Optimal reflection of diffusions and barrier options pricing under constraints

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Abstract

We introduce a new class of control problems in which the gain depends on the solution of a stochastic differential equation reflected at the boundary of a bounded domain, along directions which are controlled by a bounded variation process. We provide a PDE characterization of the associated value function. This study is motivated by applications in mathematical finance where such equations are related to the pricing of barrier options under portfolio constraints.

Keywords : Reflected diffusion, Skorokhod problem, viscosity solutions, barrier option, portfolio contraints.
MSC Classification (2000): 93E20, 60G99, 49L25, 91B28.

1 Introduction

This paper is motivated by a previous work [1] where a new class of parabolic PDE with Neumann and Dirichlet conditions is introduced. Namely, [1] discusses the problem of super-hedging a barrier option under portfolio constraints and shows that, when there is no rebate, the super-hedging price is a viscosity solution of an equation of the form

$$\begin{cases} \min\left\{-\mathcal{L}\varphi \,, \min_{e\in E} \mathcal{H}^e\varphi\right\} = 0 & \text{on} \quad [0,T)\times\mathcal{O} \\ \min\left\{\varphi \,,\ \min_{e\in E} \mathcal{H}^e\varphi\right\} = 0 & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - \hat{g} = 0 & \text{on} \quad \{T\}\times\bar{\mathcal{O}} \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is an open domain of $\mathbb{R}^d$, E is some compact subset of $\mathbb{R}^\ell$ which depends on the constraints imposed on the portfolio, $\mathcal{L}\varphi = \frac{\partial}{\partial t}\varphi + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma' D^2\varphi\right]$ is the generator of the diffusion which models the evolution of the risky assets under the risk neutral probability measure, $\mathcal{H}^e\varphi = \delta(\cdot,e)\varphi - \langle\gamma(\cdot,e), D\varphi\rangle$, $\gamma(x,e)$ is an (oblique) inward direction when $x \in \partial\mathcal{O}$ and $\hat{g}$ is a suitable function associated to the payoff function g of the option which, in a suitable sens, satisfies $\min_{e\in E} \mathcal{H}^e\hat{g} \geq 0$ and $\hat{g} \geq g$ on $\bar{\mathcal{O}}$. See [1] for details and Section 4 below for an example.

When the solution φ of the above equation is positive, it reduces to $\min_{e\in E} \mathcal{H}^e\varphi = 0$ on $[0,T) \times \partial\mathcal{O}$, and, in particular cases, see [9] and [10], the constraint $\mathcal{H}^e\varphi \geq 0$ at the parabolic boundary of $[0,T)\times\mathcal{O}$ propagates in the domain, which allows to simplify the above equation in

$$\begin{cases} -\mathcal{L}\varphi = 0 & \text{on} \quad [0,T)\times\mathcal{O} \\ \min_{e\in E} \mathcal{H}^e\varphi = 0 & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - \hat{g} = 0 & \text{on} \quad \{T\}\times\bar{\mathcal{O}}\ . \end{cases} \tag{1.2}$$

When E is a singleton $\{e_0\}$, such equations formally admit a Feynman-Kac representation of the form

$$\mathbb{E}\left[e^{-\int_t^T \delta(X(s),e_0)dL(s)}\hat{g}(X(T))\right] \tag{1.3}$$

where L is a non-decreasing process such that (X,L) solves on $[t,T]$

$$\begin{aligned} X(s) &= x + \int_t^s \sigma(X(r))dW(r) + \int_t^s \gamma(X(r),e_0)dL(r) \\ X(s) \in \bar{\mathcal{O}} \quad &\text{and} \quad L(s) = \int_t^s \mathbf{1}_{\{X(r)\in\partial\mathcal{O}\}}\, dL(r)\ \ ,\ t \leq s \leq T\ , \end{aligned} \tag{1.4}$$

and W is a standard Brownian motion, recall that $\gamma(x,e_0)$ is an inward direction for $x \in \partial\mathcal{O}$, see e.g. [4]. Thus, the pricing of the barrier option is, at least formally, related to the expectation of a functional depending on the solution of a stochastic differential equation which is reflected at the boundary of $\mathcal{O}$ along the direction $\gamma(x,e_0)$. This phenomenon was already observed in [9] in a particular setting and can be easily explained when $\hat{g} \geq 0$ and $\hat{g}$ is non-decreasing on $\mathcal{O}$, see Remark 4.4 below.

By analogy, (1.2) should be associated to the control problem

$$\sup_{\epsilon\in\mathcal{E}} \mathbb{E}\left[e^{-\int_t^T \delta(X^\epsilon(s),\epsilon(s))dL^\epsilon(s)}\hat{g}(X^\epsilon(T))\right] \tag{1.5}$$

where (X^ϵ, L^ϵ) is the solution on $[t,T]$ of

$$\begin{aligned} X^\epsilon(s) &= x + \int_t^s \sigma(X^\epsilon(r))dW(r) + \int_t^s \gamma(X^\epsilon(r),\epsilon(r))dL^\epsilon(r) \\ X^\epsilon(s) \in \bar{\mathcal{O}} \quad &\text{and} \quad L^\epsilon(s) = \int_t^s \mathbf{1}_{\{X^\epsilon(r)\in\partial\mathcal{O}\}}\, dL^\epsilon(r)\ \ ,\ t \leq s \leq T \end{aligned} \tag{1.6}$$

and $\mathcal{E}$ is a suitable set of adapted processes with values in E. The difference with (1.3) is that the direction of reflection is now controlled by the process $\epsilon \in \mathcal{E}$.

This naturally leads to the introduction of a new class of control problems of the form (1.5), which, to the best of our knowledge, have not been studied so far.

In this paper, we first show that (1.6) admits a strong solution in the case where $\mathcal{O}$ is bounded, $|\gamma| = 1$ and $(\mathcal{O}, \gamma)$ satisfies a uniform exterior sphere condition:

$$\bigcup_{0 \le \lambda \le r} B\left(x - \lambda\gamma(x,e), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } (x,e) \in \partial\mathcal{O} \times \mathbb{R}^\ell \,. \tag{1.7}$$

There is a huge literature on reflected SDEs and we refer to [5] for an overview of mains results. In the case where (X, ϵ) is the solution of a SDE with Lipschitz coefficients, the existence of a strong solution under the exterior sphere condition (1.7) is easily deduced from [4]. Indeed, it suffices to consider the extended system (X, ϵ) reflected at the boundary of $\mathcal{O} \times \tilde{E}$ for some open ball $\tilde{E} = B(0, \tilde{r})$ which contains the compact set E along a smooth direction $\tilde{\gamma}$ such that $\tilde{\gamma} = (\gamma, 0)$ on $\mathcal{O} \times E$ and $\tilde{\gamma} = (\gamma, -e/\tilde{r})/\sqrt{2}$ on $\mathcal{O} \times \partial\tilde{E}$. This system satisfies the exterior sphere condition of [4]. Since ϵ takes values in E, the reflection does not operate on this component and we therefore obtain existence of a solution to (1.6). However, this formulation is quite restrictive and we are interested by a more general class of controls.

We therefore come back to the initial deterministic Skorokhod problem and follow the steps of [4] which are inspired by [7]. The existence to the Skorokhod problem with directions of reflection controlled by a continuous function ϵ with bounded variations is deduced from [4] by using the above arguments which consists in considering an extended system. Since the problem is deterministic and the reflection does not operate on ϵ, we can add jumps to this component without any difficulty. We then use suitable estimates on a family of test functions introduced in [3] to prove the existence of a solution to (1.6) in our general setting. Moreover, by considering SDEs with random coefficients, we are able to incorporate an other control on the direction which takes the form of an Itô process, see Section 2.

We then introduce a control problem which generalizes (1.5) and prove that its value function is a viscosity solution of an equation of the form (1.2), for which we provide a comparison result. In the case where $\gamma(x, e)$ does not depend on e, it essentially follows from the results of [3]. In our general setting, we need to introduce an additional condition which is satisfied whenever (1.2) admits a non-negative subsolution and ρ is independent of x. These results are presented in

Section 3.

In the last section, we discuss the link between (1.5) and the pricing of barrier options under portfolio constraints. In a particular setting, we prove that (1.5) coincides with the super-hedging price of the option, when (1.2) admits a sufficiently smooth solution. This generalizes previous results of [9]. When E is reduced to a singleton, this leads to a natural Monte-Carlo approach for its estimation.

Notations. Given $E \subset \mathbb{R}^m$, $m \geq 1$ and $E_i \subset \mathbb{R}^{m_i}$, $m_i \geq 1$ for $i \leq I$, we denote by $C^{k_1,\ldots,k_I}(E_1 \times \cdots \times E_I, E)$ (resp. $C_b^{k_1,\ldots,k_I}(E_1 \times \cdots \times E_I, E)$) the set of continuous maps φ from $E_1 \times \cdots \times E_I$ into E that admit continuous (resp. bounded) derivatives up to order k_i in their i-th component x_i. We omit k_i when it is equal to 0 and only write $C^{k_1}(E_1 \times \cdots \times E_I, E)$ when $k_1 = k_2 = \ldots = k_I$. We omit E when $E = \mathbb{R}$, and, in this case, we denote by $D_{x_i}\varphi$ and $D^2_{x_i}\varphi$ the (partial) Jacobian and Hessian matrix with respect to x_i. We simply write $D\varphi$ and $D^2\varphi$ for $D_{x_2}\varphi$ and $D^2_{x_2}\varphi$ if $I = 2$. For $T > 0$, we define $\mathrm{BV}([0,T], E)$ as the set of cadlag maps from $[0,T]$ into E with a bounded total variation and a finite number of discontinuities. For $\epsilon \in \mathrm{BV}([0,T], E)$, we set $|\epsilon| := \sum_{i \leq m} |\epsilon^i|$ where $|\epsilon^i|(t)$ is the total variation of ϵ^i on $[0,t]$, $t \geq 0$. We write E^c for $\mathbb{R}^m \setminus E$, ∂E and $\bar{E}$ denote the boundary and the closure of E, $\mathbb{R}^m_+ = [0,\infty)^m$, $\mathbb{R}^m_- = -\mathbb{R}^m_+$. The Euclydian norm of $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$ is denoted by $|x|$, $B(x,r)$ is the open ball centered on x with radius r, $\langle \cdot, \cdot \rangle$ is the natural scalar product on $\mathbb{R}^m$. We denote by $\mathbb{M}^m$ the set of square matrices of dimension m and we extend the definition of $|\cdot|$ to $\mathbb{M}^m$ by identifying $\mathbb{M}^m$ to $\mathbb{R}^{m\times m}$. For $x \in \mathbb{R}^m$, $\mathrm{diag}\,[x]$ is the diagonal matrix of $\mathbb{M}^m$ whose i-th diagonal element is x^i, $\mathrm{Tr}\,[M]$ is the trace of $M \in \mathbb{M}^m$. All inequalities between random variables have to be taken in the a.s. sens.

2 SDEs with controlled reflecting directions

The aim of this section is to construct a stochastic differential equation wich is reflected at the boundary of some bounded open set $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 1$, along a direction which is controlled by an adapted cadlag process with bounded variations and a.s. a finite number of jumps taking values in a compact subset E of $\mathbb{R}^\ell$, $\ell \geq 1$. We follow the arguments of [4] and start with the resolution of the associated (deterministic) Skorokhod problem.

2.1 The Skorokhod problem with controlled reflecting directions

We first recall one of the main results of [4] which provides a solution to the Skorokhod problem for oblique reflection on general bounded sets.

