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A Gauss-Bonnet-like Formula on Two-Dimensional Almost-Riemannian Manifolds¹

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Abstract — We consider a generalization of Riemannian geometry that naturally arises in the framework of control theory. Let X and Y be two smooth vector fields on a two-dimensional manifold M . If X and Y are everywhere linearly independent, then they define a classical Riemannian metric on M (the metric for which they are orthonormal) and they give to M the structure of metric space. If X and Y become linearly dependent somewhere on M , then the corresponding Riemannian metric has singularities, but under generic conditions the metric structure is still well defined. Metric structures that can be defined locally in this way are called almost-Riemannian structures. They are special cases of rank-varying sub-Riemannian structures, which are naturally defined in terms of submodules of the space of smooth vector fields on M . Almost-Riemannian structures show interesting phenomena, in particular for what concerns the relation between curvature, presence of conjugate points, and topology of the manifold. The main result of the paper is a generalization to almost-Riemannian structures of the Gauss-Bonnet formula.

Keywords — Generalized Riemannian geometry, Grushin plane, rank-varying distributions, Gauss-Bonnet formula, conjugate points, optimal control

MSC-class: 49j15, 53c17

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1 Introduction

Let M be a two-dimensional smooth manifold and consider a pair of smooth vector fields X and Y on M . If the pair X, Y is Lie bracket generating, i.e., if $\text{span}\{X(q), Y(q), [X, Y](q), [X, [X, Y]](q), \dots\}$ is full-dimensional at every $q \in M$, then the control system

$$\dot{q} = uX(q) + vY(q), \quad u^2 + v^2 \leq 1, \quad q \in M, \quad (1)$$

is completely controllable and the minimum-time function defines a continuous distance d on M . When X and Y are everywhere linear independent (the only possibility for this to happen is that M is parallelizable), such distance is Riemannian and it corresponds to the metric for which (X, Y) is an orthonormal moving frame. The idea is to study the geometry obtained starting from a pair of vector fields which may become collinear. Under generic hypotheses, the set \mathcal{Z} (called *singular locus*) of points of M at which X and Y are parallel is a one-dimensional embedded submanifold of M (possibly disconnected).

Metric structures that can be defined *locally* by a pair of vector fields (X, Y) through (1) are called almost-Riemannian structures.

An almost-Riemannian structure can equivalently be seen as a locally finitely generated Lie bracket generating $\mathcal{C}^\infty(M)$ -submodule Δ of $\text{Vec}(M)$, the space of smooth vector fields on M , endowed with a bilinear, symmetric map $G : \Delta \times \Delta \rightarrow \mathcal{C}^\infty(M)$ which is positive definite (in a suitable sense). A pair of vector fields X, Y in Δ is said to be orthonormal on some open set Ω if $G(X, Y)(q) = 0$ and $G(X, X)(q) = G(Y, Y)(q) = 1$ for every $q \in \Omega$.

An almost-Riemannian structure is said to be *orientable* if there exists a *volume form*, i.e., a bilinear, skew-symmetric, non-degenerate form $\omega : \Delta \times \Delta \rightarrow \mathcal{C}^\infty(M)$. In this case it is possible to normalize ω in such that $|\omega(X, Y)| = 1$ on Ω for every open subset Ω of M and every local orthonormal frame (X, Y) on Ω .

It is interesting to notice that it is possible to define non-orientable almost-Riemannian structures on orientable manifolds and orientable almost-Riemannian structures on non-orientable manifolds.

We say that an almost-Riemannian structure is *trivializable* if Δ is globally generated by a pair of vector fields defined on M . Trivializable almost-Riemannian structures are always orientable.

The singular locus \mathcal{Z} can be defined on M as the set where the linear subspace $\Delta(q) = \{V(q) \mid V \in \Delta\}$ of T_qM is not full-rank. An almost-Riemannian structure is Riemannian if and only if $\mathcal{Z} = \emptyset$.

A famous example of genuine almost-Riemannian structure is provided by the Grushin plane, which is the almost-Riemannian structure on \mathbb{R}^2 for which the vector fields $X(x, y) = (1, 0)$ and $Y(x, y) = (0, x)$ form a pair of orthonormal generators. (See Section 3.2 and [3, 4]. The model was originally introduced in the context of hypoelliptic operator theory [8, 9, 10, 11].) Notice that the singular locus is indeed nonempty, being equal to the y -axis. Another example of (trivializable) almost-Riemannian structure has appeared in problems of control of quantum mechanical systems (see [5, 6]). In this case $M = S^2$ represents a suitable state space reduction of a three-level quantum system while the orthonormal generators X and Y are two infinitesimal rotations along two orthogonal axes, modeling the action on the system of two lasers in the rotating wave approximation (see Section 5.5).

Such examples, and the naturalness of the construction leading to the definition of almost-Riemannian structure, motivate the study of general properties of such geometry, which exhibits many interesting features. One can check, for instance, that even in the case where the Gaussian curvature is everywhere negative (where it is defined, i.e., on $M \setminus \mathcal{Z}$) geodesics may have conjugate points. For this reason it seems interesting to analyze the relations between the curvature, the presence of conjugate points, and the topology of the manifold (see also [1]). After providing a characterization of generic almost-Riemannian structures by means of local normal forms, in this paper we start this program by proving a generalization of the Gauss-Bonnet formula. Let M be compact and oriented, and endow it with an orientable almost-Riemannian structure. Denote by $K : M \setminus \mathcal{Z} \rightarrow \mathbb{R}$ the Gaussian curvature. The first difficulty in order to extend the Gauss-Bonnet formula is to give a meaning to $\int_M K dA$, the integral of K on M with respect to the Riemannian density dA induced by the Riemannian metric on $M \setminus \mathcal{Z}$. In the two examples cited above one can check, for instance, that, as q approaches \mathcal{Z} , dA diverges, while $K(q)$, which is everywhere negative, tends to $-\infty$.

The idea is to replace $K dA$ with a signed version of it. A natural choice is $K dA_s$, where dA_s is a volume form intrinsically associated with the almost-Riemannian structure on $M \setminus \mathcal{Z}$.

Our goal is to prove the existence and to assign a value to the limit

$$\lim_{\varepsilon \searrow 0} \int_{\{q \in M \mid d(q, \mathcal{Z}) > \varepsilon\}} K(q) dA_s, \quad (2)$$

where $d(\cdot, \cdot)$ is the distance globally defined by the almost-Riemannian structure on M .

The goal will be attained under the following additional assumption. Generically the singular locus \mathcal{Z} is smooth and $\Delta(q)$ is one-dimensional at every point of \mathcal{Z} . We say that $q \in \mathcal{Z}$ is a *tangency point* if $\Delta(q)$ is tangent to \mathcal{Z} . Under generic assumptions, \mathcal{Z} contains only a discrete set of tangency points. The hypothesis under which the main results of the paper are obtained is that \mathcal{Z} contains no such point.

Define M^+ (respectively, M^-) as the subset of $M \setminus \mathcal{Z}$ on which the orientation defined by dA_s coincides with (respectively, is opposite to) that of M . If M has no tangency point, then the limit (2) turns out to exist and is equal to $2\pi(\chi(M^+) - \chi(M^-))$, where χ denotes the Euler characteristic.

When the almost-Riemannian structure is trivializable, we have that $\chi(M^+) = \chi(M^-)$ and thus the limit (2) is equal to zero. Once applied to the special subclass of Riemannian structures, such result simply states that the integral of the curvature of a parallelizable compact oriented surface (i.e., the torus) is equal to zero. In a sense, in the standard Riemannian construction the topology of the surface gives a constraint on the total curvature through the Gauss-Bonnet formula, whereas for an almost-Riemannian structure induced by a single pair of vector fields the total curvature is equal to zero and the topology of the manifold constrains the metric to be singular on a suitable set.

It is interesting to notice that every oriented compact surface can be endowed with a trivializable almost-Riemannian structure satisfying the requirement that there are no tangency points.

The paper is organized as follows. In Section 2, we introduce two equivalent definitions of rank-varying sub-Riemannian structure on a manifold of any dimension, first by using the language of moduli and then by identifying it with an atlas of orthonormal frames. Rank-varying sub-Riemannian structures have already been studied, from a different perspective, in [12, 15].

A notion of orientability for rank-varying sub-Riemannian structure is then introduced. Almost-Riemannian structures are defined as rank-varying sub-Riemannian structures of maximal rank.

Starting from Section 3 we focus on the case of almost-Riemannian structures on two-dimensional manifolds. Geodesics associated with such structures are characterized in Section 3.1 using the Pontryagin Maximum Principle. In Section 3.2 we study the Grushin plane, for which we compute the cut and the conjugate loci.

In Section 4 we provide local normal forms for generic almost-Riemannian structures, which are used in Section 5 to prove a generalization of the Gauss-Bonnet formula to almost-Riemannian structures without tangency points. The formula is then specialized to the case of trivializable almost-Riemannian structures. In Section 5.5 we show that every compact orientable two dimensional manifold admits a trivializable almost-Riemannian structure with no tangency points.

2 Rank-varying distributions and sub-Riemannian structures

Let M be a n -dimensional smooth manifold. Recall that $\text{Vec}(M)$, the set of smooth vector fields on M , is naturally endowed with the structure of $\mathcal{C}^\infty(M)$ -module. Given an open subset Ω of M , a submodule Δ of $\text{Vec}(M)$ is said to be *generated on Ω* by the vector fields $\{V_1, \dots, V_m\}$ if every $V \in \Delta$ can be written as $V = a_1 V_1 + \dots + a_m V_m$ on Ω where a_1, \dots, a_m belong to $\mathcal{C}^\infty(M)$.

