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A note on maximally repeated sub-patterns of a point set

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Abstract

We answer a question raised by P. Brass on the number of maximally repeated sub-patterns in a set of \( n \) points in \( \mathbb{R}^d \). We show that this number, which was conjectured to be polynomial, is in fact \( \Theta(2^{n/2}) \) in the worst case, regardless of the dimension \( d \).

Key words: Discrete geometry, point sets, repeated configurations.

1 Introduction

Let \( S \) be a set of \( n \) points in \( \mathbb{R}^d \). A \textit{sub-pattern}, i.e. a subset, of \( S \) is repeated if it can be translated to another subset of \( S \). A sub-pattern \( P \subseteq S \) is \textit{maximally repeated} if for any subset \( Q \) such that \( P \subseteq Q \subseteq S \) there exists a translation that maps \( P \) to a subset of \( S \) without mapping \( Q \) to a subset of \( S \). In other words, a pattern is maximally repeated if it cannot be extended without losing

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Fig. 1. A pattern $S$ and all its maximally repeated sub-patterns.

Maximally repeated sub-patterns (MRSP for short) originated from the field of pattern matching to solve the following problem: given two point sets $X$ and $Y$, can $Y$ be translated to a subset of $X$? P. Brass [1, Theorem 3] gave an algorithm that answers such queries in time $O(|Y| \log |X|)$ whose preprocessing time depends on the number of distinct MRSP of $X$, where two MRSP are distinct if they are not equal up to a translation. A natural question is thus to give a theoretical bound on this number of MRSP in order to provide an upper bound on the time requirement of that algorithm. This number was conjectured [1] [2, p.267] to be $O(n^d)$ where $d$ is the dimension in which the point set is embedded.

In this note we prove that the number of MRSP of a set of $n$ points in $\mathbb{R}^d$ is actually $\Theta(2^{n/2})$ in the worst case and thus finding sub-patterns via this approach leads to exponential worst-case running time. More precisely, we show the following theorem:

**Theorem 1** A set of $n$ points has at most $16 \cdot 2^{[n/2]}$ distinct MRSP and for arbitrary large $n$ there exist sets $S$ of $n$ points with $2^{[n/2]-1}$ distinct MRSP.

Our proof is based on combinatorial rather than geometrical properties of the point set, which explains that the bound is independent of the dimension $d$ in which the points are considered.

## 2 The Proof of Theorem 1

Let us first introduce some terminology. Given a set of points $P \subseteq \mathbb{R}^d$ and a translation $t \in \mathbb{R}^d$, $P + t := \{x + t \mid x \in P\}$ is the set of translated points of
$G_k$

$S_k$

Fig. 2. A set $S_k$ of $2k$ points with at least $2^{k-1}$ distinct MRSP.

$P$ by $t$. A subset $P \subseteq S$ is a repeated sub-pattern if there exists a translation $t \neq 0$ such that $P + t \subseteq S$. $P$ is a maximally repeated sub-pattern (MRSP) if, in addition, for any subset $Q$ such that $P \subseteq Q \subseteq S$ there exists a translation $t$ such that $P + t \subseteq S$ and $Q + t \not\subseteq S$. Two MRSP are distinct if they are not equal up to a translation.

In the sequel, we present a set of $n$ points in $\mathbb{R}$ having at least $2^{\lfloor n/2 \rfloor - 1}$ distinct MRSP (Section 2.1) and then prove that any set of $n$ points in $\mathbb{R}^d$ can have at most $16 \cdot 2^{\lceil n/2 \rceil}$ distinct MRSP (Section 2.2).

### 2.1 Lower bound

We build our example on a 1-dimensional grid which can, of course, be considered as embedded in $\mathbb{R}^d$ for any $d \geq 1$. Let $k$ be an integer, $G_k$ denote the set of integers $\{1, \ldots, k\}$ and $S_k$ be $G_k \cup (G_k + (k + 1))$, that is two copies of $G_k$ separated by a gap of one point at $k + 1$ (see Figure 2).

Let $P$ be a subset of the first copy of $G_k$, $Q \subseteq S_k$ be a proper super-set of $P$ and $p^* \in Q \setminus P$. If $p^* \geq k + 2$ then $P + (k + 1) \subseteq S_k$ and $Q + (k + 1) \not\subseteq S_k$. If $p^* \leq k$ then $P + (k + 1 - p^*) \subseteq S_k$ and $Q + (k + 1 - p^*) \not\subseteq S_k$. This proves that $P$ is a MRSP. Two subsets of $G_k$ containing 1 cannot be equal up to a non-trivial translation. Thus, all subsets of $G_k$ containing 1 are distinct MRSP and $S_k$ admits at least $2^{k-1}$ MRSP. This proves the first statement of Theorem 1.