Theorem 2.1 (Dupuis and Ishii [4]) *Fix $\gamma \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ with $|\gamma| = 1$. Assume that there exists some $r \in (0,1)$ such that*

$$\bigcup_{0\le\lambda\le r} B\left(x - \lambda\gamma(x), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } x \in \partial\mathcal{O}\ . \tag{2.1}$$

Then, for all $\psi \in C([0,T], \mathbb{R}^d)$ satisfying $\psi(0) \in \bar{\mathcal{O}}$, there exists $(\phi, \eta) \in C([0,T], \bar{\mathcal{O}}) \times \mathrm{BV}([0,T], \mathbb{R}_+)$ such that

$$\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s))d\eta(s)\ ,\ \eta(t) = \int_0^t \mathbf{1}_{\{\phi(s)\in\partial\mathcal{O}\}} d|\eta|(s)\ \ ,\ t \le T\ .$$

Moreover, $(\phi(t), \eta(t)) \in \sigma(\psi(s), s \le t)$ for all $t \le T$, and uniqueness holds if $\psi \in \mathrm{BV}([0,T], \mathbb{R}^d)$.

Proof. See Theorem 4.8 and the discussion after Corollary 5.2 in [4]. □

We now fix an open bounded set $\mathcal{O} \subset \mathbb{R}^d$ and $\gamma \in C(\mathbb{R}^{d+\ell}, \mathbb{R})$ such that

$$\gamma \in C^2(\mathbb{R}^{d+\ell}, \mathbb{R}^d)\ ,\ |\gamma| = 1 \tag{2.2}$$

$$\exists\, r \in (0,1) \text{ s.t. } \bigcup_{0\le\lambda\le r} B\left(x - \lambda\gamma(x,e), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } (x,e) \in \partial\mathcal{O} \times \mathbb{R}^\ell\ . \tag{2.3}$$

Given a compact set $E \subset \mathbb{R}^\ell$, we deduce from Theorem 2.1 the following result.

Corollary 2.1 *Let the conditions (2.2) and (2.3) hold. Then, for all $\psi \in C([0,T], \mathbb{R}^d) \cap \mathrm{BV}([0,T], \mathbb{R}^d)$ satisfying $\psi(0) \in \bar{\mathcal{O}}$ and $\epsilon \in \mathrm{BV}([0,T], E)$, there exists a unique couple $(\phi, \eta) \in C([0,T], \bar{\mathcal{O}}) \times \mathrm{BV}([0,T], \mathbb{R}_+)$ such that*

$$\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s), \epsilon(s))d\eta(s) \quad \text{and} \quad \eta(t) = \int_0^t \mathbf{1}_{\{\phi(s)\in\partial\mathcal{O}\}} d|\eta|(s)\ ,\ t \le T\ . \tag{2.4}$$

Moreover, $(\phi(t), \eta(t)) \in \sigma((\psi(s), \epsilon(s)), s \le t)$ for all $t \le T$

Proof. 1. We first assume that $\epsilon \in C([0,T], E) \cap \mathrm{BV}([0,T], E)$. Fix $\tilde{r} > 0$ so that $\tilde{E} := B(0, \tilde{r})$ contains E. Fix $\phi \in C^2(\mathbb{R}^\ell, [0,1])$ such that $\phi(e) = 0$ is $e \in E$ and $\phi(e) = 1$ if $e \in \partial\tilde{E}$ and set $\tilde{\gamma}(x,e) = (\gamma(x,e), -e\phi(e)/\tilde{r})/|(\gamma(x,e), -e\phi(e)/\tilde{r})|$ on $\mathbb{R}^{d+\ell}$. Since $|\gamma| = 1$, $|(\gamma(x,e), -e\phi(e)/\tilde{r})| \ge 1$ and $\tilde{\gamma} \in C^2(\mathbb{R}^{d+\ell}, \mathbb{R}^{d+\ell})$. Moreover, since $|(\gamma(x,e), -e\phi(e)/\tilde{r})|^2 \leq 2$ on the closure of $\mathcal{O} \times \tilde{E}$, and $B(e + \lambda e/\tilde{r}, \lambda r) \cap \tilde{E} = \emptyset$ for all $e \in \partial\tilde{E}$ and $\lambda \geq 0$, recall that $r < 1$, we deduce from (2.3) that for $(x,e) \in \partial(\mathcal{O} \times \tilde{E})$ and $\lambda \in [0, r/\sqrt{2}]$

$$|(y,f) - ((x,e) - \lambda\tilde{\gamma}(x,e))|^2 \leq \lambda^2 (r/\sqrt{2})^2 \quad \Rightarrow \quad (y,f) \notin \mathcal{O} \times \tilde{E} \; .$$

We can therefore apply Theorem 2.1 to the couple (ϕ, ϵ) reflected at the boundary of $\mathcal{O} \times \tilde{E}$. Since ϵ does not reach the boundary of $\tilde{E}$, this leads to the required result.
2. The existence and uniqueness result for $\epsilon \in \mathrm{BV}([0,T], E)$ is obtained by constructing the solution to (2.4) between the jump times of ϵ and by pasting the solutions in an obvious manner. □

2.2 The stochastic Skorokhod problem with controlled reflecting direction

We now consider some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a d-dimensional standard Brownian motion W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ the natural filtration induced by W, satisfying the usual conditions, and assume that $\mathcal{F} = \mathcal{F}_T$. Given two uniformly Lipschitz functions μ and σ from $\mathbb{R}^d$ into $\mathbb{R}^d$ and $\mathbb{M}^d$ respectively, it is shown in [4] that, under the condition (2.1), there exists a unique couple (X, L) of $\mathbb{F}$-adapted continuous processes such that L is real valued, has bounded variations and

$$\begin{aligned} X(t) &= x + \textstyle\int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(X(s))dL(s) \\ X(t) &\in \bar{\mathcal{O}} \quad \text{and} \quad L(t) = \textstyle\int_0^t \mathbf{1}_{\{X(s) \in \partial\mathcal{O}\}} \, d|L|(s) \;\; , \; t \leq T \; . \end{aligned} \tag{2.5}$$

The aim of this section is to extend this result to the case where μ and σ are random, and γ is controlled by some bounded variation process with a.s. a finite number of jumps taking values in the compact set E.

In the following, given two subsets E_1 and E_2 of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, $m_1, m_2 \geq 1$, we denote by $L_{\mathbb{F}}(E_1, E_2)$ the set of measurable maps

$$f \quad : \quad (\omega, t, x) \in \Omega \times [0,T] \times E_1 \longrightarrow f_t(\omega, x) \in E_2$$

such that $t \mapsto f_t(\cdot, x)$ is progressively measurable for each $x \in E_1$, and

$$|f_t(\omega, x) - f_t(\omega, y)| \leq K|x - y| \;\; \forall \, x, y \in E_1 \;\; d\mathbb{P}(\omega) - a.s.$$

for some $K > 0$ independent of $(t, \omega) \in [0, T] \times \Omega$. In the sequel, we shall only write $f_t(x)$ for $f_t(\omega, x)$.
We denote by $\mathrm{BV}_{\mathbb{F}}(E_2)$ the set of E_2-valued nondecreasing cadlag adapted processes with bounded variations and $\mathbb{P}-$a.s. a finite number of jumps. For ease of notations, we write $\mathcal{E}$ for $\mathrm{BV}_{\mathbb{F}}(E)$ and we denote by $\mathcal{E}^b$ the set of elements of $\mathcal{E}$ whose total variation on $[0, T]$ is essentially bounded.

In the rest of this section, we fix $(\mu, \sigma) \in L_{\mathbb{F}}(\mathbb{R}^d, \mathbb{R}^d \times \mathbb{M}^d)$ and assume that the conditions (2.2) and (2.3) hold. Our first result extends Theorem 5.1 in [4].

Lemma 2.1 *Let X be a continuous semimartingale with values in $\bar{\mathcal{O}}$. Fix $\epsilon \in \mathcal{E}^b$. Assume that Y is a continuous semimartingale with values in $\bar{\mathcal{O}}$ satisfying for $t \leq T$*

$$Y(t) \quad = \quad X(0) + \int_0^t \mu_s(X(s))ds + \int_0^t \sigma_s(X(s))dW(s) + \int_0^t \gamma(Y(s), \epsilon(s))dL(s) \ ,$$

where L is an element of $\mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ such that

$$L(t) = \int_0^t \mathbf{1}_{\{Y(s) \in \partial \mathcal{O}\}} \, d|L|(s) \ \ , \ \ t \leq T \ .$$

Let X' be an other continuous semimartingales with values in $\bar{\mathcal{O}}$ and assume that (Y', L') satisfies the same properties as (Y, L) with X' in place of X. Then, there is a contant $C > 0$ such that

$$\begin{aligned} &\mathbb{E}\left[\sup_{s \leq t} |Y(s) - Y'(s)|^2 + \int_0^t \left|Y(s) - Y'(s)\right|^2 d(L + L')(s)\right] \\ &\leq C \left(|X(0) - X'(0)|^2 + \int_0^t \mathbb{E}\left[\sup_{0 \leq s \leq u} |X(s) - X'(s)|^2\right] du\right) \quad , \quad t \leq T \ . \end{aligned}$$

In order to prove Lemma 2.1, we shall appeal to the following technical result.

Lemma 2.2 *Given $\theta \in (0, 1)$ there exists a family of functions $(f_\varepsilon)_{\varepsilon > 0}$ in $C^2(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \times E)$ and a constant $K > 0$ independent of $\varepsilon > 0$ such that, for all $(x, y, e) \in$*

$\bar{\mathcal{O}} \times \bar{\mathcal{O}} \times E$,

$$\frac{|x-y|^2}{\varepsilon} \le f_\varepsilon(x,y,e) \le K\left(\varepsilon + \frac{|x-y|^2}{\varepsilon}\right)$$
$$\langle \gamma(x,e), D_x f_\varepsilon(x,y,e)\rangle \le K\frac{|x-y|^2}{\varepsilon} \quad \textit{if} \quad \langle y-x, \gamma(x,e)\rangle \ge -\theta|y-x| \ ,$$
$$\langle \gamma(x,e), D_y f_\varepsilon(x,y,e)\rangle \le K\frac{|x-y|^2}{\varepsilon} \quad \textit{if} \quad \langle x-y, \gamma(y,e)\rangle \ge -\theta|y-x| \ ,$$
$$|D_x f_\varepsilon(x,y,e) + D_y f_\varepsilon(x,y,e)| \vee |D_e f_\varepsilon(x,y,e)| \le K\frac{|x-y|^2}{\varepsilon} \ ,$$
$$|D_x f_\varepsilon(x,y,e)| \vee |D_y f_\varepsilon(x,y,e)| \le K\frac{|x-y|}{\varepsilon} \ ,$$
$$D^2_{(x,y)} f_\varepsilon(x,y,e) \le \frac{C}{\varepsilon}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + K\frac{|x-y|^2}{\varepsilon} I_{2d} \ .$$

Moreover, there is $h \in C^2(\bar{\mathcal{O}} \times E)$ *with non-negative values such that*

$$\langle D_x h(x,e), \gamma(x,e)\rangle \ge 1 \quad \textit{for all } (x,e) \in \partial\mathcal{O} \times E \ .$$

Proof. For $e \in E$, we can define the family $(f_\varepsilon(\cdot,e))_{\varepsilon>0}$ associated to $\gamma(\cdot,e)$ as in [3] and [4]. The bound on $|D_e f_\varepsilon(x,y,e)|$ follows from the construction in the proof of Theorem 4.1 in [3], see in particular page 1136. The existence of h is deduced from [3] and [4] by increasing the dimension of the reflection problem as in 1. of the proof of Corollary 2.1. $\square$

Remark 2.1 Observe that given $\theta \in (0,1)$ such that $\theta^2 > 1 - r^2$, we can find $\delta \in (0,r)$ for which $\langle y-x, \gamma(x,e)\rangle \ge -\theta|y-x|$ for all $e \in E$, $x \in \partial\mathcal{O}$ and $y \in \bar{\mathcal{O}}$ such that $|y-x| \le \delta$. This follows from (2.3).