Definition 1 *A (n, k) -rank-varying distribution is a pair (M, Δ) where M is a n -dimensional smooth manifold, Δ is a submodule of $\text{Vec}(M)$, and $k \leq n$ is such that for every $q \in M$ and every small enough neighborhood Ω^q of q , the restriction to Ω^q of Δ is generated by k vector fields and cannot be generated by less than k vector fields.*

From now on the expression $\Delta(q)$ will denote the linear subspace $\{V(q) \mid V \in \Delta\} \subset T_q M$. When the dimension of $\Delta(q)$ is independent of q , we recover the standard definition of distribution as a smooth field of linear subspaces of $T_q M$. Notice that Δ cannot be identified with the map $q \mapsto \Delta(q)$. Indeed, it can happen that two different moduli Δ_1 and Δ_2 are such that $\Delta_1(q) = \Delta_2(q)$ for every $q \in M$. Take for instance $M = \mathbb{R}$ and Δ_1, Δ_2 generated, respectively, by $F_1(x) = x$, $F_2(x) = x^2$.

Denote by $\text{Lie}(\Delta)$ the smallest Lie subalgebra of $\text{Vec}(M)$ containing Δ and let $\text{Lie}_q(\Delta) = \{V(q) \mid V \in \text{Lie}(\Delta)\}$ for every $q \in M$. We say that (M, Δ) satisfies the Lie bracket generating condition if $\text{Lie}_q(\Delta) = T_q M$ for every $q \in M$. We also introduce the flag of a rank-varying distribution (M, Δ) as the sequence of submodules $\Delta_0 = \Delta \subset \Delta_1 \subset \dots \subset \Delta_m \subset \dots$ defined through the recursive formula

$$\Delta_{k+1} = \Delta_k + [\Delta, \Delta_k]. \quad (3)$$

As above, we let $\Delta_m(q) = \{V(q) \mid V \in \Delta_m\}$.

In order to provide an example of rank-varying distribution, let us introduce the *Grushin distribution*. Take as M the plane \mathbb{R}^2 and let Δ be generated by the vector fields $F_1(x, y) = (1, 0)$ and $F_2(x, y) = (0, x)$. Then (\mathbb{R}^2, Δ) is a $(2, 2)$ -rank-varying distribution. Notice that $\Delta(q)$ is equal to $\mathbb{R} \times \{0\}$ when q is on the y -axis and to \mathbb{R}^2 elsewhere. The Grushin distribution is Lie bracket generating since $\Delta_2(q) = \mathbb{R}^2$ for every $q \in \mathbb{R}^2$.

Crucial in what follows is the notion of *generic* (n, k) -rank-varying distribution. Denote by \mathcal{W} the \mathcal{C}^2 -Whitney topology defined on $\text{Vec}(M)$ and by $(\text{Vec}(M), \mathcal{W})^k$ the product of k copies of $\text{Vec}(M)$ endowed with the corresponding product topology. We recall that if M is compact (as it is the case in most of what follows), then \mathcal{W} is the standard \mathcal{C}^2 topology.

Definition 2 A property (P) defined for (n, k) -rank-varying distributions is said to be *generic* if there exists an open and dense subset \mathcal{O} of $(\text{Vec}(M), \mathcal{W})^k$ such that (P) holds for every (n, k) -rank-varying distribution which is generated by elements of \mathcal{O} .

E.g., generically, a (n, k) -rank-varying distribution is Lie bracket generating.

2.1 Orientable rank-varying distributions

Let (M, Δ) be a (n, k) -rank-varying distribution. A k -form on (M, Δ) is a multilinear skew-symmetric map

$$\omega : \underbrace{\Delta \times \dots \times \Delta}_{k \text{ times}} \rightarrow \mathcal{C}^\infty(M).$$

We say that a k -form ω is a *volume form* if, for every $q \in M$, there exist k vector fields $F_1, \dots, F_k \in \Delta$ such that $\omega(F_1, \dots, F_k)(q) \neq 0$.

Definition 3 We say that a (n, k) -rank-varying distribution (M, Δ) is *orientable* if it admits a volume form, otherwise we say that (M, Δ) is *non-orientable*.

Notice that a rank-varying distribution can be orientable even if M is a non-orientable manifold (see example below). However, the distribution $(M, \text{Vec}(M))$ is orientable if and only if M is.

Remark 4 Thanks to its multilinearity, a volume form is completely characterized by its action on the generators. Given a trivialisable rank-varying distribution Δ and a global system of generators F_1, \dots, F_k , the equality $\omega(F_1, \dots, F_k) = 1$ uniquely defines a volume form on Δ . Hence every trivialisable rank-varying distribution is orientable.

Remark 5 Let ω be a k -form on a (n, k) -rank-varying distribution (M, Δ) . Then ω acts as a tensor on the open subset of M made of points q such that the dimension of $\Delta(q)$ is equal to k , i.e., for every $V_1, \dots, V_k \in \Delta$ the value of $\omega(V_1, \dots, V_k)(q)$ depends only on $V_1(q), \dots, V_k(q)$. Indeed, let $\{F_1, \dots, F_k\}$ be a local system of generators of Δ on a neighborhood Ω of q and take $a_{ij} \in \mathcal{C}^\infty(M)$, $1 \leq i, j \leq k$, such that $V_i = \sum_{j=1}^k a_{ij} F_j$ on Ω . The multilinearity and skew-symmetry of ω imply that $\omega(V_1, \dots, V_k)(q) = \det(a_{ij}(q)) \omega(F_1, \dots, F_k)(q)$. Therefore, $\omega(V_1, \dots, V_k)(q)$ depends on V_1, \dots, V_k only through the matrix $(a_{ij}(q))$, which is uniquely determined by $V_1(q), \dots, V_k(q)$.

Let us present some example of orientable and non-orientable rank-varying distributions. All these examples are $(2, 2)$ -rank-varying distributions, since this is our main case of interest in the following.

The Grushin distribution. Let $M = \mathbb{R}^2$ and recall that Δ is generated by the vector fields $F_1(x, y) = (1, 0)$ and $F_2(x, y) = (0, x)$. The distribution Δ is orientable. A volume form can be defined by

$$\omega(V_1, V_2)(x, y) = \lim_{x' \rightarrow x} \frac{1}{x'} dx \wedge dy (V_1(x', y), V_2(x', y)),$$

for every pair of vector fields V_1, V_2 belonging to Δ . Equivalently we could have defined ω on the generators F_1, F_2 by requiring that $\omega(F_1, F_2)(x, y) = 1$.

Notice that ω is not a tensor on the y -axis.

A non-orientable rank-varying distribution on the torus. Let M be the two-dimensional torus $[-\pi, \pi] \times [-\pi, \pi]$ with the standard identifications. Consider the open covering of M given by

$$\Omega^1 = (-\pi/2, \pi/2) \times [-\pi, \pi], \quad \Omega^2 = ([-\pi, -\pi/4] \cup (\pi/4, \pi]) \times [-\pi, \pi].$$

Let Δ be generated by the vector fields

$$\begin{aligned} F_1^1 &= (1, 0), & F_2^1 &= (0, \sin x), & \text{on } \Omega^1 \\ F_1^2 &= (1, 0), & F_2^2 &= (0, 1), & \text{on } \Omega^2. \end{aligned}$$

This rank-varying distribution is non-orientable. In fact a volume form ω should acts on the local generators as

$$\omega(F_1^1, F_2^1)(q) = f_1(q) \text{ for every } q \in \Omega^1, \quad \omega(F_1^2, F_2^2)(q) = f_2(q) \text{ for every } q \in \Omega^2,$$

where f_1 and f_2 are two never-vanishing smooth functions. On $\Omega^1 \cap \Omega^2$ we would have

$$f_1(x, y) = \omega(F_1^1, F_2^1)(x, y) = \sin(x)\omega(F_1^2, F_2^2)(x, y) = \sin(x)f_2(x, y),$$

contradicting the constant-sign assumption on f_1, f_2 . As a consequence Δ is not trivializable.

An orientable rank-varying distribution on the Klein bottle. Let M be the Klein bottle seen as the square $[-\pi, \pi] \times [-\pi, \pi]$ with the identifications $(x, -\pi) \sim (x, \pi)$, $(-\pi, y) \sim (\pi, -y)$. Consider the vector fields

$$F_1(x, y) = (1, 0), \quad F_2(x, y) = (0, \sin(2x)),$$

which are well defined on M . The distribution generated by F_1 and F_2 is orientable since it is trivializable.

2.2 Rank-varying sub-Riemannian structures

In this section we see how to introduce a a smoothly-varying Riemannian structure on every subspace $\Delta(q)$.

Definition 6 A (n, k) -rank-varying sub-Riemannian structure is a triple $\mathcal{S} = (M, \Delta, G)$, where (M, Δ) is a Lie bracket generating (n, k) -rank-varying distribution and $G : \Delta \times \Delta \rightarrow \mathcal{C}^\infty(M)$ is a symmetric, positive definite bilinear map, i.e., a map such that for every $V, W \in \Delta$ and $f \in \mathcal{C}^\infty(M)$ we have

$$\begin{aligned} G(V, W) &= G(W, V), \\ G(fV, W) &= G(V, fW) = fG(V, W), \\ G(V, V)(q) &\geq 0, \quad \text{for every } q \in M, \\ G(V, V)(q) &= 0 \quad \text{implies that } V(q) = 0. \end{aligned}$$

A (n, n) -rank-varying sub-Riemannian structure is called a n -dimensional almost-Riemannian structure.

