathcal{D}$ . Letting  $n$  tend to infinity in (A.6), we get  $\nu'_0 = i \sum_{k=1}^\infty P_\varepsilon^k(f\mu_\varepsilon)$ . In particular,  $\nu'_0(f) = i \sum_{k=1}^\infty \int f \cdot f \circ T_\varepsilon^k d\mu_\varepsilon$ , and this series converges exponentially fast. From (A.5), we obtain

$$(A.7) \quad \lambda''(0) = - \int f^2 d\mu_\varepsilon - 2 \sum_{k=1}^\infty \int f \cdot f \circ T_\varepsilon^k d\mu_\varepsilon.$$

Moreover,

$$\begin{aligned}
\int \left( \sum_{k=0}^{n-1} f \circ T_\varepsilon^k \right)^2 d\mu_\varepsilon &= n \int f^2 d\mu_\varepsilon + 2 \sum_{k=1}^n (n-k) \int f \cdot f \circ T_\varepsilon^k d\mu_\varepsilon \\
&= -n\lambda''(0) - 2 \sum_{k=1}^{\infty} k \int f \cdot f \circ T_\varepsilon^k d\mu_\varepsilon + O(\delta^n) \\
&= -n\lambda''(0) + O(1).
\end{aligned}$$

Since this integral is nonnegative, this shows that  $\lambda''(0) \leq 0$ . Hence, we can write  $\lambda''(0) = -\sigma^2$  for some  $\sigma \geq 0$ . Furthermore, if  $\lambda''(0) = 0$ , then  $S_n f$  is bounded in  $L^2$ , which implies that  $f$  can be written as  $u - u \circ T_\varepsilon$  in  $L^2$  (see e.g. [Kac96]). This proves the non-degeneracy criterion in Theorem 1.2.

Since  $\lambda(t) = -\sigma^2 t^2/2 + o(t^2)$ ,  $\lambda(t/\sqrt{n})^n \rightarrow e^{-\sigma^2 t^2/2}$ . Together with (A.1), this shows that  $S_n f/\sqrt{n}$  converges in distribution to  $\mathcal{N}(0, \sigma^2)$  and proves the central limit theorem. The local limit theorem is then easily derived from the description of  $\lambda(t)$  for small  $t$  and the control of the spectral radius of  $P_{t,\varepsilon}$  for all  $t \neq 0$ , see [HH01] for further details.

#### REFERENCES

- [Bar02] Jean-Baptiste Bardet. Limit theorems for coupled analytic maps. *Probab. Theory Related Fields*, 124(2):151–177, 2002.
- [Hen93] Hubert Hennion. Sur un théorème spectral et son application aux noyaux lipschitziens. *Proc. Amer. Math. Soc.*, 118:627–634, 1993.
- [HH01] Hubert Hennion and Loïc Hervé. *Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness*, volume 1766 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.
- [JP98] Miaohua Jiang and Yakov B. Pesin. Equilibrium measures for coupled map lattices: existence, uniqueness and finite-dimensional approximations. *Commun. Math. Phys.*, 193(3):675–711, 1998.
- [Kac96] A. G. Kachurovskii. Rates of convergence in ergodic theorems. *Russian Math. Surveys*, 51:653–703, 1996.
- [Kat66] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer, 1966.
- [KL05] Gerhard Keller and Carlangelo Liverani. A spectral gap for a one-dimensional lattice of coupled piecewise expanding interval maps. In Jean-René Chazottes and Bastien Fernandez, editors, *Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems*, volume 671, *Lecture Notes in Physics*, pages 115–151. Springer, 2005.
- [KL06] Gerhard Keller and Carlangelo Liverani. Uniqueness of the SRB measure for piecewise expanding weakly coupled map lattices in any dimension. *Comm. Math. Phys.*, 262(1):33–50, 2006.
- [MH93] Jonathan Miller and David A. Huse. Macroscopic equilibrium from microscopic irreversibility in a chaotic coupled-map lattice. *Phys. Rev. E*, 48:2528–2535, 1993.
- [PP90] William Parry and Mark Pollicott. *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, volume 187–188 of *Astérisque*. Société Mathématique de France, 1990.

J.-B. BARDET AND S. GOUÉZEL: IRMAR/UFR MATHÉMATIQUES, UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE; G. KELLER: MATHEMATISCHES INSTITUT, UNIVERSITÄT ERLANGEN-NÜRNBERG, BISMARCKSTR. 1 1/2, 91054 ERLANGEN, GERMANY  
*E-mail address:* jean-baptiste.bardet@univ-rennes1.fr, sebastien.gouezel@univ-rennes1.fr, keller@mi.uni-erlangen.de