Proof of Lemma 2.1. First observe that we can always assume that $|Y - Y'| \le \delta$ where δ is defined as in Remark 2.1, for a given $\theta \in (0,1)$. Indeed, we can always replace (X, X', Y, Y', L, L') by $(X, X', Y, Y', L, L')/\eta$ with $\eta \ge 1$ such that $B(0, \eta\delta/2) \supset \mathcal{O}$ and change (μ, σ, γ) accordingly so that the equations in Lemma 2.1 holds for these new processes. From now on, we therefore assume that $|Y - Y'| \le \delta$. Recall the definitions of h and f_ε for θ defined as above. We fix $\varepsilon, \lambda > 0$ and define the smooth function $\tilde{f}_\varepsilon$ on $\bar{\mathcal{O}} \times \bar{\mathcal{O}} \times E$ by

$$\tilde{f}_\varepsilon(x,y,e) := e^{-\lambda(h(x,e)+h(y,e))} f_\varepsilon(x,y,e) \ . \tag{2.6}$$

Fix $\bar{K} > 0$. Applying Itô's Lemma to $(e^{-\bar{K}|\epsilon|(t)}\tilde{f}_\varepsilon(Y(t), Y'(t), \epsilon(t)))_{t \leq T}$ and following the arguments of the proof of Theorem 5.1 in [4], we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\bar{K}|\epsilon|(t)}\left|Y(t) - Y'(t)\right|^2\right] \\
&\leq C_\lambda\left(\varepsilon^2 + |X(0) - X'(0)|^2 + \varepsilon\mathbb{E}\left[A_t + B_t\right]\right) \\
&+C_\lambda\mathbb{E}\left[\int_0^t e^{-\bar{K}|\epsilon|(s)}\left(\left|Y(s) - Y'(s)\right|^2 + \left|X(s) - X'(s)\right|^2\right)\right] ds \\
&+C_\lambda(C - \lambda)\mathbb{E}\left[\int_0^t e^{-\bar{K}|\epsilon|(s)}\left|Y(s) - Y'(s)\right|^2 d(L + L')(s)\right]
\end{aligned}
$$

where C_λ, C are two positive constant such that the second one does not depend on λ, and

$$
\begin{aligned}
A_t &:= \int_0^t e^{-\bar{K}|\epsilon|(s)}\left(\left|D_e\tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s))\right| - \bar{K}\tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s))\right) d|\epsilon|^c(s) \\
B_t &:= \sum_{s \leq t}\left(e^{-\bar{K}|\epsilon|(s)}\tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) - e^{-\bar{K}|\epsilon|(s-)}\tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s-))\right)
\end{aligned}
$$

where ^{c} stands for the continuous part. Using the bounds on f_ε and $D_e f_\varepsilon$ of Lemma 2.2, we observe that $A_t + B_t \leq 0$ for $\bar{K}$ large enough with respect to K and λ. Since $|\epsilon|(T)$ is uniformly bounded, it follows that for $\lambda := 2C$

$$
\begin{aligned}
&\mathbb{E}\left[\left|Y(t) - Y'(t)\right|^2\right] + \mathbb{E}\left[\int_0^t \left|Y(s) - Y'(s)\right|^2 d(L + L')(s)\right] \\
&\leq C'\left(\varepsilon^2 + |X(0) - X'(0)|^2\right) + C'\int_0^t \mathbb{E}\left[\left|Y(s) - Y'(s)\right|^2 + \left|X(s) - X'(s)\right|^2\right] ds
\end{aligned}
$$

where C' is a positive constant. The required result is then obtained by sending $\varepsilon \to 0$ and using Doob's inequality and Gronwall's Lemma. $\square$

We can now provide the main result of this section, which ensures the strong existence and uniqueness of a SDE with random coefficients and controlled reflecting directions.

Theorem 2.2 *Fix $\epsilon \in \mathcal{E}$ and $(t, x) \in [0, T] \times \bar{\mathcal{O}}$. Then, there exists a unique continuous adapted process (X, L) such that $L \in \mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ and*

$$
\begin{aligned}
X(s) &= x + \int_t^s \mu_r(X(r))dr + \int_t^s \sigma_r(X(r))dW(r) + \int_t^s \gamma(X(r), \epsilon(r))dL(r) \\
L(s) &= \int_t^s \mathbf{1}_{\{X(r) \in \partial\mathcal{O}\}}\, d|L|(r)\ , \quad t \leq s \leq T\ . \qquad (2.7)
\end{aligned}
$$

Proof. Observe that Lemma 4.7 in [4] can be easily extended to our setting by appealing to the arguments already used in the proof of Corollary 2.1. The existence when $|\epsilon|(T)$ is uniformly bounded then follows from Corollary 2.1, Lemma 2.1 and the same arguments as in [4], see the discussion after their Corollary 5.2, or as in the proof of Proposition 4.1 in [7]. In the case where $|\epsilon|(T)$ is not uniformly bounded, we use a localization argument. For each $n \geq 1$, we define $\tau_n := \inf\{s \geq t : \mid |\epsilon|(s) \mid \geq n\}$ and let (X^n, L^n) be the unique solution of (2.7) associated to $\epsilon^n(\cdot) := \epsilon(\cdot \wedge \tau_n)$. We then define (X, L) by

$$(X, L)(s) \;\; := \;\; \sum_{n \geq 0} (X^n, L^n)(s) \mathbf{1}_{\tau_n \leq s < \tau_{n+1}} \; ,$$

with the convention $\tau_0 = 0$. It solves (2.7) associated to ϵ. The same argument provides uniqueness. □

Remark 2.2 Let (a, b) be a predictable process with values in $\mathbb{M}^\ell \times \mathbb{R}^\ell$ satisfying

$$\int_0^t (|b(s)| + |a(s)|^2) < \infty$$

and assume that the process Z defined on $[t, T]$ by

$$Z(s) := z + \int_t^s b(r)dr + \int_t^s a(r)dW(r)$$

takes values in a compact set F of $\mathbb{R}^\ell$. Then, it follows from Theorem 2.2 that existence and uniqueness holds for

$$\begin{aligned} X(s) &= x + \int_t^s \mu_r(X(r))dr + \int_t^s \sigma_r(X(r))dW(r) + \int_t^s \tilde{\gamma}(X(r), Z(r), \epsilon(r))dL(r) \\ L(s) &= \int_t^s \mathbf{1}_{\{X(r) \in \partial \mathcal{O}\}} \, d|L|(r) \;\; , \quad t \leq s \leq T \end{aligned}$$

when $\tilde{\gamma} \in C^2(\mathbb{R}^d \times \mathbb{R}^\ell \times \mathbb{R}^\ell, \mathbb{R}^d)$ satisfies

$$\bigcup_{0 \leq \lambda \leq r} B\left(x - \lambda \tilde{\gamma}(x, z, e), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } (x, z, e) \in \partial \mathcal{O} \times \mathbb{R}^{2\ell} \; ,$$

for some $r \in (0, 1)$. This is easily checked by arguing as in the proof of Corollary 2.1. This allows us to introduce a new control on the direction of reflection which corresponds to an Itô process.

3 Optimal control

As in the previous section, we consider a bounded open set $\mathcal{O} \subset \mathbb{R}^d$ and $\gamma \in C^2(\mathbb{R}^{d+\ell}, \mathbb{R}^d)$ such that $|\gamma| = 1$ and (2.3) holds.

3.1 Definitions and assumptions

We fix a compact subset A of $\mathbb{R}^\ell$ and denote by $\mathcal{A}$ the set of predictable processes with values in A.

Let μ and σ be two continuous maps on $\mathbb{R}^d \times A \times E$ with values in $\mathbb{R}^d$ and $\mathbb{M}^d$ respectively. We assume that both are Lipschitz with respect to their first variable uniformly in the two other ones, so that $(\mu^{\alpha,\epsilon}, \sigma^{\alpha,\epsilon})$ defined by

$$(\mu_t^{\alpha,\epsilon}, \sigma_t^{\alpha,\epsilon})(\cdot) := (\mu, \sigma)(\cdot, \alpha(t), \epsilon(t)) \ , \ t \leq T$$

belongs to $L_{\mathbb{F}}(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{M}^d)$ for all $(\alpha, \epsilon) \in \mathcal{A} \times \mathcal{E}$. It then follows from Theorem 2.2 that, for all $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, their exists a unique solution $(X_{t,x}^{\alpha,\epsilon}, L_{t,x}^{\alpha,\epsilon})$ to (2.7) associated to $(\mu^{\alpha,\epsilon}, \sigma^{\alpha,\epsilon})$.

The aim of this section is to provide a PDE characterization for the control problem

$$v(t,x) \quad := \sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}} J(t,x;\alpha,\epsilon) \tag{3.1}$$

where

$$\begin{aligned} J(t,x;\alpha,\epsilon) &:= \mathbb{E}\left[\beta_{t,x}^{\alpha,\epsilon}(T) g\left(X_{t,x}^{\alpha,\epsilon}(T)\right) + \int_t^T \beta_{t,x}^{\alpha,\epsilon}(s) f\left(X_{t,x}^{\alpha,\epsilon}(s), \alpha(s), \epsilon(s)\right) ds\right] , \\ \beta_{t,x}^{\alpha,\epsilon}(s) &:= e^{-\int_t^s \rho(X_{t,x}^{\alpha,\epsilon}(r), \epsilon(r)) dL_{t,x}^{\alpha,\epsilon}(r)} , \end{aligned}$$

and ρ, g, f are continuous real valued maps on $\bar{\mathcal{O}} \times E$, $\bar{\mathcal{O}}$ and $\bar{\mathcal{O}} \times A \times E$ respectively. In order to ensure that J is well defined, we assume that $\rho \geq 0$. We also assume that, as a function on $\bar{\mathcal{O}} \times E$, ρ is C^1 with Lipschitz first derivative in its first variable, uniformly in the second one, and Lipschitz in its second variable, uniformly in the first one.

3.2 Dynamic programming

We first provide some useful estimates on $X_{t,x}^{\alpha,\epsilon}$ and J which will be used to derive the dynamic programming principle of Lemma 3.2 below.

Proposition 3.1 *For all $(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}^b$, there is some constant $C>0$ such that, for all $t\le t'\le T$ and $x,x'\in\bar{\mathcal{O}}$,*

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t'\le s\le T}|X^{\alpha,\epsilon}_{t,x}(s)-X^{\alpha,\epsilon}_{t',x'}(s)|^2\right] &\le C\left(|x-x'|^2+|t'-t|\right)\ , &(3.2)\\
\mathbb{E}\left[\int_{t'}^{T}|X^{\alpha,\epsilon}_{t,x}(s)-X^{\alpha,\epsilon}_{t',x'}(s)|^2 d(L^{\alpha,\epsilon}_{t',x'}(s)+L^{\alpha,\epsilon}_{t,x}(s))\right] &\le C\left(|x-x'|^2+|t'-t|\right)\ , &(3.3)\\
\mathbb{E}\left[\sup_{t\le s\le t'}|X(s)-x|^2\right]^{\frac{1}{2}}+\mathbb{E}\left[L^{\alpha,\epsilon}_{t,x}(t')\right] &\le C\ |t'-t|^{\frac{1}{2}}\ , &(3.4)\\
\mathbb{E}\left[\sup_{t'\le s\le T}|\ln(\beta^{\alpha,\epsilon}_{t,x}(s))-\ln(\beta^{\alpha,\epsilon}_{t',x'}(s))|\right] &\le C\left(|x-x'|^2+|t'-t|\right)^{\frac{1}{2}} &(3.5)
\end{aligned}
$$

Proof. We write (X,L,β) and (X',L',β') for $(X^{\alpha,\epsilon}_{t,x},L^{\alpha,\epsilon}_{t,x},\beta^{\alpha,\epsilon}_{t,x})$ and $(X^{\alpha,\epsilon}_{t',x'},L^{\alpha,\epsilon}_{t',x'},\beta^{\alpha,\epsilon}_{t',x'})$. It follows from Lemma 2.1 that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t'\le s\le T}|X(s)-X'(s)|^2+\int_{t'}^{T}|X(s)-X'(s)|^2 d(L'(s)+L(s))\right]\\
&\le C\ \mathbb{E}\left[|X(t')-x'|^2\right]
\end{aligned}
$$

where $C>0$ denotes a generic constant independent of (t,t',x,x'). Choosing some large $\bar{K}>0$, applying Itô's Lemma to $(e^{-\bar{K}|\epsilon|(t)}\tilde{f}_\varepsilon(X(t),y,\epsilon(t)))_{t\le T}$, $y\in\bar{\mathcal{O}}$ and $\tilde{f}_\varepsilon$ defined as in (2.6), and using the same arguments as in Lemma 2.1 leads to

$$
\mathbb{E}\left[\sup_{t\le s\le t'}|X(s)-y|^2+\int_{t}^{t'}|X(s)-y|^2 dL(s)\right]\ \le\ C\ (|t'-t|+|x-y|^2)\ . \qquad (3.6)
$$

This proves (3.2) and (3.3).