Let $\mathcal{S} = (M, \Delta, G)$ be a (n, k) -rank-varying sub-Riemannian structure. Reasoning as in Remark 5, we get that G is a tensor at the points q where $\dim(\Delta(q)) = k$. Although this is not necessarily the case everywhere on M , we can define, for every $q \in M$, a quadratic form \mathbf{G}_q on $\Delta(q)$ through

$$\mathbf{G}_q(v, v) = \inf\{G(V, V)(q) \mid V(q) = v, V \in \Delta\}.$$

For every $q \in M$, it is possible to find a neighborhood Ω_q of q and an *orthonormal frame* on Ω_q , i.e., a set of k vector fields $X_1, \dots, X_k \in \Delta$ such that $G(X_i, X_j) = \delta_{i,j}$ on Ω_q . One easily proves that orthonormal frames are local generators in Ω_q .

If \mathcal{S} is orientable then a volume form ω can be chosen in such a way that $|\omega(X_1, \dots, X_k)| = 1$ on every local orthonormal frame.

Let $\mathcal{S} = (M, \Delta, G)$ be a (n, k) -rank-varying sub-Riemannian structure. A curve $\gamma : [0, T] \rightarrow M$ is said to be *admissible* for \mathcal{S} if it is Lipschitz continuous and $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for almost every $t \in [0, T]$. Given an admissible curve $\gamma : [0, T] \rightarrow M$, the *length* of γ is

$$l(\gamma) = \int_0^T \sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

The *distance induced by \mathcal{S} on M* is the function

$$d(q_0, q_1) = \inf\{l(\gamma) \mid \gamma(0) = q_0, \gamma(T) = q_1, \gamma \text{ admissible}\}. \quad (4)$$

It is a standard fact that $l(\gamma)$ is invariant under reparameterization of the curve γ . Moreover, if an admissible curve γ minimizes the so-called *energy functional* $E(\gamma) = \int_0^T \mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt$ with T fixed (and fixed initial and final point) then $v = \sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}$ is constant and γ is also a minimizer of $l(\cdot)$. On the other side a minimizer γ of $l(\cdot)$ such that v is constant is a minimizer of $E(\cdot)$ with $T = l(\gamma)/v$.

A *geodesic* for \mathcal{S} is a curve $\gamma : [0, T] \rightarrow M$ such that for every sufficiently small interval $[t_1, t_2] \subset [0, T]$, $\gamma|_{[t_1, t_2]}$ is a minimizer of $E(\cdot)$. A geodesic for which $\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))$ is (constantly) equal to one is said to be parameterized by arclength.

The finiteness and the continuity of $d(\cdot, \cdot)$ with respect to the topology of M are guaranteed by the Lie bracket generating assumption on the rank-varying sub-Riemannian structure. The distance $d(\cdot, \cdot)$ gives to M the structure of metric space. The local existence of minimizing geodesics is a standard consequence of Filippov Theorem (see for instance [2]). When M is compact any two points of M are connected by a minimizing geodesic.

A convenient way to deal with a rank-varying sub-Riemannian structure is to identify it with an atlas of local orthonormal frames. In the case of an orientable rank-varying sub-Riemannian structure, one can impose that all orthonormal frames are coherently oriented. In this way one is led to the following equivalent definition.

Definition 7 *Let M be a n -dimensional smooth manifold, fix $k \in \mathbb{N}$, and consider a family*

$$\mathcal{S} = \{(\Omega^\mu, X_1^\mu, \dots, X_k^\mu)\}_{\mu \in I},$$

where $\{\Omega^\mu\}_{\mu \in I}$ is an open covering of M and, for every $\mu \in I$, $\{X_1^\mu, \dots, X_k^\mu\}$ is a family of smooth vector fields defined on M , whose restriction to Ω^μ satisfies the Lie bracket generating condition. We assume moreover that for every $\mu \in I$ and every open nonempty subset Ω of Ω^μ , the submodule of $\text{Vec}(\Omega)$ generated by X_1^μ, \dots, X_k^μ on Ω cannot be generated by less than k vector fields.

We say that \mathcal{S} is a (n, k) -rank-varying sub-Riemannian structure if, for every $\mu, \nu \in I$ and for every $q \in \Omega^\mu \cap \Omega^\nu$, there exists an orthogonal matrix $R^{\mu, \nu}(q) = (R_{i,j}^{\mu, \nu}(q)) \in O(k)$ such that

$$X_i^\mu(q) = \sum_{j=1}^k R_{i,j}^{\mu, \nu}(q) X_j^\nu(q). \quad (5)$$

We say that two rank-varying sub-Riemannian structures \mathcal{S}_1 and \mathcal{S}_2 on M are equivalent if $\mathcal{S}_1 \cup \mathcal{S}_2$ is a rank-varying sub-Riemannian structure. Given an open subset Ω of M and a set of k vector fields (X_1, \dots, X_k) , we say that $(\Omega, X_1, \dots, X_k)$ is compatible with \mathcal{S} if $\mathcal{S} \cup \{(\Omega, X_1, \dots, X_k)\}$ is equivalent to \mathcal{S} .

If \mathcal{S} is equivalent to a rank-varying sub-Riemannian structure of the form $\{(M, X_1, \dots, X_k)\}$, i.e., for which the cardinality of I is equal to one, we say that \mathcal{S} is *trivializable*.

If \mathcal{S} admits an equivalent rank-varying sub-Riemannian structure such that each $R^{\mu,\nu}(q)$ belongs to $SO(k)$, we say that \mathcal{S} is *orientable*.

Notice that $R^{\mu,\nu}(q)$ is uniquely defined by equation (5) and, moreover, is smooth as a function of q . In the following, when dealing with an orientable rank-varying sub-Riemannian structure, we always assume that the atlas of local orthonormal frames is *positive oriented*, i.e., such that each $R^{\mu,\nu}$ belongs to $SO(k)$. For such an atlas, a volume form ω can be chosen such that $\omega(X_1^\mu, \dots, X_k^\mu) = 1$ on Ω^μ , for every $\mu \in I$.

In terms of Definition 6, Δ is the module that is locally (in Ω^μ) generated by X_1^μ, \dots, X_k^μ . Moreover

$$\begin{aligned} \Delta(q) &= \text{span}\{X_1^\mu(q), \dots, X_k^\mu(q)\}, \\ \mathbf{G}_q(v, v) &= \inf \left\{ \sum_{i=1}^k \alpha_i^2 \mid v = \sum_{i=1}^k \alpha_i X_i^\mu(q) \right\}, \end{aligned}$$

for every $q \in \Omega^\mu$ and $v \in \Delta(q)$.

Definition 8 A property (P) defined for (n, k) -rank-varying sub-Riemannian structures on M is said to be *generic* if there exists an open and dense subset \mathcal{O} of $(\text{Vec}(M), \mathcal{W})^k$ such that (P) holds for every (n, k) -rank-varying sub-Riemannian structure admitting an atlas of local orthonormal frames whose elements belong to \mathcal{O} .

Given a (n, k) -rank-varying sub-Riemannian structure \mathcal{S} , the problem of finding a curve minimizing the energy between two fixed points $q_0, q_1 \in M$ is naturally formulated as the optimal control problem

$$\dot{q} = \sum_{i=1}^k u_i X_i^\mu(q), \quad u_i \in \mathbb{R}, \quad \mu \in I(q) = \{\mu \in I \mid q \in \Omega^\mu\}, \quad (6)$$

$$\int_0^T \sum_{i=1}^k u_i^2(t) dt \rightarrow \min, \quad q(0) = q_0, \quad q(T) = q_1. \quad (7)$$

Here μ, u_1, \dots, u_k are seen as controls and T is fixed. It is a standard fact that this optimal control problem is equivalent to the minimum time problem with controls u_1, \dots, u_k satisfying $u_1^2 + \dots + u_k^2 \leq 1$.

Notice that if the rank-varying sub-Riemannian structure is trivializable, then the role of μ is empty and (6), (7) can be rewritten as a classical sub-Riemannian control problem

$$\dot{q} = \sum_{i=1}^k u_i X_i(q), \quad u_i \in \mathbb{R}, \quad \int_0^T \sum_{i=1}^k u_i^2(t) dt \rightarrow \min, \quad q(0) = q_0, \quad q(T) = q_1.$$

3 Two-dimensional almost-Riemannian structures

Henceforth the paper is focused on the special case of $(2, 2)$ -rank-varying sub-Riemannian structures, i.e., two-dimensional almost-Riemannian structures (2-ARSs for short). In this case a local orthonormal frame on Ω^μ is a pair of Lie bracket generating vector fields (X^μ, Y^μ) .

Given a 2-ARS \mathcal{S} , we call *singular locus* the set $\mathcal{Z} \subset M$ of points q at which the dimension of $\Delta(q)$ is equal to one. Denote by g the restriction of the quadratic form \mathbf{G} on $M \setminus \mathcal{Z}$. By construction g is a Riemannian metric satisfying

$$g(X^\mu(q), X^\mu(q)) = 1, \quad g(X^\mu(q), Y^\mu(q)) = 0, \quad g(Y^\mu(q), Y^\mu(q)) = 1,$$

for every μ in I and every $q \in \Omega^\mu \setminus \mathcal{Z}$. Denote moreover by dA the Riemannian density associated with $(M \setminus \mathcal{Z}, g)$, which coincides with $|dX^\mu \wedge dY^\mu|$ on $\Omega^\mu \setminus \mathcal{Z}$, for every $\mu \in I$.

Finally, one can define on $M \setminus \mathcal{Z}$ the Gaussian curvature K associated with g , which is easily expressed in each open set $\Omega^\mu \setminus \mathcal{Z}$ through the formula (see for instance [2], equation (24.6))

$$K = -(\alpha^\mu)^2 - (\beta^\mu)^2 + X^\mu \beta^\mu - Y^\mu \alpha^\mu,$$

where $\alpha^\mu, \beta^\mu : \Omega^\mu \setminus \mathcal{Z} \rightarrow \mathbb{R}$ are (uniquely) defined by

$$[X^\mu, Y^\mu] = \alpha^\mu X^\mu + \beta^\mu Y^\mu,$$

and $X^\mu \beta^\mu$ (respectively, $Y^\mu \alpha^\mu$) denotes the Lie derivative of β^μ with respect to X^μ (respectively, of α^μ with respect to Y^μ).