### 2.2 Upper bound

Recall that $(x_1, \ldots, x_d) <_L (y_1, \ldots, y_d)$ in the lexicographic order on vectors of $\mathbb{R}^d$ if $x_1 < y_1$ or for some $r = 1, \ldots, d - 1$:

$$x_1 = y_1, \ldots, x_r = y_r \text{ and } x_{r+1} < y_{r+1}.$$ 

Let $S = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}^d$ be a set of $n$ points and $T \subseteq \mathbb{R}^d$ the set of translations defined by $T := S - S = \{x - y \mid (x, y) \in S^2\}$. 

3
Let $\mathcal{A}$ denote the set of MRSP $P$ such that no translation $t < L 0$ satisfies $P + t \subseteq S$. The set $\mathcal{A}$ contains exactly one representative of each equivalence class of MRSP under translation, namely the one with the smallest point. To bound $|\mathcal{A}|$, we first partition this set in the following families:

$$\mathcal{A}_{ij} = \{P \in \mathcal{A} | \{a_i, a_j\} \subseteq P \subseteq \{a_i, \ldots, a_j\}\}.$$

Informally, $\mathcal{A}_{ij}$ is the set of MRSP spanning the range $\{a_i, \ldots, a_j\}$. Since $\mathcal{A}_{11} = \{a_1\}$ and $\mathcal{A}_{ii}$ is empty for $i \geq 2$, we have

$$|\mathcal{A}| = 1 + \sum_{1 \leq i < j \leq n} |\mathcal{A}_{ij}|. \quad (1)$$

There is an injection between the MRSP of $\mathcal{A}_{ij}$ and the subsets of $\{a_{i+1}, \ldots, a_{j-1}\}$. Hence, 

$$|\mathcal{A}_{ij}| \leq 2^{j-i-1} \quad (2)$$

which will be enough to bound the number of MRSP spanning a “small” range.

To bound the number of MRSP spanning a “wide” range, we describe them by the set of translations they allow. Let $\phi$ denote the function:

$$\phi : 2^S \rightarrow 2^T$$

$$P \mapsto \{t \in T | P + t \subseteq S\}$$

If two elements of $\mathcal{A}$, $P_1$ and $P_2$, have the same image by $\phi$ then:

$$\phi(P_1 \cup P_2) = \phi(P_1) = \phi(P_2).$$

By definition of MRSP, this implies that $P_1 \cup P_2 = P_1 = P_2$. Thus, $\phi$ is an injection from $\mathcal{A}$ to the subsets of $T$. For $1 \leq i < j \leq n$, let

$$T_{ij} = \{t \in T | t \geq L 0 \text{ and } \{a_i, a_j\} + t \subseteq S\}$$

be the set of all non-negative translations compatible with $a_i$ and $a_j$. MRSP in $\mathcal{A}_{ij}$ only allow translations in $T_{ij}$, so $\phi$ is an injection from $\mathcal{A}_{ij}$ to the subsets of $T_{ij}$ and it follows that $|\mathcal{A}_{ij}| \leq 2^{|T_{ij}|}$. Any $t \in T_{ij} \setminus \{0\}$ can be identified by the element $a_y = a_j + t$. Thus, the size of $T_{ij}$ is bounded by the number of such indexes $y$, which is at most $n - j$. Finally, we obtain that

$$|\mathcal{A}_{ij}| \leq 2^{n-j}. \quad (3)$$

Combining Equations (2), (3) and (1) we obtain:

$$|\mathcal{A}| \leq 1 + \sum_{1 \leq i < j \leq n} 2^{\min(n-j,j-i-1)}.$$
Splitting the sum at $j = \lceil \frac{n+i}{2} \rceil + 1$, we have

$$|A| \leq 1 + 2 \sum_{i=1}^{n} \left( \sum_{j=i+1}^{n+i} 2^{j-i-1} \right) \leq 1 + 2 \sum_{i=1}^{n} 2^{\left\lceil \frac{n-i}{2} \right\rceil} \leq 1 + 8 \sum_{\ell=1}^{\left\lceil \frac{n}{2} \right\rceil} 2^{\ell}$$

and finally $|A| \leq 16 \cdot 2^{\left\lceil n/2 \right\rceil}$, which proves the second statement of Theorem 1.

References