We now prove (3.4). Recalling that $\gamma\in C^2(\mathbb{R}^{d+\ell},\mathbb{R}^d)$ and $|\gamma|=1$, we deduce from Itô's Lemma applied to $\langle X,\gamma(x,\epsilon)\rangle-\langle x,\gamma(x,\epsilon)\rangle$ and Cauchy-Schwartz inequality that

$$\begin{aligned}
\mathbb{E}\left[L(t')\right] &= \mathbb{E}\left[\int_t^{t'} |\gamma(X(s),\epsilon(s))|^2 dL(s)\right] \\
&= \mathbb{E}\left[\int_t^{t'} \langle \gamma(x,\epsilon(s)),\gamma(X(s),\epsilon(s))\rangle dL(s)\right] \\
&+ \mathbb{E}\left[\int_t^{t'} \langle \gamma(X(s),\epsilon(s))-\gamma(x,\epsilon(s)),\gamma(X(s),\epsilon(s))\rangle dL(s)\right] \\
&\leq C\,\mathbb{E}\left[|t'-t| + \sup_{t\leq s\leq t'} |X(s)-x|^2\right] \\
&+ C\,\mathbb{E}\left[\left(\int_t^{t'} |X(s)-x|^2 dL(s)\right)^{\frac{1}{2}} L(t')^{\frac{1}{2}}\right]
\end{aligned}$$

which in view of (3.6) and Cauchy-Schwartz inequality implies that

$$\mathbb{E}\left[L(t')\right] \leq C\left(|t'-t| + |t'-t|^{\frac{1}{2}}\mathbb{E}\left[L(t')\right]^{\frac{1}{2}}\right) .$$

This proves (3.4).

We finally prove (3.5). We first assume that $\rho \in C^{2,1}(\mathbb{R}^{d+\ell},\mathbb{R})$ and apply Itô's Lemma to $\langle X-X', \gamma(X,\epsilon)\rho(X,\epsilon)\rangle$ on $[t',T]$. Using the above estimates, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t'\leq s\leq T} |\ln\beta(s)-\ln\beta'(s)|\right] \\
&= \mathbb{E}\left[\sup_{t'\leq s\leq T} |\int_t^s (\rho|\gamma|^2)(X(s),\epsilon(s))dL(s) - \int_{t'}^s (\rho|\gamma|^2)(X'(s),\epsilon(s))dL'(s)|\right] \\
&\leq C\left(\mathbb{E}\left[L(t')\right] + \mathbb{E}\left[\sup_{t'\leq s\leq T}|X(s)-X'(s)|^2\right]^{\frac{1}{2}}\right) \\
&+ C\mathbb{E}\left[\sup_{t'\leq s\leq T} |\int_{t'}^s \langle \rho\gamma(X'(s),\epsilon(s))-\rho\gamma(X(s),\epsilon(s)),\gamma(X'(s),\epsilon(s)\rangle dL'(s)|\right]
\end{aligned}$$

where C depends on ρ only through the bounds on $|\rho|$, on the first and second derivatives in its first variable and on the first derivative in its second variable. In view of the previous estimates and Cauchy-Schwartz inequality, the result follows for ρ smooth enough. Since the estimate of (3.4) clearly does not depend on ρ, this result is easily extended to the general case by a standard approximation argument.

□

Lemma 3.1 *The following holds.*
(i) $J(\cdot;\alpha,\epsilon)$ *is continuous on* $[0,T]\times\bar{\mathcal{O}}$ *for all* $(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}^b$.
(ii) $\sup\limits_{\epsilon\in\mathcal{E}^b} J(t,x;\alpha,\epsilon)=\sup\limits_{\epsilon\in\mathcal{E}} J(t,x;\alpha,\epsilon)$ *for all* $\alpha\in\mathcal{A}$ *and* $(t,x)\in[0,T]\times\bar{\mathcal{O}}$.
(iii) v *is lower semicontinuous on* $[0,T]\times\bar{\mathcal{O}}$.

Proof. Combining the estimates of Proposition 3.1 with a dominated convergence argument leads to (i) which implies (iii). Item (ii) is proved by using a localization argument as in the proof of Theorem 2.2. □

We can now prove the following dynamic programming principle.

Lemma 3.2 *Fix* $(t,x)\in[0,T)\times\bar{\mathcal{O}}$. *For all* $[t,T]$*-valued stopping time* θ, *we have*

$$v(t,x)=\sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}}\mathbb{E}\left[\beta^{\alpha,\epsilon}_{t,x}(\theta)v\left(\theta,X^{\alpha,\epsilon}_{t,x}(\theta)\right)+\int_t^\theta\beta^{\alpha,\epsilon}_{t,x}(s)f\left(X^{\alpha,\epsilon}_{t,x}(s),\alpha(s),\epsilon(s)\right)ds\right].$$

Proof. Fix $(t_0,x_0)\in[0,T)\times\bar{\mathcal{O}}$. The fact that $v(t_0,x_0)$ is bounded from above by

$$\sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}}\mathbb{E}\left[\beta^{\alpha,\epsilon}_{t_0,x_0}(\theta)v\left(\theta,X^{\alpha,\epsilon}_{t_0,x_0}(\theta)\right)+\int_{t_0}^\theta\beta^{\alpha,\epsilon}_{t_0,x_0}(s)f\left(X^{\alpha,\epsilon}_{t_0,x_0}(s),\alpha(s),\epsilon(s)\right)ds\right]$$

follows from the Markov feature of our model. We now prove the converse inequality.
1. Let φ be a continuous map on $[0,T]\times\mathbb{R}^d$ such that

$$\varphi\ \le\ v\quad\text{on}\quad[0,T]\times\bar{\mathcal{O}}\,. \tag{3.7}$$

Let $(B_n)_{n\ge1}$ be a partition of $[0,T]\times\bar{\mathcal{O}}$ and $(t_n,x_n)_{n\ge1}$ be a sequence such that $(t_n,x_n)\in B_n$ for each $n\ge1$. It follows from Lemma 3.1 that, for each $n\ge1$, we can find $\xi^n:=(\alpha^n,\epsilon^n)\in\mathcal{A}\times\mathcal{E}^b$ such that

$$J(t_n,x_n,\xi^n)\ge v(t_n,x_n)-\varepsilon/3\,, \tag{3.8}$$

where $\varepsilon>0$ is a fix parameter. Moreover, by continuity of φ and $J(\cdot,\xi)$ for $\xi\in\mathcal{A}\times\mathcal{E}^b$, see Lemma 3.1, we can choose $(B_n,t_n,x_n)_{n\ge1}$ in such a way that

$$|\varphi-\varphi(t_n,x_n)|+|J(\cdot,\xi^n)-J(t_n,x_n,\xi^n)|\le\varepsilon/3\quad\text{on } B_n\,. \tag{3.9}$$

2. Given $\xi\in\mathcal{A}\times\mathcal{E}^b$ and θ a stopping time with values in $[t_0,T]$, we define $\bar\xi\in\mathcal{A}\times\mathcal{E}^b$ by

$$\bar\xi(t)\ \ :=\ \ \xi(t)\mathbf{1}_{t<\theta}+\mathbf{1}_{t\ge\theta}\sum_{n\ge1}\xi^n(t)\ \mathbf{1}_{\{(\theta,X^{\xi}_{t_0,x_0}(\theta))\in B_n\}}\,.$$

It follows from (3.7), (3.8), (3.9) and the Markov feature of our model that, for all $\xi \in \mathcal{A} \times \mathcal{E}^b$,

$$
\begin{aligned}
J(t_0, x_0, \bar{\xi}) &\geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) J(\theta, X^{\xi}_{t_0,x_0}(\theta), \bar{\xi}) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \xi(s)) ds\right] \\
&\geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) \varphi(\theta, X^{\xi}_{t_0,x_0}(\theta)) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \xi(s)) ds\right] - \varepsilon .
\end{aligned}
$$

By arbitrariness of $\varepsilon > 0$, this shows that

$$
v(t_0, x_0) \geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) \varphi(\theta, X^{\xi}_{t_0,x_0}(\theta)) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \xi(s)) ds\right] . \tag{3.10}
$$

3. By replacing φ by a sequence $(\varphi_k)_{k\geq 1}$ of continuous functions satisfying

$$
\varphi_k \leq v \quad \text{ and } \quad \varphi_k \to v \;\text{ on }\; [0,T] \times \bar{\mathcal{O}} ,
$$

we deduce from (3.10) and the dominated convergence Theorem that, for all $\xi \in \mathcal{A} \times \mathcal{E}^b$,

$$
v(t_0, x_0) \geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) v(\theta, X^{\xi}_{t_0,x_0}(\theta)) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \xi(s)) ds\right] .
$$

Using the lower semi-continuity of v and the same localization argument as in the proof of Theorem 2.2 shows that the above inequality actually holds for all $\xi \in \mathcal{A} \times \mathcal{E}$.

□

3.3 PDE characterization for the optimal control problem

In this section, we show that v is a solution of

$$
\mathcal{K}\varphi \;=\; 0
$$

where

$$
\mathcal{K}\varphi \;:=\; \begin{cases}
\min\limits_{(a,e)\in A\times E} (-\mathcal{L}^{a,e}\varphi - f(\cdot, a, e)) = 0 & \text{on} \quad [0,T) \times \mathcal{O} \\
\min\limits_{e\in E} \mathcal{H}^e \varphi = 0 & \text{on} \quad [0,T) \times \partial\mathcal{O} \\
\varphi - g = 0 & \text{on} \quad \{T\} \times \bar{\mathcal{O}}
\end{cases}
$$

and for a smooth function φ on $[0,T] \times \bar{\mathcal{O}}$ and $(a,e) \in A \times E$, we set

$$
\begin{aligned}
\mathcal{L}^{a,e}\varphi &:= \frac{\partial}{\partial t}\varphi + \langle \mu(\cdot, a, e), D\varphi \rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma(\cdot, a, e)\sigma(\cdot, a, e)' D^2\varphi\right] + f(\cdot, a, e) \\
\mathcal{H}^e\varphi &:= \rho(\cdot, e)\varphi - \langle \gamma(\cdot, e), D\varphi \rangle ,
\end{aligned}
$$

where σ' is the transposed matrix associated to σ.

3.3.1 Definitions

Since v may not be smooth, we need to consider the above equation in the viscosity sens. Moreover, the boundary conditions may not be satisfied in a strong sens and, as usual, we have to consider a relaxed version, see e.g. [2]. We therefore introduce the operator $\mathcal{K}_+$ and $\mathcal{K}_-$ defined as

$$\mathcal{K}_+\varphi \;:=\; \begin{cases} \mathcal{K}\varphi & \text{on} \quad [0,T]\times\mathcal{O} \\ \min\limits_{(a,e)\in A\times E} \max\left\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a,e)\;,\;\mathcal{H}^e\varphi\right\} & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - g & \text{on} \quad \{T\}\times\partial\mathcal{O} \end{cases}$$

and

$$\mathcal{K}_-\varphi \;:=\; \begin{cases} \mathcal{K}\varphi & \text{on} \quad [0,T]\times\mathcal{O} \\ \min\limits_{(a,e)\in A\times E} \min\left\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a,e)\;,\;\mathcal{H}^e\varphi\right\} & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \min\left\{\varphi - g\;,\;\mathcal{H}^e\varphi\right\} & \text{on} \quad \{T\}\times\partial\mathcal{O}\;. \end{cases}$$

Definition 3.1 *We say that a lower-semicontinuous (resp. upper-semicontinuous) function w on $[0,T]\times\bar{\mathcal{O}}$ is a viscosity supersolution (resp. subsolution) of*

$$\mathcal{K}\varphi = 0 \tag{3.11}$$

if for all $\varphi \in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ and all $(t,x)\in[0,T)\times\bar{\mathcal{O}}$ which realizes a local minimum (resp. maximum) of $w-\varphi$, we have $\mathcal{K}_+\varphi \geq 0$ (resp. $\mathcal{K}_-\varphi\leq 0$) We say that a locally bounded function w is a (discontinuous) viscosity solution of (3.11) *if w_* (resp. w^*) is a supersolution (resp. subsolution) of* (3.11) *where*

$$\begin{aligned} w^*(t,x) \;&:=\; \limsup_{(t',x')\to(t,x),\,(t',x')\in D} w(t',x') \\ w_*(t,x) \;&:=\; \liminf_{(t',x')\to(t,x),\,(t',x')\in D} w(t',x')\;\;,\;\;(t,x)\in[0,T]\times\bar{\mathcal{O}}\;, \end{aligned}$$

with $D := [0,T)\times\mathcal{O}$.