3.1 Minimizers, cut and conjugate loci

A natural tool to look for geodesics in almost-Riemannian geometry is to apply the necessary condition for optimality given by the Pontryagin Maximum Principle (see [13]). As a result we obtain the following proposition. In view of later applications in the paper, we consider as initial and final conditions not only points, but submanifolds as well.

Proposition 9 *Define on T^*M the Hamiltonian*

$$H(\lambda, q) = \frac{1}{2}(\langle \lambda, X^\mu(q) \rangle^2 + \langle \lambda, Y^\mu(q) \rangle^2), \quad q \in \Omega^\mu, \quad \lambda \in T_q^*M.$$

(Notice that H is well defined on the whole T^*M , thanks to (5).) Consider the minimization problem

$$\dot{q} \in \Delta(q), \quad \int_0^T \mathbf{G}_{q(t)}(\dot{q}(t), \dot{q}(t)) dt \rightarrow \min, \quad q(0) = M_{\text{in}}, \quad q(T) = M_{\text{fin}}, \quad (8)$$

where M_{in} and M_{fin} are two submanifolds of M and the final time $T > 0$ is fixed. Then every solution of (8) is the projection on M of a trajectory $(\lambda(t), q(t))$ of the Hamiltonian system associated with H satisfying $\lambda(0) \perp T_{q(0)}M_{\text{in}}$, $\lambda(T) \perp T_{q(T)}M_{\text{fin}}$, and $H(\lambda(t), q(t)) \neq 0$.

Remark 10 The simple form of the statement above follows from the absence of abnormal minimizers, which follows from the Lie bracket generating assumption. As a consequence a curve is a geodesic if and only if it the projection of a normal extremal.

Remark 11 Notice that H is constant along any given solution of the Hamiltonian system. Moreover, $H = 1/2$ if and only if $q(\cdot)$ is parameterized by arclength.

Fix $q \in M$. For every $\lambda \in T_q^*M$ satisfying

$$H(\lambda, q) = 1/2 \quad (9)$$

and every $t > 0$ define $E(\lambda, t)$ as the projection on M of the solution, evaluated at time t , of the Hamiltonian system associated with H , with initial condition $\lambda(0) = \lambda$ and $q(0) = q$. Notice that if $q \notin \mathcal{Z}$ then condition (9) defines an ellipse in T_q^*M ; otherwise it identifies a pair of parallel straight lines.

Definition 12 *The conjugate locus from q is the set of critical values of the map $E(\lambda, t)$. For every $\bar{\lambda}$ such that (9) holds, let $t(\bar{\lambda})$ be the first positive time, if it exists, for which the map $(\lambda, t) \mapsto E(\lambda, t)$ is singular at $(\bar{\lambda}, t(\bar{\lambda}))$. The first conjugate locus from q is the set $\{E(\bar{\lambda}, t(\bar{\lambda})) \mid t(\bar{\lambda}) \text{ exists}\}$. The cut locus from q is the set of points reached optimally by more than one geodesic, i.e., the set*

$$\{q' \in M \mid \exists \lambda_1, \lambda_2, t \text{ such that } q' = E(\lambda_1, t) = E(\lambda_2, t), \lambda_1 \neq \lambda_2, \text{ and } E(\lambda_1, \cdot), E(\lambda_2, \cdot) \text{ are optimal in } [0, t]\}.$$

Remark 13 It is a standard fact that for every $\bar{\lambda}$ satisfying (9), the set $T(\bar{\lambda}) = \{\bar{t} > 0 \mid \text{the map } (\lambda, t) \mapsto E(\lambda, t) \text{ is singular at } (\bar{\lambda}, \bar{t})\}$ is a discrete set (see for instance [2]).

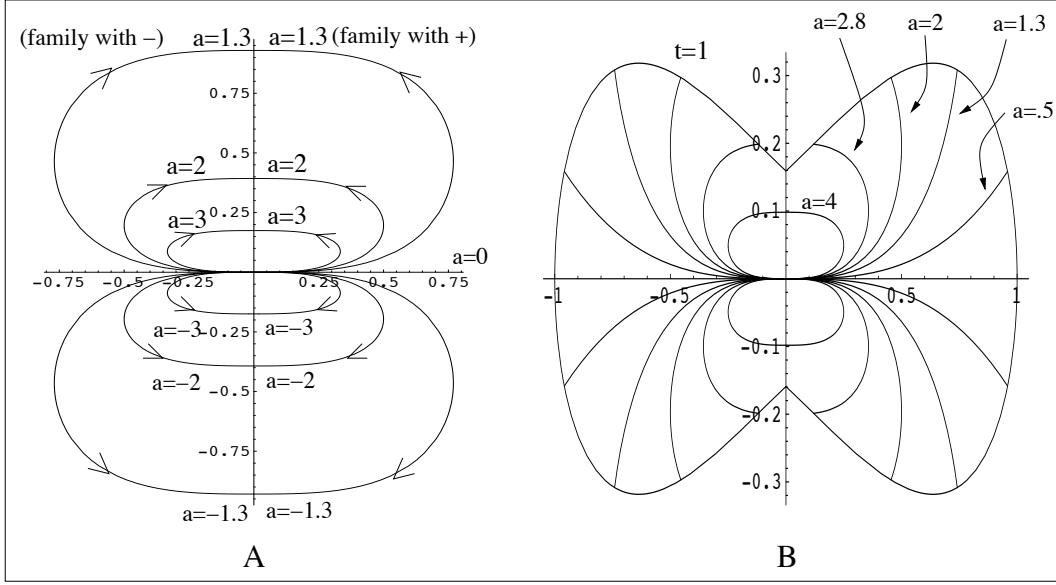


Figure 1: Geodesics and minimum time front (for $t = 1$) for the Grushin metric

3.2 An example: the Grushin almost-Riemannian structure

Consider again the Grushin distribution $X(x, y) = (1, 0)$, $Y(x, y) = (0, x)$ on the plane \mathbb{R}^2 . If we consider X and Y as an orthonormal frame, we get an almost-Riemannian structure.

As already remarked, the singular locus coincides with the y -axis. Therefore, every trajectory crossing the y -axis does it horizontally. The Riemannian metric g associated with the Grushin metric on $\mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ explodes when one is approaching the y -axis,

$$g = dx^2 + \frac{1}{x^2} dy^2.$$

Also the curvature and the Riemannian density explode while approaching the y -axis,

$$K = -\frac{2}{x^2}, \quad dA = \frac{1}{|x|} dx dy.$$

According to Proposition 9, geodesics are the projection on the (x, y) -plane of the solutions of the Hamiltonian system corresponding to

$$H = \frac{1}{2}(\lambda_x^2 + \lambda_y^2 x^2).$$

Fixing the initial condition $x(0) = 0$, $y(0) = 0$, the normalization $H = 1/2$ implies that $\lambda_x(0) = \pm 1$. Taking $a = \lambda_y(0) \in \mathbb{R}$, the geodesics starting from the origin are

$$\begin{aligned} \text{for } a = 0 & \quad \begin{cases} x_0(t) = \pm t \\ y_0(t) = 0, \end{cases} \\ \text{for } a \neq 0 & \quad \begin{cases} x_a(t) = \pm \frac{1}{a} \sin(at) \\ y_a(t) = \frac{1}{2a} t - \frac{1}{4a^2} \sin(2at). \end{cases} \end{aligned}$$

Due to the symmetries of the problem, one can easily check that the time at which a geodesic $(x_a(t), y_a(t))$ loses optimality is $\bar{t} = \pi/|a|$, for $a \neq 0$, and that $(x_a(\bar{t}), y_a(\bar{t}))$ belongs to the y -axis. The geodesics corresponding to $a = 0$ are optimal for every positive time. As a consequence the cut locus from the origin is the set $\{(0, \alpha) \mid \alpha \in \mathbb{R} \setminus \{0\}\}$.

In Figure 1A geodesics for some values of a are portrayed, while Figure 1B illustrates the set of points reached in time $t = 1$. Notice that this set has a non-smooth boundary. In contrast with what would happen in Riemannian geometry, this is the case for every positive time, as it happens in constant-rank sub-Riemannian geometry. However, this is a consequence of the fact that the initial condition belongs to \mathcal{Z} .

One can check that, even if the curvature is always negative where it is defined, a geodesic $(x_a(t), y_a(t))$, $a \neq 0$, has its first conjugate point at time $\tau/|a|$, where $\tau \sim 4.49$ is the first positive root of the equation $\tan(\tau) = \tau$. As a consequence the first conjugate locus is the parabola

$$y = \frac{x^2}{2} \left(\frac{1}{\cos \tau \sin \tau} - \frac{1}{\tau} \right).$$

One could ask whether the presence of conjugate points is the consequence of the particular initial point on the set \mathcal{Z} . In fact this is not the case. Consider as initial condition the point $x(0) = -1, y(0) = 0$. Define, for every $a \in [0, 1]$,

$$\begin{aligned} x^+(t, a) &= \frac{-(a \cos(at)) + \sqrt{1-a^2} \sin(at)}{a}, \\ x^-(t, a) &= \frac{-(a \cos(at)) - \sqrt{1-a^2} \sin(at)}{a}, \\ y^+(t, a) &= \frac{-4a\sqrt{1-a^2} + 2at + 4a\sqrt{1-a^2} \cos(at)^2 - \sin(2at) + 2a^2 \sin(2at)}{4a^2}, \\ y^-(t, a) &= \frac{4a\sqrt{1-a^2} + 2at - 4a\sqrt{1-a^2} \cos(at)^2 - \sin(2at) + 2a^2 \sin(2at)}{4a^2}. \end{aligned}$$

Then every geodesic from the point $(-1, 0)$ belongs to one of the four families

$$\begin{aligned} G1 : & (x^+(t, a), y^+(t, a)), \\ G2 : & (x^-(t, a), y^-(t, a)), \\ G3 : & (x^+(t, a), -y^+(t, a)), \\ G4 : & (x^-(t, a), -y^+(t, a)). \end{aligned}$$

The geodesics in G1 and G2 lie in the half plane $\{y \geq 0\}$, while those in G3 and G4 lie in $\{y \leq 0\}$.