Remark 3.1 Take $E = \tilde{K}_1 := \tilde{K}\cap\partial B(0,1)$ where $\tilde{K}$ is the domain of the support function δ of a closed convex set $K\subset\mathbb{R}^\ell$, i.e.

$$\delta(e) \;:=\; \sup_{y\in K}\langle y,e\rangle \;\;,\;\; e\in\mathbb{R}^\ell\;,$$

and assume that $\rho(x,e) = \delta(e)$ and $\gamma(x,e) = e$ on $\partial\mathcal{O}\times E$. Then, for $\varphi\in C^1(\bar{\mathcal{O}},(0,\infty))$, the constraint

$$\min_{e\in E}\mathcal{H}^e\varphi = \min_{e\in E}\left(\delta(e)\varphi - \langle e, D\varphi\rangle\right) \geq 0$$

means that $D\varphi/\varphi \in K$, see e.g. [8]. In this case, the term $\mathcal{H}^e\varphi \geq 0$ can be assimilated to a constraint on the gradient of the logarithm of the solution at the boundary of $\mathcal{O}$. A similar constraint appears in [1], but in the whole domain.

Remark 3.2 Assume that $\mathcal{O}$ is C^2 and that σ satisfies the non-characteristic boundary condition

$$\min_{(a,e)\in A\times E} |\sigma(x,a,e)\xi| > 0 \quad \text{for all} \quad x \in \partial\mathcal{O} \text{ and } \xi \in \mathbb{R}^d \setminus \{0\} . \tag{3.12}$$

Then, it follows from the same arguments as in 2. of the proof of Proposition 6.3 of [1] that w is a supersolution of $\mathcal{K}_+\varphi = 0$ only if it is a supersolution of $\bar{\mathcal{K}}_+\varphi = 0$ where

$$\bar{\mathcal{K}}_+\varphi \ := \ \begin{cases} \mathcal{K}_+\varphi & \text{on} \quad ([0,T]\times\mathcal{O}) \cup (\{T\}\times\bar{\mathcal{O}}) \\ \min\limits_{e\in E} \mathcal{H}^e\varphi & \text{on} \quad [0,T)\times\partial\mathcal{O} \end{cases}$$

Similarly, it follows from the same arguments as in 2. of Proposition 6.6 in [1] that w is a subsolution of $\mathcal{K}_-\varphi = 0$ only if it is a subsolution of $\bar{\mathcal{K}}_-\varphi = 0$ where

$$\bar{\mathcal{K}}_-\varphi \ := \ \begin{cases} \mathcal{K}_-\varphi & \text{on} \quad ([0,T]\times\mathcal{O}) \cup (\{T\}\times\bar{\mathcal{O}}) \\ \min\limits_{e\in E} \mathcal{H}^e\varphi & \text{on} \quad [0,T)\times\partial\mathcal{O} . \end{cases}$$

3.3.2 Super and subsolution properties

Proposition 3.2 *The function v_* is a viscosity supersolution of* (3.11)*.*

Proof. The fact that $v_* \geq g$ on $\{T\}\times\bar{\mathcal{O}}$ is a direct consequence of Proposition 3.1 and the continuity of g. Fix $(t_0,x_0) \in [0,T)\times\bar{\mathcal{O}}$ and $\varphi \in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ such that

$$0 \ = \ (v_*-\varphi)(t_0,x_0) = \min_{[0,T]\times\bar{\mathcal{O}}} (v_*-\varphi) .$$

1. We first assume that $(t_0,x_0) \in [0,T)\times\partial\mathcal{O}$ and that

$$\min_{(a,e)\in A\times E} \max\left\{-\mathcal{L}^{a,e}\varphi(t_0,x_0) - f(x_0,a,e) \ , \ \mathcal{H}^e\varphi(t_0,x_0)\right\} = -2\varepsilon < 0$$

and work toward a contradiction. Under the above assumption, we can find $(a_0,e_0) \in A\times E$ and $\delta > t_0$ for which

$$\max\left\{-\mathcal{L}^{a_0,e_0}\varphi - f(\cdot,a_0,e_0) \ , \ \mathcal{H}^{e_0}\varphi\right\} \leq -\varepsilon \tag{3.13}$$

on $\bar{B}_0 \cap \bar{D}_0$ where $B_0 := B(t_0, \delta) \times B(x_0, \delta)$ and $D_0 := (t_0 - \delta, t_0 + \delta) \times \mathcal{O}$. Observe that we can assume, without loss of generality, that (t_0, x_0) achieves a strict local minimum so that

$$\inf_{\partial_p B_0 \cap \bar{D}_0} (v_* - \varphi) =: \zeta > 0 \,, \tag{3.14}$$

where $\partial_p B_0 = ([t_0 - \delta, t_0 + \delta] \times \partial B(x_0, \delta)) \cup (\{t_0 + \delta\} \times B(x_0, \delta))$. Let $(t_k, x_k)_{k \geq 1}$ be a sequence in $B_0 \cap D_0$ satisfying

$$(t_k, x_k) \longrightarrow (t_0, x_0) \quad \text{and} \quad v(t_k, x_k) \longrightarrow v_*(t_0, x_0) \quad \text{as } k \longrightarrow \infty$$

so that

$$\eta_k := v(t_k, x_k) - \varphi(t_k, x_k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \,. \tag{3.15}$$

Let us write (X^k, L^k, β^k) for $(X^{a_0,e_0}_{t_k,x_k}, L^{a_0,e_0}_{t_k,x_k}, \beta^{a_0,e_0}_{t_k,x_k})$ where (a_0, e_0) is viewed as an element of $\mathcal{A} \times \mathcal{E}$. Set

$$\theta^k := \inf\left\{s \geq t_k \;:\; (s, X^k(s)) \notin B_0\right\} \quad , \quad \vartheta^k := \inf\left\{s \geq t_k \;:\; X^k(s) \notin \mathcal{O}\right\} \,.$$

It then follows from Itô's Lemma, (3.13) and (3.14) that

$$\begin{aligned} v(t_k, x_k) \;&\leq\; \eta_k + \mathbb{E}\left[\beta^k(\theta^k) v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s), a_0, e_0)\right] \\ &\quad - \; \mathbb{E}\left[\zeta \mathbf{1}_{\theta^k < \vartheta^k} + \left(\beta^k(\theta^k)\zeta + \varepsilon L^k(\theta^k)\right) \mathbf{1}_{\theta^k \geq \vartheta^k}\right] \end{aligned}$$

where we used the fact that $\beta^k(\theta^k) = 1$ on $\{\theta^k < \vartheta^k\}$. Let $c > 0$ be such that $|\rho| \leq c$ on $\bar{\mathcal{O}} \times E$ and observe that

$$\nu := \inf_{\ell \in [0,\infty)} e^{-c\ell}\zeta + \varepsilon \ell > 0 \,.$$

It follows that

$$v(t_k, x_k) \;\leq\; \eta_k - \zeta \wedge \nu + \mathbb{E}\left[\beta^k(\theta^k) v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s), a_0, e_0)\right]$$

which leads to a contradiction to Lemma 3.2 for k large enough, recall (3.15).
2. The case where $(t_0, x_0) \in D$ is treated similarly. It suffices to take δ small enough so that $B(x_0, \delta) \subset \mathcal{O}$ and therefore $\theta^k < \vartheta^k$. □

Proposition 3.3 *The function v^* is a viscosity subsolution of* (3.11).

Proof. Fix $(t_0,x_0) \in [0,T) \times \bar{\mathcal{O}}$ and $\varphi \in C^{1,2}([0,T] \times \bar{\mathcal{O}})$ such that

$$0 \;=\; (v_* - \varphi)(t_0,x_0) = \max_{[0,T]\times\bar{\mathcal{O}}} (v_* - \varphi) \;.$$

The case where $(t_0,x_0) \in [0,T) \times \bar{\mathcal{O}}$ is treated by similar arguments as in the proof of Proposition 3.2, see also below. We therefore assume that $t_0 = T$.

1. We first consider the case where $x_0 \in \partial\mathcal{O}$. We assume that

$$\min_{(a,e)\in A\times E} \min\left\{\varphi - g \;,\; \mathcal{H}^e\varphi\right\} := 2\varepsilon > 0 \;.$$

Set $\phi(t,x) = \varphi(t,x) + \sqrt{T-t}$ so that $(\partial/\partial t)\phi(t,x) \to -\infty$ as $t \to T$ and observe that (T,x_0) also achieves a maximum for $v^* - \phi$. Without loss of generality, we can therefore assume that $(\partial/\partial t)\varphi(t,x) \to -\infty$ as $t \to T$ and that we can find $\delta \in (t_0, T - t_0)$ for which

$$\min_{(a,e)\in A\times E} \min\left\{-\mathcal{L}^{a,e}\varphi - f(\cdot,a,e) \;,\; \varphi - g \;,\; \mathcal{H}^e\varphi\right\} \geq \varepsilon \tag{3.16}$$

on $\bar{B}_0 \cap \bar{D}_0$ where $B_0 := [t_0 - \delta, T) \times B(x_0,\delta)$ and $D_0 := (t_0 - \delta, T) \times \mathcal{O}$. Observe that we can assume, without loss of generality, that (t_0,x_0) achieves a strict local maximum so that

$$\max_{\partial_p B_0 \cap \bar{D}_0} (v^* - \varphi) =: -\zeta < 0 \;, \tag{3.17}$$

where $\partial_p B_0 = ([t_0 - \delta, T] \times \partial B(x_0,\delta)) \cup (\{T\} \times B(x_0,\delta))$. Let $(t_k,x_k)_{k\geq 1}$ be a sequence in $B_0 \cap D_0$ satisfying

$$(t_k,x_k) \;\longrightarrow\; (t_0,x_0) \;\text{ and }\; v(t_k,x_k) \;\longrightarrow\; v_*(t_0,x_0) \quad \text{as } k \longrightarrow \infty$$

so that

$$\eta_k := v(t_k,x_k) - \varphi(t_k,x_k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \;. \tag{3.18}$$

Let us write (X^k, L^k, β^k) for $(X^{\alpha,\epsilon}_{t_k,x_k}, L^{\alpha,\epsilon}_{t_k,x_k}, \beta^{\alpha,\epsilon}_{t_k,x_k})$ where (α,ϵ) is a given element of $\mathcal{A} \times \mathcal{E}$. Set

$$\theta^k := \inf\left\{s \geq t_k \;:\; (s, X^k(s)) \notin B_0\right\} \quad , \quad \vartheta^k := \inf\left\{s \geq t_k \;:\; X^k(s) \notin \mathcal{O}\right\} \;.$$

It follows from Itô's Lemma, (3.16), (3.17) and the identity $v(T,\cdot) = g$ that

$$\begin{aligned} v(t_k,x_k) \;&\geq\; \eta^k + \mathbb{E}\left[\beta^k(\theta^k) v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s),\alpha(s),\epsilon(s))ds\right] \\ &+\;\; \mathbb{E}\left[\zeta \mathbf{1}_{\theta^k < \vartheta^k} + \left(\beta^k(\theta^k)(\zeta \wedge \varepsilon) + \varepsilon L^k(\theta^k)\right) \mathbf{1}_{\theta^k \geq \vartheta^k}\right] \;. \end{aligned}$$

Arguing as in 1. of the proof of Proposition 3.2, this implies that

$$\begin{aligned}
v(t_k, x_k) &\geq \eta^k + \zeta \wedge \nu \\
&+ \mathbb{E}\left[\beta^k(\theta^k) v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s), \alpha(s), \epsilon(s)) ds\right]
\end{aligned}$$

for some $\nu > 0$. By arbitrariness of (α, ϵ) and (3.18), this leads to a contradiction to Lemma 3.2 for k large enough.
2. The case where $x_0 \in \mathcal{O}$ is treated similarly, it suffices to take δ small enough so that $B(x_0, \delta) \subset \mathcal{O}$ and therefore $\theta^k < \vartheta^k$. □

3.4 A comparison result

Proposition 3.4 *Let u (resp. w) be a bounded upper-semicontinuous viscosity subsolution (resp. lower-semicontinuous viscosity supersolution) of* (3.11)*. Assume that $u \geq 0$ on $[0, T] \times \partial\mathcal{O}$ and*

$$\hat{e} \in \arg\min\{\rho(x, e)\ ,\ e \in E\}$$

is independent of $x \in \partial\mathcal{O}$. Then, $u \leq w$ on $[0, T] \times \bar{\mathcal{O}}$.