Let us describe the cut locus from $(-1, 0)$. Consider first a geodesics $(x^+(t, a), y^+(t, a))$ belonging to the family G1. One can check that π/a is the first positive time at which $(x^+(t, a), y^+(t, a))$ intersects another geodesic, namely $(x^-(t, a), y^-(t, a))$, which belongs to G2. The situation is similar for the families G3 and G4. As a consequence the cut locus from $(-1, 0)$ is the set $\{(1, \alpha) \mid \alpha \in [\pi/2, \infty) \cup (-\infty, -\pi/2]\}$.

As above, one can also check that every geodesic (except those corresponding to $a = 0$) has a conjugate time (see Figure 2). In particular π is a conjugate time for the geodesics corresponding to $a = 1$. Notice that conjugate points appear on geodesics which have already crossed \mathcal{Z} . (Before crossing \mathcal{Z} a geodesic is Riemannian and lies in a Riemannian space with negative Gaussian curvature.)

4 Normal forms for generic 2-ARSs

The following proposition is a standard corollary of the transversality theorem. It formulates generic properties of a 2-ARS in terms of the flag of the distribution Δ (see equation (3)).

Proposition 14 *Let M be a two-dimensional smooth manifold. Generically, a 2-ARS $\mathcal{S} = \{(\Omega^\mu, X^\mu, Y^\mu)\}_{\mu \in I}$ on M satisfies the following properties: (i) \mathcal{Z} is an embedded one-dimensional smooth submanifold of M ; (ii) The points $q \in M$ at which $\Delta_2(q)$ is one-dimensional are isolated; (iii) $\Delta_3(q) = T_q M$ for every $q \in M$.*

Remark 15 Notice that properties (i), (ii), and (iii) are actually generic for every (2, 2)-rank-varying distribution, since they do not involve the metric structure.

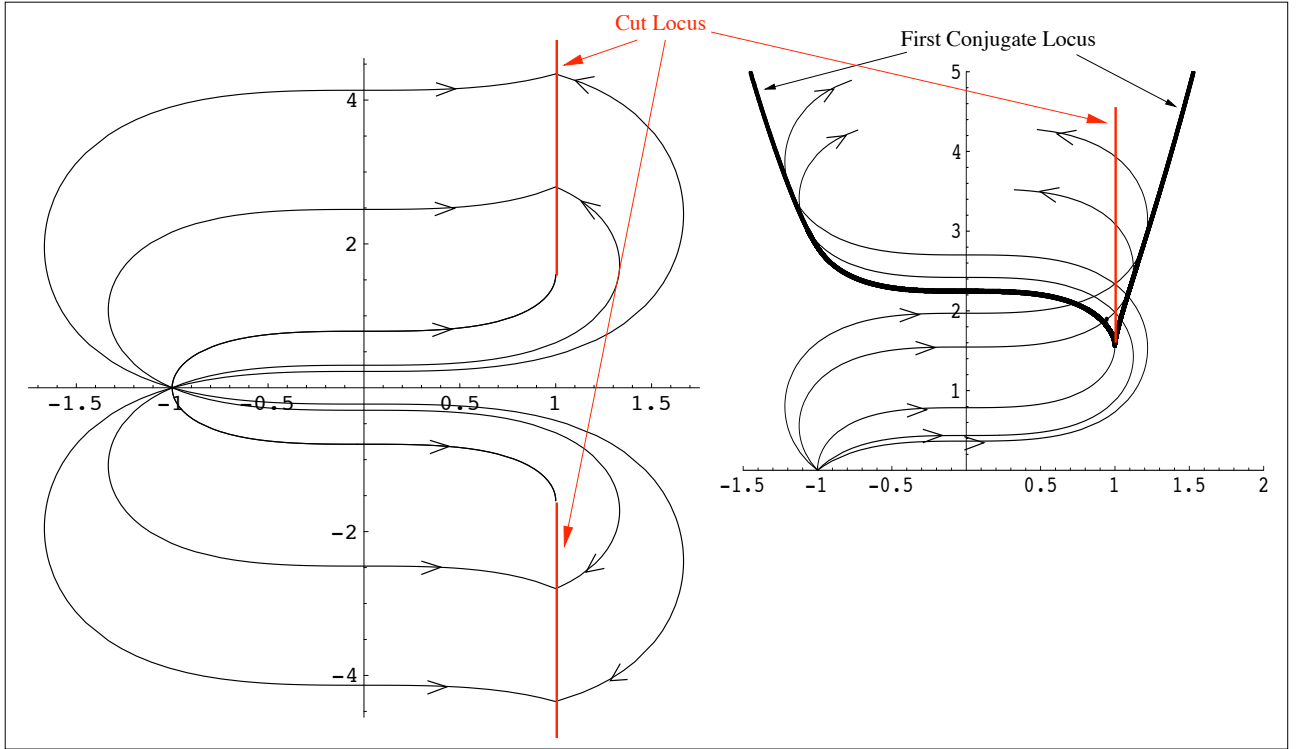


Figure 2:

As a consequence of Proposition 14, one can classify the local normal forms of a generic 2-ARS.

Theorem 16 *Generically for a 2-ARS \mathcal{S} , for every point $q \in M$ there exist a neighborhood U of q and a pair of vector fields (X, Y) on M such that (U, X, Y) is compatible with \mathcal{S} and, up to a smooth change of coordinates defined on U , $q = (0, 0)$ and (X, Y) has one of the forms*

$$\begin{aligned}
 (\text{F1}) \quad & X(x, y) = (1, 0), \quad Y(x, y) = (0, e^{\phi(x, y)}), \\
 (\text{F2}) \quad & X(x, y) = (1, 0), \quad Y(x, y) = (0, xe^{\phi(x, y)}), \\
 (\text{F3}) \quad & X(x, y) = (1, 0), \quad Y(x, y) = (0, (y - x^2\psi(x))e^{\phi(x, y)}),
 \end{aligned}$$

where ϕ and ψ are smooth real-valued functions such that $\phi(0, y) = 0$ and $\psi(0) \neq 0$.

Before proving Theorem 16 let us show the following lemma.

Lemma 17 *Let \mathcal{S} be a 2-ARS and let W be a smooth embedded one-dimensional submanifold of M . Assume that W is transversal to the distribution Δ , i.e., such that $\Delta(q) + T_qW = T_qM$ for every $q \in W$. Then, for every $q \in W$ there exists an open neighborhood U of q such that for every $\varepsilon > 0$ the set*

$$\{q' \in U \mid d(q', W) = \varepsilon\},$$

is a smooth embedded one-dimensional submanifold of U . Moreover, there exists a pair of vector fields (X, Y) such that (U, X, Y) is compatible with \mathcal{S} and, up to a smooth change of coordinates defined on U , $q = (0, 0)$ and W, X, Y have the form

$$\begin{aligned}
 W &= \{(0, h) \mid h \in \mathbb{R}\}, \\
 X(x, y) &= (1, 0), \\
 Y(x, y) &= (0, f(x, y))
 \end{aligned}$$

where $f(x, y)$ is a smooth function defined on U .

Proof. Consider a smooth regular parametrization $\alpha \mapsto w(\alpha)$ of W . Let $\alpha \mapsto \lambda_0(\alpha) \in T_{w(\alpha)}^*M$ be a smooth map satisfying $H(\lambda_0(\alpha), w(\alpha)) = 1/2$ and $\lambda_0(\alpha) \perp T_{w(\alpha)}W$.

Let $E(t, \alpha)$ be the solution at time t of the Hamiltonian system given by the Pontryagin Maximum Principle with initial condition $(q(0), \lambda(0)) = (w(\alpha), \lambda_0(\alpha))$ (see Proposition 9). Fix $q \in W$ and define $\bar{\alpha}$ by $q = w(\bar{\alpha})$. In order to prove that $E(t, \alpha)$ is a local diffeomorphism around the point $(0, \bar{\alpha})$, let us show that the two vectors

$$v_1 = \frac{\partial E}{\partial \alpha}(0, \bar{\alpha}) \quad \text{and} \quad v_2 = \frac{\partial E}{\partial t}(0, \bar{\alpha})$$

are not parallel. On one hand, since v_1 is equal to $\frac{dw}{d\alpha}(\bar{\alpha})$, then it spans T_qW . On the other hand, being H quadratic in λ ,

$$\langle \lambda_0(\bar{\alpha}), v_2 \rangle = \langle \lambda_0(\bar{\alpha}), \frac{\partial H}{\partial \lambda}(\lambda_0(\bar{\alpha}), q) \rangle = 2H(\lambda_0(\bar{\alpha}), q) = 1.$$

Thus v_2 does not belong to the orthogonal to $\lambda_0(\bar{\alpha})$, that is, to T_qW .

Therefore for a small enough neighborhood U of q we have that the set $\{q' \in U \mid d(q', W) = \varepsilon\}$ is given by the intersection of U with the images of $E(\varepsilon, \cdot)$ and $E(-\varepsilon, \cdot)$. This proves the first part of the statement. To prove the second part, let us take (t, α) as a system of coordinates on U and define the vector field X by

$$X(t, \alpha) = \frac{\partial E(t, \alpha)}{\partial t}.$$

Notice that, by construction, for every $q' \in U$ the vector $X(q')$ belongs to $\Delta(q')$ and $\mathbf{G}_{q'}(X(q'), X(q')) = 1$. In the coordinates (t, α) we have $X = (1, 0)$. Let Y be a vector field on U such that (X, Y) is compatible with \mathcal{S} . We are left to prove that the first component of Y is identically equal to zero. Indeed, were this not the case, the norm of X would not be equal to one. ■

Proof of Theorem 16. Let us start from the case in which $\Delta(q) = T_qM$. Let W be any one-dimensional submanifold passing through q . Lemma 17 provides us with a possible choice of orthonormal frame $X = (1, 0)$, $Y = (0, f(x, y))$ in a neighborhood U of q . Since, without loss of generality, X and Y are everywhere linearly independent in U , then $f(x, y) \neq 0$ for every $(x, y) \in U$. By applying a smooth coordinate transformation of the type $x \rightarrow x$, $y \rightarrow \nu(y)$ we get the new expressions $X = (1, 0)$, $Y = (0, \nu'(y)f(x, y))$. A normal form of type (F1) is obtained by choosing ν in such a way that $\nu'(y)f(0, y) = 1$.

Let now $q \in \mathcal{Z}$ and assume that $\Delta_2(q) = T_qM$. Assume, moreover, that the generic condition (i) holds true. One can easily check that $\Delta(q)$ is transversal to the submanifold \mathcal{Z} at q . Hence we can apply Lemma 17 with $W = \mathcal{Z}$. As a result we obtain a possible choice of orthonormal frame $X = (1, 0)$, $Y = (0, f(x, y))$ in a neighborhood U of q . Since X and Y are linearly dependent on \mathcal{Z} , which is identified with the y -axis, then $f(0, y) = 0$. The condition $\Delta_2(q) = T_qM$ implies that, by taking U small enough, $\partial_x f(0, y) \neq 0$. Hence f admits a representation of the type $f(x, y) = xe^{\phi(x, y)}$, with ϕ smooth. Again, a change of coordinates $x \rightarrow x$, $y \rightarrow \nu(y)$ can be used in order to ensure that $\phi(0, y) = 0$. The normal form (F2) is obtained.

Let now $q \in \mathcal{Z}$ be such that $\Delta_2(q) = \Delta(q)$. Assume that the generic conditions (i), (ii), (iii) are fulfilled. Let W be any one-dimensional submanifold passing through q and being transversal to \mathcal{Z} . Using Lemma 17 we can choose $X = (1, 0)$, $Y = (0, f(x, y))$ with f satisfying, by assumption, $\partial_x f(0, 0) = 0$, $\partial_x^2 f(0, 0) \neq 0$. Let us identify \mathcal{Z} with the graph of a smooth function $y = \Gamma(x)$. Then $f(x, y)$ can be written in the form $(y - \Gamma(x))e^{\phi(x, y)}$ with ϕ smooth. As above, without loss of generality $\phi(0, y) = 0$. The conditions on f at $(0, 0)$, moreover, justify the representation $\Gamma(x) = x^2\psi(x)$, with ψ smooth. ■

Remark 18 Because of Remark 15, for a generic distribution Δ , every 2-ARS having Δ as corresponding distribution can be locally represented by one of the normal forms (F1), (F2), (F3).

Definition 19 Let \mathcal{S} be a 2-ARS and assume that the generic conditions (i), (ii), (iii) of Proposition 14 hold true. A point $q \in M$ is said to be an ordinary point if $\Delta(q) = T_qM$, hence, if \mathcal{S} is locally described by (F1). We call q a Grushin point if $\Delta(q)$ is one-dimensional and $\Delta_2(q) = T_qM$, i.e., if the local description (F2) applies. Finally, if $\Delta(q) = \Delta_2(q)$ is of dimension one and $\Delta_3(q) = T_qM$ we say that q is a tangency point and \mathcal{S} can be described near q by the normal form (F3).

Let us take advantage of the common expressions of the normal forms (F1), (F2), (F3), which are all of the type $X(x, y) = (1, 0)$, $Y(x, y) = (0, f(x, y))$, in order to investigate the local behavior of g , K , and dA .

Lemma 20 *Let $X(x, y) = (1, 0)$ and $Y(x, y) = (0, f(x, y))$ be two smooth vector fields on \mathbb{R}^2 . Let $D = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \neq 0\}$ and g be the Riemannian metric on D having (X, Y) as an orthonormal frame. Denote by K the curvature of g and by dA the Riemannian density. We have*

$$\begin{aligned} g &= dx^2 + \frac{1}{f^2} dy^2, \\ K &= \frac{-2(\partial_x f)^2 + f \partial_x^2 f}{f^2}, \\ dA &= \frac{1}{|f|} dx dy. \end{aligned}$$

4.1 An example of tangency point

A simple example of tangency point can be observed in the 2-ARS defined by

$$X(x, y) = (1, 0), \quad Y(x, y) = (0, y - x^2).$$

The pair (X, Y) appears in the classification of planar phase portraits of pairs of vector fields given by Davydov in [7]. For this system one has $g = dx^2 + (y - x^2)^{-2} dy^2$, and

$$K = \frac{-2(3x^2 + y)}{(x^2 - y)^2}.$$

The graph of K is illustrated in Figure 3. Notice that, in contrast with the behavior of the curvature in the Grushin plane (see Section 3.2), in this case $\limsup_{q \rightarrow (0,0)} K(q) = +\infty$, while we still have $\liminf_{q \rightarrow (0,0)} K(q) = -\infty$.

5 The main result

5.1 Statement

Let M be an orientable two-dimensional manifold and let \mathcal{S} be an orientable 2-ARS on M . Chose a positive oriented atlas of orthonormal frames $\{(\Omega^\mu, X^\mu, Y^\mu)\}_{\mu \in I}$ and denote by ω the volume form on \mathcal{S} such that $\omega(X^\mu, Y^\nu) = 1$ on Ω^μ for every $\mu \in I$. As noticed in Remark 5, ω acts as a tensor on $M \setminus \mathcal{Z}$. Define a two-form dA_s on $M \setminus \mathcal{Z}$ by the rule $dA_s(V(q), W(q)) = \omega(V, W)(q)$. Notice that $dA_s = dX^\mu \wedge dY^\mu$ on $\Omega^\mu \setminus \mathcal{Z}$ for every $\mu \in I$.

Fix now an orientation Ξ of M . Recall that the choice of Ξ determines uniquely a notion of integration on $M \setminus \mathcal{Z}$ with respect to the form dA_s . More precisely, given a dA -integrable function f on $\Omega \subset M$, if for every $q \in \Omega$, Ξ and dA_s define the same orientation at q (i.e. if $\Xi(q) = \alpha dA_s(q)$ with $\alpha > 0$), then

$$\int_{\Omega} f dA_s = \int_{(\Omega, \Xi)} f dA_s = \int_{\Omega} f |dA_s| = \int_{\Omega} f dA.$$

Let

$$M^\pm = \{q \in \Omega^\mu \setminus \mathcal{Z} \mid \mu \in I, \pm \Xi(X^\mu, Y^\nu)(q) > 0\}.$$

Then $\int_{\Omega} f dA_s = \pm \int_{\Omega} f dA$ if $\Omega \subset M^\pm$.

For every $\varepsilon > 0$ let $M_\varepsilon = \{q \in M \mid d(q, \mathcal{Z}) > \varepsilon\}$, where $d(\cdot, \cdot)$ is the almost-Riemannian distance (see equation (4)). We say that K is \mathcal{S} -integrable if

$$\lim_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} K dA_s$$

exists and is finite. In this case we denote such limit by $\int K dA_s$.

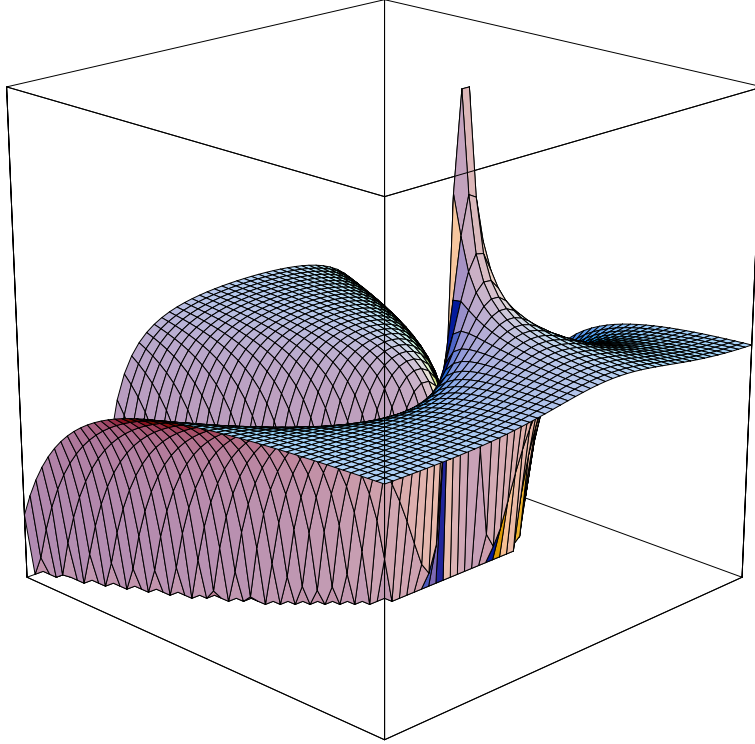


Figure 3:

Theorem 21 *Let M be a compact oriented two-dimensional manifold. For a generic oriented 2-ARS on M such that no tangency point exists, K is \mathcal{S} -integrable and*

$$\int K dA_s = 2\pi(\chi(M^+) - \chi(M^-)),$$

where χ denotes the Euler characteristic.