Proof. We argue by contradiction and assume that $\max_{\bar{D}}(u - w) > 0$, with $D := [0, T) \times \mathcal{O}$. We can then find $\varepsilon > 0$ small enough and $(t_0, x_0) \in \bar{D}$ such that

$$\max_{\bar{D}}(\tilde{u} - \tilde{w} - 2\varepsilon H) = (\tilde{u} - \tilde{w} - 2\varepsilon H)(t_0, x_0) =: \eta > 0 \tag{3.19}$$

where $\tilde{u}(t, x) = e^{rt} u(t, x)$, $\tilde{w}(t, x) = e^{rt} w(t, x)$ and $H(t, x) := e^{-rt - h(x, \hat{e})}$ where h is defined as in Lemma 2.2 and $r > 0$ is a constant parameter such that

$$-\mathcal{L}^{a,e} H \geq 0 \quad \text{on } \bar{D} \ \text{ for all } (a, e) \in A \times E\ . \tag{3.20}$$

Given $\lambda \in \mathbb{N}$, we next define

$$\Phi_\lambda(t, x, y) := \tilde{u}(t, x) - \tilde{w}(t, y) - \Psi_\lambda(t, x, y)$$

where

$$\begin{aligned}
\Psi_\lambda(t, x, y) &:= \varepsilon(H(t, x) + H(t, y)) + \rho(x_0, \hat{e}) u(t_0, x_0) \langle \gamma(x_0, \hat{e})\ ,\ x - y \rangle \\
&\quad + \frac{\lambda}{2}|x - y|^2 + |t - t_0|^2 + |x - x_0|^4\ .
\end{aligned}$$

Let $(t_\lambda, x_\lambda, y_\lambda)$ be a global maximum point for Φ_λ on $\bar{D}$. Using standard arguments, one easily checks that

$$(t_\lambda, x_\lambda) \to (t_0, x_0)\ ,\ \lambda|x_\lambda - y_\lambda|^2 \to 0\ ,\ (\tilde{u}(t_\lambda, x_\lambda), \tilde{w}(t_\lambda, y_\lambda)) \to (\tilde{u}(t_0, x_0), \tilde{w}(t_0, x_0)) \tag{3.21}$$

as $\lambda \to \infty$.

1. Assume that $x_\lambda \in \partial\mathcal{O}$ for all λ. Fix $e \in E$. Since $y_\lambda \in \bar{\mathcal{O}}$, it follows from (2.3) that $|x_\lambda - r\gamma(x_\lambda, e) - y_\lambda|^2 \geq r^2$. Since $|\gamma| = 1$, this implies

$$2\langle \gamma(x_\lambda, e)\ ,\ y_\lambda - x_\lambda \rangle \geq -r^{-1}|x_\lambda - y_\lambda|^2\ . \tag{3.22}$$

Then, it follows from the definition of $\hat{e}$, the assumption $u \geq 0$ on $[0, T] \times \partial\mathcal{O}$, (3.21) and (3.22) that

$$\begin{aligned}
&\rho(x_\lambda, e)u(t_\lambda, x_\lambda) - \langle \gamma(x_\lambda, e)\ ,\ D_x\Psi_\lambda(t_\lambda, x_\lambda, y_\lambda)\rangle \\
&\geq (\rho(x_0, e) - \rho(x_0, \hat{e}))u(t_0, x_0) + \rho(x_0, \hat{e})u(t_0, x_0)(1 - \langle \gamma(x_0, e)\ ,\ \gamma(x_0, \hat{e})\rangle) \\
&+O(\lambda^{-1}) - \langle \gamma(x_\lambda, e)\ ,\ \lambda(x_\lambda - y_\lambda) - \varepsilon Dh(t_\lambda, x_\lambda)H(t_\lambda, x_\lambda)\rangle \\
&\geq O(\lambda^{-1}) + \varepsilon H(t_0, x_0)\ .
\end{aligned}$$

Arguing as above, using the inequalities $\rho \geq 0$, $u(t_0, x_0) \geq w(t_0, x_0)$ and observing that $\langle \gamma(y_\lambda, \hat{e})\ ,\ \gamma(x_0, \hat{e})\rangle \to 1$, we also deduce that

$$\rho(y_\lambda, \hat{e})w(t_\lambda, y_\lambda) - \langle \gamma(y_\lambda, \hat{e})\ ,\ -D_y\Psi_\lambda(t_\lambda, x_\lambda, y_\lambda)\rangle \quad \leq \quad O(\lambda^{-1}) - \varepsilon H(t_0, x_0)$$

if $y_\lambda \in \partial\mathcal{O}$ for all λ.

2. We now assume that $t_\lambda = T$ for all $\lambda > 0$. In view of 1. and Ishii's Lemma, see [2] and 4. below, we must have $u(t_\lambda, x_\lambda) \leq g(x_\lambda)$ and $g(y_\lambda) \leq w(t_\lambda, y_\lambda)$, after possibly passing to a subsequence. Since g is continuous, we deduce from (3.21) that $u(t_0, x_0) \leq w(t_0, w_0)$ which contradicts the definition of (t_0, x_0).

3. Observe that $\tilde{u}$ and $\tilde{w}$ are viscosity super- and subsolutions of $\tilde{\mathcal{K}}_+\varphi = 0$ and $\tilde{\mathcal{K}}_-\varphi = 0$ where $\tilde{\mathcal{K}}_+$ and $\tilde{\mathcal{K}}_-$ are defined as $\mathcal{K}_+$ and $\mathcal{K}_-$ with $\mathcal{L}^{a,e}$ replaced by $\tilde{\mathcal{L}}^{a,e}$ defined by

$$\tilde{\mathcal{L}}^{a,e}\varphi = -r\varphi + \mathcal{L}^{a,e}\varphi\ .$$

4. The rest of the proof is standard. Using Ishii's Lemma, see Theorem 8.3 in [2], we deduce that we can find $p_{\lambda,1}, p_{\lambda,2} \in \mathbb{R}$ and two symetric matrices $X_{\eta,\lambda}$ and $Y_{\eta,\lambda}$, depending on a parameter $\eta > 0$, such that

$$\begin{aligned}
(p_{\lambda,1}, D_x\Psi_\lambda(t_\lambda, x_\lambda, y_\lambda), X_{\eta,\lambda}) &\in \bar{\mathcal{P}}^{2,+}_{\bar{\mathcal{O}}}\tilde{u}(t_\lambda, x_\lambda) \\
(p_{\lambda,2}, -D_y\Psi_\lambda(t_\lambda, x_\lambda, y_\lambda), Y_{\eta,\lambda}) &\in \bar{\mathcal{P}}^{2,-}_{\bar{\mathcal{O}}}\tilde{w}(t_\lambda, y_\lambda)
\end{aligned}$$

and

$$
\begin{aligned}
& p_{\lambda,1} - p_{\lambda,2} = 2(t_\lambda - t_0) - r\varepsilon(H(t_\lambda, x_\lambda) + H(t_\lambda, y_\lambda)) \\
& \left(\begin{array}{cc} X_{\eta,\lambda} & 0 \\ 0 & -Y_{\eta,\lambda} \end{array} \right) \le (A_\lambda + B_\lambda) + \eta\,(A_\lambda + B_\lambda)^2
\end{aligned}
$$

where

$$
\begin{aligned}
A_\lambda &:= \varepsilon \left(\begin{array}{cc} D^2 H(t_\lambda, x_\lambda) & 0 \\ 0 & D^2 H(t_\lambda, y_\lambda) \end{array} \right) + 12(x_\lambda - x_0) \otimes (x_\lambda - x_0) \\
B_\lambda &:= \lambda \left(\begin{array}{cc} I_d & -I_d \\ -I_d & I_d \end{array} \right) ,
\end{aligned}
$$

see [2] for the notations $\bar{\mathcal{P}}^{2,+}_{\bar{\mathcal{O}}}$ and $\bar{\mathcal{P}}^{2,-}_{\bar{\mathcal{O}}}$. It follows from (3.20), 1., the fact that $H(t_0, x_0) > 0$, 2. and 3. that, after possibly passing to a subsequence, we may find $(a_\lambda, e_\lambda) \in A \times E$ such that

$$
\begin{aligned}
r\left(\tilde{u}(t_\lambda, x_\lambda) - \tilde{w}(t_\lambda, y_\lambda)\right) \;&\le\; C\left(|t_\lambda - t_0| + |x_\lambda - y_\lambda|^2 + |x_\lambda - x_0|^2\right) \\
&+\; \langle \mu(x_\lambda, a_\lambda, e_\lambda) - \mu(y_\lambda, a_\lambda, e_\lambda) \;,\; x_\lambda - y_\lambda \rangle \\
&+\; \lambda C |\sigma(x_\lambda, a_\lambda, e_\lambda) - \sigma(y_\lambda, a_\lambda, e_\lambda)|^2 + \eta C (1+\lambda)^2 ,
\end{aligned}
$$

where $C > 0$ is independent of λ and η. Sending $\eta \to 0$ and using the Lipschitz continuity of μ and σ with respect to their first variable, uniformly in the two other ones, we deduce that

$$
r\left(\tilde{u}(t_\lambda, x_\lambda) - \tilde{w}(t_\lambda, y_\lambda)\right) \;\le\; O\left(|t_\lambda - t_0| + (1+\lambda)|x_\lambda - y_\lambda|^2 + |x_\lambda - x_0|^2\right) .
$$

Recalling (3.21), this leads to a contradiction to (3.19). $\Box$

Remark 3.3 1. The assumption $u \ge 0$ on $[0,T] \times \partial\mathcal{O}$ is only used in step 1. of the above proof to insure that $\rho(x_0, \hat{e}) u(t_0, x_0)$ is a minimum of $e \mapsto \rho(x_0, e) u(t_0, x_0)$ and $\rho(x_0, \hat{e}) \tilde{u}(t_0, x_0)(1 - \langle \gamma(x_0, e) \;,\; \gamma(x_0, \hat{e}) \rangle) \ge 0$ for all $e \in E$, recall that $|\gamma| = 1$.
2. If $w \ge 0$ on $[0,T] \times \partial\mathcal{O}$ then $\tilde{u}(t_0, x_0) \ge 0$ if $x_0 \in \partial\mathcal{O}$, by definition of (t_0, x_0). It follows that the assumption $u \ge 0$ on $[0,T] \times \partial\mathcal{O}$ can be replaced by $w \ge 0$ on $[0,T] \times \partial\mathcal{O}$.
3. If $f, g \ge 0$, then one easily checks that 0 is a subsolution of $\mathcal{K}\varphi = 0$. It then follows from Proposition 3.4 and the previous observation that any supersolution is non-negative. Thus, if $f, g \ge 0$, then Proposition 3.4 holds without assuming that $u \ge 0$ on $[0,T] \times \partial\mathcal{O}$.