Theorem 21 is proved in Section 5.2. For a generic trivializable 2-ARS without tangency points one can show, thanks to topological considerations (see Section 5.4), that $\chi(M^+) = \chi(M^-)$. As a consequence, we derive the following result.

Corollary 22 *Let M be a compact oriented two-dimensional manifold. For a generic trivializable 2-ARS on M without tangency points we have*

$$\int K dA_s = 0.$$

Remark 23 In the results stated above, the hypothesis that there are not tangency points seems to be essential. Technically, the difficulty comes when one tries to integrate the Hamiltonian system given by the Pontryagin Maximum Principle applied to a system written in the normal form (F3). However it is our hope to extend the Gauss-Bonnet formula even in presence of tangency points, using a more general approach.

It is anyway interesting to notice that the hypotheses of Corollary 22 are never empty, independently of M . Indeed:

Lemma 24 *Every compact orientable two-dimensional manifold admits a trivializable 2-ARS satisfying the generic conditions of Proposition 14 and having no tangency points.*

The proof of Lemma 24 is given in Section 5.5.

5.2 Proof of Theorem 21

As a consequence of the compactness of M and of Lemma 17 one easily gets:

Lemma 25 *Let M be compact and oriented. For a generic 2-ARS \mathcal{S} on M the set \mathcal{Z} is the union of finitely many curves diffeomorphic to S^1 . Moreover, there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, the set $M \setminus M_\varepsilon$ is homeomorphic to $\mathcal{Z} \times [0, 1]$. Under the additional assumption that M contains no tangency point, ε_0 can be taken in such a way that ∂M_ε is smooth for every $0 < \varepsilon < \varepsilon_0$.*

Fix M and a 2-ARS \mathcal{S} as in the statement of Theorem 21. Thus, \mathcal{S} can be described, around each point of \mathcal{Z} , by a normal form of type (F2).

Take ε_0 as in the statement of Lemma 25. For every $\varepsilon \in (0, \varepsilon_0)$, let $M_\varepsilon^\pm = M^\pm \cap M_\varepsilon$. By definition of dA_s and M^\pm ,

$$\int_{M_\varepsilon} K dA_s = \int_{M_\varepsilon^+} K dA - \int_{M_\varepsilon^-} K dA.$$

The Gauss-Bonnet formula asserts that for every compact oriented Riemannian manifold (N, g) with smooth boundary ∂N , we have

$$\int_N K dA + \int_{\partial N} k_g ds = 2\pi\chi(N),$$

where K is the curvature of (N, g) , dA is the Riemannian density, k_g is the geodesic curvature of ∂N (whose orientation is induced by the one of N), and ds is the length element.

Applying the Gauss-Bonnet formula to the Riemannian manifolds (M_ε^+, g) and (M_ε^-, g) (whose boundary smoothness is guaranteed by Lemma 25), we have

$$\int_{M_\varepsilon} K dA_s = 2\pi(\chi(M_\varepsilon^+) - \chi(M_\varepsilon^-)) - \int_{\partial M_\varepsilon^+} k_g ds + \int_{\partial M_\varepsilon^-} k_g ds.$$

Thanks again to Lemma 25, $\chi(M_\varepsilon^\pm) = \chi(M^\pm)$. We are left to prove that

$$\lim_{\varepsilon \rightarrow 0} \left(\int_{\partial M_\varepsilon^+} k_g ds - \int_{\partial M_\varepsilon^-} k_g ds \right) = 0. \quad (10)$$

Fix $q \in \mathcal{Z}$ and a (F2)-type local system of coordinates (x, y) in a neighborhood U_q of q . We can assume that U_q is given, in the coordinates (x, y) , by a rectangle $[-a, a] \times [-b, b]$, $a, b > 0$. Assume that $\varepsilon < a$. Notice that $\mathcal{Z} \cap U_q = \{0\} \times [-b, b]$ and $\partial M_\varepsilon \cap U_q = \{-\varepsilon, \varepsilon\} \times [-b, b]$.

We are going to prove that

$$\int_{\partial M_\varepsilon \cap U_q} k_g ds = O(\varepsilon). \quad (11)$$

Then (10) follows from the compactness of \mathcal{Z} . (Indeed, $\{-\varepsilon\} \times [-b, b]$ and $\{\varepsilon\} \times [-b, b]$, the horizontal edges of ∂U_q , are geodesics minimizing the length from \mathcal{Z} . Therefore, \mathcal{Z} can be covered by a finite number of neighborhoods of type U_q whose pairwise intersections have empty interior.)

Without loss of generality, we can assume that $M^+ \cap U_q = (0, a] \times [-b, b]$. Therefore, M_ε^+ induces on $\partial M_\varepsilon^+ = \{\varepsilon\} \times [-b, b]$ a downwards orientation (see Figure 4). The curve $s \mapsto c(s) = (\varepsilon, y(s))$ satisfying

$$\dot{c}(s) = -Y(c(s)), \quad c(0) = (\varepsilon, 0),$$

is an oriented parametrization by arclength of ∂M_ε^+ , making a constant angle with X . Let (θ_1, θ_2) be the dual basis to (X, Y) on $U_q \cap M^+$, i.e., $\theta_1 = dx$ and $\theta_2 = x^{-1}e^{-\phi(x, y)}dy$. According to [14, Corollary 3, p. 389, Vol. III], the geodesic curvature of ∂M_ε^+ at $c(s)$ is equal to $\lambda(\dot{c}(s))$, where $\lambda \in \Lambda^1(U_q)$ is the unique one-form satisfying

$$d\theta_1 = \lambda \wedge \theta_2, \quad d\theta_2 = -\lambda \wedge \theta_1.$$

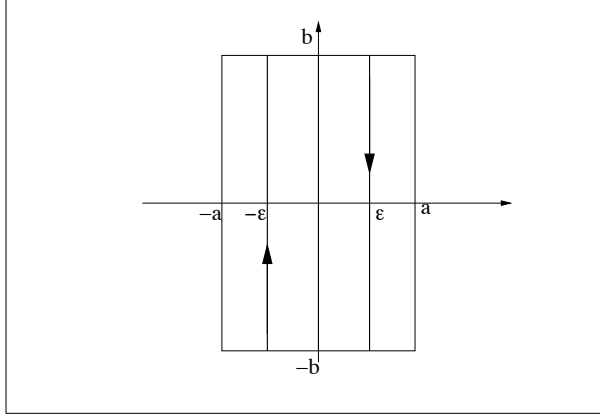


Figure 4:

A trivial computation shows that

$$\lambda = \partial_x(x^{-1}e^{-\phi(x,y)})dy.$$

Thus,

$$k_g(c(s)) = -\partial_x(x^{-1}e^{-\phi(c(s))})(dy(Y))(c(s)) = \frac{1}{\varepsilon} + \partial_x\phi(\varepsilon, y(s)).$$

Denote by L_1 and L_2 the lengths of, respectively, $\{\varepsilon\} \times [0, b]$ and $\{\varepsilon\} \times [-b, 0]$. Then,

$$\begin{aligned} \int_{\partial M_\varepsilon^+ \cap U_q} k_g ds &= \int_{-L_1}^{L_2} k_g(c(s)) ds \\ &= \int_{-L_1}^{L_2} \left(\frac{1}{\varepsilon} + \partial_x\phi(\varepsilon, (s)) \right) ds \\ &= \int_{-b}^b \left(\frac{1}{\varepsilon} + \partial_x\phi(\varepsilon, y) \right) \frac{1}{\varepsilon e^{\phi(\varepsilon, y)}} dy, \end{aligned}$$

where the last equality is obtained taking $y = y(-s)$ as new variable of integration.

We reason similarly on $\partial M_\varepsilon^- \cap U_q$, on which M_ε^- induces the upwards orientation. An orthonormal frame on $M^- \cap U_q$, oriented consistently with M , is given by $(X, -Y)$, whose dual basis is $(\theta_1, -\theta_2)$. The same computations as above lead to

$$\int_{\partial M_\varepsilon^- \cap U_q} k_g ds = \int_{-b}^b \left(\frac{1}{\varepsilon} - \partial_x\phi(-\varepsilon, y) \right) \frac{1}{\varepsilon e^{\phi(-\varepsilon, y)}} dy.$$

Define

$$F(\varepsilon, y) = (1 + \varepsilon\partial_x\phi(\varepsilon, y))e^{-\phi(\varepsilon, y)}. \quad (12)$$

Then

$$\int_{\partial M_\varepsilon^+ \cap U_q} k_g ds - \int_{\partial M_\varepsilon^- \cap U_q} k_g ds = \frac{1}{\varepsilon^2} \int_{-b}^b (F(\varepsilon, y) - F(-\varepsilon, y)) dy.$$

By Taylor expansion with respect to ε we get

$$F(\varepsilon, y) - F(-\varepsilon, y) = 2\partial_\varepsilon F(0, y)\varepsilon + O(\varepsilon^3) = O(\varepsilon^3)$$

where the last equality follows from the relation $\partial_\varepsilon F(0, y) = 0$ (see equation (12)). Therefore,

$$\int_{\partial M_\varepsilon^+ \cap U_q} k_g ds - \int_{\partial M_\varepsilon^- \cap U_q} k_g ds = O(\varepsilon),$$

and (11) is proved. ■

5.3 A counterexample in the non-generic case

In this section we justify the assumption that the 2-ARS is generic, by presenting an example of 2-ARS such that the conclusion of Theorem 21 does not hold, although \mathcal{Z} is smooth and $\Delta(q)$ is transversal to \mathcal{Z} at every point q of \mathcal{Z} .

Let M be the two-dimensional torus $[-\pi, \pi] \times [-\pi, \pi]$ with the standard identifications, and consider the trivializable 2-ARS associated with the vector fields

$$X(x, y) = (1, 0), \quad Y(x, y) = (0, 1 - \cos(x)).$$

In this case \mathcal{Z} is the circle $\{0\} \times [-\pi, \pi]$ and one among M^+ and M^- is empty (say M^-). Notice that the generic condition (ii) is not verified since $[X, Y](q) = 0$ at every $q \in \mathcal{Z}$. By Lemma 20 we have

$$\begin{aligned} dA &= \frac{1}{1 - \cos(x)} dx dy, \\ K &= \frac{\cos(x) - 2}{2 \sin(\frac{x}{2})^2}, \end{aligned}$$

on $M \setminus \mathcal{Z}$. Thus, $\int K dA_s = \int_{M \setminus \mathcal{Z}} K dA = -\infty$.

5.4 Trivializable 2-ARSs

The aim of this section is to characterize topologically trivializable 2-ARSs having no tangency point.

Lemma 26 *Let M be orientable. For a generic trivializable 2-ARS on M without tangency points the Euler characteristics of M^+ and M^- are equal.*

Proof. Let us consider on M a notion of angle, induced by any fixed, globally defined, Riemannian metric g_0 . For every θ in S^1 , denote by $R_\theta : TM \rightarrow TM$ the corresponding rotation of angle θ .

Since, by hypothesis, the map $\mathcal{Z} \ni q \mapsto \Delta(q)$ is a one-dimensional distribution everywhere transversal to the smooth submanifold \mathcal{Z} , then we can define a smooth function $\phi : \mathcal{Z} \rightarrow (0, \pi)$ such that $R_{\phi(q)}(\Delta(q)) = T_q \mathcal{Z}$ for every $q \in \mathcal{Z}$. Let $\theta : M \times [0, 1] \rightarrow S^1$ be a C^∞ map such that

$$\begin{aligned} \theta|_{M \times \{0\}} &\equiv 0 \\ \theta|_{\mathcal{Z} \times \{1\}} &= \phi. \end{aligned}$$

For every $t \in [0, 1]$ define a vector field X_t on M by the relation

$$X_t(q) = R_{\theta(q,t)} X(q).$$

Then X_1 is a smooth vector field tangent to \mathcal{Z} at every point of \mathcal{Z} . Notice that $\{q \in M \mid X_1(q) = 0\} = \{q \in M \mid X(q) = 0\} \subset \mathcal{Z}$. Moreover, under the generic assumption that the zeroes of X are non-degenerate, the same is true for those of X_1 .

Consider now the manifold \hat{M}^+ obtained by gluing smoothly two copies of M^+ along \mathcal{Z} . Since X_1 is tangent to \mathcal{Z} , the vector field \hat{X}_1 , obtained as a double copy of $X_1|_{M^+}$, is well defined, continuous, and has isolated zeroes. Thus, $\chi(\hat{M}^+)$ is equal to the sum of the indices of the zeroes of \hat{X}_1 . Notice that $\hat{X}_1(q) = 0$ if and only if $q \in \mathcal{Z}$ and $X_1(q) = 0$. Moreover, the index of \hat{X}_1 at q is equal to that of X_1 , since the latter is non-degenerate. The same reasoning on M^- and $X_1|_{M^-}$ shows that the Euler characteristic of \hat{M}^- , obtained by gluing two copies of M^- along \mathcal{Z} , is again equal to the sum of the indices of the zeroes of X_1 , i.e., to $\chi(M)$. Therefore,

$$\chi(M^+) = \frac{\chi(\hat{M}^+)}{2} = \frac{\chi(M)}{2} = \frac{\chi(\hat{M}^-)}{2} = \chi(M^-). \quad \blacksquare$$

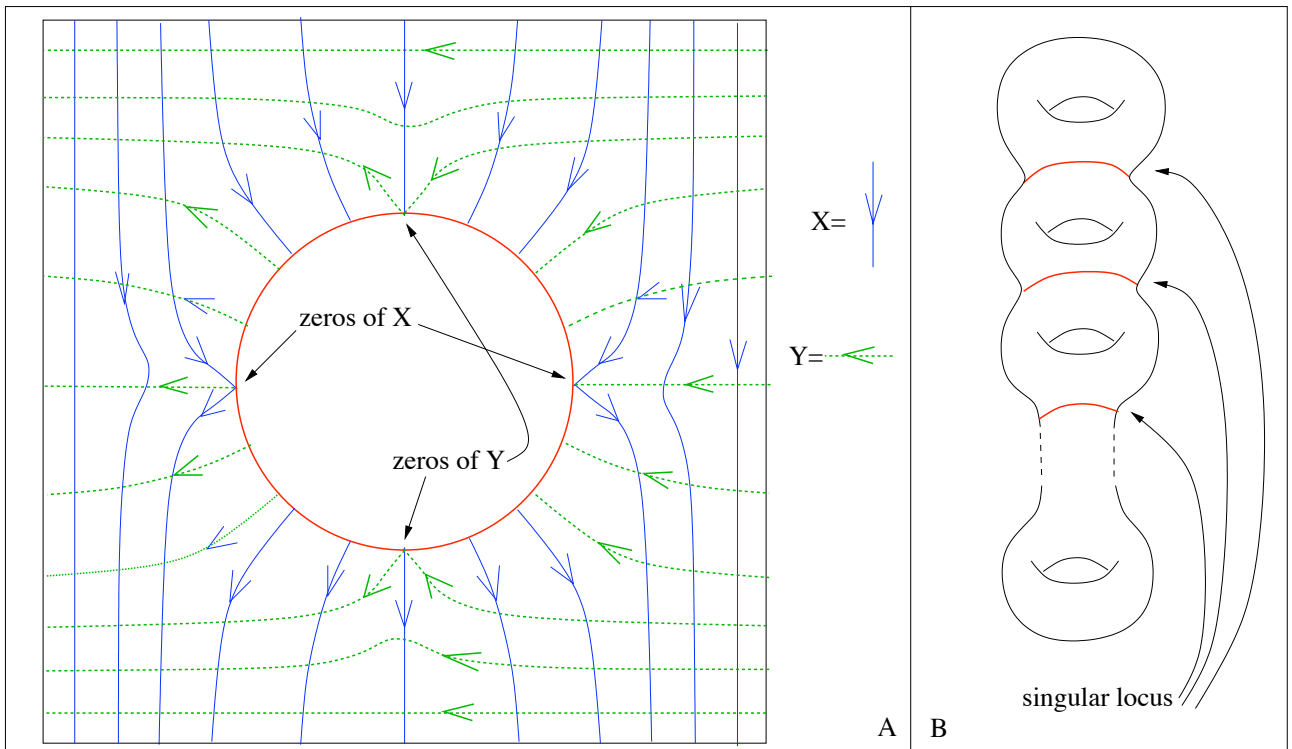


Figure 5:

5.5 Construction of trivialisable 2-ARSs with no tangency points

In this section we prove Lemma 24, by showing how to construct a trivialisable 2-ARS with no tangency points on every compact orientable two-dimensional manifold.

For the torus, an example of such structure is provided by the standard Riemannian one. The case of a connected sum of two tori can be treated by gluing together two copies of the pair of vector fields X and Y represented in Figure 5A, which are defined on a torus with a hole cut out. In the figure the torus is represented as a square with the standard identifications on the boundary. The vector fields X and Y are parallel on the boundary of the disk which has been cut out. Each vector field has exactly two zeros and the distribution spanned by X and Y is transversal to the singular locus. Examples on the connected sum of three or more tori can be constructed similarly by induction. The resulting singular locus is represented in Figure 5B.

We are left to check the existence of a trivialisable 2-ARS with no tangency points on a sphere. A simple example can be found in the literature and arises from a model of control of quantum systems (see [5, 6]). Let M be a sphere in \mathbb{R}^3 centered at the origin and take $X(x, y, z) = (y, -x, 0)$, $Y(x, y, z) = (0, z, -y)$ as orthonormal frame. Then X (respectively, Y) is an infinitesimal rotation around the third (respectively, first) axis. The singular locus is therefore given by the intersection of the sphere with the plane $\{y = 0\}$ and none of its points is tangency (see Figure 6). Notice that the generic conditions given in Proposition 14 are satisfied.

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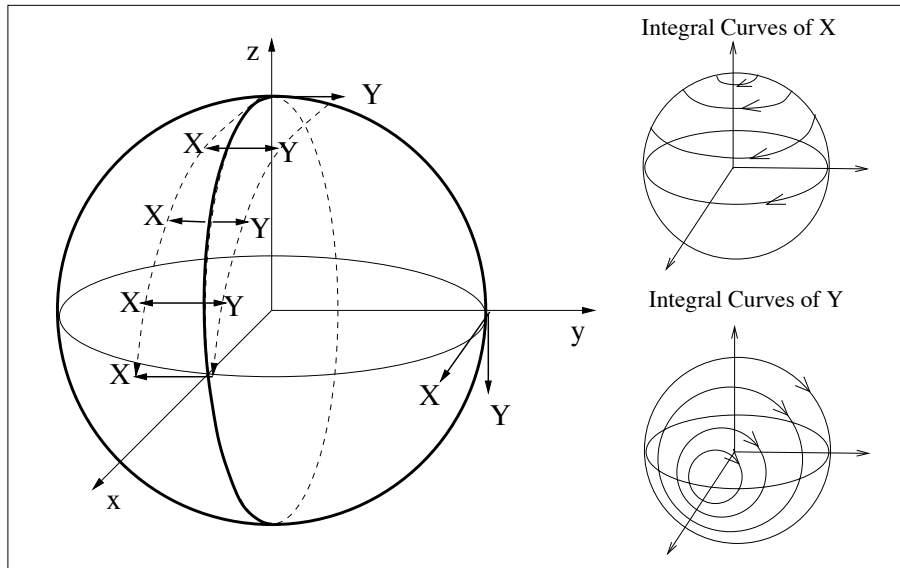


Figure 6:

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