3.5 General remarks

Remark 3.4 Assume that

$$\mu(x,a,e) = \operatorname{diag}[x]\,\bar\mu(x,a,e)\ ,\ \sigma(x,a,e) = \operatorname{diag}[x]\,\bar\sigma(x,a,e) \quad \text{on } \mathbb{R}^d_+ \times A \times E$$

and

$$\gamma(x,e) = \operatorname{diag}[x]\,\bar\gamma(x,e) \quad \text{on } (\partial\mathcal{O} \cap \mathbb{R}^d_+) \times E$$

with $\bar\mu$, $\bar\sigma$ and $\bar\gamma$ such that μ, σ and γ satisfy the general assumptions of this section. Then, the process $X^{\alpha,\epsilon}_{t,x}$ takes values in $(0,\infty)^d$ whenever $x \in (0,\infty)^d$. It is therefore natural to consider the PDE $\mathcal{K}\varphi = 0$ on $[0,T] \times (\bar{\mathcal{O}} \cap (0,\infty)^d)$, with a notion of viscosity solution similar to the one of Definition 3.1 with $\mathcal{O}$, $\partial\mathcal{O}$ and $\bar{\mathcal{O}}$ replaced by $\mathcal{O}^* := \mathcal{O} \cap (0,\infty)^d$, $\partial\mathcal{O}^* := \partial\mathcal{O} \cap (0,\infty)^d$ and $\bar{\mathcal{O}}^* := \bar{\mathcal{O}} \cap (0,\infty)^d$.
The proof of Proposition 3.2 and Proposition 3.3 are easily adapted to this context. We therefore obtain that v is a viscosity solution of $\mathcal{K}\varphi = 0$ on $[0,T] \times \bar{\mathcal{O}}^*$. Moreover, the proof of the comparison principle of Proposition 3.4 can also be extended. It suffices to add an additional penalty function of the form $k\sum_{i\le d}|x^i|^{-1}$, with $k \to \infty$, as in [1].

Remark 3.5 The smoothness assumptions on ρ and γ are only used either to construct $(X^{\alpha,\epsilon}_{t,x}, L^{\alpha,\epsilon}_{t,x})$ or to prove the dynamic programming principle of Lemma 3.2. We shall see through an example in Section 4.3 below how they can be relaxed.

4 Application to the pricing of barrier options under constraints

As already stated in the introduction, the main motivation comes from applications in mathematical finance. More precisely, [1] provides a PDE characterization of the super-hedging price of barrier options under portfolio constraints which is very similar to the equation $\mathcal{K}\varphi = 0$ up to an additional term inside the domain $\mathcal{O}$ which imposes a constraint on the gradient of the logarithm of the solution.
The aim of this section is to show that the super-hedging price of barrier options under portfolio constraints can actually admit a dual formulation in terms of an optimal control problem for a reflected diffusion in which the direction of reflection is controlled. Due to the additional term which appears in the PDE of [1], we can not expect this result to be general and we shall restrict to a Black and Scholes type model, see below.

In order to simplify the presentation, we shall work under quite restrictive conditions, assuming for instance that the equation $\mathcal{K}\varphi = 0$ admits a sufficiently smooth solution for a suitable choice of parameters. The general case is left for further research.

4.1 Problem formulation

We briefly present the hedging problem. Details can be found in [1] and the references contained in this paper.

We consider a financial market which consists of one non-risky asset, whose price process is normalized to unity, and d risky assets $S_{t,x} = (S^i_{t,x})_{i\le d}$ which solve on $[t,T]$

$$S_{t,x}(s) = x + \int_t^s \mathrm{diag}\,[S_{t,x}(r)]\,\Sigma\,dW(r)$$

where Σ is a d-dimensionnal invertible matrix. A financial strategy is described by a d-dimensional predictable process $\pi = (\pi^1, \ldots, \pi^d)$ satisfying the integrability condition

$$\int_0^T |\pi(s)|^2 ds \ < \ \infty \qquad \mathbb{P}-\text{a.s.} \tag{4.1}$$

where $\pi^i(s)$ is the proportion of wealth invested at time s in the risky asset $S^i_{t,x}$. To an initial capital $y \in \mathbb{R}$ and a financial strategy π, we associate the induced wealth process $Y^\pi_{t,y}$ which solves on $[t,T]$

$$\begin{aligned} Y(s) &= y + \int_t^s Y(r)\pi(r)'\mathrm{diag}\,[S_{t,x}(r)]^{-1}\,dS_{t,x}(r) \\ &= y + \int_t^s Y(r)\pi(r)'\Sigma\,dW(r)\ . \end{aligned} \tag{4.2}$$

In this paper, we restrict to the case where the proportion invested in the risky asset are constrained to be bounded from below. Given $m^i > 0$, $i \le d$, we set

$$K \ := \ \prod_{i=1}^d [-m^i, \infty)$$

and denote by Π_K the set of financial strategies π satisfying

$$\pi \in K \ \ dt \times d\mathbb{P} - \text{a.e.} \tag{4.3}$$

We consider an up-and-out type option. More precisely, we take $\mathcal{O}$ such that

$$\mathcal{O}^* := \mathcal{O} \cap (0,\infty)^d \quad = \quad \left\{ x \in (0,\infty)^d \ : \ \sum_{i=1}^d x^i < \kappa \right\} \ , \ \kappa > 0 \ .$$

The "pay-off" of the barrier option is a continuous map g defined on $\mathbb{R}^d_+$ satisfying

$$g \geq 0 \ \text{ on } \mathcal{O}^* \qquad \text{and} \qquad g = 0 \ \text{ on } \partial\mathcal{O}^* := \partial\mathcal{O} \cap (0,\infty)^d \ . \tag{4.4}$$

In order to apply the general results of [1], we assume that the map $\hat{g}$ defined by

$$\hat{g}(x) \quad = \quad \sup_{y \in \mathbb{R}^d_-} e^{-\delta(y)} g(x^1 e^{y^1}, \ldots, x^d e^{y^d}) \ , \ x \in \bar{\mathcal{O}}^* := \bar{\mathcal{O}} \cap (0,\infty)^d$$

is continuous. Here, δ is the support function of K, see Remark 3.1. We also assume that $\hat{g}$ is almost everywhere differentiable on $\bar{\mathcal{O}}^*$ and we denote by $D\hat{g}$ its gradient, when it is well defined.

Remark 4.1 One easily checks that

$$\hat{g}(x) \quad = \quad \sup_{y \in \mathbb{R}^d_-} e^{-\delta(y)} \hat{g}(x^1 e^{y^1}, \ldots, x^d e^{y^d}) \ , \ x \in \bar{\mathcal{O}}^* \ ,$$

see [1], which implies

$$\inf \left\{ \delta(e)\hat{g}(x) - \langle e \ , \ \mathrm{diag}\,[x]\, D\hat{g}(x) \rangle, \ e \in \tilde{K}_1 \right\} \geq 0$$

for all $x \in \bar{\mathcal{O}}^*$ where $D\hat{g}$ is well defined. Here, $\tilde{K}_1 := \mathbb{R}^d_- \cap \partial B(0,1)$ is the set of unit elements of the domain of δ, see Remark 3.1.

The option paies $g(S_{t,x}(T))$ at T if and only if $S_{t,x}$ does not exit $\mathcal{O}^*$ before T. Since $S_{t,x}$ has positive components, this corresponds to the situation where

$$\tau_{t,x} \quad := \quad \inf\{s \in [t,T] \ : \ X_{t,x}(s) \notin \mathcal{O}\} > T \ ,$$

with the usual convention $\inf \emptyset = \infty$.

The super-replication cost of the barrier option is then defined as the minimal initial dotation y such that $Y^\pi_{t,y}(T) \geq g(S_{t,x}(T))\mathbf{1}_{T<\tau_{t,x}}$ for some suitable strategy $\pi \in \Pi_K$. This leads to the introduction of the value function defined on $[0,T] \times \bar{\mathcal{O}}^*$ by

$$w(t,x) \quad := \quad \inf \left\{ y \in \mathbb{R} \ : \ Y^\pi_{t,y}(T) \geq g(S_{t,x}(T))\mathbf{1}_{T<\tau_{t,x}} \quad \text{for some } \pi \in \Pi_K \right\} \ . \tag{4.5}$$

4.2 PDE characterization

We define $\mathcal{L}$ as $\mathcal{L}^{0,0}$ with $A = \{0\}$, $\mu = 0$, $\sigma(x, \cdot) = \mathrm{diag}\,[x]\,\Sigma$ and $f = 0$. The next result is a consequence of [1].

Theorem 4.1 ([1]) *The value function w is the unique viscosity solution in the class of bounded functions on $[0,T] \times (\bar{\mathcal{O}} \cap \mathbb{R}^d_+)$ of $\mathcal{G}\varphi = 0$ where $\mathcal{G}\varphi$ equals*

$$\begin{cases} \min\left\{-\mathcal{L}\varphi(t,x)\ ,\ \min\limits_{e \in \tilde{K}_1} \left(\delta(e)\varphi(t,x) - \langle e, \mathrm{diag}\,[x]\, D\varphi(t,x)\rangle\right)\right\} & \text{on} \quad [0,T) \times \mathcal{O}^* \\ \min\left\{\varphi\ ,\ \min\limits_{e \in \tilde{K}_1} \left(\delta(e)\varphi(t,x) - \langle e, \mathrm{diag}\,[x]\, D\varphi(t,x)\rangle\right)\right\} & \text{on} \quad [0,T) \times \partial\mathcal{O}^* \\ \varphi - \hat{g} & \text{on} \quad \{T\} \times \bar{\mathcal{O}}^* \ . \end{cases}$$

In the above theorem, the notion of viscosity solution has to be taken in the classical sens.
When the equation (4.6)-(4.7)-(4.8) below admits a sufficiently smooth solution, the above equation can be simplified as follows.

Proposition 4.1 *Assume that there is a bounded non-negative $C^{1,3}([0,T) \times \mathcal{O}^*) \cap C^{0,1}([0,T) \times \bar{\mathcal{O}}^*) \cap C([0,T] \times \bar{\mathcal{O}}^*)$ function ψ such that $\partial\psi/\partial t \in C^{0,1}([0,T) \times \bar{\mathcal{O}}^*)$ and satisfying*

$$-\mathcal{L}\psi(t,x) = 0 \ \text{on}\ [0,T) \times \mathcal{O}^* \tag{4.6}$$

$$\min_{e \in \tilde{K}_1} \left(\delta(e)\psi(t,x) - \langle e, \mathrm{diag}\,[x]\, D\psi(t,x)\rangle\right) = 0 \ \text{on}\ [0,T) \times \partial\mathcal{O}^* \tag{4.7}$$

$$\psi = \hat{g} \ \text{on}\ \{T\} \times \bar{\mathcal{O}}^* \tag{4.8}$$

$$\lim_{\substack{(t',x') \to (T,x) \\ (t',x') \in [0,T) \times \mathcal{O}^*}} D\psi(t',x') = D\hat{g}(x) \ \text{almost everywhere on}\ \bar{\mathcal{O}}^* \ . \tag{4.9}$$

Then, $\psi = w$ on $[0,T) \times \mathcal{O}^*$ and ψ is the unique bounded solution to (4.6)-(4.7)-(4.8) *on* $[0,T] \times \bar{\mathcal{O}}^*$.

Proof. In view of Theorem 4.1, it suffices to show that ψ is a solution of $\mathcal{G}\varphi = 0$. Clearly, it is a subsolution. To prove that it is also a supersolution, we only have to show that

$$\min_{e \in \tilde{K}_1} \left(\delta(e)\psi(t,x) - \langle e, \mathrm{diag}\,[x]\, D\psi(t,x)\rangle\right) \geq 0 \quad \text{on} \quad [0,T) \times \mathcal{O}^* \ . \tag{4.10}$$

To see this, observe that each component $\phi^k := (D\psi)^k$ of $D\psi$ solves on $[0,T) \times \mathcal{O}^*$

$$-\frac{\partial}{\partial t}\phi^k(t,x) - \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}\,[x]\,\Sigma\Sigma'\mathrm{diag}\,[x]\, D^2\phi^k(t,x)\right] - \langle D\phi^k(t,x)'\mathrm{diag}\,[x]\,\Sigma\ ,\ \Sigma^k\rangle = 0$$

where Σ^k denotes the k-th line of Σ. Applying Itô's Lemma to $\langle e, \mathrm{diag}\,[S_{t,x}]\, D\psi(\cdot, S_{t,x})\rangle$, $e \in \tilde{K}_1$ and $(t,x) \in [0,T) \times \mathcal{O}^*$, and using (4.9), we deduce that

$$\begin{aligned}\langle e, \mathrm{diag}\,[x]\, D\psi(t,x)\rangle &= \mathbb{E}\left[\langle e, \mathrm{diag}\,[S_{t,x}(\tau_{t,x})]\, D\psi(\tau_{t,x}, S_{t,x}(\tau_{t,x}))\rangle \mathbf{1}_{\tau_{t,x}<T}\right] \\ &+ \mathbb{E}\left[\langle e, \mathrm{diag}\,[S_{t,x}(T)]\, D\hat{g}(S_{t,x}(T))\rangle \mathbf{1}_{\tau_{t,x}\geq T}\right] .\end{aligned}$$

Since by (4.6) and (4.8)

$$\psi(t,x) = \mathbb{E}\left[\psi(\tau_{t,x}, S_{t,x}(\tau_{t,x}))\mathbf{1}_{\tau_{t,x}<T} + \hat{g}(S_{t,x}(T))\mathbf{1}_{\tau_{t,x}\geq T}\right] ,$$

it follows from (4.7) and Remark 4.1 that

$$\delta(e)\psi(t,x) - \langle e, \mathrm{diag}\,[x]\, D\psi(t,x)\rangle \geq 0$$

which, by arbitrariness of e, provides the required result. □

4.3 Dual formulation

The equation (4.6)-(4.7)-(4.8) is very similar to $\mathcal{K}\varphi = 0$ with $E = \tilde{K}_1$ and

$$\rho(x,e) := \delta(e)/|\mathrm{diag}\,[x]\, e| \ , \ \gamma(x,e) = \mathrm{diag}\,[x]\, e/|\mathrm{diag}\,[x]\, e| \ .$$

However, the gradient of $\mathrm{diag}\,[x]\, e/|\mathrm{diag}\,[x]\, e|$ may blow up near $\partial(0,\infty)^d$ and it is not possible to consider a smooth extension of γ on $\mathbb{R}^{2d}$ (even on $\mathbb{R}^d_+ \times \tilde{K}_1$).
In order to surround this difficulty, we use the following construction. First we define $\mathcal{O}$ as

$$\mathcal{O} := \{x \in \mathbb{R}^d \ : \ \sum_{i=1}^{d} |x^i| < \kappa\}$$

so that $\mathcal{O}^* = \{x \in (0,\infty)^d \ : \ \sum_{i=1}^d x^i < \kappa\}$. Let $r \in (0,1/2)$ be such that $B(0,2r) \subset \mathcal{O}$. Then, given a non-decreasing $C^2(\mathbb{R},[0,1])$ function ϕ such that $\phi(y) = 1$ if $y \leq 1$ and $\phi(y) = 0$ if $y \geq 3/2$, we set, for $n \geq 1$,

$$z_n(e) \ := \ \left(e^i\phi(ne^i+2) - (1-\phi(ne^i+2))\right)_{i\leq d} \ .$$

Observe that $z(e) = e$ on $E_n := \{e \in \tilde{K}_1 \ : \ e^i \leq -n^{-1} \ \forall \ i \leq d\}$, $z(e) \in (-\infty,-1/(2n)]^d$ for all $e \in \mathbb{R}^d$, and

$$|\mathrm{diag}\,[x]\, e| \geq r/(2n) := \eta_n \quad \text{for } (x,e) \in B(0,r)^c \times (-\infty,-1/(2n)]^d \ . \qquad (4.11)$$

We then set

$$\begin{aligned}
\bar{\gamma}_n(x,e) &:= \operatorname{diag}[x]\, z_n(e)(1-\phi\left(\frac{3}{2}|\operatorname{diag}[x]\, e|/\eta_n)\right) - 1_d\phi\left(\frac{3}{2}|\operatorname{diag}[x]\, e|/\eta_n\right) \\
\gamma_n(x,e) &:= \bar{\gamma}_n(x,e)/|\bar{\gamma}_n(x,e)| \,.
\end{aligned}$$

Using (4.11), one easily checks that $\gamma_n \in C^2(\mathbb{R}^{2d}, \mathbb{R}^d)$. Moreover, $\gamma_n(x,e) = \gamma(x,e) = \operatorname{diag}[x]\, e/|\operatorname{diag}[x]\, e|$ on $B(0,r)^c \times E_n$ and (2.3) holds for $(\mathcal{O}, \gamma_n)$.

For $\epsilon \in \mathcal{E}_0 := \cup_{n\geq 1} \mathrm{BV}_{\mathbb{F}}(E_n)$ and $(t,x) \in [0,T] \times \bar{\mathcal{O}}^*$, we can then define $(X^\epsilon_{t,x}, L^\epsilon_{t,x})$ $:= (X^{0,\epsilon}_{t,x}, L^{0,\epsilon}_{t,x})$ as in Section 3 with $\mu = 0$, $\sigma(x,a,e) = \operatorname{diag}[x]\,\Sigma$ and γ defined as above. Clearly, $X^\epsilon_{t,x}$ takes values in $(0,\infty)^d$.

We next define ρ on $\mathbb{R}^d \times \tilde{K}_1$ as

$$\rho(x,e) \;=\; (\delta(e)/|\operatorname{diag}[x]\, e|)(1-\phi(|x|/r + 1/2))$$

so that ρ is continuous on $\mathbb{R}^d \times \tilde{K}_1$, satisfies the assumption of Section 3 as a function on $\mathbb{R}^d \times E_n$, for all $n \geq 1$, and

$$\rho(x,e) \;=\; \delta(e)/|\operatorname{diag}[x]\, e| \quad \text{on } \partial\mathcal{O}^* \times \tilde{K}_1 \,.$$

With this construction, we can now consider the control problem

$$v(t,x) \;:=\; \sup_{\epsilon\in\mathcal{E}_0} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)}\hat{g}\left(X^\epsilon_{t,x}(T)\right)\right] \;,\; (t,x) \in [0,T] \times \bar{\mathcal{O}}^* \,.$$

Proposition 4.2 *The function v is a bounded viscosity solution on $[0,T] \times \bar{\mathcal{O}}^*$ of (4.6)-(4.7)-(4.8).*

Proof. For $n \geq 1$ and $(t,x) \in [0,T] \times \bar{\mathcal{O}}^*$, set

$$v_n(t,x) \;:=\; \sup_{\epsilon\in\mathcal{E}_n} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)}\hat{g}\left(X^\epsilon_{t,x}(T)\right)\right]$$

where $\mathcal{E}_n := \mathrm{BV}_{\mathbb{F}}(E_n)$. It follows from the previous discussion that we can apply Lemma 3.2 to v_n. Since, $v = \sup_{n\geq 1} v_n = \lim_{n\to\infty} \uparrow v_n$, a monotone convergence argument shows that the dynamic programming principle of Lemma 3.2 holds for v. Following the arguments used in Proposition 3.2 and Proposition 3.3, and using the continuity of ρ and γ on $(B(0,r)^c \cap (0,\infty)^d) \times \tilde{K}_1 \supset \partial\mathcal{O}^* \times \tilde{K}_1$, we deduce that v is a viscosity solution of $\mathcal{K}\varphi = 0$ on $[0,T] \times \bar{\mathcal{O}}^*$, see Remark 3.4. Since

$$\delta(e)y - \langle e, \operatorname{diag}[x]\, p\rangle \geq 0 \quad\Leftrightarrow\quad |\operatorname{diag}[x]\, e|^{-1}\left(\delta(e)y - \langle e, \operatorname{diag}[x]\, p\rangle\right) \geq 0$$

for $(x,e,y,p) \in \partial\mathcal{O}^* \times \tilde{K}_1 \times \mathbb{R} \times \mathbb{R}^d$, this implies that v is a viscosity solution on $[0,T] \times \bar{\mathcal{O}}^*$ of (4.6)-(4.7)-(4.8). □

In view of Proposition 4.1, we finally obtain the main result of this section which provides a dual formulation for the super-hedging price w.

Theorem 4.2 *Let the conditions of Proposition 4.1 holds. Then, for all* $(t,x) \in [0,T] \times \mathcal{O}^*$,

$$w(t,x) = \sup_{\epsilon \in \mathcal{E}_0} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)} \hat{g}\left(X^\epsilon_{t,x}(T)\right)\right] . \tag{4.12}$$

Remark 4.2 It follows from Theorem 4.1, Proposition 4.2 and Theorem 7.1 in [1] that

$$w(t,x) \geq \sup_{\epsilon \in \mathcal{E}_0} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)} \hat{g}\left(X^\epsilon_{t,x}(T)\right)\right]$$

even if the conditions of Proposition 4.1 are not satisfied.

Remark 4.3 When $d = 1$, we retrieve the results of [9], see also [10]. In this case, $\mathcal{E}_0 = \{-1\}$ and the right hand-side quantity in (4.12) can be computed by using Monte-Carlo methods.

Remark 4.4 It follows from [1], that w admits the dual formulation

$$w(t,x) = \sup_{\vartheta \in \Theta} \mathbb{E}^\vartheta \left[e^{-\int_t^T \delta(\vartheta(s))ds} \hat{g}\left(S_{t,x}(T)\right) \mathbf{1}_{\tau_{t,x} > T}\right]$$

where Θ denotes the set of bounded adapted processes with values in $\mathbb{R}^d_-$ and $\mathbb{E}^\vartheta$ is the expectation operator under the equivalent probability measure $\mathbb{Q}^\vartheta$ under which the process W^ϑ defined by

$$W^\vartheta(t) \quad = \quad W(t) - \int_0^t \Sigma^{-1}\vartheta(s)ds \;\; t \leq T ,$$

is a Brownian motion. Letting $S^\vartheta_{t,x}$ be the solution on $[t,T]$ of

$$S^\vartheta_{t,x}(s) \quad = \quad x + \int_t^s \operatorname{diag}\left[S^\vartheta_{t,x}(r)\right] \Sigma dW(r) + \int_t^s \gamma(S^\vartheta_{t,x}(r), \vartheta(r))dr$$

with $\gamma(x,e) = \operatorname{diag}[x]\, e / |\operatorname{diag}[x]\, e|$, this is formally equivalent to

$$w(t,x) = \sup_{\vartheta \in \Theta} \mathbb{E}\left[e^{-\int_t^T \rho(S^\vartheta_{t,x}(s),\vartheta(s))ds} \hat{g}\left(S^\vartheta_{t,x}(T)\right) \mathbf{1}_{\tau^\vartheta_{t,x} > T}\right] \tag{4.13}$$

where $\rho(x,e)=\delta(e)/|\mathrm{diag}\,[x]\,e|$, $\tau^{\vartheta}_{t,x}$ is the first exit time of $S^{\vartheta}_{t,x}$ from $\mathcal{O}^*$ and we use the convention $0/0=0$.
Since $\hat{g}\geq 0$, we should seek for a control ϑ such that $\tau^{\vartheta}_{t,x}>T$, i.e. which "causes reflection" of S^{ϑ} at the boundary $\partial\mathcal{O}^*$. Moreover, the "reflection" should be optimal so that the right hand-side of (4.13) is maximal. If $d=1$ and $\hat{g}$ is non-decreasing on $\mathcal{O}^*$, the action of ϑ should be minimal since it decreases the value of $S^{\vartheta}_{t,x}(T)$ and $\rho(x,e)>0$ if $e\neq 0$. This phenomenon, which was already observed in [9] in the one dimensional case, naturally leads to the formulation (4.12).

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